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# Uniform asymptotic properties of a nonparametric regression estimator of conditional tails

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**Abstract.** We consider a nonparametric regression estimator of conditional tails introduced by Goegebeur, Y., Guillou, A., Schorgen, G. (2012). *Nonparametric regression estimation of conditional tails - the random covariate case*. It is shown that this estimator is uniformly strongly consistent on compact sets and its rate of convergence is given.

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**Keywords:** Tail-index, kernel estimation, strong uniform consistency.

## 1 Introduction

Extreme value analysis has attracted considerable attention in many fields of application, such as hydrology, biology and finance, for instance. The main result of extreme value theory asserts that the asymptotic distribution of the – properly rescaled – maximum of a sequence  $(Y_1, \dots, Y_n)$  of independent copies of a random variable  $Y$  with distribution function  $F$  is a distribution having the form

$$G_\gamma(x) = \exp(-(1 + \gamma x)_+^{-1/\gamma}) \quad \text{where } y_+ = \max(0, y)$$

for some  $\gamma \in \mathbb{R}$ , with  $G_0(x) = \exp(-e^{-x})$ . The distribution function  $F$  is then said to belong to the maximum domain of attraction of  $G_\gamma$  and the parameter  $\gamma$  is called the extreme value index. Most applications of extreme value theory stem from the case  $\gamma > 0$ , where  $F$  is a heavy-tailed distribution *i.e.* the associated survival function  $\bar{F} := 1 - F$  satisfies  $\bar{F}(x) = x^{-1/\gamma}L(x)$ , where  $\gamma$  shall now be referred to as the tail-index and  $L$  is a slowly varying function at infinity: namely,  $L$  satisfies, for all

$\lambda > 0$ ,  $L(\lambda x)/L(x) \rightarrow 1$  as  $x$  goes to infinity. In this case, the parameter  $\gamma$  clearly drives the tail behavior of  $F$ ; its estimation is in general a first step of extreme value analysis. For instance, if the idea is to estimate extreme quantiles – namely, quantiles with order  $\alpha_n > 1 - 1/n$ , where  $n$  is the sample size – then one has to extrapolate beyond the available data using an extreme value model which depends on the tail-index. For this reason, the problem of estimating  $\gamma$  has been extensively studied in the literature. Recent overviews on univariate tail-index estimation can be found in the monographs of Beirlant *et al.* [2] and de Haan and Ferreira [17].

In practice, it is often useful to link the variable of interest  $Y$  to a covariate  $X$ . In this situation, the tail-index depends on the observed value  $x$  of the covariate  $X$  and shall be referred to, in the following, as the conditional tail-index. Its estimation has been addressed in the recent extreme value literature, albeit mostly when the covariates are nonrandom. Smith [31] and Davison and Smith [10] considered a parametric regression model while Hall and Tajvidi [19] used a semi-parametric approach to estimate the conditional tail-index. Fully nonparametric methods have been considered using splines (see Chavez-Demoulin and Davison [4]), local polynomials (see Davison and Ramesh [9]), a moving window approach (see Gardes and Girard [12]), or a nearest neighbor approach (see Gardes and Girard [13]), among others. Less attention though has been paid to the random covariate case, despite its practical interest. One can recall the works of Wang and Tsai [33], based on a maximum likelihood approach in the Hall class of distribution functions (see Hall [18]), Daouia *et al.* [7] who use a fixed number of nonparametric conditional quantile estimators to estimate the conditional tail-index, later generalized in Daouia *et al.* [6] to a regression context with response distributions belonging to the general max-domain of attraction, and Goegebeur *et al.* [16] and Gardes and Stupfler [14] who both provide adaptations of Hill’s estimator (Hill [22]), the latter also studying an average of Hill-type statistics to improve the finite sample performance of the method.

In this paper, we focus on a nonparametric regression estimator of conditional tails introduced by Goegebeur *et al.* [16]. The particular structure of this estimator makes it possible to study its uniform properties. Note that uniform properties of estimators of the conditional tail-index are seldom considered in the literature. One can think of the work of Gardes and Stupfler [14], who study the uniform weak consistency of their estimator. Outside the field of conditional tail-index estimation, uniform convergence of the Parzen-Rosenblatt density estimator (Parzen [28] and Rosenblatt [29]) was first considered by Nadaraya [27]. His results were then improved by Silverman [30] and Stute [32], the latter proving a law of the iterated logarithm in this context. Analogous results on kernel regression estimators were obtained by, among others, Mack and Silverman [26], Härdle *et al.* [20] and Einmahl and Mason [11]. Uniform consistency of isotonized versions of order- $\alpha$  quantile estimators introduced in Aragon *et al.* [1] was shown in Daouia and Simar [8]. The case of estimators of the left-truncated quantiles is considered in Lemdani *et al.* [25]. Finally, the uniform strong consistency of a frontier estimator using kernel regression on high order moments was shown in Girard *et al.* [15].

The paper is organised as follows. Our main results are stated in Section 2. The estimator is shown to be uniformly strongly consistent on compact sets in a semiparametric framework. The rate of convergence is provided when a further condition on the bias is satisfied. The rate of uniform convergence is closely linked to the rate of pointwise convergence in distribution established in Goegebeur *et al.* [16]. The proofs of the main results are given in Section 3. Auxiliary results are postponed to the Appendix.

## 2 Main results

We assume that the covariate  $X$  takes its values in  $\mathbb{R}^d$  for some  $d \geq 1$ . We shall work in the following semiparametric framework:

(SP)  $X$  has a probability density function  $f$  with support  $S \subset \mathbb{R}^d$  having nonempty interior and the conditional survival function of  $Y$  given  $X = x$  is such that

$$\forall x \in S, \forall y \geq 1, \bar{F}(y|x) = y^{-1/\gamma(x)}L(y|x)$$

where  $\gamma(x) > 0$  and  $L(\cdot|x)$  is a slowly varying function at infinity.

The estimator of the conditional tail-index we shall study in this paper is defined as

$$\hat{\gamma}_n(x) := \frac{\sum_{i=1}^n K_h(x - X_i)(\log Y_i - \log \omega_{n,x})_+ \mathbb{1}_{\{Y_i > \omega_{n,x}\}}}{\sum_{i=1}^n K_h(x - X_i) \mathbb{1}_{\{Y_i > \omega_{n,x}\}}}. \quad (1)$$

Here  $K_h(u) := h^{-d}K(u/h)$  where  $K$  is a probability density function on  $\mathbb{R}^d$  and  $h := h_n$  is a positive sequence tending to 0 while for all  $x$ ,  $(\omega_{n,x})$  is a positive sequence tending to infinity. Note that  $\hat{\gamma}_n(x) = T_n^{(1,1)}(x)/T_n^{(1,0)}(x)$  where, for all  $s \geq 1$  and  $t \geq 0$ ,

$$T_n^{(s,t)}(x) := \frac{1}{n} \sum_{i=1}^n K_h^s(x - X_i)(\log Y_i - \log \omega_{n,x})_+^t \mathbb{1}_{\{Y_i > \omega_{n,x}\}}.$$

The estimator (1) is an element of the family of estimators introduced in Goegebeur *et al.* [16], which can be seen as an adaptation of the classical Hill estimator of the tail-index for univariate distributions (see Hill [22]). Note that the threshold  $\omega_{n,x}$  is local, *i.e.* it depends on the point  $x$  where the estimation is to be made, while the bandwidth  $h$  is global.

We first wish to state the uniform strong consistency of our estimator on an arbitrary compact subset  $\Omega$  of  $\mathbb{R}^d$  contained in the interior of  $S$ . To this end, we first assume that for every  $x \in S$  the slowly varying function  $L(\cdot|x)$  appearing in  $\bar{F}(\cdot|x)$  is normalised (see Bingham *et al.* [3]):

(A<sub>1</sub>) For all  $x \in S$  and  $y \geq 1$ ,

$$L(y|x) = c_L(x) \exp\left(\int_1^y \frac{\alpha(v|x)}{v} dv\right)$$

where  $c_L(x) > 0$  and  $\alpha(\cdot|x)$  is a function converging to 0 at infinity.

Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^d$ . The following classical regularity assumptions, needed to show the uniform consistency of our estimator, are introduced:

(A<sub>2</sub>) On  $S$ , the functions  $f$  and  $\gamma$  are positive Hölder continuous functions,  $\log c_L$  is a Hölder continuous function and  $\alpha(y|\cdot)$  is a Hölder continuous function uniformly in  $y \geq 1$ : for all  $x, x' \in S$ ,

$$\begin{aligned} |f(x) - f(x')| &\leq M_f \|x - x'\|^{\eta_f}, \\ |\gamma(x) - \gamma(x')| &\leq M_\gamma \|x - x'\|^{\eta_\gamma}, \\ |\log c_L(x) - \log c_L(x')| &\leq M_{c_L} \|x - x'\|^{\eta_{c_L}}, \\ \sup_{y \geq 1} |\alpha(y|x) - \alpha(y|x')| &\leq M_\alpha \|x - x'\|^{\eta_\alpha}. \end{aligned}$$

Conditions (SP) and (A<sub>1</sub>) imply that, for all  $x, x' \in S$  and all  $y, y' \geq 1$ ,

$$\log \frac{\bar{F}(y|x)}{\bar{F}(y|x')} = \left[ \frac{1}{\gamma(x')} - \frac{1}{\gamma(x)} \right] \log y + [\log c_L(x) - \log c_L(x')] + \int_1^y \frac{\alpha(v|x) - \alpha(v|x')}{v} dv$$

and

$$\log \frac{\bar{F}(y|x')}{\bar{F}(y'|x')} = \frac{1}{\gamma(x')} [\log y' - \log y] + \int_{y'}^y \frac{\alpha(v|x')}{v} dv.$$

Thus if (A<sub>2</sub>) holds then, if we introduce  $\eta := \eta_\gamma \wedge \eta_{c_L} \wedge \eta_\alpha$  and  $\bar{\alpha}(y|x) := \sup_{t \geq y} |\alpha(t|x)|$ , there exists a positive constant  $M_{\bar{F}}$  such that the function  $(x, y) \mapsto \log \bar{F}(y|x)$  has the following property: for all  $x, x' \in S$  such that  $\|x - x'\| \leq 1$  and  $y, y' \geq e$ ,

$$\left| \log \frac{\bar{F}(y|x)}{\bar{F}(y'|x')} \right| \leq M_{\bar{F}} \|x - x'\|^\eta \log y + \left( \frac{1}{\gamma(x')} + \bar{\alpha}(y \wedge y'|x') \right) |\log y - \log y'|. \quad (2)$$

Before stating our first result, let us highlight that under (A<sub>2</sub>) and since  $\Omega$  is compact,  $\bar{f} := \sup_{\Omega} f < \infty$  and  $\underline{f} := \inf_{\Omega} f > 0$ . Besides, if  $\varepsilon := \varepsilon_n$  is a positive sequence converging to 0 then, applying Lemma 1, it holds that for  $n$  large enough the ball  $B(x, \varepsilon)$  with center  $x$  and radius  $\varepsilon$  in  $\mathbb{R}^d$  is contained in  $S$  for every  $x \in \Omega$ . As a consequence, the uniform relative oscillation of  $f$  over the ball  $B(x, h)$  can be controlled as

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \left| \frac{f(z)}{f(x)} - 1 \right| = O(h^{\eta_f}) \rightarrow 0. \quad (3)$$

Second,  $\bar{\gamma} := \sup_{\Omega} \gamma < \infty$  and  $\underline{\gamma} := \inf_{\Omega} \gamma > 0$  and we thus have

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \left| \frac{\gamma(z)}{\gamma(x)} - 1 \right| = O(h^{\eta_\gamma}) \rightarrow 0. \quad (4)$$

Third, we can write for all  $x, x' \in \Omega$  and  $t \geq 1$

$$\alpha(t|x) \leq \alpha(t|x') + |\alpha(t|x) - \alpha(t|x')|$$

and the roles of  $x$  and  $x'$  are symmetric in the above inequality, so that taking the supremum over  $t \geq y$  on both sides yields

$$\forall y \geq 1, |\bar{\alpha}(y|x) - \bar{\alpha}(y|x')| \leq M_\alpha \|x - x'\|^{\eta_\alpha}. \quad (5)$$

We finally introduce the oscillation of  $x \mapsto \log \omega_{n,x}$  at a point  $x \in \mathbb{R}^d$  over the ball  $B(x, \varepsilon)$ :

$$\forall \varepsilon > 0, \Delta(\log \omega_{n,x})(\varepsilon) := \sup_{z \in B(x, \varepsilon)} |\log \omega_{n,x} - \log \omega_{n,z}|.$$

Our results are established under the following classical regularity condition on the kernel:

(K)  $K$  is a probability density function which is Hölder continuous with Hölder exponent  $\eta_K > 0$ :  
for all  $x, x' \in \mathbb{R}^d$ ,

$$|K(x) - K(x')| \leq M_K \|x - x'\|^{\eta_K}$$

and its support is included in the unit ball  $B$  of  $\mathbb{R}^d$ .

Note that (K) implies that  $K$  is bounded with compact support. Especially, for every  $s \geq 1$  the  $L^s$ -norm  $\|K\|_s$  of  $K$  is finite.

Let  $v_n(x) = \sqrt{\frac{nh^d}{\log n} \bar{F}(\omega_{n,x} | x)}$  and introduce the hypothesis

(C) For some  $b \geq 1/d + 1/2\eta_K$ , it holds that  $\limsup_{n \rightarrow \infty} \sup_{x \in \Omega} v_n(x) \Delta(\log \omega_{n,x})(n^{-b}) < \infty$ .

Our uniform strong consistency result may now be stated:

**Theorem 1.** *Assume that (SP), (K), (A<sub>1</sub>) and (A<sub>2</sub>) hold and that*

- $\inf_{x \in \Omega} v_n(x) \rightarrow \infty$ ;
- $\inf_{x \in \Omega} \omega_{n,x} \rightarrow \infty$ ;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \rightarrow 0$ ;
- $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \rightarrow 0$ ;
- $\sup_{x \in \Omega} \bar{\alpha}(y | x) \rightarrow 0$  as  $y \rightarrow \infty$ .

*Assume moreover that condition (C) is satisfied. Then it holds that*

$$\sup_{x \in \Omega} |\hat{\gamma}_n(x) - \gamma(x)| \rightarrow 0 \quad \text{almost surely as } n \rightarrow \infty.$$

Note that the hypotheses  $\inf_{x \in \Omega} \omega_{n,x} \rightarrow \infty$  and  $\sup_{x \in \Omega} \bar{\alpha}(y | x) \rightarrow 0$  as  $y \rightarrow \infty$  imply the convergence

$$\sup_{x \in \Omega} \bar{\alpha}(\omega_{n,x} | x) \rightarrow 0$$

which shall frequently be used in the proofs of our results. Besides, using the mean value theorem, it holds that  $|e^u - 1| \leq 2|u|$  for  $u \in \mathbb{R}$  such that  $|u|$  is sufficiently small. As a consequence, using the condition  $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \rightarrow 0$ , this inequality implies that for  $n$  large enough

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \left| \frac{\omega_{n,x}}{\omega_{n,z}} - 1 \right| \leq 2\Delta(\log \omega_{n,x})(h) \rightarrow 0. \quad (6)$$

Finally, the conditions

$$\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \rightarrow 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \sup_{x \in \Omega} v_n(x) \Delta(\log \omega_{n,x})(n^{-b}) < \infty$$

(for some  $b \geq 1/d + 1/2\eta_K$ ) are satisfied if for instance  $\omega_{n,x} = n^{g(x)}$  where  $g : S \rightarrow \mathbb{R}$  is a positive Hölder continuous function whose Hölder exponent is not less than  $\eta$ . In other words, Theorem 1 requires that a continuity property on  $x \mapsto \log \omega_{n,x}$  be satisfied.

Our second aim is to compute the rate of uniform strong consistency of the estimator (1):

**Theorem 2.** *Assume that (SP), (K), (A<sub>1</sub>) and (A<sub>2</sub>) hold and that*

- $\inf_{x \in \Omega} v_n(x) \rightarrow \infty$ ;
- $\inf_{x \in \Omega} \omega_{n,x} \rightarrow \infty$ ;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \rightarrow 0$ ;
- $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \rightarrow 0$ ;
- $\sup_{x \in \Omega} \bar{\alpha}(y | x) \rightarrow 0$  as  $y \rightarrow \infty$ .

*If moreover*

$$\limsup_{n \rightarrow \infty} \left( \sup_{x \in \Omega} v_n(x) \{ \bar{\alpha}(\omega_{n,x} | x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x} \vee \Delta(\log \omega_{n,x})(h) \} \right) < \infty \quad (7)$$

*then it holds that*

$$\sup_{x \in \Omega} v_n(x) |\hat{\gamma}_n(x) - \gamma(x)| = O(1) \quad \text{almost surely as } n \rightarrow \infty.$$

Let us highlight that condition (7) controls the bias of the estimator  $\hat{\gamma}_n$ . The terms  $h^{\eta_f}$  and  $h^\eta$  correspond to the bias which stems from the use of a kernel regression, while the presence of the other terms is due to the particular structure of the semiparametric model (SP). Besides, as pointed out in Goegebeur *et al.* [16], the rate of pointwise convergence of  $\hat{\gamma}_n(x)$  to  $\gamma(x)$  is  $\sqrt{nh^d \bar{F}(\omega_{n,x} | x)}$ . Up to the term  $\sqrt{\log n}$ , the rate of uniform convergence of  $\hat{\gamma}_n$  to  $\gamma$  is therefore the infimum (over  $\Omega$ ) of the rate of pointwise convergence of  $\hat{\gamma}_n(x)$  to  $\gamma(x)$ . Finally, note that, if  $f$ ,  $\gamma$ ,  $\log c_L$  and  $\alpha(y | \cdot)$  are all assumed to be Lipschitz functions – namely,  $\eta_f = \eta_\gamma = \eta_{c_L} = \eta_\alpha = 1$  – condition (7) is in fact

$$\limsup_{n \rightarrow \infty} \left( \sup_{x \in \Omega} v_n(x) \{ \bar{\alpha}(\omega_{n,x} | x) \vee h \log \omega_{n,x} \vee \Delta(\log \omega_{n,x})(h) \} \right) < \infty.$$

### 3 Proofs of the main results

The key idea to show Theorem 1 is to prove uniform laws of large numbers for  $T_n^{(1,0)}(x)$  and  $T_n^{(1,1)}(x)$ .

**Proposition 1.** *Assume that (SP), (K), (A<sub>1</sub>) and (A<sub>2</sub>) hold and that*

- $\inf_{x \in \Omega} v_n(x) \rightarrow \infty$  ;
- $\inf_{x \in \Omega} \omega_{n,x} \rightarrow \infty$  ;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \rightarrow 0$  ;

- $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \rightarrow 0$  ;
- $\sup_{x \in \Omega} \bar{\alpha}(y|x) \rightarrow 0$  as  $y \rightarrow \infty$ .

Assume moreover that condition (C) holds. Then for every  $t \in \{0, 1\}$  and for every sequence of positive numbers  $(\delta_n)$  converging to 0 such that  $\delta_n \inf_{x \in \Omega} v_n(x) \rightarrow \infty$ ,

$$\delta_n \sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - 1 \right| \rightarrow 0 \text{ almost surely as } n \rightarrow \infty.$$

In particular,

$$\sup_{x \in \Omega} \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - 1 \right| \rightarrow 0 \text{ almost surely as } n \rightarrow \infty.$$

**Proof of Proposition 1.** The proof is based on that of Lemma 1 in Härdle and Marron [21]: we shall in fact show complete convergence in the sense of Hsu and Robbins [24]. Since  $\Omega$  is a compact subset of  $\mathbb{R}^d$ , we may, for every  $n \in \mathbb{N} \setminus \{0\}$ , find a finite subset  $\Omega_n$  of  $\Omega$  such that:

$$\forall x \in \Omega, \exists \chi(x) \in \Omega_n, \|x - \chi(x)\| \leq n^{-b} \quad \text{and} \quad \exists c > 0, |\Omega_n| = O(n^c),$$

where  $b \geq 1/d + 1/2\eta_K$  is given by condition (C) and  $|\Omega_n|$  stands for the cardinality of  $\Omega_n$ . Notice that, since  $nh^d \rightarrow \infty$ , one has

$$\frac{n^{-b}}{h} = n^{-b+1/d} \left[ \frac{1}{nh^d} \right]^{1/d} \rightarrow 0$$

so that one can assume that eventually  $\chi(x) \in B(x, h)$  for all  $x \in \Omega$ . Besides, since  $h \rightarrow 0$ , we can pick  $n$  so large that  $h \leq 1$  and, using Lemma 1, such that  $B(x, h) \subset S$  for all  $x \in \Omega$ . Remark that

$$\left| \frac{v_n(x)}{v_n(\chi(x))} - 1 \right| = \left| \sqrt{\frac{\bar{F}(\omega_{n,x}|x)}{\bar{F}(\omega_{n,\chi(x)}|\chi(x))}} - 1 \right|; \quad (8)$$

as  $\|x - \chi(x)\| \leq n^{-b} \leq h \leq 1$ , and noting that since  $n^{-b} \leq h$  the convergences

$$n^{-b\eta} \sup_{x \in \Omega} \log \omega_{n,x} \leq h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \rightarrow 0 \quad \text{and} \quad \sup_{x \in \Omega} \Delta(\log \omega_{n,x})(n^{-b}) \leq \sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \rightarrow 0$$

hold, Lemma 2 and (8) entail

$$\sup_{x \in \Omega} \left| \frac{v_n(x)}{v_n(\chi(x))} - 1 \right| \rightarrow 0. \quad (9)$$

Using together (9) and the triangular inequality shows that for  $n$  large enough

$$\sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - 1 \right| \leq \sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - \frac{T_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(\chi(x))} \right| + 2 \sup_{x \in \Omega} v_n(\chi(x)) \left| \frac{T_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(\chi(x))} - 1 \right|.$$

Picking  $\varepsilon > 0$ , the triangular inequality then yields

$$\mathbb{P} \left( \delta_n \sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - 1 \right| > \varepsilon \right) \leq R_{1,n} + R_{2,n}$$



where

$$R_{1,n} := \sum_{z \in \Omega_n} \mathbb{P} \left( \delta_n v_n(z) \left| \frac{T_n^{(1,t)}(z)}{\mu_n^{(1,t)}(z)} - 1 \right| > \frac{\varepsilon}{4} \right)$$

and

$$R_{2,n} := \mathbb{P} \left( \delta_n \sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - \frac{T_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(\chi(x))} \right| > \frac{\varepsilon}{2} \right).$$

The goal of the proof is now to show that the series  $\sum_n R_{1,n}$  and  $\sum_n R_{2,n}$  converge. The first convergence in Proposition 1 shall then be an easy consequence of Borel-Cantelli's lemma. The second convergence is a consequence of the straightforward inequalities

$$\sup_{x \in \Omega} \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - 1 \right| \leq \left\{ \delta_n \inf_{x \in \Omega} v_n(x) \right\} \sup_{x \in \Omega} \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - 1 \right| \leq \delta_n \sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - 1 \right|$$

which hold true for  $n$  large enough.

We start by controlling  $R_{1,n}$ . To this end, apply Lemma 4 to get that there exists a positive constant  $\kappa$  such that for  $n$  large enough,

$$\forall z \in \Omega_n, \mathbb{P} \left( \delta_n v_n(z) \left| \frac{T_n^{(1,t)}(z)}{\mu_n^{(1,t)}(z)} - 1 \right| > \frac{\varepsilon}{4} \right) \leq 2 \exp \left[ -\frac{\kappa}{16} \varepsilon^2 \frac{nh^d \bar{F}(\omega_{n,z} | z)}{\delta_n^2 v_n^2(z)} \right].$$

Use now the definition of  $v_n(z)$  to get

$$R_{1,n} = O \left( n^c \exp \left[ -\frac{\kappa}{16} \varepsilon^2 \frac{\log n}{\delta_n^2} \right] \right).$$

Hence  $\sum_n R_{1,n}$  converges.

We now turn to  $R_{2,n}$ . Using the triangular inequality gives

$$\left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - \frac{T_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(\chi(x))} \right| \leq S_{1,n}(x) + S_{2,n}(x)$$

where

$$S_{1,n}(x) := \frac{1}{n} \sum_{i=1}^n \left| \frac{K_h(x - X_i)}{\mu_n^{(1,t)}(x)} - \frac{K_h(\chi(x) - X_i)}{\mu_n^{(1,t)}(\chi(x))} \right| (\log Y_i - \log \omega_{n,\chi(x)})_+^t \mathbb{1}_{\{Y_i > \omega_{n,\chi(x)}\}},$$

$$S_{2,n}(x) := \frac{1}{n} \sum_{i=1}^n \frac{K_h(x - X_i)}{\mu_n^{(1,t)}(x)} \left| (\log Y_i - \log \omega_{n,x})_+^t \mathbb{1}_{\{Y_i > \omega_{n,x}\}} - (\log Y_i - \log \omega_{n,\chi(x)})_+^t \mathbb{1}_{\{Y_i > \omega_{n,\chi(x)}\}} \right|.$$

As a consequence

$$R_{2,n} \leq \mathbb{P} \left( \delta_n \sup_{x \in \Omega} v_n(x) S_{1,n}(x) > \frac{\varepsilon}{4} \right) + \mathbb{P} \left( \delta_n \sup_{x \in \Omega} v_n(x) S_{2,n}(x) > \frac{\varepsilon}{4} \right) =: R_{3,n} + R_{4,n}$$

and it is enough to show that the series  $\sum_n R_{3,n}$  and  $\sum_n R_{4,n}$  converge.

To deal with  $\sum_n R_{3,n}$  use once again the triangular inequality to obtain

$$\begin{aligned} \mu_n^{(1,t)}(\chi(x)) \left| \frac{K_h(x - X_i)}{\mu_n^{(1,t)}(x)} - \frac{K_h(\chi(x) - X_i)}{\mu_n^{(1,t)}(\chi(x))} \right| &\leq |K_h(x - X_i) - K_h(\chi(x) - X_i)| \\ &+ \left| \frac{\mu_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(x)} - 1 \right| K_h(x - X_i). \end{aligned}$$

Using hypothesis (K) and Lemma 5, there exists a positive constant  $M$  such that for  $n$  large enough:

$$\forall x \in \Omega, \mu_n^{(1,t)}(\chi(x)) \left| \frac{K_h(x - X_i)}{\mu_n^{(1,t)}(x)} - \frac{K_h(\chi(x) - X_i)}{\mu_n^{(1,t)}(\chi(x))} \right| \leq \frac{M}{h^d} \left\{ \left[ \frac{n^{-b}}{h} \right]^{\eta_K} \vee \Delta(\log \omega_{n,x})(n^{-b}) \right\}.$$

Besides

$$\tilde{\mathbf{m}}_n^{(1,t)}(z) := \frac{1}{n} \sum_{i=1}^n \mathcal{K}_{2h}(z - X_i) (\log Y_i - \log \omega_{n,z})_+^t \mathbb{1}_{\{Y_i > \omega_{n,z}\}}$$

is the empirical analogue of  $\mathbf{m}_n^{(1,t)}(z)$  defined before Lemma 5; since the support of the random variable  $K_h(x - X_i)$  is included in  $B(\chi(x), 2h)$ , one has for  $n$  large enough

$$\forall x \in \Omega, v_n(x) S_{1,n}(x) \leq 2^d \mathcal{V} M v_n(x) \left\{ \left[ \frac{n^{-b}}{h} \right]^{\eta_K} \vee \Delta(\log \omega_{n,x})(n^{-b}) \right\} \frac{\tilde{\mathbf{m}}_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(\chi(x))}.$$

Moreover, since  $\tilde{\mathbf{m}}_n^{(1,t)}(z)$  is a kernel estimator of  $m_n^{(1,t)}(z, z)$  for which the conditions of Lemma 3 are satisfied, we get for  $n$  large enough:

$$\forall z \in \Omega_n, \delta_n v_n(z) \frac{\tilde{\mathbf{m}}_n^{(1,t)}(z)}{\mu_n^{(1,t)}(z)} \leq 2 \delta_n v_n(z) \left[ 1 + \left| \frac{\tilde{\mathbf{m}}_n^{(1,t)}(z)}{\mu_n^{(1,t)}(z)} - 1 \right| \right].$$

The fact that  $b \geq 1/d + 1/2\eta_K$  gives

$$\sup_{z \in \Omega_n} v_n(z) \left[ \frac{n^{-b}}{h} \right]^{\eta_K} \leq \sqrt{n} \left[ \frac{n^{-b}}{h} \right]^{\eta_K} \leq \left[ \frac{1}{nh^d} \right]^{\eta_K/d} \rightarrow 0.$$

Using this convergence together with hypothesis (C) and (9) entails for  $n$  large enough:

$$R_{3,n} \leq \sum_{z \in \Omega_n} \mathbb{P} \left( \delta_n v_n(z) \left| \frac{\tilde{\mathbf{m}}_n^{(1,t)}(z)}{\mu_n^{(1,t)}(z)} - 1 \right| > \varepsilon \right).$$

Finally, apply Lemma 4 to get

$$\sum_{z \in \Omega_n} \mathbb{P} \left( \delta_n v_n(z) \left| \frac{\tilde{\mathbf{m}}_n^{(1,t)}(z)}{\mu_n^{(1,t)}(z)} - 1 \right| > \varepsilon \right) = \mathcal{O} \left( n^c \exp \left[ -\kappa' \varepsilon^2 \frac{\log n}{\delta_n^2} \right] \right)$$

where  $\kappa'$  is a positive constant. Hence  $\sum_n R_{3,n}$  converges.

To control  $\sum_n R_{4,n}$  first note that

$$\frac{\mathbf{m}_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(x)} = \frac{\mathbf{m}_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(\chi(x))} \frac{\mu_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(x)}$$

and use Lemmas 3(iv) and 5 to get, for  $n$  large enough

$$\sup_{x \in \Omega} \frac{\mathbf{m}_n^{(1,t)}(\chi(x))}{\mu_n^{(1,t)}(x)} \leq 2.$$

Therefore, since the support of the random variable  $K_h(x - X_i)$  is included in  $B(\chi(x), 2h)$ , one has for  $n$  large enough and all  $x \in \Omega$

$$S_{2,n}(x) \leq 2^{d+1} \mathcal{V} \|K\|_\infty S_{3,n}(x)$$

where  $\|K\|_\infty := \sup_B K$  and

$$S_{3,n}(x) := \frac{1}{n} \sum_{i=1}^n \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathbf{m}_n^{(1,t)}(\chi(x))} \left| (\log Y_i - \log \omega_{n,x})_+^t \mathbb{1}_{\{Y_i > \omega_{n,x}\}} - (\log Y_i - \log \omega_{n,\chi(x)})_+^t \mathbb{1}_{\{Y_i > \omega_{n,\chi(x)}\}} \right|.$$

We then get

$$R_{4,n} \leq \mathbb{P} \left( \delta_n \sup_{x \in \Omega} v_n(x) S_{3,n}(x) > \frac{\varepsilon}{2^{d+3} \mathcal{V} \|K\|_\infty} \right) =: R_{5,n}$$

and it is enough to control  $\sum_n R_{5,n}$ . We start by considering the case  $t = 0$ . In this case,  $S_{3,n}(x)$  reduces to

$$S_{3,n}(x) = \frac{1}{n} \sum_{i=1}^n \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathbf{m}_n^{(1,0)}(\chi(x))} \mathbb{1}_{\{\omega_{n,x} \wedge \omega_{n,\chi(x)} < Y_i \leq \omega_{n,x} \vee \omega_{n,\chi(x)}\}}.$$

Letting  $\rho_{n,x} := 2\Delta(\log \omega_{n,x})(n^{-b})$  and using (6), we have  $\sup_{x \in \Omega} \rho_{n,x} \rightarrow 0$  and for  $n$  large enough

$$\forall x \in \Omega, (1 - \rho_{n,\chi(x)})\omega_{n,\chi(x)} \leq \omega_{n,x} \leq (1 + \rho_{n,\chi(x)})\omega_{n,\chi(x)}.$$

As a consequence, for  $n$  large enough it holds that

$$\forall x \in \Omega, S_{3,n}(x) \leq \frac{1}{n} \sum_{i=1}^n \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathbf{m}_n^{(1,0)}(\chi(x))} \mathbb{1}_{\{(1 - \rho_{n,\chi(x)})\omega_{n,\chi(x)} < Y_i < (1 + \rho_{n,\chi(x)})\omega_{n,\chi(x)}\}}.$$

Similarly to Lemma 6, let

$$\begin{aligned} M_n^{(1,0)}(x) &:= \mathbb{E}(\mathcal{K}_{2h}(x - X) \mathbb{1}_{\{(1 - \rho_{n,x})\omega_{n,x} < Y < (1 + \rho_{n,x})\omega_{n,x}\}}) \\ \text{and } U_n^{(1,0)}(x) &:= \frac{1}{n} \sum_{i=1}^n \mathcal{K}_{2h}(x - X_i) \mathbb{1}_{\{(1 - \rho_{n,x})\omega_{n,x} < Y_i < (1 + \rho_{n,x})\omega_{n,x}\}}. \end{aligned}$$

Write

$$\forall x \in \Omega, \delta_n v_n(x) S_{3,n}(x) \leq \delta_n v_n(x) \frac{M_n^{(1,0)}(\chi(x))}{\mathbf{m}_n^{(1,0)}(\chi(x))} \left[ 1 + \left| \frac{U_n^{(1,0)}(\chi(x))}{M_n^{(1,0)}(\chi(x))} - 1 \right| \right].$$

Use together Lemmas 3(iv) and 6 along with (9) to get for  $n$  large enough

$$\forall x \in \Omega, \delta_n v_n(x) S_{3,n}(x) \leq \frac{4}{\gamma(\chi(x))} \delta_n v_n(\chi(x)) \rho_{n,\chi(x)} \left[ 1 + \left| \frac{U_n^{(1,0)}(\chi(x))}{M_n^{(1,0)}(\chi(x))} - 1 \right| \right].$$

Recall that  $\rho_{n,x} = 2\Delta(\log \omega_{n,x})(n^{-b})$  and that condition (C) is satisfied to obtain

$$\delta_n \sup_{z \in \Omega_n} v_n(z) \rho_{n,z} \rightarrow 0.$$

Therefore, since  $0 < \underline{\gamma} \leq \gamma(\chi(x))$ , the triangular inequality implies that

$$R_{5,n} \leq \sum_{z \in \Omega_n} \mathbb{P} \left( \delta_n v_n(z) \rho_{n,z} \left| \frac{U_n^{(1,0)}(z)}{M_n^{(1,0)}(z)} - 1 \right| > \frac{\varepsilon \underline{\gamma}}{2^{d+6} \mathcal{V} \|K\|_\infty} \right)$$

for  $n$  large enough. Lemma 6 now makes it clear that

$$R_{5,n} = O \left( n^c \sup_{z \in \Omega_n} \exp \left( -\kappa'' \frac{\varepsilon \underline{\gamma}}{2^{d+6} \mathcal{V} \|K\|_\infty} v_n(z) \frac{\log n}{\delta_n} \right) \right) = o \left( n^c \exp \left( -\kappa'' \varepsilon \frac{\log n}{\delta_n} \right) \right)$$

which proves that  $\sum_n R_{5,n}$  converges in this case.

If now  $t = 1$ , we recall (49) in the proof of Lemma 5 to get for  $n$  large enough and for all  $x \in \Omega$

$$S_{3,n}(x) = \left| \log \frac{\omega_{n,x}}{\omega_{n,\chi(x)}} \right| \frac{\mathbf{m}_n^{(1,0)}(\chi(x))}{\mathbf{m}_n^{(1,1)}(\chi(x))} \frac{1}{n} \sum_{i=1}^n \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathbf{m}_n^{(1,0)}(\chi(x))} \mathbb{1}_{\{Y_i > \omega_{n,x} \wedge \omega_{n,\chi(x)}\}}.$$

Use (6) and Lemma 3(iv) to get for  $n$  large enough

$$\begin{aligned} \forall x \in \Omega, S_{3,n}(x) &\leq \frac{2}{\gamma} \Delta(\log \omega_{n,x})(n^{-b}) \frac{1}{n} \sum_{i=1}^n \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathbf{m}_n^{(1,0)}(\chi(x))} \mathbb{1}_{\{Y_i > \omega_{n,\chi(x)}/2\}} \\ &\leq \frac{2}{\gamma} \Delta(\log \omega_{n,x})(n^{-b}) \frac{\nu_n^{(1,0)}(\chi(x))}{\mathbf{m}_n^{(1,0)}(\chi(x))} \left[ 1 + \left| \frac{V_n^{(1,0)}(\chi(x))}{\nu_n^{(1,0)}(\chi(x))} - 1 \right| \right] \end{aligned} \quad (10)$$

where

$$\nu_n^{(1,0)}(x) := \mathbb{E}(\mathcal{K}_{2h}(x - X) \mathbb{1}_{\{Y > \omega_{n,x}/2\}}) \quad \text{and} \quad V_n^{(1,0)}(x) := \frac{1}{n} \sum_{i=1}^n \mathcal{K}_{2h}(x - X_i) \mathbb{1}_{\{Y_i > \omega_{n,x}/2\}}.$$

The family of sequences  $(\omega_{n,x}/2)$  clearly satisfies the hypotheses of Lemmas 3 and 4: in particular

$$\sup_{x \in \Omega} \left| \frac{\nu_n^{(1,0)}(x)}{\mathbf{m}_n^{(1,0)}(x)} \frac{\bar{F}(\omega_{n,x} | x)}{\bar{F}(\omega_{n,x}/2 | x)} - 1 \right| \rightarrow 0 \quad (11)$$

and there exists a positive constant  $\kappa'''$  such that for  $n$  large enough

$$\forall x \in \Omega, \mathbb{P} \left( \left| \frac{V_n^{(1,0)}(x)}{\nu_n^{(1,0)}(x)} - 1 \right| > \varepsilon \right) \leq 2 \exp(-\kappa''' \varepsilon^2 n h^d \bar{F}(\omega_{n,x} | x)) \quad (12)$$

where the inequality  $\bar{F}(\omega_{n,x}/2 | x) \geq \bar{F}(\omega_{n,x} | x)$  was used. We conclude by noting that according to (2)

$$\left| \log \frac{\bar{F}(\omega_{n,x} | x)}{\bar{F}(\omega_{n,x}/2 | x)} \right| \leq \left( \frac{1}{\gamma(x)} + \bar{\alpha}(\omega_{n,x}/2 | x) \right) \log 2 \rightarrow \frac{\log 2}{\gamma(x)}$$

uniformly in  $x \in \Omega$ , so that

$$\limsup_{n \rightarrow \infty} \sup_{x \in \Omega} \left| \log \frac{\bar{F}(\omega_{n,x} | x)}{\bar{F}(\omega_{n,x}/2 | x)} \right| < \infty \Rightarrow 0 < \limsup_{n \rightarrow \infty} \sup_{x \in \Omega} \frac{\bar{F}(\omega_{n,x} | x)}{\bar{F}(\omega_{n,x}/2 | x)} < \infty.$$

From this we obtain that

$$0 < \limsup_{n \rightarrow \infty} \sup_{x \in \Omega} \frac{\nu_n^{(1,0)}(x)}{\mathbf{m}_n^{(1,0)}(x)} < \infty.$$

This property entails the convergences

$$\delta_n \sup_{x \in \Omega} \frac{\nu_n^{(1,0)}(x)}{\mathbf{m}_n^{(1,0)}(x)} \rightarrow 0 \quad \text{and} \quad \sup_{x \in \Omega} \Delta(\log \omega_{n,x})(n^{-b}) \frac{\nu_n^{(1,0)}(x)}{\mathbf{m}_n^{(1,0)}(x)} \rightarrow 0. \quad (13)$$

Reporting (11) along with (13) into (10), recalling condition (C) and using the triangular inequality together with (9) shows that for  $n$  large enough,

$$R_{5,n} \leq \sum_{z \in \Omega_n} \mathbb{P} \left( \delta_n \nu_n(z) \left| \frac{V_n^{(1,0)}(z)}{\nu_n^{(1,0)}(z)} - 1 \right| > \varepsilon \right).$$

Use now (12) to obtain

$$R_{5,n} = O \left( n^c \exp \left( -\kappa''' \varepsilon^2 \frac{\log n}{\delta_n^2} \right) \right)$$

so that  $\sum_n R_{5,n}$  converges in this case as well. This completes the proof of Proposition 1.  $\blacksquare$

With Proposition 1 at hand, we can now prove Theorem 1 and Theorem 2.

**Proof of Theorem 1.** Notice that

$$\widehat{\gamma}_n(x) = \frac{\mu_n^{(1,1)}(x) T_n^{(1,1)}(x) \mu_n^{(1,0)}(x)}{\mu_n^{(1,0)}(x) \mu_n^{(1,1)}(x) T_n^{(1,0)}(x)}. \quad (14)$$

Applying Proposition 1 twice yields

$$\sup_{x \in \Omega} \left| \frac{T_n^{(1,1)}(x) \mu_n^{(1,0)}(x)}{\mu_n^{(1,1)}(x) T_n^{(1,0)}(x)} - 1 \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (15)$$

Moreover, since

$$\frac{\mu_n^{(1,1)}(x)}{\mu_n^{(1,0)}(x)} = \frac{f(x)\overline{F}(\omega_{n,x} | x)}{\mu_n^{(1,0)}(x)} \left[ \frac{\mu_n^{(1,1)}(x)}{f(x)\overline{F}(\omega_{n,x} | x)} - \gamma(x) \right] + \gamma(x) \left[ \frac{f(x)\overline{F}(\omega_{n,x} | x)}{\mu_n^{(1,0)}(x)} - 1 \right] + \gamma(x)$$

and recalling that  $\gamma$  is continuous and therefore bounded on the compact set  $\Omega$ , using Lemma 3(i) and (iv) twice entails

$$\sup_{x \in \Omega} \left| \frac{\mu_n^{(1,1)}(x)}{\mu_n^{(1,0)}(x)} - \gamma(x) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (16)$$

The result follows by reporting (15) and (16) into (14). ■

**Proof of Theorem 2.** Note that because  $nh^d \rightarrow \infty$ , the hypothesis

$$\limsup_{n \rightarrow \infty} \sup_{x \in \Omega} v_n(x) \Delta(\log \omega_{n,x})(h) < \infty$$

entails condition (C). We can then apply Proposition 1 and Lemma 7 twice to get

$$\sup_{x \in \Omega} v_n(x) \left| \frac{\mu_n^{(1,0)}(x)}{T_n^{(1,0)}(x)} - 1 \right| = O(1) \quad \text{and} \quad \sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(1,1)}(x)}{\mu_n^{(1,1)}(x)} - 1 \right| = O(1) \quad (17)$$

almost surely as  $n \rightarrow \infty$ . Moreover, Lemma 3 (iv) gives

$$\begin{aligned} \sup_{x \in \Omega} \frac{1}{\overline{\alpha}(\omega_{n,x} | x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x}} \left| \frac{\mu_n^{(1,0)}(x)}{f(x)\overline{F}(\omega_{n,x} | x)} - 1 \right| &= O(1), \\ \sup_{x \in \Omega} \frac{1}{\overline{\alpha}(\omega_{n,x} | x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x}} \left| \frac{\mu_n^{(1,1)}(x)}{f(x)\overline{F}(\omega_{n,x} | x)} - \gamma(x) \right| &= O(1) \end{aligned}$$

so that, using condition (7),

$$\sup_{x \in \Omega} v_n(x) \left| \frac{\mu_n^{(1,1)}(x)}{\mu_n^{(1,0)}(x)} - \gamma(x) \right| = O(1). \quad (18)$$

The result follows by reporting (17) and (18) into (14). ■

## Appendix: Auxiliary results and proofs

The first lemma of this section is a topological result which shall be needed in several proofs.

**Lemma 1.** *Let  $S$  be the support of  $f$ . Assume that  $S$  has nonempty interior, and let  $\Omega$  be a compact set of  $\mathbb{R}^d$  contained in the interior of  $S$ . Then there exists  $\beta > 0$  such that for every  $x \in \Omega$ ,  $B(x, \beta) \subset S$ .*

**Proof of Lemma 1.** Let  $U$  denote the interior of  $S$  and  $\partial S = S \setminus U$  be the (topological) boundary of  $S$ . Note that  $\partial S$  is a closed set since it is the intersection of two closed sets in  $\mathbb{R}^d$ ; since  $\Omega$  is a compact set and  $\partial S$  is a closed set with  $\Omega \cap \partial S = \emptyset$ , it holds that

$$\exists \beta > 0, \quad d(\Omega, \partial S) := \inf_{x \in \Omega} \inf_{s \in \partial S} \|x - s\| = 2\beta > 0. \quad (19)$$

We shall now prove the result. Pick  $x \in \Omega$ . If one could find  $y \in B(x, \beta) \cap S^c$  – where  $S^c$  is the complement of the set  $S$  – then the real number

$$t_0 = \inf\{t \in [0, 1] \mid z_t := (1-t)x + ty \notin S\}$$

would belong to  $(0, 1)$  since  $x \in U$  and  $y \in S^c$  which are both open sets. Therefore, because for every  $t \in (0, t_0)$ ,  $z_t \in S$  and there exists a nonincreasing sequence  $(t_k)$  converging to  $t_0$  such that  $(z_{t_k}) \subset S^c \subset U^c$  which is a closed set, one has

$$z_{t_0} = \lim_{t \uparrow t_0} z_t \in S \quad \text{and} \quad z_{t_0} = \lim_{k \rightarrow \infty} z_{t_k} \in U^c.$$

Hence  $z_{t_0} \in S \cap U^c = \partial S$ , but  $\|x - z_{t_0}\| = t_0\|x - y\| < \beta$ , which contradicts (19): Lemma 1 is proven. ■

The second lemma of this section is a technical result that gives an upper bound for the oscillation of the log-conditional survival function.

**Lemma 2.** *Assume that (SP),  $(A_1)$  and  $(A_2)$  hold. Let moreover  $\varepsilon := \varepsilon_n$  and  $\varepsilon' := \varepsilon'_n$  be two positive sequences tending to 0 and assume that*

- $\inf_{x \in \Omega} \omega_{n,x} \rightarrow \infty$  ;
- $\varepsilon'^\eta \sup_{x \in \Omega} \log \omega_{n,x} \rightarrow 0$  ;
- $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(\varepsilon) \rightarrow 0$  ;
- $\sup_{x \in \Omega} \bar{\alpha}(y \mid x) \rightarrow 0$  as  $y \rightarrow \infty$ .

Then it holds that, for  $n$  large enough,

$$\forall x, x' \in \Omega, \forall (z, z') \in B(x, \varepsilon) \times B(x', \varepsilon'), \quad \left| \log \frac{\bar{F}(\omega_{n,z} \mid z')}{\bar{F}(\omega_{n,x} \mid x')} \right| \leq M_{\bar{F}} \varepsilon'^\eta \log \omega_{n,z} + \frac{2}{\underline{\gamma}} \Delta(\log \omega_{n,x})(\varepsilon).$$

In particular,

$$\sup_{x, x' \in \Omega} \sup_{z \in B(x, \varepsilon)} \sup_{z' \in B(x', \varepsilon')} \frac{1}{\varepsilon'^\eta \log \omega_{n,x} \vee \Delta(\log \omega_{n,x})(\varepsilon)} \left| \frac{\bar{F}(\omega_{n,z} \mid z')}{\bar{F}(\omega_{n,x} \mid x')} - 1 \right| = O(1).$$

**Proof of Lemma 2.** Lemma 1 shows that for  $n$  large enough,  $B(x, \varepsilon)$  and  $B(x, \varepsilon')$  are contained in  $S$  for every  $x \in \Omega$ . Pick  $x, x' \in \Omega$  and  $(z, z') \in B(x, \varepsilon) \times B(x', \varepsilon')$ . Use (2) to get for  $n$  large enough

$$\left| \log \frac{\bar{F}(\omega_{n,z} | z')}{\bar{F}(\omega_{n,x} | x')} \right| \leq M_{\bar{F}} \|x' - z'\|^\eta \log \omega_{n,z} + \left( \frac{1}{\gamma(x')} + \bar{\alpha}(\omega_{n,z} \wedge \omega_{n,x} | x') \right) |\log \omega_{n,x} - \log \omega_{n,z}|.$$

Since  $(z, z') \in B(x, \varepsilon) \times B(x', \varepsilon')$ , one has

$$\|x' - z'\| \leq \varepsilon' \quad \text{and} \quad |\log \omega_{n,x} - \log \omega_{n,z}| \leq \Delta(\log \omega_{n,x})(\varepsilon).$$

Besides, using (6) with  $\varepsilon$  instead of  $h$ , we get  $\inf_{x \in \Omega} \inf_{z \in B(x, \varepsilon)} \omega_{n,z} \wedge \omega_{n,x} = \inf_{x \in \Omega} \omega_{n,x} (1 + o(1)) \rightarrow \infty$ , so that

$$\sup_{x, x' \in \Omega} \sup_{z \in B(x, \varepsilon)} \bar{\alpha}(\omega_{n,z} \wedge \omega_{n,x} | x') \rightarrow 0.$$

Epecially, since  $0 < \underline{\gamma} \leq \gamma(x')$ , we obtain for  $n$  large enough:

$$\forall x, x' \in \Omega, \forall (z, z') \in B(x, \varepsilon) \times B(x', \varepsilon'), \left| \log \frac{\bar{F}(\omega_{n,z} | z')}{\bar{F}(\omega_{n,x} | x')} \right| \leq M_{\bar{F}} \varepsilon'^\eta \log \omega_{n,z} + \frac{2}{\underline{\gamma}} \Delta(\log \omega_{n,x})(\varepsilon)$$

which is the first part of the result. To prove the second part, note that because  $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(\varepsilon) \rightarrow 0$  it holds that for  $n$  large enough

$$\forall x, x' \in \Omega, \forall (z, z') \in B(x, \varepsilon) \times B(x', \varepsilon'), \left| \log \frac{\bar{F}(\omega_{n,z} | z')}{\bar{F}(\omega_{n,x} | x')} \right| \leq 2M_{\bar{F}} \varepsilon'^\eta \log \omega_{n,x} + \frac{2}{\underline{\gamma}} \Delta(\log \omega_{n,x})(\varepsilon).$$

Consequently

$$\sup_{x, x' \in \Omega} \sup_{z \in B(x, \varepsilon)} \sup_{z' \in B(x', \varepsilon')} \frac{1}{\varepsilon'^\eta \log \omega_{n,x} \vee \Delta(\log \omega_{n,x})(\varepsilon)} \left| \log \frac{\bar{F}(\omega_{n,z} | z')}{\bar{F}(\omega_{n,x} | x')} \right| = O(1).$$

Using the equivalent  $e^u - 1 = u(1 + o(1))$  therefore completes the proof of Lemma 2.  $\blacksquare$

The third lemma examines the behavior of the conditional moment

$$m_n^{(t)}(x, z) := \mathbb{E}((\log Y - \log \omega_{n,x})_+^t \mathbb{1}_{\{Y > \omega_{n,x}\}} | X = z)$$

and that of its smoothed version  $\mu_n^{(s,t)}(x) := \mathbb{E}(K_h^s(x-X) m_n^{(t)}(x, X))$ . Let  $\Gamma$  be Euler's Gamma function:

$$\forall t > 0, \Gamma(t) := \int_0^{+\infty} v^{t-1} e^{-v} dv.$$

**Lemma 3.** Assume that (SP), (A<sub>1</sub>) and (A<sub>2</sub>) hold. Pick  $s \geq 1, t \geq 0$  and assume that  $K$  is a bounded probability density function on  $\mathbb{R}^d$  with support included in  $B$ . If moreover

- $\inf_{x \in \Omega} \omega_{n,x} \rightarrow \infty$  ;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \rightarrow 0$  ;
- $\sup_{x \in \Omega} \bar{\alpha}(y | x) \rightarrow 0$  as  $y \rightarrow \infty$

then, as  $n \rightarrow \infty$ , the following estimations hold:

$$(i) \sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{\bar{\alpha}(\omega_{n,x} | x) \vee h^{\eta_\alpha}} \left| \frac{m_n^{(t)}(x, z)}{\gamma^t(z) \Gamma(t+1) \bar{F}(\omega_{n,x} | z)} - 1 \right| = O(1).$$

$$(ii) \sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{\bar{\alpha}(\omega_{n,x} | x) \vee h^\eta \log \omega_{n,x}} \left| \frac{m_n^{(t)}(x, z)}{m_n^{(t)}(x, x)} - 1 \right| = O(1).$$

$$(iii) \sup_{x \in \Omega} \frac{1}{\bar{\alpha}(\omega_{n,x} | x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x}} \left| \frac{h^{d(s-1)} \mu_n^{(s,t)}(x)}{\|K\|_s^s f(x) m_n^{(t)}(x, x)} - 1 \right| = O(1).$$

$$(iv) \sup_{x \in \Omega} \frac{1}{\bar{\alpha}(\omega_{n,x} | x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x}} \left| \frac{h^{d(s-1)} \mu_n^{(s,t)}(x)}{\|K\|_s^s f(x) \gamma^t(x) \Gamma(t+1) \bar{F}(\omega_{n,x} | x)} - 1 \right| = O(1).$$

**Proof of Lemma 3.** Using Lemma 1, we can pick  $n$  large enough such that  $B(x, h) \subset S$  for all  $x \in \Omega$ .

(i) When  $t = 0$ , there is nothing to prove, since  $m_n^{(0)}(x, z) = \bar{F}(\omega_{n,x} | z)$  and  $\Gamma(1) = 1$ . In the case  $t > 0$ , an integration by parts yields

$$m_n^{(t)}(x, z) = \int_{\omega_{n,x}}^{+\infty} t \frac{(\log y - \log \omega_{n,x})^{t-1}}{y} \bar{F}(y | z) dy = t \bar{F}(\omega_{n,x} | z) \int_1^{+\infty} (\log r)^{t-1} \frac{\bar{F}(r\omega_{n,x} | z)}{r \bar{F}(\omega_{n,x} | z)} dr.$$

From (SP) and (A<sub>1</sub>), one has

$$\left| \frac{\bar{F}(r\omega_{n,x} | z)}{r \bar{F}(\omega_{n,x} | z)} - r^{-1/\gamma(z)-1} \right| = r^{-1/\gamma(z)-1} \left| \exp \left( \int_{\omega_{n,x}}^{r\omega_{n,x}} \frac{\alpha(v | z)}{v} dv \right) - 1 \right|. \quad (20)$$

For all  $y \in \mathbb{R}$ , the mean value theorem yields  $|e^y - 1| \leq |y|e^{|y|}$ . Meanwhile,

$$\left| \int_{\omega_{n,x}}^{r\omega_{n,x}} \frac{\alpha(v | z)}{v} dv \right| \leq \bar{\alpha}(\omega_{n,x} | z) \log r. \quad (21)$$

Since (5) gives  $\bar{\alpha}(\omega_{n,x} | z) \leq \bar{\alpha}(\omega_{n,x} | x) + M_\alpha h^{\eta_\alpha}$  for all  $x \in \Omega$  and  $z \in B(x, h)$ , it holds that

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \bar{\alpha}(\omega_{n,x} | z) \leq \sup_{x \in \Omega} \bar{\alpha}(\omega_{n,x} | x) + M_\alpha h^{\eta_\alpha} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (22)$$

Choosing  $n$  so large that  $\sup_{x \in \Omega} \sup_{z \in B(x, h)} \bar{\alpha}(\omega_{n,x} | z) < 1/2\bar{\gamma}$ , (20) and (21) together imply that, for all  $x \in \Omega$  and  $z \in B(x, h)$ ,

$$\left| \int_1^{+\infty} (\log r)^{t-1} \left[ \frac{\bar{F}(r\omega_{n,x} | z)}{r \bar{F}(\omega_{n,x} | z)} - r^{-1/\gamma(z)-1} \right] dr \right| \leq (\bar{\alpha}(\omega_{n,x} | x) + M_\alpha h^{\eta_\alpha}) \int_1^{+\infty} (\log r)^t r^{-1/2\bar{\gamma}-1} dr$$

which, since the integral on the right-hand side of this inequality converges, gives

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{\bar{\alpha}(\omega_{n,x} | x) \vee h^{\eta_\alpha}} \left| \int_1^{+\infty} (\log r)^{t-1} \left[ \frac{\bar{F}(r\omega_{n,x} | z)}{r \bar{F}(\omega_{n,x} | z)} - r^{-1/\gamma(z)-1} \right] dr \right| = O(1)$$

as  $n \rightarrow \infty$ . Rewriting with an elementary change of variables

$$\int_1^{+\infty} (\log r)^{t-1} r^{-1/\gamma(z)-1} dr = \gamma^t(z) \Gamma(t)$$

and using the well-known equality  $t\Gamma(t) = \Gamma(t+1)$ , we get

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{\bar{\alpha}(\omega_{n,x} | x) \vee h^{\eta_\alpha}} \left| \frac{m_n^{(t)}(x, z)}{\bar{F}(\omega_{n,x} | z)} - \gamma^t(z) \Gamma(t+1) \right| = O(1)$$



as  $n \rightarrow \infty$  and (i) is proven.

(ii) Since for all  $x \in \Omega$ ,  $0 < \underline{\gamma} \leq \gamma(x) \leq \bar{\gamma} < \infty$ , applying (i) entails

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{\bar{\alpha}(\omega_{n, x} | x) \vee h^{\eta_\alpha}} \left| \frac{m_n^{(t)}(x, z)}{\gamma^t(z) \Gamma(t+1) \bar{F}(\omega_{n, x} | z)} \frac{\gamma^t(x) \Gamma(t+1) \bar{F}(\omega_{n, x} | x)}{m_n^{(t)}(x, x)} - 1 \right| = O(1). \quad (23)$$

Moreover, using the mean value theorem:

$$\left| \frac{\gamma^t(x)}{\gamma^t(z)} - 1 \right| \leq \frac{1}{\underline{\gamma}^t} \sup_{\underline{\gamma} \leq r \leq \bar{\gamma}} |tr^{t-1}| |\gamma(x) - \gamma(z)|.$$

Recalling that  $0 < \underline{\gamma} \leq \bar{\gamma} < \infty$  and that  $(A_2)$  holds, this implies the estimation

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \left| \frac{\gamma^t(x)}{\gamma^t(z)} - 1 \right| = O(h^{\eta_\gamma}). \quad (24)$$

Besides, using Lemma 2 gives

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{h^\eta \log \omega_{n, x}} \left| \frac{\bar{F}(\omega_{n, x} | x)}{\bar{F}(\omega_{n, x} | z)} - 1 \right| = O(1). \quad (25)$$

Note finally that since  $\eta \leq \eta_\gamma \wedge \eta_\alpha$  and  $\inf_{x \in \Omega} \omega_{n, x} \rightarrow \infty$  one has

$$\sup_{x \in \Omega} \frac{h^{\eta_\gamma} \vee h^{\eta_\alpha}}{h^\eta \log \omega_{n, x}} \rightarrow 0.$$

Using then (24) and (25) together with (23) yields (ii).

(iii) Let us remark that for all  $x \in \Omega$ :

$$\frac{h^{d(s-1)} \mu_n^{(s, t)}(x)}{f(x) m_n^{(t)}(x, x)} = \int_B K^s(u) \frac{f(x - hu)}{f(x)} \frac{m_n^{(t)}(x, x - hu)}{m_n^{(t)}(x, x)} du.$$

From (3) and (ii) it follows that

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{\bar{\alpha}(\omega_{n, x} | x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n, x}} \left| \frac{f(z)}{f(x)} \frac{m_n^{(t)}(x, z)}{m_n^{(t)}(x, x)} - 1 \right| \rightarrow 0$$

as  $n \rightarrow \infty$ , which yields (iii).

(iv) Write for all  $x \in \Omega$ :

$$\frac{h^{d(s-1)} \mu_n^{(s, t)}(x)}{\|K\|_s^s f(x) \gamma^t(x) \Gamma(t+1) \bar{F}(\omega_{n, x} | x)} = \frac{h^{d(s-1)} \mu_n^{(s, t)}(x)}{\|K\|_s^s f(x) m_n^{(t)}(x, x)} \frac{m_n^{(t)}(x, x)}{\gamma^t(x) \Gamma(t+1) \bar{F}(\omega_{n, x} | x)}$$

and use (i) and (iii) together to prove (iv). ■

The fourth lemma is essential to prove Proposition 1. It gives a uniform exponential bound for large deviations of  $T_n^{(1,0)}$  and  $T_n^{(1,1)}$ .

**Lemma 4.** *Assume that (SP),  $(A_1)$  and  $(A_2)$  hold. Assume that  $K$  is a bounded probability density function on  $\mathbb{R}^d$  with support included in  $B$ . If moreover*

- $\inf_{x \in \Omega} \omega_{n, x} \rightarrow \infty$  ;

- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \rightarrow 0$  ;
- $\sup_{x \in \Omega} \bar{\alpha}(y | x) \rightarrow 0$  as  $y \rightarrow \infty$

then there exists a positive constant  $\kappa$  such that for all  $n$  large enough, one has for  $t \in \{0, 1\}$  and every  $\varepsilon > 0$  small enough:

$$\forall x \in \Omega, \mathbb{P} \left( \left| \frac{T_n^{(1,t)}(x)}{\mu_n^{(1,t)}(x)} - 1 \right| > \varepsilon \right) \leq 2 \exp(-\kappa \varepsilon^2 n h^d \bar{F}(\omega_{n,x} | x)).$$

**Proof of Lemma 4.** Using Lemma 1, we can pick  $n$  large enough such that  $B(x, h) \subset S$  for all  $x \in \Omega$ .

We start by considering  $T_n^{(1,0)}(x)$ . For every  $x \in \Omega$ :

$$\mathbb{P} \left( \left| \frac{T_n^{(1,0)}(x)}{\mu_n^{(1,0)}(x)} - 1 \right| > \varepsilon \right) = \mathbb{P} \left( \left| h^d T_n^{(1,0)}(x) - h^d \mu_n^{(1,0)}(x) \right| > \varepsilon h^d \mu_n^{(1,0)}(x) \right).$$

Notice now that if  $W_{n,i}(x) := h^d K_h(x - X_i) \mathbb{1}_{\{Y_i > \omega_{n,x}\}}$  then

$$h^d T_n^{(1,0)}(x) - h^d \mu_n^{(1,0)}(x) = \frac{1}{n} \sum_{i=1}^n [W_{n,i}(x) - \mathbb{E}(W_{n,i}(x))]$$

is a mean of bounded, centered, independent and identically distributed random variables. Define

$$\tau_n(x) := \frac{\varepsilon}{\|K\|_\infty} n h^d \mu_n^{(1,0)}(x) \quad \text{and} \quad \lambda_n(x) := \varepsilon \|K\|_\infty h^d \mu_n^{(1,0)}(x) \frac{1}{\text{Var}(W_{n,1}(x))}.$$

Bernstein's inequality (see Hoeffding [23]) yields, for all  $\varepsilon > 0$ :

$$\mathbb{P} \left( \left| \frac{T_n^{(1,0)}(x)}{\mu_n^{(1,0)}(x)} - 1 \right| > \varepsilon \right) \leq 2 \exp \left( -\frac{\tau_n(x) \lambda_n(x)}{2(1 + \lambda_n(x)/3)} \right).$$

Applying Lemma 3(iii) yields for  $n$  large enough:

$$\inf_{x \in \Omega} \frac{\tau_n(x)}{n h^d \bar{F}(\omega_{n,x} | x)} \geq \frac{\varepsilon f}{2 \|K\|_\infty}. \quad (26)$$

Moreover, for all  $x \in \Omega$ :

$$\frac{1}{\lambda_n(x)} = \frac{\mathbb{E}(W_{n,1}^2(x)) - [\mathbb{E}(W_{n,1}(x))]^2}{\varepsilon \|K\|_\infty h^d \mu_n^{(1,0)}(x)}.$$

Since  $W_{n,1}(x)$  is bounded by  $\|K\|_\infty$ , it follows that

$$\sup_{x \in \Omega} \frac{1}{\lambda_n(x)} \leq \sup_{x \in \Omega} \frac{\mathbb{E}(W_{n,1}^2(x)) - [\mathbb{E}(W_{n,1}(x))]^2}{\varepsilon \|K\|_\infty h^d \mu_n^{(1,0)}(x)} \leq \frac{1}{\varepsilon}. \quad (27)$$

Finally, it holds that

$$\frac{\tau_n(x) \lambda_n(x)}{2(1 + \lambda_n(x)/3)} \geq \left\{ \inf_{x \in \Omega} \frac{\tau_n(x)}{n h^d \bar{F}(\omega_{n,x} | x)} \right\} \left\{ \inf_{x \in \Omega} \frac{1}{2(1/\lambda_n(x) + 1/3)} \right\} n h^d \bar{F}(\omega_{n,x} | x).$$

Using (26), (27) and the fact that the function  $t \mapsto 1/[2(t+1/3)]$  is decreasing on  $\mathbb{R}_+$ , it is then clear that for all  $n$  large enough, if  $\varepsilon > 0$  is small enough, there exists a positive constant  $\kappa_1$  that is independent of  $\varepsilon$  such that

$$\forall x \in \Omega, \mathbb{P} \left( \left| \frac{T_n^{(1,0)}(x)}{\mu_n^{(1,0)}(x)} - 1 \right| > \varepsilon \right) \leq 2 \exp[-\kappa_1 \varepsilon^2 n h^d \bar{F}(\omega_{n,x} | x)].$$

We now turn to  $T_n^{(1,1)}(x)$ . For every  $x \in \Omega$ , it holds that

$$\mathbb{P} \left( \left| \frac{T_n^{(1,1)}(x)}{\mu_n^{(1,1)}(x)} - 1 \right| > \varepsilon \right) = \mathbb{P} \left( \frac{T_n^{(1,1)}(x)}{\mu_n^{(1,1)}(x)} - 1 > \varepsilon \right) + \mathbb{P} \left( \frac{T_n^{(1,1)}(x)}{\mu_n^{(1,1)}(x)} - 1 < -\varepsilon \right) =: u_{1,n}(x) + u_{2,n}(x).$$

We shall then give a uniform Chernoff-type exponential bound (see Chernoff [5]) for both terms on the right-hand side of the above inequality. We start by considering  $u_{1,n}(x)$ . Write for all  $q > 0$ :

$$u_{1,n}(x) = \mathbb{P} \left( \exp \left( q \frac{T_n^{(1,1)}(x)}{\mu_n^{(1,1)}(x)} \right) > \exp(q[\varepsilon + 1]) \right).$$

Letting

$$\varphi_n(s, x) := \mathbb{E}(\exp(sK_h(x - X)(\log Y - \log \omega_{n,x}) + \mathbb{1}_{\{Y > \omega_{n,x}\}}))$$

be the moment generating function of the random variable  $K_h(x - X)(\log Y - \log \omega_{n,x}) + \mathbb{1}_{\{Y > \omega_{n,x}\}}$ , Markov's inequality entails

$$u_{1,n}(x) \leq \exp \left( -q[\varepsilon + 1] + n \log \varphi_n \left( \frac{q}{n\mu_n^{(1,1)}(x)}, x \right) \right). \quad (28)$$

Our goal is now to use inequality (28) with a suitable value  $q^*(\varepsilon, x)$  for  $q$ . To this end, notice that

$$\begin{aligned} \varphi_n(s, x) &= \int_{\mathbb{R}^d} \psi_n(sK_h(x - z) | x, z) f(z) dz \\ &= \int_{\mathbb{R}^d \setminus B(x, h)} f(z) dz + \int_{B(x, h)} \psi_n(sK_h(x - z) | x, z) f(z) dz \end{aligned}$$

where

$$\psi_n(s | x, z) := \mathbb{E}(\exp(s(\log Y - \log \omega_{n,x}) + \mathbb{1}_{\{Y > \omega_{n,x}\}}) | X = z)$$

is the conditional moment generating function of the random variable  $(\log Y - \log \omega_{n,x}) + \mathbb{1}_{\{Y > \omega_{n,x}\}}$  given  $X = z$ . In particular, since  $f$  is a probability density function on  $\mathbb{R}^d$ ,

$$\varphi_n(s, x) = 1 + \int_{B(x, h)} [\psi_n(sK_h(x - z) | x, z) - 1] f(z) dz. \quad (29)$$

This equality makes it clear that it is enough to study the behavior of  $\psi_n(\cdot | x, z)$ . One has

$$\psi_n(s | x, z) = 1 - \bar{F}(\omega_{n,x} | z) + \mathbb{E} \left( \left[ \frac{Y}{\omega_{n,x}} \right]^s \mathbb{1}_{\{Y > \omega_{n,x}\}} | X = z \right). \quad (30)$$

We then work on the last term, which equals

$$\mathbb{E} \left( \left[ \frac{Y}{\omega_{n,x}} \right]^s \mathbb{1}_{\{Y > \omega_{n,x}\}} | X = z \right) = \bar{F}(\omega_{n,x} | z) + \mathbb{E} \left( \int_1^{+\infty} st^{s-1} \mathbb{1}_{\{Y > t\omega_{n,x}\}} dt | X = z \right)$$

and since the integrand is a positive measurable function, switching the expectation and the integral sign implies that

$$\mathbb{E} \left( \left[ \frac{Y}{\omega_{n,x}} \right]^s \mathbb{1}_{\{Y > \omega_{n,x}\}} | X = z \right) = \bar{F}(\omega_{n,x} | z) + \int_1^{+\infty} st^{s-1} \bar{F}(t\omega_{n,x} | z) dt. \quad (31)$$

Using (30) and (31) together yields

$$\psi_n(s | x, z) = 1 + \bar{F}(\omega_{n,x} | z) \int_1^{+\infty} st^s \frac{\bar{F}(t\omega_{n,x} | z)}{t\bar{F}(\omega_{n,x} | z)} dt.$$

A use of (20) and (21) therefore entails, for all  $s < 1/\bar{\gamma}$ ,

$$\psi_n(s | x, z) = 1 + s\bar{F}(\omega_{n,x} | z) \left( \left[ \frac{1}{\gamma(z)} - s \right]^{-1} + R_n(s | x, z) \right) \quad (32)$$

where  $R_n(s | x, z)$  satisfies, for all  $\delta > 0$ , if  $n$  is large enough,

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} |R_n(s | x, z)| \leq \sup_{x \in \Omega} \sup_{z \in B(x, h)} \bar{\alpha}(\omega_{n,x} | z) \int_1^{+\infty} v^{s-1/\bar{\gamma}-1+\delta} \log v \, dv.$$

Since by (22) it holds that  $\sup_{x \in \Omega} \sup_{z \in B(x, h)} \bar{\alpha}(\omega_{n,x} | z) \rightarrow 0$  we get, for all  $\delta > 0$ :

$$\sup_{s < 1/\bar{\gamma}-\delta} \sup_{x \in \Omega} \sup_{z \in B(x, h)} |R_n(s | x, z)| \rightarrow 0 \quad (33)$$

as  $n \rightarrow \infty$ . We shall now derive a suitable value for the parameter  $q$ . Given  $X = x$ , if the remainder term  $R_n$  were identically 0, then one would have  $m_n^{(1)}(x, x) = \gamma(x)\bar{F}(\omega_{n,x} | x)$  and thus an optimal value of  $q$  would be obtained by minimizing the function

$$q \mapsto -q[1 + \varepsilon] + n \log \left[ 1 + \frac{q}{n} \left[ 1 - \frac{q}{n\bar{F}(\omega_{n,x} | x)} \right]^{-1} \right].$$

Straightforward but cumbersome computations lead to the optimal value

$$q_{c,+}^*(\varepsilon) := n\bar{F}(\omega_{n,x} | x) \frac{[2 - \bar{F}(\omega_{n,x} | x)] - \sqrt{[2 - \bar{F}(\omega_{n,x} | x)]^2 - \frac{4\varepsilon}{\varepsilon + 1} [1 - \bar{F}(\omega_{n,x} | x)]}}{2 [1 - \bar{F}(\omega_{n,x} | x)]}. \quad (34)$$

Since we are mostly interested in what happens in the limit  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ , we may examine the behavior of  $q_{c,+}^*(\varepsilon)$  in this case. Using (34), we get the following asymptotic equivalent

$$q_{c,+}^*(\varepsilon) = n\bar{F}(\omega_{n,x} | x) \frac{\varepsilon}{2(\varepsilon + 1)}.$$

Note that since  $q_{c,+}^*(\varepsilon)/[nm_n^{(1)}(x, x)] = \varepsilon/[2\gamma(x)(\varepsilon + 1)]$  is positive and converges to 0 as  $\varepsilon \rightarrow 0$ , the moment generating function  $\psi_n(\cdot | x, x)$  at  $q_{c,+}^*(\varepsilon)/[nm_n^{(1)}(x, x)]$  is well-defined and finite for  $\varepsilon$  small enough and therefore this choice of  $q$  is valid. Back to our original context, taking into account the presence of the covariate  $X$  motivates the following value for  $q$ :

$$q_{n,+}^*(\varepsilon, x) := \frac{M\varepsilon}{\varepsilon + 1} nh^d f(x) \bar{F}(\omega_{n,x} | x)$$

where  $M$  is a positive constant to be chosen later. For  $\varepsilon$  small enough and for  $n$  so large that the quantity  $\varphi_n \left( q_{n,+}^*(\varepsilon, x) / (n\mu_n^{(1,1)}(x)), x \right)$  is well-defined and finite for all  $x \in \Omega$ , replacing  $q$  by  $q_{n,+}^*(\varepsilon, x)$  in the right-hand side of (28) gives

$$\forall x \in \Omega, u_{1,n}(x) \leq \exp \left( -M\varepsilon nh^d f(x) \bar{F}(\omega_{n,x} | x) + n \log \varphi_n \left( \frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1,1)}(x)}, x \right) \right). \quad (35)$$

Using the classical inequality  $\log(1 + r) \leq r$  for all  $r > 0$  together with (29), we obtain

$$\log \varphi_n(s, x) \leq \int_{B(x, h)} [\psi_n(sK_h(x - z) | x, z) - 1] f(z) \, dz.$$

Applying (32) entails

$$\log \varphi_n(s, x) \leq \int_{B(x, h)} s K_h(x-z) \overline{F}(\omega_{n,x} | z) \left( \left[ \frac{1}{\gamma(z)} - s K_h(x-z) \right]^{-1} + R_n(s K_h(x-z) | x, z) \right) f(z) dz.$$

According to Lemma 3(iv),

$$\frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1,1)}(x)} = \frac{M\varepsilon}{\varepsilon+1} h^d f(x) \frac{\overline{F}(\omega_{n,x} | x)}{\mu_n^{(1,1)}(x)} = M \frac{\varepsilon h^d}{\gamma(x)(\varepsilon+1)} [1 + r_{1,n}(x)] \quad (36)$$

where  $\sup_{x \in \Omega} |r_{1,n}(x)| \rightarrow 0$  as  $n$  goes to infinity. As a consequence, using an elementary Taylor expansion, we get, for all  $z \in B(x, h)$ ,

$$\left[ \frac{1}{\gamma(z)} - \frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1,1)}(x)} K_h(x-z) \right]^{-1} = \gamma(z) \left[ 1 + \frac{\gamma(z)}{\gamma(x)} \frac{M\varepsilon}{\varepsilon+1} h^d [1 + r_{1,n}(x)] K_h(x-z) + k \left( \frac{\gamma(z)}{\gamma(x)} \frac{M\varepsilon}{\varepsilon+1} h^d [1 + r_{1,n}(x)] K_h(x-z) \right) \right]$$

where  $k(r)/r \rightarrow 0$  as  $r$  goes to 0. Letting

$$p_n(x, z) := \frac{\gamma(z)}{\gamma(x)} [1 + r_{1,n}(x)] h^d K_h(x-z)$$

and using (4), the uniform convergence of  $r_{1,n}$  to 0 and the fact that  $K$  is bounded yields

$$p_n(x, z) = h^d K_h(x-z) + r_{2,n}(x, z) \quad \text{where} \quad \sup_{x \in \Omega} \sup_{z \in B(x, h)} |r_{2,n}(x, z)| \rightarrow 0$$

as  $n$  goes to infinity. Especially,

$$\left[ \frac{1}{\gamma(z)} - \frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1,1)}(x)} K_h(x-z) \right]^{-1} = \gamma(z) \left[ 1 + \frac{M\varepsilon}{\varepsilon+1} h^d K_h(x-z) + \varepsilon r_{3,n}(\varepsilon, x, z) \right] \quad (37)$$

where  $r_{3,n}(\varepsilon, x, z) \rightarrow 0$  as  $\varepsilon$  goes to 0 and  $n$  goes to infinity, uniformly in  $x \in \Omega$  and  $z \in B(x, h)$ .

Besides, since for every  $\varepsilon_0 > 0$

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in \Omega} \sup_{z \in B(x, h)} \left| \frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1,1)}(x)} K_h(x-z) - \frac{M\varepsilon}{\varepsilon+1} \frac{h^d K_h(x-z)}{\gamma(x)} \right| \rightarrow 0$$

as  $n$  goes to infinity and

$$\sup_{n \in \mathbb{N}} \sup_{x \in \Omega} \sup_{z \in B(x, h)} \left| \frac{M\varepsilon}{\varepsilon+1} \frac{h^d K_h(x-z)}{\gamma(x)} \right| \rightarrow 0$$

as  $\varepsilon$  goes to 0, (33) yields for  $\varepsilon$  small enough

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \left| R_n \left( \frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1,1)}(x)} K_h(x-z) | x, z \right) \right| \rightarrow 0 \quad (38)$$

as  $n$  goes to infinity. Using together (4), (36), (37) and (38) entails that there exist functions  $r_{4,n} = r_{4,n}(x, z)$  and  $r_{5,n} = r_{5,n}(\varepsilon, x, z)$  satisfying

$$\begin{aligned} \sup_{x \in \Omega} \sup_{z \in B(x, h)} |r_{4,n}(x, z)| &\rightarrow 0 \quad \text{as } n \rightarrow \infty \\ \text{and } \sup_{x \in \Omega} \sup_{z \in B(x, h)} |r_{5,n}(\varepsilon, x, z)| &\rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \text{and } n \rightarrow \infty \end{aligned}$$

such that

$$\begin{aligned} \log \varphi_n \left( \frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1,1)}(x)}, x \right) &\leq \int_{B(x,h)} \frac{M\varepsilon}{\varepsilon+1} h^d \left[ 1 + \frac{M\varepsilon}{\varepsilon+1} h^d K_h(x-z) \right] \bar{F}(\omega_{n,x} | z) K_h(x-z) f(z) dz \\ &\quad + \frac{M\varepsilon}{\varepsilon+1} h^d \int_{B(x,h)} \bar{F}(\omega_{n,x} | z) [r_{4,n}(x, z) + \varepsilon r_{5,n}(\varepsilon, x, z)] K_h(x-z) f(z) dz. \end{aligned}$$

Recalling (3) and (25), we get, for  $n$  large enough and  $\varepsilon$  small enough, the inequality

$$\forall x \in \Omega, \log \varphi_n \left( \frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1,1)}(x)}, x \right) \leq \frac{M\varepsilon}{\varepsilon+1} \left[ 1 + 2 \frac{M\varepsilon}{\varepsilon+1} \|K\|_2^2 \right] h^d f(x) \bar{F}(\omega_{n,x} | x).$$

Using this result together with (35) and recalling that  $0 < \underline{f} \leq f(x)$  entails, for  $n$  large enough and  $\varepsilon$  small enough,

$$\forall x \in \Omega, u_{1,n}(x) \leq \exp \left( \underline{f} \left[ -M\varepsilon + \frac{M\varepsilon}{\varepsilon+1} \left[ 1 + 2 \frac{M\varepsilon}{\varepsilon+1} \|K\|_2^2 \right] \right] nh^d \bar{F}(\omega_{n,x} | x) \right).$$

A straightforward computation shows that the optimal value for  $M$  in the above inequality is

$$M_+^* := \frac{\varepsilon+1}{4\|K\|_2^2}$$

for which

$$\forall x \in \Omega, u_{1,n}(x) \leq \exp \left( -\frac{\varepsilon^2}{8\|K\|_2^2} \underline{f} nh^d \bar{F}(\omega_{n,x} | x) \right) = \exp \left( -\kappa_2 \varepsilon^2 nh^d \bar{F}(\omega_{n,x} | x) \right)$$

where  $\kappa_2$  is a positive constant independent of  $\varepsilon$ .

Providing a uniform exponential bound for  $u_{2,n}(x)$  starts by noticing that, for all  $q > 0$ ,

$$u_{2,n}(x) \leq \exp \left( -q[\varepsilon-1] + n \log \varphi_n \left( -\frac{q}{n\mu_n^{(1,1)}(x)}, x \right) \right).$$

Considering again the conditional framework, if the remainder term  $R_n$  in (32) were identically 0, an optimal value of  $q$  would be obtained by minimizing the function

$$q \mapsto -q[\varepsilon-1] + n \log \left[ 1 - \frac{q}{n} \left[ 1 + \frac{q}{n\bar{F}(\omega_{n,x} | x)} \right]^{-1} \right].$$

Burdensome computations lead to the optimal value

$$q_{c,-}^*(\varepsilon) := n\bar{F}(\omega_{n,x} | x) \frac{-[2 - \bar{F}(\omega_{n,x} | x)] + \sqrt{[2 - \bar{F}(\omega_{n,x} | x)]^2 + \frac{4\varepsilon}{1-\varepsilon} [1 - \bar{F}(\omega_{n,x} | x)]}}{2[1 - \bar{F}(\omega_{n,x} | x)]}$$

which yields the following asymptotic equivalent as  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ :

$$q_{c,-}^*(\varepsilon) = n\bar{F}(\omega_{n,x} | x) \frac{\varepsilon}{2(1-\varepsilon)}.$$

Adapting this in our framework motivates the following value for  $q$ :

$$q_{n,-}^*(\varepsilon, x) := \frac{M\varepsilon}{1-\varepsilon} nh^d f(x) \bar{F}(\omega_{n,x} | x)$$

where  $M$  is a positive constant to be chosen later. Recall (29) to write, for all  $s > 0$ ,

$$\varphi_n(-s, x) = 1 - \int_{B(x, h)} [1 - \psi_n(-sK_h(x-z) | x, z)] f(z) dz.$$

Using the well-known inequality  $\log(1-r) \leq -r$  for all  $r > 0$ , we get

$$\log \varphi_n(-s, x) \leq \int_{B(x, h)} [\psi_n(-sK_h(x-z) | x, z) - 1] f(z) dz.$$

Replacing  $s$  by  $q_{n,-}^*(\varepsilon, x)/n\mu_n^{(1,1)}(x)$ , the ideas developed to control  $u_{1,n}(x)$  entail, for  $n$  large enough,

$$\forall x \in \Omega, u_{2,n}(x) \leq \exp \left( f \left[ M\varepsilon - \frac{M\varepsilon}{1-\varepsilon} \left[ 1 - 2\frac{M\varepsilon}{1-\varepsilon} \|K\|_2^2 \right] \right] nh^d \bar{F}(\omega_{n,x} | x) \right).$$

A straightforward computation shows that the optimal value for  $M$  in the above inequality is

$$M_-^* := \frac{1-\varepsilon}{4\|K\|_2^2}$$

for which

$$\forall x \in \Omega, u_{2,n}(x) \leq \exp(-\kappa_2 \varepsilon^2 nh^d \bar{F}(\omega_{n,x} | x)).$$

Setting  $\kappa = \kappa_1 \wedge \kappa_2$  completes the proof of Lemma 4. ■

The fifth lemma of this section establishes a uniform control of the relative oscillation of  $x \mapsto \mu_n^{(s,t)}(x)$ .

Before stating this result, we let

$$\mathbf{m}_n^{(s,t)}(x) := \mathbb{E}(\mathcal{K}_{2h}^s(x-X)m_n^{(t)}(x, X))$$

where  $\mathcal{K} := \mathbb{1}_B/\mathcal{V}$  is the uniform kernel on  $\mathbb{R}^d$ , with  $\mathcal{V}$  being the volume of the unit ball of  $\mathbb{R}^d$ ; let further  $\mathcal{K}_h(u) := h^{-d}\mathcal{K}(u/h)$ .

**Lemma 5.** *Assume that (SP), (K), (A<sub>1</sub>) and (A<sub>2</sub>) hold. Pick  $s \geq 1$ ,  $t \in \{0, 1\}$  and let  $\varepsilon := \varepsilon_n$  be a sequence of positive real numbers such that  $\varepsilon \leq h$ . If moreover*

- $\inf_{x \in \Omega} \omega_{n,x} \rightarrow \infty$  ;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \rightarrow 0$  ;
- $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(\varepsilon) \rightarrow 0$  ;
- $\sup_{x \in \Omega} \bar{\alpha}(y | x) \rightarrow 0$  as  $y \rightarrow \infty$

then

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{[\varepsilon/h]^{\eta\kappa} \vee \Delta(\log \omega_{n,x})(\varepsilon)} \left| \frac{\mu_n^{(s,t)}(z)}{\mu_n^{(s,t)}(x)} - 1 \right| = \mathcal{O}(1).$$

**Proof of Lemma 5.** Using Lemma 1, we can pick  $n$  large enough such that  $B(x, 2h) \subset S$  for all  $x \in \Omega$ .

We start by the case  $t = 0$ . For all  $x \in \Omega$  and  $z \in B(x, \varepsilon)$ , we have

$$\begin{aligned} \left| \mu_n^{(s,0)}(x) - \mu_n^{(s,0)}(z) \right| &\leq \mathbb{E}(|K_h^s(x-X) - K_h^s(z-X)| \mathbb{1}_{\{Y > \omega_{n,x}\}}) \\ &\quad + \mathbb{E}(K_h^s(z-X) |\mathbb{1}_{\{Y > \omega_{n,x}\}} - \mathbb{1}_{\{Y > \omega_{n,z}\}}|) \\ &=: R_{1,n}^{(0)}(x, z) + R_{2,n}^{(0)}(x, z) \end{aligned} \tag{39}$$

and we shall handle both terms in the right-hand side separately. Since by the mean value theorem, for all  $0 \leq a < b$

$$|a^s - b^s| \leq \max_{t \in [a, b]} |st^{s-1}| |a - b|,$$

hypothesis (K) and the inclusion  $B(z, h) \subset B(x, 2h)$  entail that there exists a constant  $c_{K,s} > 0$  such that

$$|K_h^s(x - X) - K_h^s(z - X)| \leq \frac{c_{K,s}}{h^{sd}} \left[ \frac{\varepsilon}{h} \right]^{\eta_K} \mathbb{1}_{\{X \in B(x, 2h)\}}. \quad (40)$$

From (40), we get

$$\sup_{z \in B(x, \varepsilon)} R_{1,n}^{(0)}(x, z) \leq c_{K,s} (2^d \mathcal{V})^s \mathbf{m}_n^{(s,0)}(x) \left[ \frac{\varepsilon}{h} \right]^{\eta_K}. \quad (41)$$

Because  $\mathcal{K}$  is a probability density function on  $\mathbb{R}^d$  with support included in  $B$ , applying Lemma 3(iii) with  $\mathcal{K}$  instead of  $K$  implies that

$$\sup_{x \in \Omega} \left| \frac{\mathbf{m}_n^{(s,0)}(x)}{(2h)^{-d(s-1)} \|\mathcal{K}\|_s^s f(x) m_n^{(0)}(x, x)} - 1 \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Applying Lemma 3(iii) once again then gives

$$\sup_{x \in \Omega} \left| \frac{\mathbf{m}_n^{(s,0)}(x)}{\mu_n^{(s,0)}(x)} - 2^{-d(s-1)} \frac{\|\mathcal{K}\|_s^s}{\|K\|_s^s} \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (42)$$

which, together with (41), yields

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \left[ \frac{\varepsilon}{h} \right]^{-\eta_K} \frac{R_{1,n}^{(0)}(x, z)}{\mu_n^{(s,0)}(x)} = O(1). \quad (43)$$

We now turn to the second term. One has

$$R_{2,n}^{(0)}(x, z) = \mathbb{E} \left( K_h^s(z - X) \left| \overline{F}(\omega_{n,x} | X) - \overline{F}(\omega_{n,z} | X) \right| \right). \quad (44)$$

Furthermore, using Lemma 2 with  $\varepsilon' = 0$  and the inclusion  $B(x, 2h) \subset S$  for all  $x \in \Omega$  entails

$$\sup_{x \in \Omega} \sup_{x' \in B(x, 2h)} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \left| \frac{\overline{F}(\omega_{n,z} | x')}{\overline{F}(\omega_{n,x} | x')} - 1 \right| = O(1). \quad (45)$$

Besides, hypothesis (K) and the inclusion  $B(z, h) \subset B(x, 2h)$  imply that

$$\mathbb{E}(K_h^s(z - X) m_n^{(0)}(x, X)) \leq c_{K,s} (2^d \mathcal{V})^s \mathbf{m}_n^{(s,0)}(x) \quad (46)$$

where  $c_{K,s}$  was introduced earlier. Using the obvious identity

$$\left| \overline{F}(\omega_{n,x} | X) - \overline{F}(\omega_{n,z} | X) \right| = m_n^{(0)}(x, X) \left| \frac{\overline{F}(\omega_{n,z} | X)}{\overline{F}(\omega_{n,x} | X)} - 1 \right| \quad (47)$$

and recalling that the support of the random variable  $K_h^s(z - X)$  is contained in  $B(z, h) \subset B(x, 2h)$ , (44) and (45) yield:

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(0)}(x, z)}{\mathbf{m}_n^{(s,0)}(x)} = O(1),$$



and (42) entails

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(0)}(x, z)}{\mu_n^{(s,0)}(x)} = O(1). \quad (48)$$

Applying (39) together with (43) and (48) gives

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{[\varepsilon/h]^{\eta_K} \vee \Delta(\log \omega_{n,x})(\varepsilon)} \left| \frac{\mu_n^{(s,0)}(z)}{\mu_n^{(s,0)}(x)} - 1 \right| = O(1)$$

which shows Lemma 5 in this case.

We now turn to the case  $t = 1$ . Note that for all real numbers  $a, b \geq 1$  such that  $a \neq b$  one has

$$\forall y \geq 1, |(\log y - \log a)_+ \mathbb{1}_{\{y > a\}} - (\log y - \log b)_+ \mathbb{1}_{\{y > b\}}| \leq |\log b - \log a| \mathbb{1}_{\{y > a \wedge b\}}. \quad (49)$$

Inequality (49) then implies, for all  $x \in \Omega$  and  $z \in B(x, \varepsilon)$ :

$$\begin{aligned} \left| \mu_n^{(s,1)}(x) - \mu_n^{(s,1)}(z) \right| &\leq \mathbb{E} \left( |K_h^s(x - X) - K_h^s(z - X)| (\log Y - \log \omega_{n,x})_+ \mathbb{1}_{\{Y > \omega_{n,x}\}} \right) \\ &\quad + \left| \log \frac{\omega_{n,x}}{\omega_{n,z}} \right| \mathbb{E} \left( K_h^s(z - X) \mathbb{1}_{\{Y > \omega_{n,x} \wedge \omega_{n,z}\}} \right) \\ &=: R_{1,n}^{(1)}(x, z) + R_{2,n}^{(1)}(x, z) \end{aligned} \quad (50)$$

and we shall once again take care of both terms in the right-hand side of this inequality. Start by using (40) to get

$$\sup_{z \in B(x, \varepsilon)} R_{1,n}^{(1)}(x, z) \leq c_{K,s} (2^d \mathcal{V})^s \mathbf{m}_n^{(s,1)}(x) \left[ \frac{\varepsilon}{h} \right]^{\eta_K}. \quad (51)$$

We now use the same idea developed to control  $R_{1,n}^{(0)}(x, z)$ : since  $\mathcal{K}$  is a probability density function on  $\mathbb{R}^d$  with support included in  $B$ , applying Lemma 3(iii) with  $\mathcal{K}$  instead of  $K$  implies that

$$\sup_{x \in \Omega} \left| \frac{\mathbf{m}_n^{(s,1)}(x)}{(2h)^{-d(s-1)} \|\mathcal{K}\|_s^s f(x) m_n^{(1)}(x, x)} - 1 \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Applying Lemma 3(iii) gives

$$\sup_{x \in \Omega} \left| \frac{\mathbf{m}_n^{(s,1)}(x)}{\mu_n^{(s,1)}(x)} - 2^{-d(s-1)} \frac{\|\mathcal{K}\|_s^s}{\|K\|_s^s} \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

which, together with (51), yields

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \left[ \frac{\varepsilon}{h} \right]^{-\eta_K} \frac{R_{1,n}^{(1)}(x, z)}{\mu_n^{(s,1)}(x)} = O(1). \quad (52)$$

To control the second term, write

$$\sup_{z \in B(x, \varepsilon)} R_{2,n}^{(1)}(x, z) \leq \Delta(\log \omega_{n,x})(\varepsilon) \sup_{z \in B(x, \varepsilon)} \mathbb{E} \left( K_h^s(z - X) \mathbb{1}_{\{Y > \omega_{n,x} \wedge \omega_{n,z}\}} \right).$$

Note that since  $\omega_{n,x} \wedge \omega_{n,z}$  is either equal to  $\omega_{n,x}$  or  $\omega_{n,z}$ , we can write, for all  $z \in B(x, \varepsilon)$

$$\mathbb{E} \left( K_h^s(z - X) \mathbb{1}_{\{Y > \omega_{n,x} \wedge \omega_{n,z}\}} \right) \leq \mathbb{E} \left( K_h^s(z - X) m_n^{(0)}(x, X) \right) \vee \mathbb{E} \left( K_h^s(z - X) m_n^{(0)}(z, X) \right).$$

Recall now (45) and (47) to obtain, for  $n$  large enough, uniformly in  $x \in \Omega$  and  $z \in B(x, \varepsilon)$ ,

$$\mathbb{E} \left( K_h^s(z - X) \mathbb{1}_{\{Y > \omega_{n,x} \wedge \omega_{n,z}\}} \right) \leq 2\mathbb{E} \left( K_h^s(z - X) m_n^{(0)}(x, X) \right). \quad (53)$$

Finally, using (46) and (53) yields:

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(1)}(x, z)}{\mathbf{m}_n^{(s,0)}(x)} = O(1),$$

and (42) entails

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(1)}(x, z)}{\mu_n^{(s,0)}(x)} = O(1)$$

so that Lemma 3(iv) gives

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(1)}(x, z)}{\mu_n^{(s,1)}(x)} = O(1). \quad (54)$$

Applying (50) together with (52) and (54) implies that

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{[\varepsilon/h]^{\eta_K} \vee \Delta(\log \omega_{n,x})(\varepsilon)} \left| \frac{\mu_n^{(s,1)}(z)}{\mu_n^{(s,1)}(x)} - 1 \right| = O(1)$$

which completes the proof of Lemma 5.  $\blacksquare$

The sixth lemma of this section provides a uniform control of both the difference of two versions of  $\mu_n^{(1,0)}(x)$  for two families of thresholds that are uniformly asymptotically equivalent and the empirical analogue of this quantity.

**Lemma 6.** *Assume that (SP), (A<sub>1</sub>) and (A<sub>2</sub>) hold. Assume that  $K$  is a bounded probability density function on  $\mathbb{R}^d$  with support included in  $B$  and that*

- $\inf_{x \in \Omega} \omega_{n,x} \rightarrow \infty$  ;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \rightarrow 0$  ;
- $\sup_{x \in \Omega} \bar{\alpha}(y | x) \rightarrow 0$  as  $y \rightarrow \infty$ .

For an arbitrary family of positive sequences  $(\rho_{n,x})$  such that  $\sup_{x \in \Omega} \rho_{n,x} \rightarrow 0$  as  $n \rightarrow \infty$ , let

$$\begin{aligned} M_n^{(1,0)}(x) &:= \mathbb{E}(K_h(x - X) \mathbb{1}_{\{(1-\rho_{n,x})\omega_{n,x} < Y \leq (1+\rho_{n,x})\omega_{n,x}\}}) \\ \text{and } U_n^{(1,0)}(x) &:= \frac{1}{n} \sum_{i=1}^n K_h(x - X_i) \mathbb{1}_{\{(1-\rho_{n,x})\omega_{n,x} < Y_i \leq (1+\rho_{n,x})\omega_{n,x}\}}. \end{aligned}$$

Then

$$\sup_{x \in \Omega} \left| \frac{\gamma(x) M_n^{(1,0)}(x)}{2f(x) \rho_{n,x} \bar{F}(\omega_{n,x} | x)} - 1 \right| \rightarrow 0$$

and there exists a positive constant  $\kappa$  such that for all  $n$  large enough, one has for every  $\varepsilon > 0$  small enough:

$$\forall x \in \Omega, \mathbb{P} \left( \rho_{n,x} \left| \frac{U_n^{(1,0)}(x)}{M_n^{(1,0)}(x)} - 1 \right| > \varepsilon \right) \leq 2 \exp(-\kappa \varepsilon n h^d \bar{F}(\omega_{n,x} | x)).$$

**Proof of Lemma 6.** Using Lemma 1, we can pick  $n$  large enough such that  $B(x, h) \subset S$  for all  $x \in \Omega$ .

We start by noting that

$$M_n^{(1,0)}(x) = \mathbb{E} \left( K_h(x - X) \rho_{n,x} \bar{F}(\omega_{n,x} | X) \left[ \frac{\bar{F}((1 - \rho_{n,x})\omega_{n,x} | X)}{\rho_{n,x} \bar{F}(\omega_{n,x} | X)} - \frac{\bar{F}((1 + \rho_{n,x})\omega_{n,x} | X)}{\rho_{n,x} \bar{F}(\omega_{n,x} | X)} \right] \right).$$

Use then  $(SP)$  and  $(A_1)$  to get, for an arbitrary  $z \in B(x, h)$ ,

$$\frac{\bar{F}((1 - \rho_{n,x})\omega_{n,x} | z)}{\rho_{n,x} \bar{F}(\omega_{n,x} | z)} = \frac{(1 - \rho_{n,x})^{-1/\gamma(z)}}{\rho_{n,x}} \exp \left( - \int_{(1 - \rho_{n,x})\omega_{n,x}}^{\omega_{n,x}} \frac{\alpha(v | z)}{v} dv \right) \quad (55)$$

$$\text{and } \frac{\bar{F}((1 + \rho_{n,x})\omega_{n,x} | z)}{\rho_{n,x} \bar{F}(\omega_{n,x} | z)} = \frac{(1 + \rho_{n,x})^{-1/\gamma(z)}}{\rho_{n,x}} \exp \left( \int_{\omega_{n,x}}^{(1 + \rho_{n,x})\omega_{n,x}} \frac{\alpha(v | z)}{v} dv \right). \quad (56)$$

Remark that using a Taylor expansion of the exponential function in a neighborhood of 0, there exists a function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  converging to 0 at 0 such that for all  $h > 0$ :

$$(1 + h)^{-1/\gamma(z)} = \exp \left( - \frac{\log(1 + h)}{\gamma(z)} \right) = 1 - \frac{\log(1 + h)}{\gamma(z)} \left( 1 + \varphi \left( \frac{\log(1 + h)}{\gamma(z)} \right) \right);$$

since  $0 < \underline{\gamma} \leq \gamma(z)$  and  $\sup_{x \in \Omega} \rho_{n,x} \rightarrow 0$ , this yields

$$\frac{(1 - \rho_{n,x})^{-1/\gamma(z)}}{\rho_{n,x}} = \frac{1}{\rho_{n,x}} + \frac{1}{\gamma(z)} (1 + r_{1,n}(x, z)) \quad (57)$$

$$\text{and } \frac{(1 + \rho_{n,x})^{-1/\gamma(z)}}{\rho_{n,x}} = \frac{1}{\rho_{n,x}} - \frac{1}{\gamma(z)} (1 + r_{2,n}(x, z)) \quad (58)$$

where  $r_{1,n}(x, z)$  and  $r_{2,n}(x, z)$  satisfy

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} |r_{1,n}(x, z)| \rightarrow 0 \quad \text{and} \quad \sup_{x \in \Omega} \sup_{z \in B(x, h)} |r_{2,n}(x, z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Besides, for all  $h > 0$ ,

$$\left| \int_{(1-h)\omega_{n,x}}^{\omega_{n,x}} \frac{\alpha(v | z)}{v} dv \right| \leq \bar{\alpha}((1-h)\omega_{n,x} | z) |\log(1-h)| \quad (59)$$

so that, because  $\inf_{x \in \Omega} \omega_{n,x} \rightarrow \infty$ ,  $\sup_{x \in \Omega} \rho_{n,x} \rightarrow 0$  and  $\sup_{x \in \Omega} \bar{\alpha}(y | x) \rightarrow 0$  as  $y \rightarrow \infty$ :

$$\exp \left( - \int_{(1 - \rho_{n,x})\omega_{n,x}}^{\omega_{n,x}} \frac{\alpha(v | z)}{v} dv \right) = 1 - \int_{(1 - \rho_{n,x})\omega_{n,x}}^{\omega_{n,x}} \frac{\alpha(v | z)}{v} dv (1 + r_{3,n}(x, z))$$

where  $r_{3,n}(x, z)$  satisfies

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} |r_{3,n}(x, z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Similarly

$$\left| \int_{\omega_{n,x}}^{(1+h)\omega_{n,x}} \frac{\alpha(v | z)}{v} dv \right| \leq \bar{\alpha}(\omega_{n,x} | z) |\log(1+h)| \quad (60)$$

and therefore

$$\exp \left( \int_{\omega_{n,x}}^{(1 + \rho_{n,x})\omega_{n,x}} \frac{\alpha(v | z)}{v} dv \right) = 1 + \int_{\omega_{n,x}}^{(1 + \rho_{n,x})\omega_{n,x}} \frac{\alpha(v | z)}{v} dv (1 + r_{4,n}(x, z))$$

where  $r_{4,n}(x, z)$  satisfies

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} |r_{4,n}(x, z)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Moreover, (59) and (60) yield

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{\rho_{n,x}} \left| \int_{(1-\rho_{n,x})\omega_{n,x}}^{\omega_{n,x}} \frac{\alpha(v|z)}{v} dv \right| \rightarrow 0 \text{ and } \sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{\rho_{n,x}} \left| \int_{\omega_{n,x}}^{(1+\rho_{n,x})\omega_{n,x}} \frac{\alpha(v|z)}{v} dv \right| \rightarrow 0$$

as  $n \rightarrow \infty$ . Plugging this together with (57) and (58) into (55) and (56) and recalling that  $0 < \underline{\gamma} \leq \gamma(z)$  entails

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \left| \frac{\gamma(z)}{2} \left[ \frac{\overline{F}((1-\rho_{n,x})\omega_{n,x}|z)}{\rho_{n,x}\overline{F}(\omega_{n,x}|z)} - \frac{\overline{F}((1+\rho_{n,x})\omega_{n,x}|z)}{\rho_{n,x}\overline{F}(\omega_{n,x}|z)} \right] - 1 \right| \rightarrow 0$$

as  $n \rightarrow \infty$ . Consequently,

$$\sup_{x \in \Omega} \left| \frac{M_n^{(1,0)}(x)}{2\mathbb{E}(K_h(x-X)\rho_{n,x}\overline{F}(\omega_{n,x}|X)/\gamma(X))} - 1 \right| \rightarrow 0. \quad (61)$$

Write then

$$\mathbb{E}(K_h(x-X)\overline{F}(\omega_{n,x}|X)/\gamma(X)) = \int_B K(u)\overline{F}(\omega_{n,x}|x-hu) \frac{f(x-hu)}{\gamma(x-hu)} du.$$

Recalling (3) and (4), we get

$$\sup_{x \in \Omega} \left| \frac{\mathbb{E}(K_h(x-X)\overline{F}(\omega_{n,x}|X)/\gamma(X))}{\mu_n^{(1,0)}(x)/\gamma(x)} - 1 \right| \rightarrow 0.$$

It only remains to recall (61) and to apply Lemma 3(iv) to obtain

$$\sup_{x \in \Omega} \left| \frac{\gamma(x)M_n^{(1,0)}(x)}{2f(x)\rho_{n,x}\overline{F}(\omega_{n,x}|x)} - 1 \right| \rightarrow 0. \quad (62)$$

We proceed by controlling  $U_n^{(1,0)}(x)$ . For every  $x \in \Omega$ ,

$$\mathbb{P} \left( \rho_{n,x} \left| \frac{U_n^{(1,0)}(x)}{M_n^{(1,0)}(x)} - 1 \right| > \varepsilon \right) = \mathbb{P} \left( \left| h^d U_n^{(1,0)}(x) - h^d M_n^{(1,0)}(x) \right| > \varepsilon \frac{h^d M_n^{(1,0)}(x)}{\rho_{n,x}} \right).$$

Notice now that if  $Z_{n,i}(x) := h^d K_h(x - X_i) \mathbb{1}_{\{(1-\rho_{n,x})\omega_{n,x} < Y_i \leq (1+\rho_{n,x})\omega_{n,x}\}}$ , then

$$h^d U_n^{(1,0)}(x) - h^d M_n^{(1,0)}(x) = \frac{1}{n} \sum_{i=1}^n [Z_{n,i}(x) - \mathbb{E}(Z_{n,i}(x))]$$

is a mean of bounded, centered, independent and identically distributed random variables. Define

$$\tau_n(x) := \frac{\varepsilon}{\|K\|_\infty} \frac{nh^d M_n^{(1,0)}(x)}{\rho_{n,x}} \text{ and } \lambda_n(x) := \varepsilon \|K\|_\infty \frac{h^d M_n^{(1,0)}(x)}{\rho_{n,x}} \frac{1}{\text{Var}(Z_{n,1}(x))}.$$

Bernstein's inequality (see Hoeffding [23]) yields, for all  $\varepsilon > 0$ ,

$$\mathbb{P} \left( \rho_{n,x} \left| \frac{U_n^{(1,0)}(x)}{M_n^{(1,0)}(x)} - 1 \right| > \varepsilon \right) \leq 2 \exp \left( - \frac{\tau_n(x)\lambda_n(x)}{2(1+\lambda_n(x)/3)} \right).$$

Applying (62) yields, for  $n$  large enough,

$$\inf_{x \in \Omega} \frac{\tau_n(x)}{nh^d \overline{F}(\omega_{n,x}|x)} \geq \frac{\varepsilon \underline{f}}{\overline{\gamma} \|K\|_\infty}. \quad (63)$$

Moreover, for all  $x \in \Omega$ ,

$$\frac{1}{\lambda_n(x)} = \rho_{n,x} \frac{\mathbb{E}(Z_{n,1}^2(x)) - [\mathbb{E}(Z_{n,1}(x))]^2}{\varepsilon \|K\|_\infty h^d M_n^{(1,0)}(x)}.$$

Since  $Z_{n,1}(x)$  is bounded by  $\|K\|_\infty$ , it follows that

$$\sup_{x \in \Omega} \frac{1}{\lambda_n(x)} \leq \sup_{x \in \Omega} \rho_{n,x} \frac{\mathbb{E}(Z_{n,1}^2(x)) - [\mathbb{E}(Z_{n,1}(x))]^2}{\varepsilon \|K\|_\infty h^d \mu_n^{(1,0)}(x)} \leq \frac{1}{\varepsilon} \sup_{x \in \Omega} \rho_{n,x} \rightarrow 0 \quad (64)$$

as  $n \rightarrow \infty$ . Finally, it holds that

$$\frac{\tau_n(x)\lambda_n(x)}{2(1 + \lambda_n(x)/3)} \geq \left\{ \inf_{x \in \Omega} \frac{\tau_n(x)}{nh^d \bar{F}(\omega_{n,x} | x)} \right\} \left\{ \inf_{x \in \Omega} \frac{1}{2(1/\lambda_n(x) + 1/3)} \right\} nh^d \bar{F}(\omega_{n,x} | x).$$

Using (63), (64) and the fact that the function  $t \mapsto 1/[2(t+1/3)]$  is decreasing on  $\mathbb{R}_+$ , it is then clear that, for all  $n$  large enough, if  $\varepsilon > 0$  is small enough, there exists a positive constant  $\kappa$  that is independent of  $\varepsilon$  such that

$$\forall x \in \Omega, \mathbb{P} \left( \rho_{n,x} \left| \frac{U_n^{(1,0)}(x)}{M_n^{(1,0)}(x)} - 1 \right| > \varepsilon \right) \leq 2 \exp [-\kappa \varepsilon nh^d \bar{F}(\omega_{n,x} | x)].$$

This completes the proof of Lemma 6. ■

The final lemma is the last step in the proof of Theorem 2.

**Lemma 7.** *Let  $(X_n)$  be a sequence of positive real-valued random variables such that for every positive nonrandom sequence  $(\delta_n)$  converging to 0, the random sequence  $(\delta_n X_n)$  converges to 0 almost surely. Then*

$$\mathbb{P} \left( \limsup_{n \rightarrow \infty} X_n = +\infty \right) = 0 \quad \text{i.e.} \quad X_n = O(1) \quad \text{almost surely.}$$

**Proof of Lemma 7.** Assume that there exists  $\varepsilon > 0$  such that  $\mathbb{P} \left( \limsup_{n \rightarrow \infty} X_n = +\infty \right) \geq \varepsilon$ . Since by definition  $\limsup_{n \rightarrow \infty} X_n = \lim_{n \rightarrow \infty} \sup_{p \geq n} X_p$  is the limit of a nonincreasing sequence, one has

$$\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, \mathbb{P} \left( \bigcup_{p \geq n} \{X_p \geq k\} \right) \geq \varepsilon.$$

From this we deduce

$$\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, \exists n' \geq n, \mathbb{P} \left( \bigcup_{p=n}^{n'} \{X_p \geq k\} \right) \geq \varepsilon/2. \quad (65)$$

We now build a sequence  $(N_k)$  by induction: start by using (65) with  $k = n = 1 =: N_1$  to obtain  $N_2 > N_1$  such that

$$\mathbb{P} \left( \bigcup_{p=N_1}^{N_2-1} \{X_p \geq 1\} \right) \geq \varepsilon/2.$$

Then for an arbitrary  $k \geq 1$ , if  $N_k$  is given, apply (65) to get  $N_{k+1} > N_k$  such that

$$\mathbb{P} \left( \bigcup_{p=N_k}^{N_{k+1}-1} \{X_p \geq k\} \right) \geq \frac{\varepsilon}{2}.$$

The sequence  $(N_k)$  is thus an increasing sequence of integers. Let  $\delta_n = 1/k$  if  $N_k \leq n < N_{k+1}$ . It is clear that  $(\delta_n)$  is a positive sequence which converges to 0. Besides, for all  $k \in \mathbb{N} \setminus \{0\}$  it holds that

$$\mathbb{P}\left(\sup_{p \geq N_k} \delta_p X_p \geq 1\right) = \mathbb{P}\left(\bigcup_{p \geq N_k} \{\delta_p X_p \geq 1\}\right) \geq \mathbb{P}\left(\bigcup_{p=N_k}^{N_{k+1}-1} \{\delta_p X_p \geq 1\}\right) = \mathbb{P}\left(\bigcup_{p=N_k}^{N_{k+1}-1} \{X_p \geq k\}\right) \geq \frac{\varepsilon}{2}.$$

This entails

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(\sup_{p \geq n} \delta_p X_p \geq 1\right) \geq \varepsilon/2 > 0.$$

Hence  $(\delta_n X_n)$  does not converge almost surely to 0, from which the result follows.  $\blacksquare$

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