



HAL
open science

Parameter maximum likelihood estimation problem for time-periodic-drift Langevin type stochastic differential equations

Dominique Dehay

► **To cite this version:**

Dominique Dehay. Parameter maximum likelihood estimation problem for time-periodic-drift Langevin type stochastic differential equations. 2013. hal-00794615v1

HAL Id: hal-00794615

<https://hal.science/hal-00794615v1>

Preprint submitted on 26 Feb 2013 (v1), last revised 16 Apr 2015 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Parameter maximum likelihood estimation problem
for time-periodic-drift Langevin type
stochastic differential equations

Dominique Dehay

Institut de Recherche Mathématique de Rennes, CNRS umr 6625

Université Rennes 2

email : dominique.dehay@univ-rennes2.fr

January 15, 2013

Abstract In this paper we investigate the large-sample behavior of the maximum likelihood estimate (MLE) of the unknown parameter θ for processes following the model

$$d\xi_t = \theta f(t)\xi_t dt + dB_t$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with period, say $P > 0$, and which is observed through continuous time interval $[0, T]$ as $T \rightarrow \infty$. Here the periodic function $f(\cdot)$ is assumed known. We establish the consistency of the MLE and we point out its minimax efficiency. These results comply with the well-established case when the function $f(\cdot)$ is constant non null. However the case when $\int_0^P f(t)dt = 0$ and $f(\cdot)$ is not identically null presents some particularities. For instance in this case whatever is the value of θ , the rate of convergence of the MLE is T as in the case when $\theta = 0$ and $\int_0^P f(t)dt \neq 0$. Furthermore when $\int_0^P f(t)dt = 0$, the MLE is locally efficient for the quadratic risk.

Keywords Time-inhomogeneous diffusion process . Periodicity, Ornstein-Uhlenbeck process . Efficiency . Local asymptotic minimax bound.

Mathematics Subject Classification (2000) 62F12, 60J60

1 Introduction

The non-stationary and seasonal behavior is quite common for many random phenomena observed in time. The periodicity can be hidden in the structure of the process, for instance in the covariance structure for a non-stationary periodically correlated processes called also cyclostationary signals in signal theory (Gardner et al. 2006), or as in the present paper, in the coefficients of a time-inhomogeneous diffusion process (Höpfner and Kutoyants 2010). Many applications can be found in mechanics, communication theory, climatology, econometrics, biology to name but a few (see e.g. Antoni 2009; Collet and Martinez 2008; Gardner et al. 2006; Höpfner 2007; Serpedin et al. 2005).

The main purpose of the paper is the maximum likelihood estimation problem for the unknown parameter $\theta \in \mathbb{R}$ in the so called P -periodic Langevin SDE

$$d\xi_t = \theta f(t)\xi_t dt + dB_t, \quad \xi_0, \quad t \geq 0, \quad (1)$$

from the observation of a continuous sample path of the P -OU-process. Here $\{B_t, t \geq 0\}$ is a standard Brownian motion which is independent of the initial random variable ξ_0 , and $f : \mathbb{R} \rightarrow \mathbb{R}$ is some known P -periodic continuous function not identically null. For simplicity of exposure we assume that the diffusion coefficient is equal to 1. Such a SDE admits a unique solution for which we exhibit a completely explicit expression (3). This permits us to develop the following analysis. In this paper a solution of SDE (1) is called *P -periodic Ornstein-Uhlenbeck type process*, and for brevity noted *P -OU-process*. Parameter estimation problem for models of SDE with drift depending in time have been considered by many authors (for instance see Mishra and Prakasa Rao 1985; Liptser and Shiryaev 2001, and recently, Barczy and Pap 2010, Höpfner and Kutoyants 2010, Dehling et al 2010). None considers the case of SDE (1) which cannot be reduced to known models.

In Section 2 we present the main features about this model which will be useful in this paper (see Dehay 2013). More precisely the periodic Markov structure is described with the help of the associated P -segments chain $\{\mathbf{X}_n := \{\xi_{nP+t}, t \in [0, P]\} : n \in \mathbb{N}\}$ as defined by Höpfner and Kutoyants (2010). According the signum of $F(P)$, the model has a recurrent or transient asymptotic behavior. The case where $F(P) = 0$ will be of particular interest.

Next we take with the problem of estimation of the parameter θ in SDE (1) by the maximum likelihood method, from the observation of a sample path of the process along the finite interval $[0, T]$, as $T \rightarrow \infty$. Liptser and Shiryaev (2001, Theorem 17.2) have given conditions for evaluating the bias and the quadratic error of the maximum likelihood estimator (MLE) of the parameter θ for a large class of models including the P -periodic Langevin SDE (1). Although this result can be applied successfully to model (1) with $f(\cdot) \equiv 1$ (Liptser and Shiryaev 2001, Theorem 17.3), to carry out any checks when $f(\cdot)$ is not constant require untractable

computations. In a recent paper Barczy and Pap (2010) has investigated the MLE for a time-inhomogeneous diffusion process given by the SDE

$$d\xi_t = \theta f(t)\xi_t dt + \sigma(t)dB_t, \quad \xi_0, \quad t \geq 0,$$

where $f(\cdot)$ and $\sigma(\cdot)$ are known continuous functions. However due to the periodicity of the function $f(\cdot)$ and so the periodic structure of the process, their results do not apply to our context except to the ergodic case where $\theta F(P) < 0$. In Section 3, thanks to the asymptotic behavior of the P -OU-processes, we establish the strong consistency of the MLE $\widehat{\theta}_T$ whatever is the value of $\theta F(P)$ (Theorem 1). We also study the limit distribution of the scaled error $\delta_T(\theta) \left(\widehat{\theta}_T - \theta \right)$. According to the signum of $\theta F(P)$, the normalizing factor $\delta_T(\theta)$ and the limit distribution have different expressions (Section 3.3). When $\theta F(P) < 0$ or $\theta F(P) > 0$ the results are plain generalization of the classical case where $f(\cdot) \equiv 1$. However when $\theta F(P) > 0$, applying a non-random normalizing factor, the limit distribution is defined in $\mathcal{C}[0, P]$, the space of real-valued continuous functions on $[0, P]$ (Theorem 5). When $F(P) = 0$, the limit distribution of $T(\widehat{\theta}_T - \theta)$ coincides with the law of

$$\frac{\int_0^1 B_u dB'_u}{\int_0^1 B_u^2 du}$$

up to a factor $c(\theta) > 0$ whatever is the value of $\theta \in \mathbb{R}$, $\{B_t, t \in [0, 1]\}$ and $\{B'_t, t \in [0, 1]\}$ being two independent standard Brownian motions (Theorem 3), while when $\theta = 0$ and the function $f(\cdot) \equiv 1$, the limit distribution of $T\widehat{\theta}_T$ coincides with the law of

$$\frac{\int_0^1 B_u dB_u}{\int_0^1 B_u^2 du} = \frac{B_1^2 - 1}{2 \int_0^1 B_u^2 du}.$$

Furthermore we investigate the case when $\theta = 0$ and $f(\cdot)$ is not identically null, establishing a connection between the two previous models (Theorem 4).

Finally we analyse the efficiency of the MLE in the sense of asymptotic local minimax property (see Ibragimov and Has'minskiĭ, 1981; Jeganathan 1995; Le Cam and Yang 1990; see also Hájek 1972; Le Cam 1969 and 1986). When $\theta F(P) < 0$, the Harris positive recurrence of the P -OU-process entails that the model is locally asymptotically normal (LAN) the scale being \sqrt{T} (regular case), and the MLE is locally asymptotically minimax for (Theorem 6). In the transient case $\theta F(P) > 0$, we establish that the model is locally asymptotically mixed normal (LAMN), and the MLE $\widehat{\theta}_T$ has also some asymptotically local efficiency (Theorem 7). The scale is not standard and depends on the parameter θ . When $F(P) = 0$ we state the model is still locally asymptotically quadratic (LAQ), even locally asymptotically Brownian functional (LABF) according the denomination of Jeganathan (1995). Furthermore the MLE is locally asymptotically minimax for the quadratic mean risk function (Theorem 8).

For an easier reading and understanding of the statements of the paper, the proofs are collected in the last section (Section 5).

2 Background : Orsntein Ulhenbeck type process

Here we present some usefull properties of the solution of the following P-periodic Langevin SDE

$$d\xi_t = f(t)\xi_t dt + dB_t, \quad \xi_0, \quad t \geq 0, \quad (2)$$

Since the function $f(\cdot)$ is periodic and continuous, the usual conditions for the existence and the unicity of the strong solution of SDE (2) when the initial value is fixed, are satisfied (Liptser and Shiryaev 2001). Moreover the strong solution can be expressed easily

$$\xi_t = e^{F(t)} \left(\xi_0 + \int_0^t e^{-F(u)} dB_u \right) \quad (3)$$

where $F(t) := \int_0^t f(u)du$. Remark that as the coefficient function $f(\cdot)$ is P -periodic, $F(nP+t) = nF(P) + F(t)$ for any $t > 0$ and any integer n .

2.1 P -segments-homogeneous Markov sequence

For $0 \leq s \leq t$, we have

$$\xi_t = e^{F(t)-F(s)} \left(\xi_s + \int_s^t e^{-(F(u)-F(s))} dB_u \right) = e^{F(t)-F(s)} \xi_s + \int_s^t e^{F(t)-F(u)} dB_u. \quad (4)$$

Thus the process $\{\xi_t, t \geq 0\}$ is inhomogeneous Markovian with transition probability density $p_{s,t}(x, \cdot)$, the density of the normal law $\mathcal{N}\left(xe^{F(t)-F(s)}, \int_s^t e^{2(F(t)-F(u))} du\right)$.

Note that the transition probability density is periodic in time in the sense that: $p_{s+P,t+P}(x, y) = p_{s,t}(x, y)$, for all s, t, x and y . Furthermore for each t ,

$$\xi_{P+t} = e^{F(P)} \left(\xi_t + \int_t^{P+t} e^{-(F(u)-F(t))} dB_u \right) = e^{F(P)} \xi_t + \int_t^{P+t} e^{F(P+t)-F(u)} dB_u,$$

and the chain $\{\xi_{nP+t}\}_{n \in \mathbb{N}}$ is homogeneous Markovian with one-step transition probability density $q_t(x, \cdot)$, the density of the Gaussian law $\mathcal{N}\left(xe^{F(P)}, \int_0^P e^{2(F(t+P)-F(t+u))} du\right)$.

Let $B_t^{(nP)} = B_{nP+t} - B_{nP}$ for $t \in [0, P]$ and $n \in \mathbb{N}$. The processes $B^{(nP)} := \{B_t^{(nP)}, t \in [0, P]\}$, $n \in \mathbb{N}$, are independent Brownian motions on $[0, P]$, and ξ_{nP+t} can be decomposed as follows

$$\xi_{nP+t} = e^{F(P)} \xi_{(n-1)P+t} + e^{F(P+t)} \int_t^P e^{-F(u)} dB_u^{((n-1)P)} + e^{F(t)} \int_0^t e^{-F(u)} dB_u^{(nP)}.$$

Following Höpfner and Kutoyants (2010, Section 2) we define the P -segments sequence $\{\mathbf{X}_n\}_{n \in \mathbb{N}}$, $\mathbf{X}_n := \{\xi_{nP+t}, t \in [0, P]\}$. Then the previous decomposition can be written as

$$\mathbf{X}_n(t) = e^{F(P)} \mathbf{X}_{n-1}(t) + e^{F(P)+F(t)} (\mathbf{Z}_{n-1}(P) - \mathbf{Z}_{n-1}(t)) + e^{F(t)} \mathbf{Z}_n(t)$$

where

$$\mathbf{Z}_n(t) := \int_0^t e^{-F(u)} dB_u^{(nP)} = e^{nF(P)}(M_{nP+t} - M_{nP}).$$

Notice that $\mathcal{L}[\mathbf{Z}_n(t)] = \mathcal{N}(0, G(t))$, where $G(t) := \int_0^t e^{-2F(u)} du$ for any t . Since $\mathcal{C}[0, P]$ -valued random variables \mathbf{X}_{n-1} , $\mathbf{Z}_{n-1}(P) - \mathbf{Z}_{n-1}$ and \mathbf{Z}_n are independent, the sequence $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is a homogeneous Markov chain with state space $\mathcal{C}[0, P]$, the Banach space of real-valued continuous functions on $[0, P]$ endowed with the uniform norm.

Thanks to (Höpfner and Kutoyants 2010, Theorem 2.1) we deduce that if $e^{F(P)} < 1$, so $F(P) < 0$, then the P -segments chain \mathbf{X} is positive Harris recurrent, its invariant marginal law is given in Proposition 1 below. If $F(P) = 0$ then the function $F(\cdot)$ is periodic, and the transition probability density $q_t(x, \cdot)$ of the homogeneous Markov chain $\{\xi_{nP+t}\}_{n \in \mathbb{N}}$ is the density of the Gaussian law $\mathcal{N}(x, e^{2F(t)}G(P))$ for any $t \in [0, P]$. In this case the P -segments chain \mathbf{X} is neither positive Harris recurrent nor transient. If $F(P) > 0$, the P -segments chain \mathbf{X} and the P -OU process $\{\xi_t, t \geq 0\}$ are transient.

2.2 Moment structure

Consider the P -OU process $\{\xi_t, t \geq 0\}$ solution of P -periodic Langevin SDE (2). As $\mathbb{E}[\xi_t] = e^{F(t)} \mathbb{E}[\xi_0]$

$$\mathbb{E}[\xi_{nP+t}] = e^{nF(P)} \mathbb{E}[\xi_t]$$

for all $t \geq 0$ and $n \in \mathbb{N}$. Hence the first moment function $t \mapsto \mathbb{E}[\xi_t]$ is periodic if and only if $\mathbb{E}[\xi_0] = 0$ or $F(P) = 0$. Futhermore the equality $\mathbb{E}[e^{-F(t+\tau)} \xi_{t+\tau} | \xi_t] = e^{-F(t)} \xi_t$ implies that

$$\text{cov}[\xi_t, \xi_{t+\tau}] = e^{F(t+\tau)-F(t)} \text{var}[\xi_t],$$

for all t and τ . Thus the regression coefficient of $\xi_{t+\tau}$ on ξ_t is equal to $e^{F(t+\tau)-F(t)}$, and it is P -periodic in t for any τ . We can also readily deduce the following expression which will be usefull in the paper

$$\mathbb{E}[\xi_{nP+t}^2] = \begin{cases} e^{2F(t)} \left(\mathbb{E}[\xi_0^2] + nG(P) + G(t) \right) & \text{if } F(P) = 0 \\ e^{2F(t)} \left(e^{2nF(P)} \mathbb{E}[\xi_0^2] + \frac{e^{2nF(P)} - 1}{1 - e^{-2F(P)}} G(P) + G(t) \right) & \text{if } F(P) \neq 0. \end{cases} \quad (5)$$

2.3 Asymptotic behavior of the sample paths of $\{\xi_t, t \geq 0\}$

Next we state some asymptotic behaviors of the sample paths of the process which are usefull in this paper. Let $t \in [0, P]$ and $n \in \mathbb{N}$ be fixed. From equality (4) we readily obtain by induction a basic decomposition of ξ_{nP+t}

$$\xi_{nP+t} = e^{F(t)} \left(e^{nF(P)} \xi_0 + \sum_{k=0}^{n-1} e^{(n-k)F(P)} Z_k + \mathbf{Z}_n(t) \right). \quad (6)$$

For each integer n , the P -segment process $\mathbf{Z}_n = \{\mathbf{Z}_n(t), t \in [0, P]\}$ is a continuous martingale with respect to the filtration $\{\mathcal{F}_t^{(nP)}, t \in [0, 1]\}$ generated by the Brownian motion $B^{(nP)} = \{B_{nP+t} - B_{nP}, t \in [0, P]\}$, and Doob maximal equality for martingales entails that

$$\mathbb{E} \left[\max_{t \in [0, 1]} Z_n(t)^2 \right] \leq 4 \int_0^P e^{-2F(u)} du. \quad (7)$$

Futhermore the processes \mathbf{Z}_n , $n \in \mathbb{N}$, are independent and identically distributed in $\mathcal{C}[0, P]$. then we can state the asymptotic behavior of ξ_t as $t \rightarrow \infty$ according to the signum of $F(P)$.

Proposition 1 *If $F(P) < 0$ then*

$$\lim_{n \rightarrow \infty} \mathcal{L}[\mathbf{X}_n] = \lim_{n \rightarrow \infty} \mathcal{L}[\xi_{nP+\cdot}] = \mathcal{L} \left[e^{F(\cdot)}(\zeta_1 + \mathbf{Z}) \right] \quad \text{in } \mathcal{C}[0, P]$$

where ζ_1 is a real-valued Gaussian variable, $\mathcal{L}[\zeta_1] = \mathcal{N} \left(0, \frac{G(P)}{e^{-2F(P)} - 1} \right)$, $\mathbf{Z} := \{\mathbf{Z}(t), t \in [0, P]\}$ is a Gaussian process with representation $\mathbf{Z}(t) = \int_0^t e^{-F(u)} dB'_u$, $\{B'_u, u \in [0, P]\}$ is a Brownian motion on $[0, P]$, and the random variable ζ_1 and the Gaussian process \mathbf{Z} are independent.

We have seen for $F(P) < 0$ that the P -segments homogeneous Markov chain $\{\mathbf{X}_n\}_{n \in \mathbb{N}}$ is positive Harris recurrent. Then Proposition 1 entails that the law of $e^{F(\cdot)}(\zeta_1 + \mathbf{Z})$ in $\mathcal{C}[0, P]$ is its invariant (stationary) law.

Proposition 2 *If $F(P) = 0$ then*

$$\lim_{n \rightarrow \infty} \frac{\mathbf{X}_n}{n} = \lim_{n \rightarrow \infty} \frac{\xi_{nP+\cdot}}{n} = 0 \quad \text{a.e. in } \mathcal{C}[0, P]$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \mathbb{E} \left[\max_{t \in [0, P]} \xi_{nP+t}^2 \right] = 0,$$

and

$$\lim_{n \rightarrow \infty} \mathcal{L} \left[\frac{\mathbf{X}_n}{\sqrt{n}} \right] = \lim_{n \rightarrow \infty} \mathcal{L} \left[\frac{\xi_{nP+\cdot}}{\sqrt{n}} \right] = \mathcal{L} \left[e^{-F(\cdot)} \zeta_2 \right] \quad \text{in } \mathcal{C}[0, P]$$

as $n \rightarrow \infty$, where ζ_2 being a real-valued Gaussian variable which does not depend on t , $\mathcal{L}[\zeta_2] = \mathcal{N}(0, G(P))$.

When $F(P) = 0$ we can establish that the process $\{\xi_t, t \geq 0\}$ and its P -segments Markov chain $\{\mathbf{X}_n\}_{n \in \mathbb{N}}$ are recurrent (Dehay 2013), but with Proposition 2 we see that $\{\mathbf{X}_n\}_{n \in \mathbb{N}}$ is not positive Harris recurrent.

Proposition 3 *If $F(P) > 0$ then*

$$\lim_{n \rightarrow \infty} e^{-nF(P)} \mathbf{X}_n = \lim_{n \rightarrow \infty} e^{-nF(P)} \xi_{nP+\cdot} = e^{F(\cdot)}(\xi_0 + \zeta_3) \quad \text{a.e. in } \mathcal{C}[0, P],$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\max_{t \in [0, P]} \left| e^{-nF(P)} \xi_{nP+t} - e^{F(t)} (\xi_0 + \zeta_3) \right|^2 \right] = 0,$$

for some real-valued Gaussian variable ζ_3 , $\mathcal{L}[\zeta_3] = \mathcal{N}\left(0, \frac{G(P)}{1 - e^{-2F(P)}}\right)$, ζ_3 and ξ_0 being independent.

Hence for $F(P) > 0$, the Markov process $\{\xi_t, t \geq 0\}$ and its P -segments Markov chain $\{\mathbf{X}_n\}_{n \in \mathbb{N}}$ are transient.

3 Parameter estimation

From now on we deal with the parameter estimation problem for model (1). Consider a diffusion process observed on the interval $[0, T]$ following the P -periodic Langevin SDE (1) with parameter $\theta \in \mathbb{R}$ and with initial variable ξ_0 independent on $\{\mathbf{B}_t, t \geq 0\}$ and on the parameter θ for simplicity of exposure. Assume that the periodic continuous function $f(\cdot)$ is known and not identically null. Then we turn to the problem of estimating the unknown parameter $\theta \in \mathbb{R}$ on the basis of the observation $\{\xi_t, t \in [0, T]\}$. More precisely we study the asymptotic behavior of the maximum likelihood estimator (MLE) of θ as the observation time T goes to infinity.

Throughout the following we assume that the drift coefficient function $f(\cdot)$ is any periodic continuous function with period P , except for specified cases. Then the strong solution of the P -periodic SDE (1) exists, is unique and admits an explicit expression

$$\xi_t := \xi_t^{(\theta)} = e^{\theta F(t)} \left(\xi_0 + \int_0^t e^{-\theta F(u)} d\mathbf{B}_u \right) = e^{\theta F(t)} \xi_0 + \int_0^t e^{\theta(F(t) - F(u))} d\mathbf{B}_u,$$

for $t \in [0, \infty)$. Notice that $\xi_t^{(0)} = \xi_0 + \mathbf{B}_t$. For simplification we write ξ_t for $\xi_t^{(\theta)}$ when there is no possibility of confusion. Next in the notations \mathbb{P}_θ , \mathbb{E}_θ and \mathcal{L}_θ , the index indicates that we consider that the true value of the parameter is θ . We need also the following functions

$$G_\theta(t) := \int_0^t e^{-2\theta F(u)} du \quad \text{and} \quad H_\theta(t) := \int_0^t f(u)^2 e^{2\theta F(u)} du$$

for $t \in [0, \infty)$ and $\theta \in \mathbb{R}$.

3.1 MLE for Ornstein-Uhlenbeck processes

First we recall some well-established results on the maximum likelihood parameter estimation for Ornstein-Uhlenbeck processes. Assume that the coefficient function $f(\cdot)$ is identically equal to 1. The problem of estimation of the parameter θ for this model have been subject of a very large amount of contributions. The MLE of θ is equal to

$$\hat{\theta}_T = \frac{\int_0^T \xi_u d\xi_u}{\int_0^T \xi_u^2 du} = \theta + \frac{\int_0^T \xi_u d\mathbf{B}_u}{\int_0^T \xi_u^2 du}.$$

and its behavior $T \rightarrow \infty$ is well-known (see e.g. Basawa and Scott 1983; Bishwal 2008; Brown and Hewitt 1975; Feigin 1979; Kutoyants 2004; Liptser and Shiryaev 1977).

i) For $\theta < 0$, the process $\{\xi_t, t \geq 0\}$ is positive recurrent, ergodic with invariant measure $\mathcal{N}\left(0, \frac{1}{2|\theta|}\right)$, and

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[\sqrt{T}(\widehat{\theta}_T - \theta) \right] = \mathcal{N}(0, 2|\theta|).$$

ii) For $\theta = 0$, the process $\{\xi_t, t \geq 0\}$ is null recurrent

$$\lim_{n \rightarrow \infty} \mathcal{L}_0 \left[T\widehat{\theta}_T \right] = \mathcal{L} \left[\frac{\int_0^1 B_t dB_t}{\int_0^1 B_t^2 dt} \right] = \mathcal{L} \left[\frac{B_1^2 - 1}{2 \int_0^1 B_t^2 dt} \right]. \quad (8)$$

iii) For $\theta > 0$, the process is transient : $|\xi_t| \rightarrow \infty$ with probability 1

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\frac{1}{\sqrt{2\theta}} e^{\theta T} (\widehat{\theta}_T - \theta) \right] = \mathcal{L} \left[\frac{\nu}{\xi_0 + \zeta^{(\theta)}} \right]$$

on $\{\xi_0 + \zeta^{(\theta)} \neq 0\}$, where $\mathcal{L}[\nu] = \mathcal{N}(0, 1)$ and $\mathcal{L}[\zeta^{(\theta)}] = \mathcal{N}\left(0, \frac{1}{2\theta}\right)$. The random variables ξ_0 , ν and $\zeta^{(\theta)}$ are independent.

3.2 MLE for P -OU processes

Now we consider the general P -periodic model (1). Since the function $f(\cdot)$ is continuous and almost every sample path of the process $\{\xi_t, t \geq 0\} = \{\xi_t^{(\theta)}, t \geq 0\}$ is also continuous we have

$$\int_0^T f(u)^2 \xi_u^2 du < \infty \quad \text{P}_\theta\text{-a.e.}$$

for any $\theta \in \mathbb{R}$ and any $T > 0$. Thus

$$\text{P}_\theta \left[\int_0^T f(u)^2 \xi_u^2 du < \infty \right] = \text{P}_0 \left[\int_0^T f(u)^2 (\xi_0 + B_u)^2 du < \infty \right] = 1.$$

Thanks to (Liptser and Shiryaev 2001, Theorem 7.7 and formula (17.24)) the distribution P_θ^T of $\{\xi_t = \xi_t^{(\theta)}, t \in [0, T]\}$ is absolutely continuous with respect to the distribution P_0^T of $\{\xi_t^{(0)} = \xi_0 + B_t, t \in [0, T]\}$, and the log-likelihood ratio $\Lambda_T(\theta)$ verifies

$$\Lambda_T(\theta) = \theta \int_0^T f(u) \xi_u d\xi_u - \frac{\theta^2}{2} \int_0^T f(u)^2 \xi_u^2 du = \theta \int_0^T f(u) \xi_u dB_u + \frac{\theta^2}{2} \int_0^T f(u)^2 \xi_u^2 du. \quad (9)$$

So, as

$$\int_0^T f(u)^2 \xi_u^2 du > 0 \quad \text{P}_\theta\text{-a.e.},$$

the maximum likelihood estimator $\widehat{\theta}_T$ of θ is well defined and equal to

$$\widehat{\theta}_T = \frac{\int_0^T f(u) \xi_u d\xi_u}{\int_0^T f(u)^2 \xi_u^2 du} = \theta + \frac{\int_0^T f(u) \xi_u dB_u}{\int_0^T f(u)^2 \xi_u^2 du}. \quad (10)$$

3.3 Consistency of the MLE

Consider that $\theta \in \mathbb{R}$ be fixed. Let

$$U_t := \int_0^t f(u) \xi_u dB_u \quad (11)$$

and let $\{\mathcal{F}_t, t \geq 0\}$ be the filtration generated by ξ_0 and the Brownian motion $\{B_t, t \geq 0\}$. Then $\{U_t, t \geq 0\}$ is a zero-mean continuous martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$. Its quadratic variation is equal to

$$J_t := \int_0^t f(u)^2 \xi_u^2 du$$

and

$$\mathbb{E}_\theta [U_t^2] = \mathbb{E}_\theta [J_t] = \int_0^t f(u)^2 \mathbb{E}_\theta [\xi_u^2] du < \infty.$$

The strong law of large numbers for martingales (see Liptser and Shiryaev 2001, Lemma 17.4) entails that

$$\lim_{T \rightarrow \infty} J_T^{-1} U_T = 0 \quad \text{P}_\theta\text{-a.e. on } \{J_\infty = \infty\}$$

where $J_\infty := \lim_{T \rightarrow \infty} J_T$. Hence to prove the strong consistency of the MLE $\hat{\theta}_T$ it suffices to show that $\text{P}_\theta [J_\infty < \infty] = 0$. This is done thanks to the results of Section 2.3.

Theorem 1 *For any $\theta \in \mathbb{R}$, the MLE $\hat{\theta}_T$ is strongly consistent*

$$\lim_{T \rightarrow \infty} \hat{\theta}_T = \theta \quad \text{P-a.e.}$$

Hence the MLE $\hat{\theta}_T$ is a consistent estimator of θ

$$\lim_{T \rightarrow \infty} \text{P}_\theta \left[\left| \hat{\theta}_T - \theta \right| > \epsilon \right] = 0$$

for any $\epsilon > 0$, whatever is the value of $F(P)$. Actually, in the following we will see that this consistency is uniform with respect to θ varying in any compact set contained in $\{\theta' \in \mathbb{R} : \theta' F(P) < 0\}$ or in $\{\theta' \in \mathbb{R} : \theta' F(P) > 0\}$ if $F(P) \neq 0$, and in any compact subset of \mathbb{R} if $F(P) = 0$, the function $f(\cdot)$ being not identically null.

Next we deal with the asymptotic behavior of the deviation $\hat{\theta}_T - \theta$ according the signum of $\theta F(P)$.

3.3.1 Positive recurrent case : $\theta F(P) < 0$

In this case the model is regular. The Markov P -OU process $\{\xi_t, t \geq 0\}$ has a positive Harris recurrent associated P -segments Markov chain, so the maximum likelihood estimator $\hat{\theta}_T$ has a standard asymptotic behavior. Here the result complies will the well-known ones for the Ornstein-Ulhenbeck stationary processes.

Theorem 2 Let K_- be any compact subset of \mathbb{R} contained in $\{\theta : \theta F(P) < 0\}$. Then

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\sqrt{J_T} (\hat{\theta}_T - \theta) \right] = \mathcal{N}(0, 1)$$

and

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\sqrt{T} (\hat{\theta}_T - \theta) \right] = \mathcal{N}(0, I(\theta)^{-1}) \quad (12)$$

uniformly in θ varying in K_- . The Fisher information $I(\theta)$ is equal to

$$I(\theta) := \frac{G_\theta(P)H_\theta(P)}{P(e^{-2\theta F(P)} - 1)} + \frac{1}{P} \int_0^P f(u)^2 e^{2\theta F(u)} G_\theta(u) du > 0. \quad (13)$$

3.3.2 Case $\theta F(P) = 0$

When $\theta F(P) = 0$, the Markov P -OU process $\{\xi_t, t \geq 0\}$ has a null recurrent behavior. The maximum likelihood estimator $\hat{\theta}_T$ is consistent. In the following theorem, we see that the rate of convergence is T and the asymptotic distribution of the scaled error $T(\hat{\theta}_T - \theta)$ is not standard. However using the random scale $\sqrt{J_T}$ instead of T , the limit distribution is parameter free: it does not depend on the model.

Theorem 3 Assume that $F(P) = 0$ and the function $f(\cdot)$ is not identically null. Then

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\sqrt{J_T} (\hat{\theta}_T - \theta) \right] = \mathcal{L} \left[\frac{\int_0^1 B_u dB'_u}{\sqrt{\int_0^1 B_u^2 du}} \right]$$

and

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[T(\hat{\theta}_T - \theta) \right] = \mathcal{L} \left[\frac{P}{\sqrt{G_\theta(P)H_\theta(P)}} \frac{\int_0^1 B_u dB'_u}{\int_0^1 B_u^2 du} \right]$$

where $\{B_t, t \in [0, 1]\}$ and $\{B'_t, t \in [0, 1]\}$ are two independent standard Brownian motions. The limits being uniform with respect to θ varying in any compact subset of \mathbb{R} .

It is worth to notice that in Theorem 3 for $F(P) = 0$ and $f(\cdot)$ not identically null we need to introduce two independent Brownian motions for the limit distribution even when $\theta = 0$. While for $\theta = 0$ and $f(\cdot) \equiv 1$, so $F(P) \neq 0$, we have the well-established limit (8) expressed with only one Brownian motion. The following theorem gives another view on this limit (8) for $\theta = 0$, placing it in the more general context of any non identically null periodic function $f(\cdot)$ whatever is the value of $F(P)$. We recall that here the basic model is the P -periodic Langevin SDE (1) and the MLE $\hat{\theta}_T$ is defined by relation (10).

Theorem 4 Assume that $\theta = 0$ and the function $f(\cdot)$ is not identically null. Then

$$\begin{aligned} \lim_{T \rightarrow \infty} \mathcal{L}_0 \left[T\hat{\theta}_T \right] &= \mathcal{L} \left[\frac{\sqrt{PH_0(P) - F(P)^2} \int_0^1 B_u dB'_u + F(P) \int_0^1 B_u dB_u}{H_0(P) \int_0^1 B_u^2 du} \right] \\ &= \mathcal{L} \left[\frac{\sqrt{PH_0(P) - F(P)^2} \int_0^1 B_u dB'_u}{H_0(P) \int_0^1 B_u^2 du} + \frac{F(P)(B_1^2 - 1)}{2H_0(P) \int_0^1 B_u^2 du} \right] \end{aligned}$$

where $\{B_t, t \in [0, 1]\}$ and $\{B'_t, t \in [0, 1]\}$ are two independent Brownian motions. Recall that $F(P) = \int_0^P f(u) du$ and $H_0(P) = \int_0^P f(u)^2 du$.

Thus, when $F(P) = 0$ and the function $f(\cdot)$ is not identically null, we obtain the limit distribution stated in Theorem 3 with $\theta = 0$. When the function $f(\cdot)$ is constant non null, Theorem 4 reduces to the classical usual limit (8).

3.3.3 Transient case : $\theta F(P) > 0$

The following result generalizes the well-known case when $f(\cdot) \equiv 1$ and $\theta > 0$. The random local scale gives a parameter free limit distribution.

Theorem 5 *Let K_+ be any compact subset of \mathbb{R} contained in $\{\theta : \theta F(P) > 0\}$. Then*

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\sqrt{J_T} (\hat{\theta}_T - \theta) \right] = \mathcal{N}(0, 1)$$

and

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[e^{n\theta F(P)} (\hat{\theta}_{nP+} - \theta) \right] = \mathcal{L} \left[\frac{\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot)}{K_\theta(\cdot) (\xi_0 + \zeta_3^{(\theta)})} \right] \quad \text{in } \mathcal{C}[0, P]$$

uniformly with respect to θ varying in the compact set K_+ . Here $\zeta_3^{(\theta)}$ and $\zeta_4^{(\theta)}$ are two real-valued Gaussian variables with zero-mean and variances respectively equal to

$$\mathbb{E} \left[\left(\zeta_3^{(\theta)} \right)^2 \right] = \frac{G_\theta(P)}{1 - e^{-2\theta F(P)}}, \quad \mathbb{E} \left[\left(\zeta_4^{(\theta)} \right)^2 \right] = \frac{H_\theta(P)}{e^{2\theta F(P)} - 1},$$

and the process $\mathbf{Y}^{(\theta)} := \{\mathbf{Y}^{(\theta)}(t), t \in [0, P]\}$ is Gaussian and admits the representation

$$\mathbf{Y}^{(\theta)}(t) = \int_0^t f(u) e^{\theta F(u)} dB'_u$$

for some Brownian motion $\{B'_t, t \in [0, P]\}$. Moreover $\xi_0, \zeta_3^{(\theta)}, \zeta_4^{(\theta)}$ and $\mathbf{Y}^{(\theta)}$ are independent. Here

$$K_\theta(t) := \sqrt{\text{var} \left[\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(t) \right]} = \sqrt{\frac{H_\theta(P)}{e^{2\theta F(P)} - 1} + H_\theta(t)}.$$

4 Efficiency

In this part we tackle the issue of the efficiency for the maximum likelihood estimator $\hat{\theta}_T$. When the size of the sample path observation T is large, we would like to know how good or optimal is the maximum likelihood procedure for the estimation of the parameter θ . In the following denote by P_θ^T , the law of the observed process $\{\xi_t^{(\theta)}, t \in [0, T]\}$ when the value of the parameter is θ . Recall that for simplicity we assume that the law of the random variable ξ_0 does not depend on the parameter θ .

The asymptotical efficiency we consider here is understood in the sense of asymptotic local minimax Hájek-Le Cam lower bound on the risk of $\{\bar{\theta}_T, T > 0\}$ for the estimation of θ , that is

$$R_\theta(\{\bar{\theta}_T\}) := \lim_{\epsilon \rightarrow 0} \liminf_{T \rightarrow \infty} \sup_{|\theta - \theta'| < \epsilon} \mathbb{E}_{\theta'} [L(\delta_T(\theta)^{-1}(\bar{\theta}_T - \theta'))]$$

where $\bar{\theta}_T$ is any statistic function of the observation $\{\xi_t, t \in [0, T]\}$, $\delta_T(\theta)$ is some coefficient called *local scale*, and $L(\cdot)$ is some loss function in \mathbb{R} , $L(\cdot) \in \mathcal{L}$, \mathcal{L} being the set of Borel functions $L : \mathbb{R} \rightarrow [0, \infty)$ symmetric, continuous at 0 with $L(0) = 0$, non identically null and such that the set $\{u : L(u) < \alpha\}$ is convex for any $\alpha > 0$ (called bowl-shaped) (see Ibragimov and Has'minskii 1981, pp.18; see also Jeganathan 1995, pp.838; Le Cam and Yang 1990; Kutoyants 2004, Section 2.1). It is well known that the likelihood ratio is a convenient tool in the study of the behavior of statistics under "contiguous alternative". This is the case for the model (1) where the distributions \mathbb{P}_θ^T , $\theta \in \mathbb{R}$ are equivalent (relation (9)).

Let $\theta \in \mathbb{R}$ be fixed, and let $\delta_T(\theta) \rightarrow 0$ as $T \rightarrow \infty$. Considering local efficiency, denote by $\Lambda_T^{(\theta, u)}$ the log-likelihood ratio function of the distribution $\mathbb{P}_{\theta + \delta_T(\theta)u}^T$ of $\{\xi_t^{(\theta + \delta_T(\theta)u)}, t \in [0, T]\}$ with respect to the distribution \mathbb{P}_θ^T of $\{\xi_t^{(\theta)}, t \in [0, T]\}$. Thanks to relation (9), we have

$$\begin{aligned} \Lambda_T^{(\theta, u)} &:= \ln \left(\frac{d\mathbb{P}_{\theta + \delta_T(\theta)u}}{d\mathbb{P}_\theta} \right) (\xi_t) \\ &= \Lambda_T(\theta + \delta_T(\theta)u) - \Lambda_T(\theta) \\ &= u\delta_T(\theta) \int_0^T f(t)\xi_t dB_t - \frac{u^2\delta_T(\theta)^2}{2} \int_0^T f(t)^2 \xi_t^2 dt \\ &= u\delta_T(\theta)U_T - \frac{1}{2}u^2\delta_T(\theta)^2 J_T. \end{aligned} \tag{14}$$

In the light of the previous results, it is not surprising to see in the followings that the log-likelihood ratio $\Lambda_T^{(\theta, u)}$ has a quite different behavior according to the signum of $\theta F(P)$.

4.1 Positive recurrent case, $\theta F(P) < 0$: LAN property

When $\theta F(P) < 0$, thanks to Theorem 2 take $\delta_T(\theta) := T^{-1/2}$ as local scale (which does not dependent on θ). Then the log-likelihood ratio $\Lambda_T^{(\theta, u)}$ can be decomposed as

$$\Lambda_T^{(\theta, u)} = u\Delta_T^{(\theta)} - \frac{1}{2}u^2 I(\theta) + r_T(\theta, u)$$

where the score $\Delta_T^{(\theta)} := \frac{1}{\sqrt{T}} \int_0^T f(t)\xi_t dB_t$ is asymptotically normal

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\Delta_T^{(\theta)} \right] = I(\theta)^{-1/2} \mathcal{N}(0, 1),$$

the Fisher information $I(\theta)$ is defined by relation (13) and

$$\mathbb{P}_\theta - \lim_{T \rightarrow \infty} r_T(\theta, u) = 0$$

uniformly with respect to θ varying in any compact subset of $\{\theta : \theta F(P) < 0\}$. Thus the families of distributions $\{P_\theta^T : \theta F(P) < 0\}$, $T > 0$, is LAN (locally asymptotically normal) at each point $\theta \in \mathbb{R}$ such that $\theta F(P) < 0$. (see e.g. Hájek 1972 ; Le Cam and Yang 1990, Chapter 5; Jeganathan 1995, Definition 3; Kutoyants 2004, Section 2.1.1). Hájek-Le Cam asymptotic lower bound result (Hájek 1972, theorem 4.1) entails that

$$\lim_{\epsilon \rightarrow 0} \liminf_{T \rightarrow \infty} \sup_{|\theta - \theta'| < \epsilon} E_{\theta'} \left[L \left(\sqrt{T}(\bar{\theta}_T - \theta') \right) \right] \geq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} L \left(y I(\theta)^{-1/2} \right) e^{-y^2/2} dy \quad (15)$$

for any loss function $L(\cdot) \in \mathcal{L}$, and for any family $\{\bar{\theta}_T, T > 0\}$ of estimators of θ , that is $\bar{\theta}_T$ is measurable with respect to the observation $\{\xi_t, t \in [0, T]\}$ (see Le Cam and Yang 1990, Section 5.6 Theorem 1). A family $\{\bar{\theta}_T^*, T > 0\}$ of estimators is said to be *locally asymptotically minimax* for the loss function $L(\cdot)$ when its asymptotic minimax risk $R_\theta(\bar{\theta}^*)$ is equal to the lower bound right hand side of inequality (15) for any θ such $\theta F(P) < 0$.

Besides Theorem 2 gives

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\sqrt{T}(\hat{\theta}_T - \theta) \right] = I(\theta)^{-1/2} \mathcal{N}(0, 1)$$

uniformly with respect to θ varying in any compact subset of $\{\theta : \theta F(P) < 0\}$. So thanks to the convergence theorem of Ibragimov and Has'minskii (1981, Appendix I Theorem 8) the next statement follows immediately.

Theorem 6 *Assume that $F(P) \neq 0$. Then the MLE $\hat{\theta}_T$ is locally asymptotically minimax with local scale $T^{-1/2}$ at any θ such that $\theta F(P) < 0$, and for any bounded loss function $L(\cdot) \in \mathcal{L}$.*

Then for a loss function $L(\cdot) \in \mathcal{L}$ whose growth as $|u| \rightarrow \infty$ is lower than any one of the functions $e^{\epsilon|u|}$, $\epsilon > 0$, we readily deduce that

$$\lim_{b \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \liminf_{T \rightarrow \infty} \sup_{|\theta - \theta'| < \epsilon} E_{\theta'} \left[b \wedge L \left(\sqrt{T}(\hat{\theta}_T - \theta') \right) \right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} L \left(y I(\theta)^{-1/2} \right) e^{-y^2/2} dy.$$

4.2 Transient case, $\theta F(P) > 0$: LAMN property

From Lemma 2 and the proof of Theorem 5, we know that

$$\begin{aligned} \lim_{n \rightarrow \infty} e^{-2n\theta F(P)} J_{nP+} &= \left(K_\theta(\cdot) \left(\xi_0 + \zeta_3^{(\theta)} \right) \right)^2 \quad \text{in } L^1(P_\theta) \\ \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[\left(e^{-2n\theta F(P)} J_{nP+}, e^{-n\theta F(P)} U_{nP+} \right) \right] \\ &= \mathcal{L}_\theta \left[\left(K_\theta(\cdot)^2 \left(\xi_0 + \zeta_3^{(\theta)} \right)^2, \left(\xi_0 + \zeta_3^{(\theta)} \right) \left(\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot) \right) \right) \right] \quad \text{in } \mathcal{C}[0, P] \times \mathcal{C}[0, P] \end{aligned}$$

the limits being uniform with respect to θ varying in any compact subset of $\{\theta : \theta F(P) > 0\}$. The random variable $\xi_0 + \zeta_3^{(\theta)}$ is independent on the process $\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot)$ which is zero mean Gaussian with $\text{var} \left[\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(t) \right] = K_\theta(t)$ for any $t \in [0, P]$. Thus taking the local scale

$\delta_{nP+t}(\theta) = e^{-n\theta F(P)}$, we obtain that the random limit Fisher information of the model is equal to

$$\mathcal{I}^{(\theta)}(t) = K_\theta(t)^2 \left(\xi_0 + \zeta_3^{(\theta)} \right)^2.$$

Thus, from equality (14)

$$\Lambda_{nP+t}^{(\theta,u)} = u\Delta_{nP+t}^{(\theta)} - \frac{1}{2}u^2\mathcal{I}^{(\theta)}(t) + r_{nP+t}(\theta, u)$$

where $\Delta_{nP+t}^{(\theta)} := e^{-n\theta F(P)}U_{nP+t}$

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[\Delta_{nP+t}^{(\theta)} \right] = \mathcal{L} \left[\left(\xi_0 + \zeta_3^{(\theta)} \right) \left(\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot) \right) \right] = \mathcal{L} \left[\mathcal{I}^{(\theta)}(t)^{-1/2} Z \right]$$

where Z is some random variable independent on $\mathcal{I}^{(\theta)}(t)$ with law $\mathcal{N}(0,1)$. We deduce that for each $t \in [0, P]$, the sequence of families of distributions $\{\mathbf{P}_\theta^{nP+t} : \theta F(P) > 0\}$, $n \geq 1$, has a likelihood ratio which is LAMN (locally asymptotically mixing normal) (see e.g. Davies 1985; Jeganathan 1982). Following (Jeganathan 1995, Theorem 8; Le Cam and Yang 1990, Section 5.6), for each sequence $\{\bar{\theta}_{nP+t}\}$ of estimators of θ

$$\begin{aligned} \lim_{M \rightarrow \infty} \liminf_{n \rightarrow \infty} \sup_{|\theta - \theta'| < M e^{-n\theta F(P)}} \mathbf{E}_{\theta'} \left[L \left(e^{n\theta F(P)} (\bar{\theta}_{nP+t} - \theta') \right) \right] \\ \geq \mathbf{E}_\theta \left[L \left(\frac{\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(t)}{K_\theta(t) \left(\xi_0 + \zeta_3^{(\theta)} \right)} \right) \right] \end{aligned} \quad (16)$$

for any $t \in [0, P]$ and any $L \in \mathcal{L}$, each estimator $\bar{\theta}_{nP+t}$ being measurable with respect the observation $\{\xi_s : s \in [0, nP + t]\}$. Thanks to Theorem 5 and to (Ibragimov and Has'minskiĭ 1981, Appendix I Theorem 8), we can assert that

Theorem 7 *Assume that $F(P) \neq 0$. Then for any $t \in [0, T]$, the MLE $\hat{\theta}_{nP+t}$ is locally asymptotically minimax with local scale $e^{-n\theta F(P)}$ at any θ such $\theta F(P) > 0$, and for any bounded loss function $L(\cdot) \in \mathcal{L}$. That is we have equality in relation (16) for the MLE $\hat{\theta}_{nP+t}$.*

4.3 $F(P) = 0$, $f(\cdot)$ non constant : LAQ property

When $F(P) = 0$ the likelihood ratio $\Lambda_T^{(\theta,u)}$ can be expressed as

$$\Lambda_T^{\theta,u} = u\delta_T(\theta)U_T - \frac{u^2\delta_T(\theta)^2}{2}J_T.$$

In Lemma 1 we state that

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\left(\frac{J_T}{T^2}, \frac{U_T}{T} \right) \right] = \mathcal{L} \left[\left(\frac{G_\theta(P)H_\theta(P)}{P^2} \int_0^1 B_u^2 du, \frac{\sqrt{G_\theta(P)H_\theta(P)}}{P} \int_0^1 B_u dB'_u \right) \right]$$

in \mathbb{R}^2 , uniformly with respect to θ varying in any compact subset of \mathbb{R} . So here we take the scale $\delta_T := T^{-1}$.

We deduce that the model has a locally asymptotically quadratic (LAQ) likelihood ratio at any θ , and following (Jeganathan 1995, Definition 2) the MLE $\widehat{\theta}_T = U_T/J_T$ is an asymptotically centering estimator of the parameter θ . However the model is neither locally asymptotically normal nor locally asymptotically mixed normal. In the following result we state some local asymptotic efficiency of the MLE $\widehat{\theta}_T$ using a Bayesian version of the Cramér-Rao lower bound due to van Trees inequality (Gill and Levit 1995). By this way we only consider the quadratic mean risk function : $L(x) = x^2$

Theorem 8 *Assume that $F(P) = 0$, $f(\cdot)$ non constant. Then for each $\theta \in \mathbb{R}$*

$$\lim_{\epsilon \rightarrow 0} \liminf_{T \rightarrow \infty} \sup_{|\theta' - \theta| < \epsilon} \mathbb{E}_{\theta'} \left[T^2 (\bar{\theta}_T - \theta')^2 \right] \geq \frac{2P^2}{G_\theta(P)H_\theta(P)} \quad (17)$$

where $\bar{\theta}_T$ is any real-valued random variable which is measurable with respect to the observation $\{\xi_t, t \in [0, T]\}$. Furthermore the MLE $\widehat{\theta}_T$ is asymptotically minimax for the quadratic lost function.

5 Proofs

5.1 Proofs of results in Section 3

5.1.1 Proof of Theorem 1

It is only sufficient to prove that $J_\infty = \infty$ a.e.. Recall that the function $f(\cdot)$ is not identically null and notice that the quadratic variation J_T increases to J_∞ a.e. as $T \rightarrow \infty$.

i) First assume that $\theta F(P) < 0$. From Section 2.1 we know that the P -segments Markov sequence $\{\mathbf{X}_k\}$ is Harris positive recurrent in this case. Then the ergodic theorem (Höpfner and Kutoyants 2010, Theorem 2.1) and Proposition 1 entail that

$$\lim_{T \rightarrow \infty} \frac{J_T}{T} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(u)^2 \xi_u^2 du = I(\theta) > 0 \quad \mathbb{P}_\theta\text{-a.e.} \quad (18)$$

where $I(\theta)$ is defined in relation (13). This implies that $J_\infty = \infty$ a.e.

ii) Assume now that $\theta F(P) = 0$. For any $n > 0$ we have

$$J_{nP} \geq \int_0^P f(u)^2 \xi_{nP+u}^2 du.$$

In another hand Proposition 2 implies that

$$\lim_{n \rightarrow \infty} \mathbb{P}_\theta \left[\frac{1}{n} \int_0^P f(u)^2 \xi_{nP+u}^2 du \geq A \right] = \mathbb{P} \left[H_\theta(P) \left(\zeta_2^{(\theta)} \right)^2 \geq A \right]$$

for any $A \geq 0$, and where $\mathcal{L} \left(\zeta_2^{(\theta)} \right) = \mathcal{N}(0, G_\theta(P))$. Since the function $f(\cdot)$ is not identically null, we have $G_\theta(P)H_\theta(P) > 0$ and we obtain that J_{nP} converges in \mathbb{P}_θ -probability to infinity as $n \rightarrow \infty$. As J_{nP} increases to J_∞ , \mathbb{P}_θ -a.e. as $T \rightarrow \infty$, we deduce that $\mathbb{P}_\theta[J_\infty < \infty] = 0$.

iii) Finally assume that $\theta F(P) > 0$. According to Proposition 3,

$$\lim_{n \rightarrow \infty} e^{-2n\theta F(P)} \int_0^{\cdot} f(u)^2 \xi_{nP+u}^2 du = H_\theta(\cdot) \left(\xi_0 + \zeta_3^{(\theta)} \right)^2 \quad \text{P}_\theta\text{-a.e. in } \mathcal{C}[0, P]$$

where $\zeta_3^{(\theta)}$ is the sum of the P-a.e. convergent series $\sum_k e^{-k\theta F(P)} Z_k$. Thus $\mathcal{L} \left(\zeta_3^{(\theta)} \right) = \mathcal{N} \left(0, \frac{G_\theta(P)}{1 - e^{-2\theta F(P)}} \right)$ and the real-valued random variables ξ_0 and $\zeta_3^{(\theta)}$ are independent. Besides J_{nP} can be expressed as

$$J_{nP} = \sum_{k=0}^{n-1} \int_0^P f(u)^2 \xi_{kP+u}^2 du = \sum_{k=0}^{n-1} e^{2k\theta F(P)} e^{-2k\theta F(P)} \int_0^P f(u)^2 \xi_{kP+u}^2 du$$

for any positive integer n . Applying Toeplitz lemma on series convergence we obtain

$$\lim_{n \rightarrow \infty} e^{-2n\theta F(P)} J_{nP} = \frac{H_\theta(P)}{e^{2\theta F(P)} - 1} \left(\xi_0 + \zeta_3^{(\theta)} \right)^2 \quad \text{P}_\theta\text{-a.e. in } \mathbb{R}.$$

Since ξ_0 and $\zeta_3^{(\theta)}$ are independent and $\zeta_3^{(\theta)}$ is non-degenerated Gaussian, we have $\text{P}_\theta \left[\xi_0 + \zeta_3^{(\theta)} = 0 \right] = 0$. Hence $J_\infty = \infty$ P $_\theta$ -a.e. This completes the proof of the theorem. ■

5.1.2 Proof of Theorem 2

Recall that $J_T(\widehat{\theta}_T - \theta) = U_T$ where $\{J_t, t \geq 0\}$ is the quadratic variation process of the martingale $\{U_t, t \geq 0\}$ defined by relation (11). The recurrence property of the process $\{\xi_t, t > 0\}$, implies that J_T converges almost surely to $I(\theta)$ as $T \rightarrow \infty$. Then the central limit theorem for integrale with respect to Brownian motion (see e.g. Kutoyants 2004, Theorem 1.19; see also Feigin 1976; Barzcy and Pap 2010) entails the convergence in law

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\sqrt{J_T} \left(\widehat{\theta}_T - \theta \right) \right] = \mathcal{N}(0, 1)$$

for any θ such that $\theta F(P) < 0$. To get the uniform convergence with respect to θ , we need a little more computations and study concerning the asymptotic behavior of J_T/T .

Let $n \in \mathbb{N}$ and $t \in [0, P]$. Then

$$J_{nP+t} = \int_0^{nP+t} f(u)^2 \xi_u^2 du = \sum_{k=0}^{n-1} \int_0^P f(u)^2 \xi_{kP+u}^2 du + \int_0^t f(u)^2 \xi_{nP+u}^2 du \quad (19)$$

Thanks to decomposition (6), we know that

$$\xi_{kP+u} = e^{\theta F(u)} \left(e^{k\theta F(P)} \left(\xi_0 + S_{k-1} \right) + \mathbf{Z}_k(u) \right) \quad (20)$$

where

$$\mathbf{Z}_k(u) := \int_0^u e^{-\theta F(v)} dB_v^{(kP)}, \quad Z_k := \mathbf{Z}_k(P), \quad (21)$$

$$S_{-1} := 0 \quad \text{and} \quad S_{k-1} := \sum_{j=0}^{k-1} e^{-j\theta F(P)} Z_j = \int_0^{kP} e^{-\theta F(v)} dB_v \quad (22)$$

for any $k = 1, \dots$. Thus we can write J_{nP+t} as

$$\begin{aligned}
J_{nP+t} &= H_\theta(P) \sum_{k=0}^{n-1} e^{2k\theta F(P)} (\xi_0 + S_{k-1})^2 + \int_0^P f(u)^2 e^{2\theta F(u)} \sum_{k=0}^{n-1} \mathbf{Z}_k(u)^2 du \\
&\quad + 2 \sum_{k=0}^{n-1} e^{k\theta F(P)} (\xi_0 + S_{k-1}) \int_0^P f(u)^2 e^{2\theta F(u)} \mathbf{Z}_k(u) du + H_\theta(t) e^{2n\theta F(P)} (\xi_0 + S_{n-1})^2 \\
&\quad + \int_0^t f(u)^2 e^{2\theta F(u)} \mathbf{Z}_n(u)^2 du + 2e^{n\theta F(P)} (\xi_0 + S_{n-1}) \int_0^t f(u)^2 e^{2\theta F(u)} \mathbf{Z}_n(u) du \\
&:= A_1 + A_2 + A_3 + A_4 + A_5 + A_6. \tag{23}
\end{aligned}$$

We study the asymptotic behavior of each term. Using the facts that the processes \mathbf{Z} s are independent, and each of them is zero-mean Gaussian with independent increments, $\text{var}_\theta[\mathbf{Z}_k(u)] = G_\theta(u)$ for $u \in [0, P]$, we can prove that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{E}_\theta \left[\left(\frac{A_1}{n} - \frac{G_\theta(P)H_\theta(P)}{e^{-2\theta F(P)} - 1} \right)^2 \right] &= 0 \\
\lim_{n \rightarrow \infty} \mathbb{E}_\theta \left[\left(\frac{A_2}{n} - \int_0^P f(u)^2 e^{2\theta F(u)} G_\theta(u) du \right)^2 \right] &= 0
\end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \frac{A_3}{n} = \lim_{n \rightarrow \infty} \frac{A_4}{n} = \lim_{n \rightarrow \infty} \frac{A_5}{n} = \lim_{n \rightarrow \infty} \frac{A_6}{n} = 0 \quad \text{in } \mathbb{P}_\theta\text{-quadratic mean.}$$

Each convergence being uniform with respect to $t \in [0, P]$ and to θ in any compact subset of $\{\theta \in \mathbb{R} : \theta F(P) < 0\}$. Notice that in the parametric model, inequation (7) becomes

$$\mathbb{E}_\theta \left[\max_{t \in [0, P]} \mathbf{Z}_n(t)^2 \right] \leq 2 \int_0^P e^{-2\theta F(u)} du. \tag{24}$$

Thus we obtain that

$$\lim_{T \rightarrow \infty} \frac{J_T}{T} = I(\theta)$$

in \mathbb{P}_θ -quadratic mean uniformly with respect to θ varying in any compact subset of $\{\theta : \theta F(P) < 0\}$. Thanks to the uniform central limit theorem for integrals with respect to Brownian motion (Kutoyants 2004, Theorem 1.20) we deduce the assertions of the theorem. ■

5.1.3 Proof of Theorem 3

As we do not prove the convergence of J_T/T^2 in probability, the results in (Barczy and Pap 2010) cannot be applied here to establish Theorem 3. In fact this theorem is a direct consequence of the following result.

Lemma 1 *Assume that $F(P) = 0$ and $f(\cdot)$ not identically null. Then*

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left[\left(\frac{J_T}{T^2}, \frac{U_T}{T} \right) \right] = \mathcal{L} \left[\left(\frac{G_\theta(P)H_\theta(P)}{P^2} \int_0^1 B_u^2 du, \frac{\sqrt{G_\theta(P)H_\theta(P)}}{P} \int_0^1 B_u dB'_u \right) \right] \tag{25}$$

uniformly with respect to θ varying in any compact subset of \mathbb{R} , where $\{B_t, t \in [0, 1]\}$ and $\{B'_t, t \in [0, 1]\}$ are two independent Brownian motions.

Proof The proof is based on the method presented by Phillips (1987) who studied the asymptotic approach of an autoregressive time series with a root near the unit by an Ornstein-Uhlenbeck time-continuous process. First we establish that the asymptotic behaviors of J_{nP+t} and of U_{nP+t} coincide with those of sums of time series (formulae (26) and (28)). Then we apply Phillips' method to conclude.

i) J_{nP+t} . Let $n \in \mathbb{N}$ and $t \in [0, P]$ and consider decomposition (23) with $F(P) = 0$.

The random variables $Z_k, k = 0, \dots, n$, are independent with the same law $\mathcal{N}(0, G_\theta(P))$, so thanks to inequality (24), the first term is equal to

$$A_1 = H_\theta(P) \sum_{k=0}^{n-1} S_{k-1}^2 + o_{P_\theta}(n^2)$$

as $n \rightarrow \infty$, and A_4 is $o_{P_\theta}(n^2)$. Furthermore inequality (24) also entails that A_2 is $\mathcal{O}_{P_\theta}(n)$, and A_5 is $\mathcal{O}_{P_\theta}(1)$. With the previous arguments we easily obtain that A_3 and A_6 are $o_{P_\theta}(n)$. Hence we deduce that

$$\frac{J_{nP+t}}{n^2} = \frac{H_\theta(P)}{n^2} \sum_{k=0}^{n-1} S_{k-1}^2 + o_{P_\theta}(1) \quad (26)$$

as $n \rightarrow \infty$. Notice that $o_{P_\theta}(1)$ is uniform with respect to $t \in [0, P]$ and with respect to θ varying in any compact subset of \mathbb{R} .

ii) U_{nP+t} . Thanks to decomposition (20) we can express U_{nP+t} as

$$U_{nP+t} = \sum_{k=0}^{n-1} (Y_k (\xi_0 + S_{k-1}) + W_k) + \mathbf{Y}_n(t) (\xi_0 + S_{n-1}) + \mathbf{W}_n(t) \quad (27)$$

where $Y_k := \mathbf{Y}_k(P)$, $W_k := \mathbf{W}_k(P)$,

$$\begin{aligned} \mathbf{Y}_k(t) &:= \int_0^t f(u) e^{\theta F(u)} d\mathbf{B}_u^{(kP)} \quad \text{and} \\ \mathbf{W}_k(t) &:= \int_0^t f(u) e^{\theta F(u)} \mathbf{Z}_k(u) d\mathbf{B}_u^{(kP)} = \int_0^t \left(\int_0^u f(u) e^{\theta(F(u)-F(v))} d\mathbf{B}_v^{(kP)} \right) d\mathbf{B}_u^{(kP)} \end{aligned}$$

for any $k \in \mathbb{N}$ and any $t \in [0, P]$. The random variables $Y_k, k = 0, \dots$, are independent with the same law $\mathcal{N}(0, H_\theta(P))$. The random variables $W_k, k = 0, \dots$, are also independent with the same law which is with zero mean and finite variance. Then the law of large numbers entails that the first term of expression (27) is equal to

$$U_{nP} = \sum_{k=0}^{n-1} Y_k S_{k-1} + o_{P_\theta}(n)$$

as $n \rightarrow \infty$ uniformly with respect to θ varying in any compact subset of \mathbb{R} . For each integer n , the processes $\{\mathbf{Y}_n(t), t \in [0, P]\}$ and $\{\mathbf{W}_n(t), t \in [0, P]\}$ are continuous martingales with respect to the filtration $\{\mathcal{F}_t^{(nP)}, t \in [0, P]\}$ generated by the standard Brownian motion $\{B_t^{(nP)}, t \in [0, P]\}$. Thanks to Doob maxima inequality for continuous martingales

$$\begin{aligned} \mathbb{E}_\theta \left[\max_{t \in [0, P]} \mathbf{Y}_n(t)^2 \right] &\leq 4\mathbb{E}_\theta [\mathbf{Y}_n(P)^2] = 4H_\theta(P) < \infty \\ \mathbb{E}_\theta \left[\max_{t \in [0, P]} \mathbf{W}_n(t)^2 \right] &\leq 4\mathbb{E}_\theta [\mathbf{W}_n(P)^2] = 4 \int_0^P f(u)^2 e^{2\theta F(u)} G_\theta(u) du < \infty. \end{aligned}$$

Thus the second term of expression (27) is of order $o_{P_\theta}(n)$ and the third term is $\mathcal{O}_{P_\theta}(1)$, as $n \rightarrow \infty$. Hence

$$\frac{U_{nP+t}}{n} = \frac{1}{n} \sum_{k=0}^{n-1} Y_k S_{k-1} + o_{P_\theta}(1). \quad (28)$$

uniformly with respect to $t \in [0, P]$ and to θ varying in any compact subset of \mathbb{R} as $n \rightarrow \infty$.

iii) From the previous computations, it remains to study the asymptotic behavior of the random vector

$$\left(\frac{1}{n^2} \sum_{k=1}^{n-1} S_{k-1}^2, \frac{1}{n} \sum_{k=1}^{n-1} Y_k S_{k-1} \right) = \left(\frac{1}{n^2} \sum_{k=1}^{n-1} \left(\sum_{j=0}^{k-1} Z_j \right)^2, \frac{1}{n} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Y_k Z_j \right).$$

For that we are going to construct another random vector with the same law and which converges in probability as $n \rightarrow \infty$ to the required limit (see e.g. Phillips 1987). Since

$$Y_j = \int_{jP}^{(j+1)P} f(u) e^{\theta F(u)} dB_u, \quad \text{and} \quad Z_j = \int_{jP}^{(j+1)P} e^{-\theta F(u)} dB_u$$

for any $j \in \mathbb{N}$, the random vector $(Y_0, \dots, Y_{n-1}, Z_0, \dots, Z_{n-1})$ is Gaussian and $\mathbb{E}_\theta[Y_j Y_k] = \mathbb{E}_\theta[Y_j Z_k] = \mathbb{E}_\theta[Z_j Z_k] = 0$, for all integers $j \neq k$. Furthermore by hypothesis

$$\mathbb{E}_\theta[Y_j Z_j] = \int_{jP}^{(j+1)P} f(u) e^{\theta F(u)} \times e^{-\theta F(u)} du = \int_{jP}^{(j+1)P} f(u) du = F(P) = 0$$

for any integer j . The last equality is due to the hypothesis that $F(P) = 0$. Thanks to the properties of Gaussian vectors, we deduce that the random variables $Y_0, \dots, Y_{n-1}, Z_0, \dots, Z_{n-1}$ are independent, $\mathcal{L}_\theta(Y_j) = \mathcal{N}(0, H_\theta(P))$ and $\mathcal{L}_\theta(Z_j) = \mathcal{N}(0, G_\theta(P))$ for any integer j . Consider $\{B_t, t \in [0, 1]\}$ and $\{B'_t, t \in [0, 1]\}$ two independent standard Brownian motions. Let

$$Y_j^{(\theta)} := \sqrt{nH_\theta(P)} \left(B'_{\frac{j+1}{n}} - B'_{\frac{j}{n}} \right), \quad \text{and} \quad Z_j^{(\theta)} := \sqrt{nG_\theta(P)} \left(B_{\frac{j+1}{n}} - B_{\frac{j}{n}} \right)$$

for any $j = 0, \dots, n-1$. Then the random variables $Y_0^{(\theta)}, \dots, Y_{n-1}^{(\theta)}, Z_0^{(\theta)}, \dots, Z_{n-1}^{(\theta)}$ are independent and $\mathcal{L}_\theta(Y_0, \dots, Y_{n-1}, Z_0, \dots, Z_{n-1}) = \mathcal{L}(Y_0^{(\theta)}, \dots, Y_{n-1}^{(\theta)}, Z_0^{(\theta)}, \dots, Z_{n-1}^{(\theta)})$. As a consequence,

$$\mathcal{L}_\theta \left[\left(\frac{1}{n^2} \sum_{k=1}^{n-1} \left(\sum_{j=0}^{k-1} Z_j \right)^2, \frac{1}{n} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Y_k Z_j \right) \right] = \mathcal{L} \left[\left(\frac{1}{n^2} \sum_{k=1}^{n-1} \left(\sum_{j=0}^{k-1} Z_j^{(\theta)} \right)^2, \frac{1}{n} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Y_k^{(\theta)} Z_j^{(\theta)} \right) \right].$$

Furthermore

$$\frac{1}{n^2} \sum_{k=1}^{n-1} \left(\sum_{j=0}^{k-1} Z_j^{(\theta)} \right)^2 = \frac{G_\theta(P)}{n} \sum_{k=1}^{n-1} \left(B_{\frac{k}{n}} \right)^2$$

and

$$\frac{1}{n} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Y_k^{(\theta)} Z_j^{(\theta)} = \sqrt{G_\theta(P)H_\theta(P)} \sum_{k=1}^{n-1} \left(B'_{\frac{k+1}{n}} - B'_{\frac{k}{n}} \right) B_{\frac{k}{n}}.$$

Since we consider a standard Brownian motion $\{B_t, t \in [0, 1]\}$, so a version of the Brownian motion such that P-almost all its sample paths are continuous, the following P-almost-everywhere limit exists and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(B_{\frac{k}{n}} \right)^2 = \int_0^1 B_u^2 du \quad \text{P - a.e.} \quad (29)$$

where the right-hand side integrale has to be understood path by path and with respect to Lebesgue measure on $[0, 1]$. Thanks to the theory of integration with respect to standard Brownian motion and since $\int_0^1 \mathbb{E} [B_u^2] du = 1/2 < \infty$, the process $\{B_t, t \in [0, 1]\}$ is integrable with respect to the Brownian motion $\{B'_t, t \in [0, 1]\}$ and the following in-probability limit exists

$$\text{P - } \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} B_{\frac{k}{n}} \left(B'_{\frac{k+1}{n}} - B'_{\frac{k}{n}} \right) = \int_0^1 B_u dB'_u. \quad (30)$$

Hence

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[\left(\frac{1}{n^2} \sum_{k=1}^{n-1} S_{k-1}^2, \frac{1}{n} \sum_{k=1}^{n-1} Y_k S_{k-1} \right) \right] = \mathcal{L} \left[\left(G_\theta(P) \int_0^1 B_u^2 du, \sqrt{G_\theta(P)H_\theta(P)} \int_0^1 B_u dB'_u \right) \right]$$

and the convergence is uniform in θ varying in any compact subset of \mathbb{R} . Then with relations (26) and (28), we deduce that

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[\frac{J_{nP+\cdot}}{n^2}, \frac{U_{nP+\cdot}}{n} \right] = \mathcal{L} \left[G_\theta(P)H_\theta(P) \int_0^1 B_u^2 du, \sqrt{G_\theta(P)H_\theta(P)} \int_0^1 B_u dB'_u \right]$$

in $\mathcal{C}[0, P] \times \mathcal{C}[0, P]$. This gives convergence (25) uniformly with respect to θ varying in any compact subset of \mathbb{R} . ■

5.1.4 Proof of Theorem 4

Here the true value of the parameter θ is equal to 0 and the function $f(\cdot)$ is non identically null. The observation is $\{\xi_t, t \in [0, T]\} = \{\xi_0 + B_t, t \in [0, T]\}$. With respect to the P-periodic Langevin model (1), the maximum likelihood estimator is still defined by relation (10). Following the same argument as in the proof of Lemma 1, we obtain that

$$J_{nP+t} = H_0(P) \sum_{k=1}^{n-1} \left(\sum_{j=0}^{k-1} Z_j \right)^2 + o_{\text{P}_0}(n^2)$$

and

$$U_{nP+t} = \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Y_k Z_j + o_{P_0}(n)$$

uniformly in $t \in [0, P]$, where

$$Z_j = B_{(j+1)P} - B_{jP}, \quad Y_k = \int_{kP}^{(k+1)P} f(u) dB_u \quad \text{and} \quad H_0(t) = \int_0^t f(u)^2 du.$$

Unlike the case where $F(P) = 0$, here $F(P)$ is not necessarily null and the random variables Z_j and Y_j are not necessarily independent:

$$\text{cov}_0 [Y_j, Z_j] = E_0 [Y_j Z_j] = F(P)$$

for any $j \in \mathbb{N}$. As $\text{var}_0[Z_j] = P$, the residual part of the linear regression of Y_j on Z_j is equal to $Y_j^* := Y_j - \frac{F(P)}{P} Z_j$. Then the random vector $(Y_0^*, \dots, Y_{n-1}^*, Z_0, \dots, Z_{n-1})$ is a Gaussian vector, its components are independent and $\text{var}_0[Y_j^*] = H^* := H_0(P) - \frac{F(P)^2}{P}$. Notice also that

$$\sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Y_k Z_j = \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Y_k^* Z_j + \frac{F(P)}{P} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Z_k Z_j$$

Now following again the argument of the proof of Lemma 1, consider $\{B_t : t \in [0, 1]\}$ and $\{B'_t : t \in [0, 1]\}$ two independent standard Brownian motions. Let

$$Y_j^{(0)} = \sqrt{nH^*} \left(B'_{\frac{j+1}{n}} - B'_{\frac{j}{n}} \right) \quad \text{and} \quad Z_j^{(0)} = \sqrt{nP} \left(B_{\frac{j+1}{n}} - B_{\frac{j}{n}} \right)$$

for any $j = 0, \dots, n-1$. Then $\mathcal{L}_0 [Y_0^*, \dots, Y_{n-1}^*, Z_0, \dots, Z_{n-1}] = \mathcal{L} [Y_0^{(0)}, \dots, Y_{n-1}^{(0)}, Z_0^{(0)}, \dots, Z_{n-1}^{(0)}]$ and the law of the random vector

$$\left(\sum_{k=1}^{n-1} \left(\sum_{j=0}^{k-1} Z_j \right)^2, \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Y_k^* Z_j, \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Z_k Z_j \right)$$

coincides with the law of

$$\left(\sum_{k=1}^{n-1} \left(\sum_{j=0}^{k-1} Z_j^{(0)} \right)^2, \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Y_k^{(0)} Z_j^{(0)}, \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} Z_k^{(0)} Z_j^{(0)} \right)$$

thus with the law of

$$\left(nP \sum_{k=1}^{n-1} \left(B_{\frac{k}{n}} \right)^2, n\sqrt{H^*P} \sum_{k=1}^{n-1} \left(B'_{\frac{k+1}{n}} - B'_{\frac{k}{n}} \right) B_{\frac{k}{n}}, nP \sum_{k=1}^{n-1} \left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \right) B_{\frac{k}{n}} \right).$$

Thanks to convergences (29) and (30) as well as

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \right) B_{\frac{k}{n}} = \int_0^1 B_u dB_u = \frac{B_1^2 - 1}{2} \quad \text{in probability}$$

we deduce that

$$\lim_{T \rightarrow \infty} \mathcal{L}_0 \left[\left(\frac{J_T}{T^2}, \frac{U_T}{T} \right) \right] = \mathcal{L} \left[\left(\frac{H_0(P)}{P} \int_0^1 B_u^2 du, \sqrt{\frac{H^*}{P}} \int_0^1 B_u dB'_u + \frac{F(P)}{P} \int_0^1 B_u dB_u \right) \right].$$

where $H^* = H_0(P) - F(P)^2/P$. Thus

$$\lim_{T \rightarrow \infty} \mathcal{L}_0 \left[T \widehat{\theta}_T \right] = \mathcal{L} \left[\frac{\sqrt{PH_0(P) - F(P)^2} \int_0^1 B_u dB'_u + F(P) \int_0^1 B_u dB_u}{H_0(P) \int_0^1 B_u^2 du} \right].$$

■

5.1.5 Proof of Theorem 5

We first study the asymptotic behavior of the quadratic variation $\{J_t, t \geq 0\}$ of the martingale $\{U_t, t \geq 0\}$.

Lemma 2

$$\lim_{n \rightarrow \infty} e^{-2n\theta F(P)} J_{nP+} = K_\theta(\cdot)^2 \left(\xi_0 + \zeta_3^{(\theta)} \right)^2 \quad \text{P}_\theta\text{-a.e. in } \mathcal{C}[0, P] \quad (31)$$

where $K_\theta(\cdot)$ is defined in Theorem 5 and $\zeta_3^{(\theta)}$ is some real-valued Gaussian variable independent on ξ_0 . Furthermore the convergence is also in $L^1(\text{P}_\theta)$ uniformly with respect to $t \in [0, P]$ and to θ varying in any compact subset of $\{\theta : \theta F(P) > 0\}$ that is

$$\lim_{n \rightarrow \infty} \sup_{\theta \in K_+} \sup_{t \in [0, P]} \text{E}_\theta \left[\left| e^{-2n\theta F(P)} J_{nP+t} - K_\theta(t)^2 \left(\xi_0 + \zeta_3^{(\theta)} \right)^2 \right| \right] = 0$$

for any compact set $K_+ \subset \{\theta \in \mathbb{R} : \theta F(P) > 0\}$.

Proof According to Proposition 3, we know that

$$\lim_{n \rightarrow \infty} e^{-2n\theta F(P)} \int_0^{\cdot} f(u)^2 \xi_{nP+u}^2 du = H_\theta(\cdot) \left(\xi_0 + \zeta_3^{(\theta)} \right)^2 \quad \text{P}_\theta\text{-a.e. in } \mathcal{C}[0, P]$$

where $\zeta_3^{(\theta)}$ is some real-valued zero-mean Gaussian variable independent on ξ_0 , and with variance equal to $\frac{G_\theta(P)}{1 - e^{2\theta F(P)}}$. Besides J_{nP} can be expressed as

$$J_{nP} = \sum_{k=0}^{n-1} \int_0^P f(u)^2 \xi_{kP+u}^2 du = \sum_{k=0}^{n-1} e^{2k\theta F(P)} e^{-2k\theta F(P)} \int_0^P f(u)^2 \xi_{kP+u}^2 du$$

for any integer $n > 0$. Then Toeplitz lemma on series convergence implies the almost sure convergence (31).

To prove the uniform $L^1(\text{P}_\theta)$ -convergence, let K_+ be any compact subset of $\{\theta \in \mathbb{R} : \theta F(P) > 0\}$ and consider decomposition (23) which is also valid for $\theta F(P) > 0$.

Since the random variables Z_k s are independent with the same law $\mathcal{N}(0, G_\theta(P))$, the hypothesis $\theta F(P) > 0$ entails that S_n converges in quadratic mean uniformly with respect to θ varying in K_+ as $n \rightarrow \infty$. It is easy to see that the limit is equal to $\zeta_3^{(\theta)}$ defined above. So $e^{-2n\theta F(P)} A_4 = e^{-2n\theta F(P)} H_\theta(t) (\xi_0 + S_n)^2$ converges in $L^1(P_\theta)$ uniformly with respect to $t \in [0, P]$ and to θ varying in K_+ as $n \rightarrow \infty$ to $H_\theta(t) (\xi_0 + \zeta_3^{(\theta)})^2$. Furthermore, applying Toeplitz lemma on series convergence in $L^1(P_\theta)$ we deduce that $e^{-2n\theta F(P)} A_1$ converges in $L^1(P_\theta)$ uniformly to θ varying in K_+ as $n \rightarrow \infty$ to

$$\frac{H_\theta(P)}{e^{2\theta F(P)} - 1} (\xi_0 + \zeta_3^{(\theta)})^2.$$

Inequality (24) implies that $e^{-2n\theta F(P)} (A_2 + A_5)$ converges to 0 in $L^1(P_\theta)$ uniformly with respect to $t \in [0, P]$ and to θ varying in K_+ as $n \rightarrow \infty$. Cauchy-Schwarz inequality involves that

$$(\mathbb{E}_\theta [|(\xi_0 + S_n) \mathbf{Z}_n(t)|])^2 \leq \mathbb{E}_\theta [|\xi_0 + S_n|^2] \mathbb{E}_\theta [|\mathbf{Z}_n(t)|^2],$$

and thanks to inequality (24) we obtain that $e^{-2n\theta F(P)} (A_3 + A_6)$ converge to 0 in L^1 uniformly with θ varying in K_+ as $n \rightarrow \infty$. Hence the proof of the lemma is achieved. ■

Remark that

$$\lim_{T \rightarrow \infty} J_T = \infty \quad \mathbb{P}_\theta\text{-a.e. in } \mathbb{R}$$

when $\theta F(P) > 0$. Then the central limit theorem for martingale (see e.g. Kutoyants 2004, Theorem 1.19), applies and

$$\lim_{T \rightarrow \infty} \mathcal{L}_\theta \left(J_T^{-1/2} \int_0^T f(u) \xi_u dB_u \right) = \mathcal{N}(0, 1).$$

However to get the uniform convergence we need to refine the analysis. In Lemma 2 we have obtained the convergence of $e^{-2n\theta F(P)} J_{nP+}$. To prove Theorem 5, first we go further into detail of the asymptotic behavior of $e^{-n\theta F(P)} U_{nP+t}$. Next we can achieve the proof of the theorem.

Lemma 3

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[e^{-n\theta F(P)} U_{nP+} \right] = \mathcal{L} \left[\left(\xi_0 + \zeta_3^{(\theta)} \right) \left(\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot) \right) \right] \quad \text{in } \mathcal{C}[0, P]$$

uniformly with respect to θ in any compact subset of $\{\theta \in \mathbb{R} : \theta F(P) > 0\}$, where $\xi_0, \zeta_3^{(\theta)}, \zeta_4^{(\theta)}$ and $\mathbf{Y}^{(\theta)}$ are defined in Theorem 5.

Proof The proof is motivated by the proof of this result in the case where the drift $f(\cdot)$ is identically equal to 1 as presented by Kutoyants (2004, Proposition 3.46) (see also Basawa and Scott 1983).

Mainly consider the Gaussian process $\{\psi_t, t \geq 0\}$ defined by

$$\psi_t := \int_0^t f(u) e^{\theta F(u)} dB_u.$$

It is zero-mean and the variance of ψ_{nP+t} is equal to

$$\int_0^{nP+t} f(u)^2 e^{2\theta F(u)} du = \frac{(e^{2n\theta F(P)} - 1) H_\theta(P)}{e^{2\theta F(P)} - 1} + e^{2n\theta F(P)} H_\theta(t), \quad (32)$$

for all $t \in [0, P]$ and $n \in \mathbb{N}$. Thanks to Itô formula we have for any $T > 0$

$$\begin{aligned} U_T &= \xi_0 \psi_T + \int_0^T M_u d\psi_u \\ &= (\xi_0 + M_T) \psi_T - \int_0^T \psi_u dM_u - \frac{1}{2} [\psi, M]_T \end{aligned} \quad (33)$$

where $\{M_t := e^{\theta F(t)} \xi_t - \xi_0, t \geq 0\}$ is a martingale, and the quadratic covariation $[\psi, M]_T$ is equal to

$$[\psi, M]_T = \int_0^T f(u) e^{-\theta F(u)} e^{\theta F(u)} du = \int_0^T f(u) du = F(T).$$

Now we are going to study the asymptotic behavior of each term of expression (33). Let K_+ any compact subset of $\{\theta \in \mathbb{R} : \theta F(P) > 0\}$.

i) The periodicity of the function $f(\cdot)$ implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n} [\psi, M]_{nP+t} = F(P)$$

uniformly with respect to $t \in [0, P]$ and $\theta \in K_+$.

ii) The second term of the right hand side of expression (33) can be decomposed as

$$\begin{aligned} \int_0^T \psi_u dM_u &= \int_0^{nP} \psi_u dM_u + \int_{nP}^{nP+t} e^{-\theta F(u)} \psi_u dB_u \\ &= \int_0^{nP} \psi_u dM_u + e^{-n\theta F(P)} \mathbf{V}_n(t), \end{aligned} \quad (34)$$

the $\mathcal{C}[0, P]$ -valued random sequence $(\mathbf{V}_n)_{n \in \mathbb{N}}$ being defined by

$$\mathbf{V}_n(t) := \int_0^t e^{-\theta F(u)} \psi_{nP+u} dB_u^{(nP)}.$$

The variance of the first term of the right hand side of equality (34) verifies

$$\mathbb{E}_\theta \left[\left(\int_0^{nP} \psi_u dM_u \right)^2 \right] = \mathbb{E}_\theta \left[\left(\int_0^{nP} e^{-\theta F(u)} \psi_u dB_u \right)^2 \right] = \int_0^{nP} e^{-2\theta F(u)} \mathbb{E}_\theta [\psi_u^2] du.$$

In the following using again the fact that $F(kP + u) = kF(P) + F(u)$, and replacing $\mathbb{E}_\theta[\psi_{kP+u}^2]$ by the corresponding expression (32), we obtain

$$\begin{aligned} \int_0^{nP} e^{-2\theta F(u)} \mathbb{E}_\theta [\psi_u^2] du &= \sum_{k=0}^{n-1} \int_0^P e^{-2\theta(kF(P)+F(u))} \mathbb{E}_\theta [\psi_{kP+u}^2] du \\ &= \frac{H_\theta(P)G_\theta(P)}{e^{2\theta F(P)} - 1} \left(n - \frac{1 - e^{-2n\theta F(P)}}{1 - e^{-2\theta F(P)}} \right) + n \int_0^P H_\theta(u) e^{-2\theta F(u)} du. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{nP} e^{-2\theta F(u)} \mathbb{E}_\theta [\psi_u^2] du = \frac{H_\theta(P)G_\theta(P)}{e^{2\theta F(P)} - 1} + \int_0^P H_\theta(u) e^{-2\theta F(u)} du$$

and $\frac{1}{n} \int_0^{nP} \psi_u dM_u$ converges to 0 in P_θ -quadratic mean uniformly with respect to $\theta \in K_+$ as $n \rightarrow \infty$.

As for the second term of the right hand side of equality (34), notice that for each integer n we have

$$\mathbf{V}_n(t) = \mathbf{V}_n(s) + \psi_{s+nP} \int_s^t e^{-\theta F(u)} dB_u^{(nP)} + \int_s^t e^{-\theta F(u)} \left(\int_{s+nP}^{nP+u} f(v) e^{\theta F(v)} dB_v \right) dB_u^{(nP)}$$

when $0 \leq s \leq t \leq P$, which entails that the process $\mathbf{V}_n := \{\mathbf{V}_n(t), t \in [0, P]\}$ is a continuous martingale with respect to the filtration $\{\mathcal{F}_t^{(nP)} : 0 \leq t \leq P\}$ generated by $B^{(nP)} = \{B_t^{(nP)}, t \in [0, P]\}$. It is zero-mean, and by equality (32) it verifies

$$\begin{aligned} \mathbb{E}_\theta [\mathbf{V}_n(t)^2] &= \int_0^t e^{-2\theta F(u)} \mathbb{E}_\theta [\psi_{nP+u}^2] du \\ &= \frac{(e^{2n\theta F(P)} - 1) H_\theta(P)G_\theta(t)}{e^{2\theta F(P)} - 1} + e^{2n\theta F(P)} \int_0^t e^{-2\theta F(u)} H_\theta(u) du. \end{aligned}$$

The Doob maxima inequality for continuous martingales implies that

$$P_\theta \left[\max_{t \in [0, P]} |\mathbf{V}_n(t)| > \epsilon \right] \leq \frac{\mathbb{E}_\theta [\mathbf{V}_n(P)^2]}{\epsilon^2}$$

thus

$$P_\theta \left[e^{-n\theta F(P)} \max_{t \in [0, P]} |\mathbf{V}_n(t)| > \epsilon \right] \leq \frac{1}{\epsilon^2} \left(\frac{H_\theta(P)G_\theta(P)}{e^{2\theta F(P)} - 1} + \int_0^P e^{-2\theta F(u)} H_\theta(u) du \right)$$

and

$$\lim_{n \rightarrow \infty} \frac{e^{-n\theta F(P)}}{n} \mathbf{V}_n = 0 \quad \text{in } P_\theta\text{-probability in } \mathcal{C}[0, P]$$

uniformly with respect to $\theta \in K_+$. Hence the following convergence to 0 of the second term of expression (33)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{nP+} \psi_u dM_u = 0 \quad \text{in } P_\theta\text{-probability in } \mathcal{C}[0, P]$$

uniformly with respect to $\theta \in K_+$.

iii) By the definition of the process $\{\psi_t, t \geq 0\}$, and by the periodicity of the function $f(\cdot)$, we have

$$\psi_{nP+t} = \int_0^{nP} f(u)e^{\theta F(u)} dB_u + e^{n\theta F(P)} \int_0^t f(u)e^{\theta F(u)} dB_u^{(nP)},$$

the two terms being Gaussian and independent. We readily deduce that

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[e^{-n\theta F(P)} \psi_{nP+\cdot} \right] = \mathcal{L} \left[\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot) \right] \quad \text{in } \mathcal{C}[0, P]$$

uniformly with respect to $\theta \in K_+$ where $\zeta_4^{(\theta)}$ is a real-valued Gaussian variable with zero-mean and variance equal to $H_\theta(P) (e^{2\theta F(P)} - 1)^{-1}$, and the P -segment process $\mathbf{Y}^{(\theta)} := \{\mathbf{Y}^{(\theta)}(t), t \in [0, P]\}$ is Gaussian and admits the following representation

$$\mathbf{Y}^{(\theta)}(t) = \int_0^t f(u)e^{\theta F(u)} dB'_u$$

for some standard Brownian motion $\{B'_t, t \in [0, P]\}$ such that the random variable $\zeta_4^{(\theta)}$ and the process $\mathbf{Y}^{(\theta)}$ are independent.

iv) Following the proof of Proposition 3 we can see that M_T converges in P_θ -probability to some Gaussian variable $\zeta_3^{(\theta)}$ as $T \rightarrow \infty$, uniformly with respect to $\theta \in K_+$. So it remains to establish the joint convergence of the terms of expression (33), and the independence between the different random limits to conclude the proof of Lemma 3.

First recall that by hypothesis the initial value ξ_0 is independent on the Brownian motion $\{B_t, t \geq 0\}$. Now let $n_1 < n$. Since

$$M_{n_1 P+t} = \int_0^{n_1 P+t} e^{-\theta F(u)} dB_u \quad \text{and} \quad \psi_{nP+t} - \psi_{(n_1+1)P} = \int_{(n_1+1)P}^{nP+t} f(u)e^{\theta F(u)} dB_u$$

for any $t \in [0, P]$, the random variables ξ_0 , $\psi_{nP} - \psi_{(n_1+1)P}$, and the processes $\{M_{n_1 P+t}, t \in [0, P]\}$, $\{\psi_{nP+t} - \psi_{nP}, t \in [0, P]\}$ are independent. Consequently, thanks to the convergence results for M_T and ψ_T , the law

$$\mathcal{L}_\theta \left[\left(\xi_0, M_{\cdot+n_1 P}, e^{-n\theta F(P)} \left(\psi_{nP} - \psi_{(n_1+1)P} \right), e^{-n\theta F(P)} \left(\psi_{nP+\cdot} - \psi_{nP} \right) \right) \right]$$

converges to the law $\mathcal{L} \left[\left(\xi_0, \zeta_3^{(\theta)}, \zeta_4^{(\theta)}, \mathbf{Y}^{(\theta)}(\cdot) \right) \right]$ in $\mathbb{R} \times \mathcal{C}[0, P] \times \mathbb{R} \times \mathcal{C}[0, P]$, as n_1 and n go to infinity as well as $n - n_1$. We deduce that the random variables ξ_0 , $\zeta_3^{(\theta)}$, $\zeta_4^{(\theta)}$ and the process $\{\mathbf{Y}^{(\theta)}(t), t \in [0, P]\}$ are independent. Furthermore from expression (32), we easily verify that $e^{-n\theta F(P)} \psi_{(n_1+1)P}$ converges in P_θ -quadratic mean to 0 uniformly with respect to $\theta \in K_+$ as $n - n_1$ goes to infinity. Then thanks to Slutsky lemma, we readily deduce that

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[\left(\xi_0, M_{nP+\cdot}, e^{-n\theta F(P)} \psi_{nP+\cdot} \right) \right] = \mathcal{L} \left[\left(\xi_0, \zeta_3^{(\theta)}, \zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot) \right) \right]$$

uniformly with respect to $\theta \in K_+$ as $n \rightarrow \infty$. Finally, Itô formula (33) and the convergence results of the previous parts (i), (ii) and (iii) entail that

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[e^{-n\theta F(P)} U_{nP+} \right] = \mathcal{L} \left[\left(\xi_0 + \zeta_3^{(\theta)} \right) \left(\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot) \right) \right] \quad \text{in } \mathcal{C}[0, P]$$

uniformly with respect to $\theta \in K_+$. ■

Proof of Theorem 5 Since

$$e^{n\theta F(P)} \left(\widehat{\theta}_{nP+t} - \theta \right) = \frac{e^{-n\theta F(P)} U_{nP+t}}{e^{-2n\theta F(P)} J_{nP+t}},$$

it suffices to establish the convergence of $\mathcal{L} \left(e^{-2n\theta F(P)} J_{nP+t}, e^{-n\theta F(P)} U_{nP+t} \right)$ in $\mathcal{C}[0, P] \times \mathcal{C}[0, P]$ as $n \rightarrow \infty$. For this purpose we follow the idea of part (iv) of the proof of Lemma 3, and we use the same notations. Let $n_1 < n$. By definition

$$M_{n_1 P+t} = \int_0^{n_1 P+t} e^{-\theta F(u)} dB_u \quad \text{and} \quad J_{n_1 P+t} = \int_0^{n_1 P+t} f(u)^2 \xi_u^2 dB_u.$$

The fact that the processes $\{\xi_{n_1 P+t}, t \in [0, P]\}$ and $\{B_{(n_1+1)P+t} - B_{(n_1+1)P}, t \geq 0\}$ are independent, implies that the random variable $\psi_{nP} - \psi_{(n_1+1)P}$ and the processes $\{(M_{n_1 P+t}, J_{n_1 P+t}), t \in [0, P]\}$, $\{\psi_{nP+t} - \psi_{nP}, t \in [0, P]\}$ are independent. According the convergence results for M_{nP+t} , J_{nP+t} and ψ_{nP+t} , the law

$$\mathcal{L}_\theta \left[\left(\xi_0, M_{nP+}, e^{-2n\theta F(P)} J_{nP+}, e^{-n\theta F(P)} \psi_{nP+} \right) \right]$$

converges to $\mathcal{L} \left[\left(\xi_0, \zeta_3^{(\theta)}, K_\theta(\cdot)^2 \left(\xi_0 + \zeta_3^{(\theta)} \right)^2, \zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot) \right) \right]$ in $\mathbb{R} \times \mathcal{C}[0, P] \times \mathcal{C}[0, P] \times \mathcal{C}[0, P]$.

Hence

$$\lim_{n \rightarrow \infty} \mathcal{L}_\theta \left[\left(e^{-2n\theta F(P)} J_{nP+}, e^{-n\theta F(P)} U_{nP+} \right) \right] = \mathcal{L} \left[K_\theta(\cdot)^2 \left(\xi_0 + \zeta_3^{(\theta)} \right)^2, \left(\xi_0 + \zeta_3^{(\theta)} \right) \left(\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(\cdot) \right) \right]$$

in $\mathcal{C}[0, P] \times \mathcal{C}[0, P]$. The convergences being uniform with respect to θ varying in any compact subset of $\{\theta \in \mathbb{R} : \theta F(P) > 0\}$. As $\mathbb{P} \left[\xi_0 + \zeta_3^{(\theta)} = 0 \right] = 0$, we readily achieve the proof of the theorem. ■

5.2 Proofs of results in Section 4

5.2.1 Proof of Theorem 6

This theorem is a direct consequence of the uniform convergence in distribution in Theorem 2, and of the convergence theorem of Ibragimov and Has'minskii (1981, Appendix I Theorem 8).

■

5.2.2 Proof of Theorem 7

Thanks to the uniform convergence in distribution in Theorem 5, the convergence theorem of Ibragimov and Has'minskiĭ (1981, Appendix I Theorem 8) entails that for each θ such that $\theta F(P) > 0$,

$$\liminf_{n \rightarrow \infty} \sup_{|\theta - \theta'| < M e^{-n\theta F(P)}} \mathbf{E}_{\theta'} \left[L \left(e^{n\theta' F(P)} (\bar{\theta}_{nP+t} - \theta') \right) \right] = \mathbf{E} \left[L \left(\frac{\zeta_4^{(\theta)} + \mathbf{Y}^{(\theta)}(t)}{K_\theta(t) (\xi_0 + \zeta_3^{(\theta)})} \right) \right]$$

for any $t \in [0, P]$ and any $M > 0$. So to prove the theorem, it is sufficient to establish that the normalizing coefficient $e^{n\theta' F(P)}$ can be replaced by $e^{n\theta F(P)}$. In fact, since $\theta F(P) > 0$, we have

$$e^{-n \left| 1 - \frac{\theta'}{\theta} \right| \theta F(P)} \leq e^{n(\theta - \theta') F(P)} \leq e^{n \left| 1 - \frac{\theta'}{\theta} \right| \theta F(P)}.$$

So if $|\theta - \theta'| < M e^{-n\theta F(P)}$ then

$$e^{-\frac{M}{|\theta|} n\theta F(P) e^{-n\theta F(P)}} \leq e^{n(\theta - \theta') F(P)} \leq e^{\frac{M}{|\theta|} n\theta F(P) e^{-n\theta F(P)}}.$$

However

$$\lim_{n \rightarrow \infty} e^{-\frac{M}{|\theta|} n\theta F(P) e^{-n\theta F(P)}} = 1$$

therefore

$$\begin{aligned} & \lim_{n \rightarrow \infty} \max \left\{ e^{n(\theta - \theta') F(P)} : |\theta - \theta'| < M e^{-n\theta F(P)} \right\} \\ &= \lim_{n \rightarrow \infty} \min \left\{ e^{n(\theta - \theta') F(P)} : |\theta - \theta'| < M e^{-n\theta F(P)} \right\} = 1 \end{aligned}$$

and we can readily deduce the theorem. ■

5.2.3 Proof Theorem 8

In order to evaluate the local asymptotic lower bound of the quadratic risk (8) we first study the asymptotic behavior of $\mathbf{E}_\theta [J_T] / T^2$

Lemma 4 *If $F(P) = 0$ and $f(\cdot)$ is non constant, then*

$$\lim_{T \rightarrow \infty} \frac{\mathbf{E}_\theta [J_T]}{T^2} = \frac{G_\theta(P) H_\theta(P)}{2P^2}$$

uniformly with respect to θ varying in any compact subset of \mathbb{R} .

Proof We know that

$$\mathbf{E}_\theta [J_{nP+t}] = \int_0^{nP+t} f(u)^2 \mathbf{E}_\theta [\xi_u^2] du = \mathbf{E}_\theta [J_{nP}] + \int_0^t f(u)^2 \mathbf{E}_\theta [\xi_{nP+u}^2] du$$

for all $n \in \mathbb{N}$ and $t \in [0, P]$. Thanks to relation (5)

$$\mathbb{E}_\theta \left[\xi_{nP+u}^2 \right] = nG_\theta(P) + e^{2\theta F(u)} \left(\mathbb{E} \left[\xi_0^2 \right] + G_\theta(u) \right).$$

Thus

$$\begin{aligned} \mathbb{E}_\theta [J_{nP}] &= \int_0^P f(u)^2 \sum_{k=0}^{n-1} \mathbb{E}_\theta \left[\xi_{kP+u}^2 \right] du \\ &= \frac{n(n-1)}{2} \int_0^P \int_0^P f(u)^2 e^{2\theta(F(u)-F(v))} dudv + \\ &\quad + n\mathbb{E}_\theta \left[\xi_0^2 \right] \int_0^P f(u)^2 e^{2\theta F(u)} du + n \int_0^P \int_0^u f(u)^2 e^{2\theta(F(u)-F(v))} dudv \\ &= \frac{n(n-1)}{2} H_\theta(P)G_\theta(P) + n\mathbb{E}_\theta \left[\xi_0^2 \right] H_\theta(P) + \int_0^P f(u)^2 e^{2\theta F(u)} G_\theta(u) du. \end{aligned}$$

Consequently

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}_\theta [J_{nP}]}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n^2} \int_0^{nP} f(u)^2 \mathbb{E}_\theta \left[\xi_u^2 \right] du = \frac{1}{2} H_\theta(P)G_\theta(P)$$

and, using again relation (5), the lemma is proved. ■

Now we can state the asymptotic lower bound (17) for the mean square error using van Trees inequality.

Let $\epsilon > 0$ such that $0 \notin [\theta - \epsilon, \theta + \epsilon]$, and let $p_\epsilon(\cdot)$ a density probability function with support in $[\theta - \epsilon, \theta + \epsilon]$ and which is continuous differentiable in \mathbb{R} . Then for any real-valued random variable $\bar{\theta}_T$ measurable with respect to the observation $\{\xi_t, t \in [0, T]\}$, van Trees inequality (see Gill and Levit, 1995) entails that

$$\int_{\theta-\epsilon}^{\theta+\epsilon} \mathbb{E}_{\theta'} \left[(\bar{\theta} - \theta')^2 \right] p_\epsilon(\theta') d\theta' \geq \left(\int_{\theta-\epsilon}^{\theta+\epsilon} \mathbb{E}_{\theta'} [J_T] p_\epsilon(\theta') d\theta' + I(p_\epsilon) \right)^{-1}$$

where $I(p_\epsilon)$ is the Fisher information for a location parameter in $p_\epsilon(\cdot)$

$$I(p_\epsilon) = \int_{\theta-\epsilon}^{\theta+\epsilon} \left(\frac{d}{d\theta'} \ln p_\epsilon(\theta') \right)^2 p_\epsilon(\theta') d\theta' = \int_{\theta-\epsilon}^{\theta+\epsilon} \frac{\left(\frac{d}{d\theta'} p_\epsilon(\theta') \right)^2}{p_\epsilon(\theta')} d\theta'.$$

Thus

$$\inf_{\theta_T} \sup_{|\theta' - \theta| < \epsilon} \mathbb{E}_{\theta'} \left[T^2 (\bar{\theta}_T - \theta')^2 \right] \geq \left(\int_{\theta-\epsilon}^{\theta+\epsilon} \mathbb{E}_{\theta'} \left[\frac{J_T}{T^2} \right] p_\epsilon(\theta') d\theta' + \frac{I(p_\epsilon)}{T^2} \right)^{-1}.$$

From Lemma 4, we deduce that

$$\begin{aligned} \liminf_{T \rightarrow \infty} \inf_{\theta_T} \sup_{|\theta' - \theta| < \epsilon} \mathbb{E}_{\theta'} \left[T^2 (\bar{\theta}_T - \theta')^2 \right] &\geq \lim_{T \rightarrow \infty} \left(\int_{\theta-\epsilon}^{\theta+\epsilon} \mathbb{E}_{\theta'} \left[\frac{J_T}{T^2} \right] p_\epsilon(\theta') d\theta' \right)^{-1} \\ &= \left(\int_{\theta-\epsilon}^{\theta+\epsilon} \frac{G_{\theta'}(P)H_{\theta'}(P)}{2P^2} p_\epsilon(\theta') d\theta' \right)^{-1} \end{aligned}$$

With $\epsilon \rightarrow 0$, we deduce the lower bound for the mean square error (17).

Thanks to the uniform convergence in θ of $T(\hat{\theta}_T - \theta)$ in Theorem 3 and to (Ibragimov and Has'minskii 1981, Appendix I Theorem 8), we readily deduce the second assertion of Theorem 8.

■

Acknowledgements

The author would like to thank Pr. Yury Kutoyants for several helpful discussions.

References

- Antoni J. (2009) Cyclostationarity by examples, *Mechan. System. Signal Process.* 23, 987–1036.
- Barczy M. and Pap G. (2010) Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes, *Journal of Statistical Planning and Inference* 140, 1576–1593.
- Bishwal J.P. N. (2008) *Parameter Estimation in Stochastic Differential Equations*, Springer-Verlag, Berlin.
- Basawa I.V. and Scott D.J. (1983) *Asymptotic Optimal Inference for Non-ergodic Models*, Lectures Notes in Statistics, vol 17, Springer-Verlag, New-York.
- Brown B.M. and Hewitt J.I. (1975) Asymptotic likelihood theory for diffusion processes, *J. Appl. Prob.* 12, 228–238.
- Collet P. and Martinez S. (2008) Asymptotic velocity of one dimensional diffusions with periodic drift, *J. Math. Biol.* 56, 765–792.
- Davies R.B. (1985) Asymptotic inference when the amount of information is random: In Le Cam L. and Olshen R. (eds), *Proc. Berkley Conf. in honor of J. Neyman and J. Kiefer*, vol. 2, Wadsworth, Monterey, pp.841–864.
- Dehay D. (2013) Time-periodic-drift Langevin type stochastic differential equations, *submitted*
- Dehling H., Franke B. and Kott T. (2010) Drift estimation for a periodic mean reversion process, *Statist. Inference Stoch. Process.* 13, 175–192.
- Feigin P.D. (1976) Maximum likelihood estimation for continuous time stochastic processes, *Adv. Appl. Prob.* 8, 712–736.

- Feigin P.D. (1979) Some comments concerning a curious singularity, *J. Appl. Prob.* 16, 440–444.
- Gardner W.A., Napolitano A. and Paura L. (2006) Cyclostationarity : half a century of research, *Signal Processing* 86, 639–697.
- Gill R.D. and Levit B.Y. (1995) Applications of the van Trees inequality : a Bayesian Cramér-Rao bound, *Bernoulli* 1, 59–79.
- Gladyshev E.G. (1963) Periodically and almost periodically correlated processes, *Theory Probab. Appl.* 8, 173–177.
- Hájek J. (1972) Local asymptotic minimax and admissibility in estimation, in *Proc. 6-th Berkeley Symposium Math. Statist. Probab.*, vol 1, 175–194, Univ. California Press.
- Has'minskii (1980) *Stochastic stability of differential equations*, Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands.
- Höpfner R. (2007) On a set of data for the membrane potential in neurona, *Math. Biosci.* 207, 275–301.
- Höpfner R. and Kutoyants Y. (2010) Estimating discontinuous periodic signals in a time inhomogeneous diffusion, *Statist. Inference Stoch. Process.* 13, 193–230.
- Ibragimov I.A. and Has'minskii R.Z. (1981) *Statistical Estimation : Asymptotic Theory*, Springer, New-York.
- Jeganathan P. (1982) On the asymptotic theory of estimation when the limit of the likelihood is mixed normal, *Sankhya Ser. A* 44, 173–212.
- Jeganathan P. (1995) Some aspect of asymptotic theory with applications to time series models, *Econometric Theory* 11, 818–887.
- Kutoyants Yu. A. (2004) *Statistical Inference for Ergodic Diffusion Processes*, Springer, London.
- Le Cam L. (1969) *Théorie Asymptotique de la décision statistique*, Univ. of Montréal Press, Montréal.
- Le Cam L. (1986) *Asymptotics Methods in Statistical Decision Theory*, Springer-Verlag, New-York.
- Le Cam L. and Yang G.L. (1990) *Asymptotics in Statistics : Some Basic Concepts*, Springer-Verlag, New-York.
- Liptser R. and Shiryaev A. (2001) *Statistics of Random Processes*, vol. I+II, (2nd edition) Springer, New-York.

Mishra M.N. and Prakasa Rao B.L.S. (1985) Asymptotic study of the maximum likelihood estimator for non-homogeneous diffusion processes, *Statistics and Decisions* 3, 193–203.

Phillips P.C.B. (1987) Towards a unified asymptotic theory for autoregression, *Biometrika* 74 (3), 535–547.

Serpedin E., Pandura F., Sari I. and Giannakis G.B. (2005) Bibliography on cyclostationarity, *Signal Processing* 85, 2233–2303.