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Hossein Abbaspour

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# On algebraic structures of the Hochschild complex

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**Abstract.** We first review various known algebraic structures on the Hochschild (co)homology of a differential graded algebras under weak Poincaré duality hypothesis, such as Calabi-Yau algebras, derived Poincaré duality algebras and closed Frobenius algebras. This includes a BV-algebra structure on  $HH^*(A, A^\vee)$  or  $HH^*(A, A)$ , which in the latter case is an extension of the natural Gerstenhaber structure on  $HH^*(A, A)$ . As an example, after proving that the chain complex of the Moore loop space of a manifold  $M$  is a CY-algebra and using Burgheloa-Fiedorowicz-Goodwillie theorem we obtain a BV-structure on the homology of the free space.

In Sections 6 we prove that these BV/coBV structures can be indeed defined for the Hochschild homology of a symmetric open Frobenius DG-algebras. In particular we prove that the Hochschild homology and cohomology of a symmetric open Frobenius algebra is a BV and coBV-algebra. In Section 7 we exhibit a BV structure on the shifted relative Hochschild homology of a symmetric commutative Frobenius algebra. The existence of a BV-structure on the relative Hochschild homology was expected in the light of Chas-Sullivan and Goresky-Hingston results for free loop spaces. In Section 8 we present an action of Sullivan diagrams on the Hochschild (co)chain complex of a closed Frobenius DG-algebra. This recovers Tradler-Zeinalian [TZ06] result for closed Frobenius algebras using the isomorphism  $C^*(A, A) \simeq C^*(A, A^\vee)$ .

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## 1 Introduction

In this article we study the algebraic structures of Hochschild homology and cohomology of differential graded associative algebras over a field  $\mathbf{k}$  in four settings: Calabi-Yau algebras, derived Poincaré duality algebras, open Frobenius algebras and closed Frobenius algebras. For instance we prove the existence of a Batalin-Vilkovisky (BV) algebra structure on the Hochschild cohomology  $HH^*(A, A)$  in the first two cases, and on the Hochschild cohomology  $HH^*(A, A^\vee)$  in the last two cases. Let us explain the main motivation of the results presented in this chapter. One knows from Chen [Che77] and Jones [Jon87] work that the homology of  $LM = C^\infty(S^1, M)$ , the free loop space of a simply connected manifold  $M$ , can be computed by

$$H_*(LM) \simeq HH^*(A, A^\vee), \quad (1.1)$$

where  $A = C^*(M)$  is the singular cochain algebra of  $M$ . Jones also proved an equivariant version  $H^{S^1}(LM) \simeq HC^*(A)$ . Starting with Chas-Sullivan's work many different algebraic structures of  $H_*(LM)$  have been discovered. This includes a BV-algebra structure on  $H_*(LM)$  [CS]; and an action of Sullivan chord diagrams on  $H_*(LM)$  which in particular implies that  $H_*(LM)$  is an open Frobenius algebra. In order to find an algebraic model of these structures using the Hochschild complex and the isomorphism above, one has to equip the cochain algebra  $A$  with further structures.

In order to find the Chas-Sullivan BV structure on  $HH^*(A, A^\vee)$ , one should take into account the Poincaré duality for  $M$ . Over a field  $\mathbf{k}$ , we have a quasi-isomorphism  $A \rightarrow C_*(M) \simeq A^\vee$  given by capping with the fundamental class of  $M$ . Therefore one can use the result of Section 5 to find a BV-algebra structure on  $HH^*(A, A^\vee) \simeq HH^*(A, A)$ . Said more explicitly,  $H_*(LM)$  is isomorphic to  $HH^*(A, A)$  as a BV-algebra where the underlying Gerstenhaber structure of the BV structure on  $HH^*(A, A)$  is the standard one (see Theorem 2.1). This last statement which is true over a field, is a result to which many authors have contributed: [CJ02, FT08, Tra08, Mer04]. The statement is not proved as yet for integer coefficients.

As we will see in Section 4.2, an alternative way to find an algebraic model for the BV-structure of  $H_*(LM)$  is via the Burghelea-Fiedorowicz-Goodwillie isomorphism  $H_*(LM) \simeq HH_*(C_*(\Omega M), C_*(\Omega M))$ , where  $\Omega M$  is the based loop space of  $M$ . This approach has the advantage of working for all closed manifolds and it does not require  $M$  to be simply connected. Moreover there is not much of a restriction on the coefficients [Mal].

Now we turn our attention to the action of the Sullivan chord diagrams and the open Frobenius algebra structure of  $H_*(LM)$  (see [CG04]). For that one has to assume that the cochain algebra has some additional structures. The results of Section 8 show that in order to have an action of Sullivan chord diagram on  $HH^*(A, A^\vee)$  and  $HH_*(A, A)$  we have to start with a closed Frobenius algebra structure on  $A$ . As far as we know, such structure is not known on  $C^*(M)$  but only on the differential forms  $\Omega^*(M)$  (see

[LS08]). Therefore the isomorphism (1.1) is an isomorphism of algebras over the PROP of Sullivan chord diagrams if we work with real coefficients (see also [CTZ]).

Here is a brief description of the organization of the chapter. In Section 2 we introduce the Hochschild homology and cohomology of a differential graded algebra and various classical operations such as cup and cap product. We also give the definition of Gerstenhaber and BV-algebras. In particular we give an explicit description of the Gerstenhaber algebra structure on the Hochschild cohomology of  $A$  with coefficient in  $A$ ,  $HH^*(A, A)$ . In Section 3 we explain how we can see Hochschild (co)homology as a derived functor. The section includes a quick review of model categories which can be skipped by the reader.

In Sections 4 and 5 we work with algebras satisfying a sort of derived Poincaré duality condition rather than being equipped with an inner product. In these two sections, we introduce a BV structure on  $HH^*(A, A)$ , whose underlying Gerstenhaber structure is the standard one (see Section 2).

In Section 6, we show that the Hochschild homology  $HH_*(A, A)$  and cohomology  $HH^*(A, A^\vee)$  of a symmetric open Frobenius algebra  $A$ , are BV-algebras. Note that we don't find a BV-algebra structure on  $HH^*(A, A)$  although it is naturally a Gerstenhaber algebra (see Section 2). However if the open Frobenius algebra  $A$  is a closed one, then via the induced isomorphism  $HH^*(A, A) \simeq HH^*(A, A^\vee)$  we obtain a BV-algebra structure on  $HH^*(A, A)$  whose underlying Gerstenhaber algebra structure is the standard one.

In Section 7 we present a BV-structure on relative Hochschild homology on the relative Hochschild homology  $\widetilde{HH}_*(A)$  (after a shift in degree) for a symmetric commutative open Frobenius algebra  $A$ . We believe that these provide an algebraic model for the dual of the similar string topology operations discussed in [CS04] and [GH09] for the (relative) homology of free loop space.

In Section 8, we describe an action of the Sullivan chord diagrams on the Hochschild chains of a closed Frobenius algebra  $A$ , which induces an action of the homology of the moduli space of curves (Section 2.10 in [WW]). In particular there is a BV and coBV structure on  $HH_*(A, A)$  and on the dual theory  $HH^*(A, A^\vee)$ . Our construction is inspired by [TZ06] and equivalent to those given in [TZ06] and [KS09] for closed Frobenius algebras. This formulation is very much suitable for an action of the moduli space in the spirit of Costello's construction of such action for Calabi-Yau categories [Cos07], as it is given in [WW] in this particular case. Finally in Section 8 we will show how a BV structure on Hochschild (co)homology induces a graded Lie algebra, and even better a gravity algebra structure on the cyclic (co)homology.

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## 2 Hochschild Complex

Throughout this paper  $\mathbf{k}$  is a field. Let  $A = \mathbf{k} \oplus \bar{A}$  be an augmented unital differential  $\mathbf{k}$ -algebra with  $\deg d_A = +1$ ,  $\bar{A} = A/\mathbf{k}$  or  $\bar{A}$  is the kernel of the augmentation  $\epsilon : A \rightarrow \mathbf{k}$ .

A differential graded  $(A, d)$ -module, or  $A$ -module for short, is a  $\mathbf{k}$ -complex  $(M, d)$

together with an (left)  $A$ -module structure  $\cdot : A \times M \rightarrow M$  such that  $d_M(am) = d_A(a)m + (-1)^{|a|}ad_M(m)$ . The multiplication map is of degree zero *i.e.*  $\deg(am) = \deg a + \deg m$ . In particular, the identity above implies that the differential of  $M$  has to be of degree 1.

Similarly for a  $(M, d_M)$  a graded differential  $(A, d)$  – *bimodule*, we have

$$d_M(amb) = d_A(a)mb - (-1)^{|a|}ad_M(m)b + (-1)^{|a|+|m|}amd_Ab,$$

or equivalently,  $M$  is a  $(A^e := A \otimes A^{op}, d_A \otimes 1 + 1 \otimes d_A)$  DG-module where  $A^{op}$  is the algebra whose underlying graded vector space is  $A$  with the opposite multiplication of  $A$ , *i.e.*  $a \overset{op}{\cdot} b = (-1)^{|a| \cdot |b|} b \cdot a$ . From now on  $Mod(A)$  denotes the category of (left or right) (differential)  $A$ -modules and  $Mod(A^e)$  denotes the category of differential  $A$ -bimodules. All modules considered in this article are differential modules. We will also drop the indices from the differential when there is no possibility of confusion.

We recall that the *two-sided bar construction* ([CE56, ML63]) is given by  $B(A, A, A) := A \otimes T(s\bar{A}) \otimes A$  equipped with the differential  $d = d_0 + d_1$  where

$$\begin{aligned} d_1(a[a_1, \dots, a_n]b) &= (-1)^{|a|}aa_1[a_2, \dots, a_n]b + \\ &\sum_{i=1}^{n-1} (-1)^{\epsilon_i} a[a_1, \dots, a_i a_{i+1}, \dots, a_n]b \\ &- (-1)^{\epsilon_n} a[a_1, \dots, a_{n-1}]a_n b \end{aligned} \quad (2.1)$$

and  $d_0$  is the internal differential for the tensor product complex  $A \otimes T(s\bar{A}) \otimes A$ . Here  $\epsilon_i = |a_1| + \dots + |a_i| - i$ . The degree on  $B(A, A, A)$  is defined by  $\deg(a[a_1, \dots, a_n]b) = \sum_{i=1}^n |s(a_i)| = |a| + |b| + \sum_{i=1}^n |a_i| - n$ , therefore  $\deg(d_0 + d_1) = +1$ . We recall that  $sA$  stands for the suspension of  $A$ , *i.e.* the shift in degree by  $-1$ .

We equip  $A$  and  $A \otimes_{\mathbf{k}} A$ , or  $A \otimes A$  for short, with the *outer*  $A$ -bimodule structure that is  $a(b_1 \otimes b_2)c = (ab_1) \otimes (b_2c)$ . Similarly  $B(A, A, A)$  is equipped with the outer  $A$ -bimodule structure. This is a free resolution of  $A$  as an  $A$ -bimodule which allows us to define *Hochschild chains and cochains* of  $A$  with coefficients in  $M$ . Then (*normalized*) *Hochschild chain complex* with coefficients in  $M$  is

$$C_*(A, M) := M \otimes_{A^e} B(A, A, A) = M \otimes T(s\bar{A}) \quad (2.2)$$

and comes equipped with a degree  $+1$  differential  $D = d_0 + d_1$ . We recall that  $TV = \bigoplus_{n \geq 0} V^{\otimes n}$  denotes the tensor algebra of a  $\mathbf{k}$ -module  $V$ .

The internal differential is given by

$$\begin{aligned} d_0(m[a_1, \dots, a_n]) &= \sum_{i=1}^{n-1} (-1)^{\epsilon_i} m[a_1, \dots, d_A a_i, \dots, a_n] \\ &- (-1)^{\epsilon_n} d_M m[a_1, \dots, a_n], \end{aligned} \quad (2.3)$$

and the external differential is

$$\begin{aligned} d_1(m[a_1, \dots, a_n]) &= ma_1[a_2, \dots, a_n] + \\ &\sum_{i=1}^{n-1} (-1)^{\epsilon_i} m[a_1, \dots, a_i a_{i+1}, \dots, a_n] \\ &- (-1)^{\epsilon_n} a_n m[a_1, \dots, a_{n-1}], \end{aligned} \quad (2.4)$$

with  $\epsilon_i = |a_1| + \dots + |a_i| - i$ . Note that the degree of  $m[a_1, \dots, a_n]$  is  $\sum_{i=1}^n |a_i| - n + |m|$ .

When  $M = A$ , by definition  $(C_*(A), D = d_0 + d_1) := (C_*(A, A), D = d_0 + d_1)$  is the *Hochschild chain complex* of  $A$  and  $HH_*(A, A) := \ker D / \text{im } D$  is the Hochschild homology of  $A$ .

Similarly we define the  $M$ -valued *Hochschild cochain* of  $A$  to be the dual complex

$$C^*(A, M) := \text{Hom}_{A^e}(B(A, A, A), M) = \text{Hom}_{\mathbf{k}}(T(s\bar{A}), M).$$

For a homogenous cochain  $f \in C^n(A, M)$ , the degree  $|f|$  is defined to be the degree of the linear map  $f : (s\bar{A})^{\otimes n} \rightarrow M$ . In the case of Hochschild cochains, the external differential of  $f \in \text{Hom}(s\bar{A}^{\otimes n}, M)$  is

$$\begin{aligned} d_1(f)(a_1, \dots, a_n) &= -(-1)^{(|a_1|+1)|f|} a_1 f(a_2, \dots, a_n) + \\ &\quad - \sum_{i=2}^n (-1)^{\epsilon_i} f(a_1, \dots, a_{i-1} a_i, \dots, a_n) + (-1)^{\epsilon_n} f(a_1, \dots, a_{n-1}) a_n, \end{aligned} \quad (2.5)$$

where  $\epsilon_i = |f| + |a_1| + \dots + |a_{i-1}| - i + 1$ . The internal differential of  $f \in C^*(A, M)$  is

$$d_0 f(a_1, \dots, a_n) = d_M f(a_1, \dots, a_n) - \sum_{i=1}^n (-1)^{\epsilon_i} f(a_1, \dots, d_A a_i \dots, a_n). \quad (2.6)$$

**Gerstenhaber bracket and cup product:** When  $M = A$ , for  $x \in C^m(A, A)$  and  $y \in C^n(A, A)$  one defines the *cup product*  $x \cup y \in C^{m+n}(A, A)$  and the *Gerstenhaber bracket*  $[x, y] \in C^{m+n-1}(A, A)$  by

$$(x \cup y)(a_1, \dots, a_{m+n}) := (-1)^{|y|(\sum_{i \leq m} |a_i| + 1)} x(a_1, \dots, a_m) y(a_{m+1}, \dots, a_{m+n}), \quad (2.7)$$

and

$$[x, y] := x \circ y - (-1)^{(|x|+1)(|y|+1)} y \circ x, \quad (2.8)$$

where

$$(x \circ_j y)(a_1, \dots, a_{m+n-1}) = (-1)^{(|y|+1) \sum_{i \leq j} (|a_i| + 1)} x(a_1, \dots, a_j, y(a_{j+1}, \dots, a_{j+m}), \dots).$$

and

$$x \circ y = \sum_j x \circ_j y \quad (2.9)$$

Note that this is not an associative product. It turns out that the operations  $\cup$  and  $[-, -]$  are chain maps, hence they define two well-defined operations on  $HH^*(A, A)$ . Moreover,

**Theorem 2.1.** (*Gerstenhaber [Ger63]*)  $(HH^*(A, A), \cup, [-, -])$  is a Gerstenhaber algebra that is:

- (1)  $\cup$  is an associative and graded commutative product,
- (2)  $[x, y \cup z] = [x, y] \cup z + (-1)^{(|x|-1)|y|} y \cup [x, z]$  (*Leibniz rule*),
- (3)  $[x, y] = (-1)^{(|x|-1)(|y|-1)} [y, x]$ ,
- (4)  $[[x, y], z] = [x, [y, z]] - (-1)^{(|x|-1)(|y|-1)} [y, [x, z]]$  (*Jacobi identity*).

The homotopy for the commutativity of the cup product  $x \cup y$  is given by  $x \circ y$ .

In this article we show that under some kind of Poincaré duality condition this Gerstenhaber structure is part of a BV structure.

**Definition 2.2.** (Batalin-Vilkovisky algebra) A BV-algebra is a Gerstenhaber algebra  $(A^*, \cdot, [-, -])$  with a degree one operator  $\Delta : A^* \rightarrow A^{*+1}$  whose deviation from being a derivation for the product  $\cdot$  is the bracket  $[-, -]$ , *i.e.*

$$[a, b] := (-1)^{|a|} \Delta(ab) - (-1)^{|a|} \Delta(a)b - a\Delta(b),$$

and  $\Delta^2 = 0$ .

It follows from  $\Delta^2 = 0$  that  $\Delta$  is a derivation for the bracket. In fact the Leibniz identity for  $[-, -]$  is equivalent to the 7-term relation [Get94]

$$\begin{aligned} \Delta(abc) &= \Delta(ab)c + (-1)^{|a|} a\Delta(bc) + (-1)^{(|a|-1)|b|} b\Delta(ac) \\ &\quad - \Delta(a)bc - (-1)^{|a|} a\Delta(b)c - (-1)^{|a|+|b|} ab\Delta c. \end{aligned} \quad (2.10)$$

Definition 2.2 is equivalent to the following one:

**Definition 2.3.** A BV-algebra is a graded commutative associative algebra  $(A^*, \cdot)$  equipped with a degree one operator  $\Delta : A^* \rightarrow A^{*+1}$  which satisfies the 7-term relation (2.10) and  $\Delta^2 = 0$ . It follows from the 7-term relation that  $[a, b] := (-1)^{|a|} \Delta(ab) - (-1)^{|a|} \Delta(a)b - a\Delta(b)$  is a Gerstenhaber bracket for the graded commutative associative algebra  $(A^*, \cdot)$ .

As we said before the Leibniz identity is equivalent to the 7-term identity and the Jacobi identity follows from  $\Delta^2 = 0$  and the 7-term identity. We refer the reader interested in the homotopic aspects of BV-algebras to [DCV].

For  $M = A^\vee := \text{Hom}_{\mathbf{k}}(A, \mathbf{k})$ , by definition  $(C^*(A), D = d_0 + d_1) := (C^*(A, A^\vee), d_0 + d_1)$  is the Hochschild cochain complex of  $A$  and  $HH^*(A) := \ker D / \text{im } D$  is the Hochschild cohomology of  $A$ . It is clear that  $C^*(A)$  and  $\text{Hom}_{\mathbf{k}}(C_*(A), \mathbf{k})$  are isomorphic as  $k$ -complexes, therefore the Hochschild cohomology  $A$  is the dual theory of the Hochschild homology of  $A$ . The Hochschild homology and cohomology of an algebra have an extra feature and that is the Connes operator  $B$  ([Con85]). On the chains we have

$$B(a_0[a_1, a_2 \cdots, a_n]) = \sum_{i=1}^{n+1} (-1)^{\epsilon_i} 1[a_{i+1} \cdots a_n, a_0, \cdots, a_i] \quad (2.11)$$

and on the dual theory  $C^*(A) = \text{Hom}_{\mathbf{k}}(T(s\bar{A}), A^\vee) = \text{Hom}(A \otimes T(s\bar{A}), \mathbf{k})$  is given by

$$(B^\vee \phi)(a_0[a_1, a_2 \cdots, a_n]) = (-1)^{|\phi|} \sum_{i=1}^{n+1} (-1)^{\epsilon_i} \phi(1[a_{i+1} \cdots a_n, a_0, \cdots, a_i])$$

where  $\phi \in C^{n+1}(A) = \text{Hom}(A \otimes (s\bar{A})^{\otimes n+1}, \mathbf{k})$  and  $\epsilon_i = (|a_0| + \dots + |a_{i-1}| - i)(|a_i| + \dots + |a_n| - n + i - 1)$ . In other words

$$B^\vee(\phi) = (-1)^{|\phi|} \phi \circ B.$$

Note that  $\text{deg}(B) = -1$  and  $\text{deg } B^\vee = +1$ .

**Warning:** The degree  $k$  of a cycle  $x \in HH_k(A, M)$ , is not given by the number terms in a tensor product but by the total degree.

**Remark 2.4.** In this article we use normalized Hochschild chains and cochains. It turns out that they are quasi-isomorphic to the non-normalized Hochschild chains and cochains. The proof is the same as the one on page 46 of [Lod92] for the algebras. One only has to modify the proof to the case of simplicial objects in the category of differential graded algebras. The proof of the Lemma 1.6.6 of [Lod92] works in this setting since the degeneracy maps commute with the internal differential of a simplicial differential graded algebra.

**Chain and cochain pairings and noncommutative calculus** Here we borrow some definitions and facts from noncommutative calculus [CST04]. Roughly said, one should think of  $HH^*(A, A)$  and  $HH_*(A, A)$  respectively as multi-vector fields and differential forms, and of  $B$  as the de Rham differential.

**1. Contraction or cap product:** The pairing between  $a_0[a_1, \dots, a_n] \in C_n(A, A)$  and  $f \in C^k(A, A)$ ,  $n \geq k$  is given by

$$i_f(a_0[a_1, \dots, a_n]) = (-1)^{|f|(\sum_{i=1}^k (|a_i|+1))} a_0 f(a_1, \dots, a_k)[a_{k+1}, \dots, a_n] \in C_{n-k}(A, A). \quad (2.12)$$

It is a chain map  $C^*(A, A) \otimes C_*(A, A) \rightarrow C_*(A, A)$  and it induces a pairing at cohomology and homology level.

**2. Lie derivative:** The next operation is the infinitesimal Lie algebra action of  $HH^*(A, A)$  on  $HH_*(A, A)$  and is given by Cartan's formula

$$L_f = [B, i_f]. \quad (2.13)$$

Note that the Gerstenhaber bracket on  $HH^*(A, A)$  becomes a (graded) Lie bracket after a shift of degrees by one. This explains also the sign convention below. The triple  $i_f$ ,  $L_f$  and  $B$  form a calculus [CST04] that is,

$$L_f = [i_f, B] \quad (2.14)$$

$$i_{[f, g]} = [L_f, i_g] \quad (2.15)$$

$$i_{f \cup g} = i_f \circ i_g \quad (2.16)$$

$$L_{[f, g]} = [L_f, L_g] \quad (2.17)$$

$$L_{fg} = [L_f, i_g] \quad (2.18)$$

As  $HH^*(A, A)$  acts on  $HH_*(A) = HH_*(A, A)$  by contraction, it also acts on the dual theory *i.e.*  $HH^*(A) = HH^*(A, A^\vee)$ . More explicitly,  $i_f(\phi) \in C^n(A, A^\vee)$  is given by

$$i_f(\phi)(a_0[a_1, \dots, a_n]) := (-1)^{|f|(|\phi| + \sum_{i=0}^k (|a_i|+1))} \phi(a_0[f(a_1, \dots, a_k), a_{k+1}, \dots, a_n]) \quad (2.19)$$

where  $\phi \in C^{n-k}(A, A^\vee)$  and  $f \in C^k(A, A)$ , in other words

$$i_f(\phi) := (-1)^{|f||\phi|} \phi \circ i_f.$$



### 3 Derived category of DGA and derived functors

Now we try to present the Hochschild (co)homology in a more conceptual way *i.e.* as a derived functor on the category of  $A$ -bimodules. We must first introduce an appropriate class of objects which can approximate all  $A$ -bimodules. This is done properly using the concept of model category introduced by Daniel Quillen [Qui67]. It is also the right language for constructing homological invariants of homotopic categories. It will naturally lead us to the construction of derived categories as well.

#### 3.1 A quick review of model categories and derived functors

The classical references for this subject are Hovey's book [Hov99] and the Dwyer-Spalinsky manuscript [DS95]. The reader who gets to know the notion of model category for the first time, should not worry about the word "closed" which now has only a historical bearing. From now on we drop the word "closed" from closed model category.

**Definition 3.1.** Let  $\mathbf{C}$  be a category with three classes of morphisms  $\mathcal{C}$  (cofibrations),  $\mathcal{F}$  (fibrations) and  $\mathcal{W}$  (weak equivalences) such that:

(MC1)  $\mathbf{C}$  is closed under finite limits and colimits.

(MC2) Let  $f, g \in \text{Mor}(\mathbf{C})$  such that  $fg$  is defined. If any two among  $f, g$  and  $fg$  are in  $\mathcal{W}$ , then the third one is in  $\mathcal{W}$ .

(MC3) Let  $f$  be a retract of  $g$ . If  $g \in \mathcal{C}$  (resp.  $\mathcal{F}$  or  $\mathcal{W}$ ), then  $f \in \mathcal{C}$  (resp.  $\mathcal{F}$  or  $\mathcal{W}$ ).

(MC4) For a commutative diagram, as below, with  $i \in \mathcal{C}$  and  $p \in \mathcal{F}$ , the morphism  $f$  making the diagram commutative exists if

- (1)  $i \in \mathcal{W}$  (left lifting property (LLP) of fibrations  $f \in \mathcal{F}$  with respect to acyclic cofibrations  $i \in \mathcal{W} \cap \mathcal{C}$ ).
- (2)  $p \in \mathcal{W}$  (right lifting property (RLP) of cofibrations  $i \in \mathcal{C}$  with respect to acyclic fibrations  $p \in \mathcal{W} \cap \mathcal{F}$ ).

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 \downarrow i & \nearrow f & \downarrow p \\
 B & \longrightarrow & Y
 \end{array} \tag{3.1}$$

The reader should have noticed that we call the elements of  $\mathcal{W} \cap \mathcal{C}$  (resp.  $\mathcal{W} \cap \mathcal{F}$ ) acyclic cofibrations (resp. fibrations).

(MC5) Any morphism  $f : A \rightarrow B$  can be written as one of the following:

- (1)  $f = pi$  where  $p \in \mathcal{F}$  and  $i \in \mathcal{C} \cap \mathcal{W}$ ;
- (2)  $f = pi$  where  $p \in \mathcal{F} \cap \mathcal{W}$  and  $i \in \mathcal{C}$ .

In fact in a model category the lifting properties characterize the fibrations and cofibrations:

**Proposition 3.2.** *In a model category:*

- (i) *The cofibrations are the morphisms which have the RLP with respect to acyclic fibrations.*
- (ii) *The acyclic cofibrations are the morphisms which have the RLP with respect to fibrations.*

- (iii) The fibrations are the morphisms which have the LLP with respect to acyclic cofibrations.
- (iv) The acyclic fibrations in  $\mathbf{C}$  are the maps which have the LLP with respect to cofibrations.

It follows from (MC1) that a model category  $\mathbf{C}$  has an initial object  $\emptyset$  and a terminal object  $*$ . An object  $A \in \text{Obj}(\mathbf{C})$  is called *cofibrant* if the morphism  $\emptyset \rightarrow A$  is a cofibration and is said to be *fibrant* if the morphism  $A \rightarrow *$  is a fibration.

**Example 1:** For any unital associative ring  $R$ , let  $\mathbf{CH}(R)$  be the category of non-negatively graded chain complexes of left  $R$ -modules. The following three classes of morphisms endow  $\mathbf{CH}(R)$  with a model category structure:

- (1) Weak equivalences  $\mathcal{W}$  are the quasi-isomorphisms i.e. maps of  $R$ -complexes  $f = \{f_k\}_{k \geq 0} : \{M_k\}_{k \in \mathbb{Z}} \rightarrow \{N_k\}_{k \geq 0}$  inducing an isomorphism  $f_* : H_*(M) \rightarrow H_*(N)$  in homology.
- (2) Fibrations  $\mathcal{F}$ :  $f$  is a fibration if it is (componentwise) surjective i.e. for all  $k \geq 1$ ,  $f_k : M_k \rightarrow N_k$  is surjective.
- (3) Cofibrations  $\mathcal{C}$ :  $f = \{f_k\}$  is a cofibration if for all  $k \geq 0$ ,  $f_k : M_k \rightarrow N_k$  is injective with a projective  $R$ -module as its cokernel. Here projective is the standard notion i.e. a direct summand of free  $R$ -module.

**Example 2:** The category  $\mathbf{Top}$  of topological spaces can be given the structure of a model category by defining a map  $f : X \rightarrow Y$  to be

- (i) a weak equivalence if  $f$  is a homotopy equivalence;
- (ii) a cofibration if  $f$  is a Hurewicz cofibration;
- (iii) a fibration if  $f$  is a Hurewicz fibration.

Let  $A$  be a closed subspace of a topological space  $B$ . We say that the inclusion  $i : A \hookrightarrow B$  is a *Hurewicz cofibration* if it has the homotopy extension property that is for all maps  $f : B \rightarrow X$ , any homotopy  $F : A \times [0, 1] \rightarrow X$  of  $f|_A$  can be extended to a homotopy of  $f : B \rightarrow X$ .

$$\begin{array}{ccc}
 B \cup (A \times [0, 1]) & \xrightarrow{f \cup F} & X \\
 \downarrow \text{id} \times 0 \cup (i \times \text{id}) & \nearrow & \\
 B \times [0, 1] & & 
 \end{array}$$

A *Hurewicz fibration* is a continuous map  $E \rightarrow B$  which has the homotopy lifting property with respect to all continuous maps  $X \rightarrow B$ , where  $X \in \mathbf{Top}$ .

**Example 3:** The category  $\mathbf{Top}$  of topological spaces can be given the structure of a model category by defining  $f : X \rightarrow Y$  to be

- (i) a weak equivalence when it is a weak homotopy equivalence.
- (ii) a cofibration if it is a retract of a map  $X \rightarrow Y'$  in which  $Y'$  is obtained from  $X$  by attaching cells,
- (iii) a fibration if it is a Serre fibration.

We recall that a *Serre fibration* is a continuous map  $E \rightarrow B$  which has the homotopy lifting property with respect to all continuous maps  $X \rightarrow B$  where  $X$  is a CW-complex (or equivalently cubes).

**Cylinder, path objects and homotopy relation.** After setting up the general framework, we define the notion of homotopy. A *cylinder object* for  $A \in \text{obj}(\mathbf{C})$  is an object  $A \wedge I \in \text{obj}(\mathbf{C})$  with a *weak equivalence*  $\sim: A \wedge I \rightarrow A$  which factors the natural map  $id_A \sqcup id_A : A \amalg A \rightarrow A$ :

$$id_A \sqcup id_A : A \amalg A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

Here  $A \amalg A \in \text{obj}(\mathbf{C})$  is the colimit, for which one has two structural maps  $in_0, in_1 : A \rightarrow A \amalg A$ . Let  $i_0 = i \circ in_0$  and  $i_1 = i \circ in_1$ . A cylinder object  $A \wedge I$  is said to be *good* if  $A \amalg A \rightarrow A \wedge I$  is a cofibration. By (MC5), every  $A \in \text{obj}(\mathbf{C})$  has a good cylinder object.

**Definition 3.3.** Two maps  $f, g : A \rightarrow B$  are said to be *left homotopic*  $f \stackrel{l}{\sim} g$  if there is a cylinder object  $A \wedge I$  and  $H : A \wedge I \rightarrow B$  such that  $f = H \circ i_0$  and  $g = H \circ i_1$ . A left homotopy is said to be *good* if the cylinder object  $A \wedge I$  is good. It turns out that every left homotopy relation can be realized by a good cylinder object. In addition one can prove that if  $B$  is a fibrant object, then a left homotopy for  $f$  and  $g$  can be refined into a *very good one* i.e.  $A \wedge I \rightarrow A$  is a fibration.

It is easy to prove the following:

**Lemma 3.4.** *If  $A$  is cofibrant, then left homotopy  $\stackrel{l}{\sim}$  is an equivalence relation on  $\text{Hom}_{\mathbf{C}}(A, B)$ .*

Similarly, we introduce the notion of path objects which will allow us to define right homotopy relation. A *path object* for  $A \in \text{obj}(\mathbf{C})$  is an object  $A^I \in \text{obj}(\mathbf{C})$  with a weak equivalence  $A \xrightarrow{\sim} A^I$  and a morphism  $p : A^I \rightarrow A \times A$  which factors the diagonal map

$$(id_A, id_A) : A \xrightarrow{\sim} A^I \xrightarrow{p} A \times A$$

Let  $pr_0, pr_1 : A \times A \rightarrow A$  be the structural projections. Define  $p_i = pr_i \circ p$ . A path object  $A^I$  is said to be *good* if  $A^I \rightarrow A \times A$  is a fibration. By (MC5) every  $A \in \text{obj}(\mathbf{C})$  has a good path object.

**Definition 3.5.** Two maps  $f, g : A \rightarrow B$  are said to be *right homotopic*  $f \stackrel{r}{\sim} g$  if there is a path object  $B^I$  and  $H : A \rightarrow B^I$  such that  $f = p_0 \circ H$  and  $g = p_1 \circ H$ . A right homotopy is said to be *good* if the path object  $B^I$  is good. It turns out that every right homotopy relation can be refined into a good one. In addition one can prove that if  $B$  is a cofibrant object then a right homotopy for  $f$  and  $g$  can be refined into a *very good one* i.e.  $B \rightarrow B^I$  is a cofibration.

**Lemma 3.6.** *If  $B$  is fibrant, then right homotopy  $\stackrel{r}{\sim}$  is an equivalence relation on  $\text{Hom}_{\mathbf{C}}(A, B)$ .*

One naturally asks whether being right and left homotopic are related. The following result answers this question.

**Lemma 3.7.** *Let  $f, g : A \rightarrow B$  be two morphisms in a model category  $\mathbf{C}$ .*

- (1) *If  $A$  is cofibrant then  $f \stackrel{l}{\sim} g$  implies  $f \stackrel{r}{\sim} g$*
- (2) *If  $B$  is fibrant then  $f \stackrel{r}{\sim} g$  implies  $f \stackrel{l}{\sim} g$ .*

**Cofibrant and Fibrant replacement and homotopy category.** By applying (MC5) to the canonical morphism  $\emptyset \rightarrow A$ , there is a cofibrant object (not unique)  $QA$  and an *acyclic fibration*  $p : QA \xrightarrow{\sim} A$  such that  $\emptyset \rightarrow QA \xrightarrow{p} A$ . If  $A$  is cofibrant we can choose  $QA = A$ .

**Lemma 3.8.** *Given a morphism  $f : A \rightarrow B$  in  $\mathbf{C}$ , there is a morphism  $\tilde{f} : QA \rightarrow QB$  such that the following diagram commutes:*

$$\begin{array}{ccc} QA & \xrightarrow{\tilde{f}} & QA \\ \downarrow p_A & & \downarrow p_B \\ A & \xrightarrow{f} & B \end{array} \quad (3.2)$$

*The morphism  $\tilde{f}$  depends on  $f$  up to left and right homotopy, and is a weak equivalence if and only if  $f$  is. Moreover, if  $B$  is fibrant then the right or left homotopy class of  $\tilde{f}$  depends only on the left homotopy class of  $f$ .*

Similarly one can introduce a fibrant replacement by applying (MC5) to the terminal morphism  $A \rightarrow *$  and obtain a fibrant object  $RA$  with an *acyclic cofibration*  $i_A : A \rightarrow RA$ .

**Lemma 3.9.** *Given a morphism  $f : A \rightarrow B$  in  $\mathbf{C}$ , there is a morphism  $\tilde{f} : RA \rightarrow RB$  such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i_A & & \downarrow i_B \\ RA & \xrightarrow{\tilde{f}} & RB \end{array} \quad (3.3)$$

*The morphism  $\tilde{f}$  depends on  $f$  up to left and right homotopy, and is a weak equivalence if and only if  $f$  is. Moreover, if  $A$  is cofibrant then right or left homotopy class of  $\tilde{f}$  depends only on the right homotopy class of  $f$ .*

**Remark 3.10.** For a cofibrant object  $A$ ,  $RA$  is also cofibrant because the trivial morphism  $(\emptyset \rightarrow RA) = (\emptyset \rightarrow A \xrightarrow{i_A} RA)$  can be written as the composition of two cofibrations, therefore is a cofibration. In particular, for any object  $A$ ,  $RQA$  is fibrant and cofibrant. Similarly,  $QRA$  is a fibrant and cofibrant object.

Putting the last three lemmas together, one can make the following definition:

**Lemma 3.11.** *Suppose that  $f : A \rightarrow X$  is a map in  $\mathbf{C}$  between objects  $A$  and  $X$  which are both fibrant and cofibrant. Then  $f$  is a weak equivalence if and only if  $f$  has a homotopy inverse, i.e., if and only if there exists a map  $g : X \rightarrow A$  such that the composites  $gf$  and  $fg$  are homotopic to the respective identity maps.*

**Definition 3.12.** The *homotopy category*  $\text{Ho}(\mathbf{C})$  of a model category  $\mathbf{C}$  has the same objects as  $\mathbf{C}$  and the morphism set  $\text{Hom}_{\text{Ho}(\mathbf{C})}(A, B)$  consists of the (right or left) homotopy classes of the morphism  $\text{Hom}_{\mathbf{C}}(RQA, RQB)$ . Note that since  $RQA$  and  $RQB$  are fibrant and cofibrant, the left and right homotopy relations are the same. There is a natural functor  $H_{\mathbf{C}} : \mathbf{C} \rightarrow \text{Ho}(\mathbf{C})$  which is the identity on the objects and sends a morphism  $f : A \rightarrow B$  to the homotopy class of the morphism obtained in  $\text{Hom}_{\mathbf{C}}(RQA, RQB)$  by applying consecutively Lemma 3.8 and Lemma 3.9.

**Localization functor.** Here we give a brief conceptual description of the homotopy category of a model category. This description relies only on the class of weak equivalences and suggests that the weak equivalences encode most of the homotopic properties of the category. Let  $W$  be a subset of the morphisms in a category  $\mathbf{C}$ . A functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  is said to be a *localization* of  $\mathbf{C}$  with respect to  $W$  if the elements of  $W$  are sent to isomorphisms and if  $F$  is universal for this property *i.e.* if  $G : \mathbf{C} \rightarrow \mathbf{D}'$  is another localizing functor then  $G$  factors through  $F$  via a functor  $G' : \mathbf{D} \rightarrow \mathbf{D}'$  for which  $G'F = G$ . It follows from Lemma 3.11 and a little work that:

**Theorem 3.13.** *For a model category  $\mathbf{C}$ , the natural functor  $H_{\mathbf{C}} : \mathbf{C} \rightarrow \text{Ho}(\mathbf{C})$  is a localization of  $\mathbf{C}$  with respect to the weak equivalences.*

**Derived and total derived functors:** In this section we introduce the notions of *left derived*  $LF$  and *right derived*  $RF$  of a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  of model categories. In particular, we spell out the necessary conditions for the existence of  $LF$  and  $RF$  which provide us a factorization of  $F$  via the homotopy categories. All functors considered here are covariant, however see Remark 3.17.

**Definition 3.14.** For a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  on a model category  $\mathbf{C}$ , we consider all pairs  $(G, s)$  where  $G : \text{Ho}(\mathbf{C}) \rightarrow \mathbf{D}$  is a functor and  $s : GH_{\mathbf{C}} \rightarrow F$  is a natural transformation. The *left derived functor* of  $F$  is such a pair  $(LF, t)$  which is universal from left *i.e.* for another such pair  $(G, s)$  there is a unique natural transformation  $t' : G \rightarrow LF$  such that  $t'(t'H_{\mathbf{C}}) : GH_{\mathbf{C}} \rightarrow F$  is  $s$ .

Similarly one can define the right derived functor  $RF : \text{Ho}(\mathbf{C}) \rightarrow \mathbf{D}$  which provides a factorization of  $F$  and satisfies the usual universal property from the right. A right derived functor for  $F$  is a pair  $(RF, t)$  where  $RF : \text{Ho}(\mathbf{C}) \rightarrow \mathbf{D}$  and  $t$  is a natural transformation  $t : F \rightarrow RFH_{\mathbf{C}}$  such that for any such pair  $(G, s)$  there is a unique natural transformation  $t' : RF \rightarrow G$  such that  $t'H_{\mathbf{C}}t : F \rightarrow GH_{\mathbf{C}}$  is  $s$ .

The reader can easily check that the derived functors of  $F$  are unique up to canonical equivalence.

The following result tells us when do derived functors exist.

**Proposition 3.15.** (1) *Suppose that  $F : \mathbf{C} \rightarrow \mathbf{D}$  is a functor from a model categories  $\mathbf{C}$  to a category  $\mathbf{d}$ , which sends acyclic cofibration between cofibrant objects to isomorphisms. Then  $(LF, t)$  the left derived functor of  $F$  exists. Moreover for any cofibrant object  $X$  the map  $t_x : LF(X) \rightarrow F(X)$  is an isomorphism.*

(2) *Suppose that  $F : \mathbf{C} \rightarrow \mathbf{D}$  is a functor between two model categories which sends acyclic fibrations between fibrant objects to isomorphisms. Then  $(RF, t)$  the right derived functor of  $F$  exists. Moreover for all fibrant object  $X$  the map  $t_x : RF(X) \rightarrow F(X)$  is an isomorphism.*

**Definition 3.16.** Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor between two model categories. The *total left derived functor*  $\mathbb{L}F : \text{Ho}(\mathbf{C}) \rightarrow \text{Ho}(\mathbf{D})$  is a left derived functor of  $H_{\mathbf{D}}F : \mathbf{C} \rightarrow \text{Ho}(\mathbf{D})$ . Similarly one defines the total right derived functor  $\mathbb{R}F : \mathbf{C} \rightarrow \mathbf{D}$  to be the right derived functor of  $H_{\mathbf{D}}F : \mathbf{C} \rightarrow \text{Ho}(\mathbf{D})$ .

**Remark 3.17.** Till now we have defined and discussed the derived functor for covariant functors. We can defined the derived functors for contravariant functors as well, for that we have to only work with the opposite category of the source of the functor. A morphism

$A \rightarrow B$  in the opposite category is a cofibration (resp. fibration, weak equivalence) if the corresponding morphism  $B \rightarrow A$  is a fibration (resp. cofibration, weak equivalence).

We finish this section with an example.

**Example 4:** Consider the model category  $\mathbf{CH}(R)$  of Example 1 in Section 3 and let  $M$  be a fixed  $R$ -module. One defines the functor  $F_M : \mathbf{CH}(R) \rightarrow \mathbf{CH}(\mathbb{Z})$  given by  $F_M(N_*) = M \otimes_R N_*$  where  $N_* \in \mathbf{CH}(R)$  is a complex of  $R$ -modules. Let us check that  $F = H_{\mathbf{CH}(R)} F_M : \mathbf{CH}(R) \rightarrow \mathbf{CH}(\mathbb{Z})$  satisfies the conditions of Proposition 3.15.

Note that in  $\mathbf{CH}(R)$  every object is fibrant and a complex  $A_*$  is cofibrant if for all  $k$ ,  $A_k$  is a projective  $R$ -module. We have to show that an acyclic cofibration  $f : A_* \rightarrow B_*$  between cofibrant objects  $A$  and  $B$  is sent by  $F$  to an isomorphism. So for all  $k$ , we have a short exact sequence  $0 \rightarrow A_* \rightarrow B_* \rightarrow B_*/A_* \rightarrow 0$  where for all  $k$ ,  $B_k/A_k$  is also projective. Since  $f$  is a quasi-isomorphism the homology long exact sequence of this short exact sequence tells us that the complex  $B_*/A_*$  is acyclic. The lemma below shows that  $B_*/A_*$  is in fact a projective complex. Therefore we have  $B_* \simeq A_* \oplus B_*/A_*$ . So  $F_M(B_*) \simeq F_M(A_*) \oplus F_M(B_*/A_*) \simeq F_M(A_*) \oplus \bigoplus_n F_M(D(Z_{n-1}(B_*/A_*), n))$ . Here  $Z_*(X_*) := \ker(d : X_* \rightarrow X_{*+1})$  stands for the graded module of the cycles in a given complex  $X_*$ , and the complex  $D(X, n)_*$  is defined as follows: To any  $R$ -module  $X$  and a positive integer  $n$ , one can associate a complex  $\{D(X, n)_k\}_{k \geq 0}$ ,

$$D(X, n)_k = \begin{cases} 0, & \text{if } k \neq n, n-1, \\ X, & \text{if } k = n, n-1, \end{cases}$$

where the only nontrivial differential is the identity map.

It is a direct check that each  $F_M(D(Z_{n-1}(B_*/A_*), n))$  is acyclic therefore  $H_{\mathbf{CH}(\mathbb{Z})}(F(B))$  is isomorphic to  $H_{\mathbf{CH}(\mathbb{Z})}(F_M(A))$  in the homotopy category  $\text{Ho}(\mathbf{CH}(\mathbb{Z}))$ .

**Lemma 3.18.** *Let  $\{C_k\}_{k \geq 0}$  be an acyclic complex where each  $C_k$  is projective  $R$ -module. Then  $\{C_k\}_{k \geq 0}$  is a projective complex i.e. any level-wise surjective chain complex map  $D_* \rightarrow E_*$  can be lifted via any chain complex map  $C_* \rightarrow E_*$ .*

*Proof.* It is easy to check that if  $X$  is a projective  $R$ -module then  $D_n(X)$  is a projective complex. Let  $C_*^{(m)}$  be the complex

$$C_k^{(m)} = \begin{cases} C_k, & \text{if } k \geq m \\ Z_k(C), & \text{if } k = m-1 \\ 0 & \text{otherwise} \end{cases}$$

Here,  $Z_k(C)$  denotes the space of cycles in  $C_k$ , and  $B_k(C)$  is the space of boundary elements in  $C_k$ . The acyclicity condition implies that  $C_*^{(m)}/C_*^{(m+1)} \simeq D(Z_{m-1}(C), m)$ . Note that  $Z_0(C) = C_0$  is a projective  $R$ -module and  $C_* = C^{(1)} = C^{(2)} \oplus D_1(Z_0(C))$ . Now  $D_1(Z_0(C))$  is a projective complex and  $C^{(2)}$  also satisfies the assumption of the lemma and vanishes in degree zero. Therefore by applying the same argument one sees that  $C^{(2)} = C^{(3)} \oplus D(Z_1(C), 2)$ . Continuing this process one obtains  $C_* = D(Z_0(C), 1) \oplus D(Z_1(C), 2) \cdots \oplus D(Z_{k-1}, k) \oplus \cdots$  where each factor is a projective complex, thus proving the statement.  $\square$

We finish this example by computing the left derived functor. For any  $R$ -module  $N$  let  $K(N, 0)$  be the chain complex concentrated in degree zero where there is a copy of  $N$ . Since every object is fibrant, a fibrant-cofibrant replacement of  $K(N, 0)$  is simply a

cofibrant replacement. A cofibrant replacement  $P_*$  of  $K(N, 0)$  is exactly a projective resolution (in the usual sense) of  $N$  in the category of  $R$ -modules. In the homotopy category of  $\mathbf{CH}(R)$ ,  $K(N, 0)$  and  $P$  are isomorphic because by definition  $\text{Hom}_{\text{Ho}(\mathbf{CH}(R))}(K(N, 0), P)$  consists of the homotopy classes of  $\text{Hom}_{\mathbf{CH}(R)}(RQK(N, 0), RQP_*) = \text{Hom}_{\mathbf{CH}(R)}(P_*, P_*)$  which contains the identity map. Therefore  $\mathbb{L}F(K(N, 0)) \simeq \mathbb{L}F(P_*)$  and  $\mathbb{L}F(P_*)$  by Proposition 3.15 and the definition of total derived functor is isomorphic to  $H_{\mathbf{CH}(R)}F(P_*) = M \otimes_R P_*$ . In particular,

$$H_*(\mathbb{L}F(K(N, 0))) = \text{Tor}_*^R(N, M),$$

where  $\text{Tor}_*^R$  is the usual  $\text{Tor}_R$  in homological algebra. We usually denote the derived functor  $\mathbb{L}F(N) = N \otimes_R^L M$ . Similarly one can prove that the contravariant functor  $N_* \mapsto \text{Hom}_R(N_*, M)$  has a total right derived functor, denoted by  $\text{RHom}_R(N_*, M)$  and

$$H^*(\text{RHom}_R(K(N, 0), M)) \simeq \text{Ext}_R^*(N, M),$$

is just the usual  $\text{Ext}$  functor (see Remark 3.17).

### 3.2 Hinich's theorem and Derived category of DG module

The purpose of this section is to introduce a model category and derived functors of DG-modules over a fixed differential graded  $\mathbf{k}$ -algebra. From now on we assume that  $\mathbf{k}$  is a field. The main result is essentially due to Hinich [Hin97] who introduced a model category structure for algebras over a vast class of operads.

Let  $\mathbf{C}(\mathbf{k})$  be the category of (unbounded) complexes over  $\mathbf{k}$ . For  $d \in \mathbb{Z}$  let  $M_d \in \mathbf{C}(\mathbf{k})$  be the complex

$$\cdots \rightarrow 0 \rightarrow \mathbf{k} = \mathbf{k} \rightarrow 0 \rightarrow 0 \cdots$$

concentrated in degree  $d$  and  $d + 1$ .

**Theorem 3.19.** (V. Hinich) *Let  $\mathbf{C}$  be a category which admits finite limits and arbitrary colimits and is endowed with two right and left adjoint functors  $(\#, F)$*

$$\# : \mathbf{C} \rightleftarrows \mathbf{C}(\mathbf{k}) : F \tag{3.4}$$

*such that for all  $A \in \text{obj}(\mathbf{C})$  the canonical map  $A \rightarrow A \coprod F(M_d)$  induces a quasi-isomorphism  $A^\# \rightarrow (A \coprod F(M_d))^\#$ . Then there is a model category structure on  $\mathbf{C}$  where the three distinct classes of morphisms are:*

- (1) *Weak equivalences  $\mathcal{W}$ :  $f \in \text{Mor}(\mathbf{C})$  is in  $\mathcal{W}$  iff  $f^\#$  is quasi-isomorphism.*
- (2) *Fibrations  $\mathcal{F}$ :  $f \in \text{Mor}(\mathbf{C})$  is in  $\mathcal{F}$  if  $f^\#$  is (component-wise) surjective.*
- (3) *Cofibrations  $\mathcal{C}$ :  $f \in \text{Mor}(\mathbf{C})$  is a cofibration if it satisfies the LLP property with respect to all acyclic fibrations  $\mathcal{W} \cap \mathcal{F}$ .*

As an application of Hinich's theorem, one obtains a model category structure on the category  $\text{Mod}(A)$  of (left) differential graded modules over a differential graded algebra  $A$ . Here  $\#$  is the forgetful functor and  $F$  is given by tensoring  $F(M) = A \otimes_{\mathbf{k}} M$ .

**Corollary 3.20.** *The category  $\text{Mod}(A)$  of DG  $A$ -modules is endowed with a model category structure where*

- (i) weak equivalences are the quasi-isomorphisms.
- (ii) fibrations are level-wise surjections. Therefore all objects are fibrant.
- (iii) cofibrations are the maps have the left lifting property with respect to all acyclic fibrations.

In what follows we give a description of cofibrations and cofibrant objects. An excellent reference for this part is [FHT95].

**Definition 3.21.** An  $A$ -module  $P$  is called a *semi-free* extension of  $M$  if  $P$  is a union of an increasing family of  $A$ -modules  $M = P(-1) \subset P(0) \subset \dots$  where each  $P(k)/P(k-1)$  is a free  $A$ -modules generated by cycles. In particular  $P$  is said to be a *semi-free*  $A$ -module if it is a semi-free extension of the 0. A *semi-free resolution* of an  $A$ -module morphism  $f : M \rightarrow N$  is a semi-free extension  $P$  of  $M$  with a quasi-isomorphism  $P \rightarrow N$  which extends  $f$ .

In particular a *semi-free* resolution of an  $A$ -module  $M$  is a semi-free resolution of the trivial map  $0 \rightarrow M$ .

The notion of semi-free modules can be traced back to [GM74], and [Dri04] is another nice reference for the subject. A  $\mathbf{k}$ -complex  $(M, d)$  is called semi-free, if it is semi-free as a differential  $\mathbf{k}$ -module. Here  $\mathbf{k}$  is equipped with the trivial differential. In the case of a field  $\mathbf{k}$ , every positively graded  $\mathbf{k}$ -complex is semi-free. It is clear from the definition that a finitely generated semi-free  $A$ -module is obtained through a finite sequence of extensions of some free  $A$ -modules of the form  $A[n]$ ,  $n \in \mathbb{Z}$ . Here is  $A[n]$  is  $A$  after in shift in degree by  $-n$ .

**Lemma 3.22.** *Let  $M$  be an  $A$ -module with a filtration  $F_0 \subset F_1 \subset F_2 \dots$  such that  $F_0$  and all  $F_{i+1}/F_i$  are semifree  $A$ -modules. Then  $M$  is semifree.*

*Proof.* Since  $F_k/F_{k-1}$  is semifree, it has a filtration  $\dots P_l^k \subset P_{l+1}^k \dots$  such that  $P_l^k/P_{l+1}^k$  is generated as an  $(A, d)$ -module by cycles. So one can write  $F_k/F_{k-1} = \bigoplus_l (A \otimes Z'_k(l))$  where  $Z'_k(l)$  are free (graded)  $\mathbf{k}$ -modules such that  $d(Z_k(l)) \subset \bigoplus_{j < l} Z_k(j)$ . Therefore there are free  $\mathbf{k}$ -modules  $Z_k(l)$  such that

$$F_k = F_{k-1} \bigoplus_{l \geq 0} Z'_k(l)$$

and

$$d(Z_k(l)) \subset F_{k-1} \bigoplus_{j < l} A \otimes Z_k(j).$$

In particular  $M$  is the free  $\mathbf{k}$ -module generated by the union of all basis elements  $\{z_\alpha\}$  of  $Z_k(l)$ 's. Now consider the filtration  $P_0 \subset P_1 \dots$  of free  $\mathbf{k}$ -modules constructed inductively as follows:  $P_0$  is generated as  $\mathbf{k}$ -module by the  $z_\alpha$ ' which are cycles, *i.e.*  $dz_\alpha = 0$ . Then  $P_k$  is generated by those  $z_\alpha$ 's such that  $dz_\alpha \in A \cdot P_{k-1}$ . This is clearly a semifree resolution if we prove that  $M = \bigcup_k P_k$ . For that, we show by induction on degree that for all  $\alpha$ ,  $z_\alpha$  belongs to some  $P_k$ . Suppose that  $z_\alpha \in Z_k(l)$ , then  $dz_\alpha \in \bigoplus A \cdot Z_i(j)$  where  $i < k$  or  $i = k$  and  $j < l$ . By induction hypothesis all  $z_\beta$ 's in the sum  $dz_\alpha$  are in some  $P_{m_\beta}$ . Therefore  $z_\alpha \in P_m$  where  $m = \max_\beta m_\beta$  and this finishes the proof.  $\square$



**Remark 3.23.** If we had not assumed that  $\mathbf{k}$  is a field but only a commutative ring then we could still put a model category on  $\text{Mod}(A)$ . This is a special case of the Schwede-Shipely theorem (Theorem 4.1 [SS00]). More details are provided in pages 503-504 of [SS00].

**Proposition 3.24.** *In the model category of  $A$ -modules, a maps  $f : M \rightarrow N$  is a cofibration if and only if it is a retract of a semi-free extension  $M \hookrightarrow P$ . In particular an  $A$ -module  $M$  is cofibrant iff it is a retract of a semi-free  $A$ -module, in other words it is a direct summand of a semi-free  $A$ -module.*

Here is a list of properties of semi-free modules which allow us to define the derived functor by means of semi-free resolutions.

**Proposition 3.25.** (i) *Any morphism  $f : M \rightarrow N$  of  $A$ -modules has a semi-free resolution. In particular every  $A$ -module has a semi-free resolution.*

(ii) *If  $P$  is a semi-free  $A$ -module then  $\text{Hom}_A(P, -)$  preserves the quasi-isomorphisms.*

(iii) *Let  $P$  and  $Q$  be semi-free  $A$ -modules and  $f : P \rightarrow Q$  a quasi-isomorphism. Then*

$$g \otimes f : M \otimes_A P \rightarrow N \otimes_A Q$$

*is a quasi-isomorphism if  $g : M \rightarrow N$  is a quasi-isomorphism.*

(iv) *Let  $P$  and  $Q$  be semi-free  $A$ -modules and  $f : P \rightarrow Q$  a quasi-isomorphism. Then*

$$\text{Hom}_R(g, f) : \text{Hom}_A(Q, M) \rightarrow \text{Hom}_A(P, N)$$

*is a quasi-isomorphism if  $g : M \rightarrow N$  is a quasi-isomorphism.*

The second statement in the proposition above implies that a quasi-isomorphism  $f : M \rightarrow N$  between semi-free  $A$ -modules is a homotopy equivalence *i.e* there is a map  $f' : N \rightarrow M$  such that  $ff' - id_N = [d_N, h']$  and  $f'f - id_M = [d_M, h]$  for some  $h : M \rightarrow N$  and  $h' : N \rightarrow M$ . In fact part (iii) and (iv) follow easily from this observation.

The properties listed above imply that the functors  $- \otimes_A M$  and  $\text{Hom}_A(-, M)$  preserves enough weak equivalences, ensuring that the derived functors  $\otimes_A^L$  and  $\text{RHom}_A(-, M)$  exist for all  $A$ -modules  $M$ .

Since we are interested in Hochschild and cyclic (co) homology, we switch to the category of DG  $A$ -bimodules. This category is the same as the category of DG  $A^e$ -modules. Therefore one can endow  $A$ -bimodules with a model category structure and define the derived functors  $- \otimes_{A^e}^L M$  and  $\text{RHom}_{A^e}(-, M)$  by mean of fibrant-cofibrant replacements.

More precisely, for two  $A$ -bimodule  $M$  and  $N$  we have

$$\text{Tor}_*^{A^e}(M, N) = H_*(P \otimes_{A^e} N)$$

and

$$\text{Ext}_{A^e}^*(M, N) = H^*(\text{Hom}_{A^e}(P, N))$$

where  $P$  is cofibrant replacement for  $M$ .

By Proposition 3.25 every  $A^e$ -module has a semi-free resolution for which there is an explicit construction using the two-sided bar construction. For right and left  $A$ -modules  $P$  and  $M$ , let

$$B(P, A, M) = \bigoplus_{k \geq 0} P \otimes (s\bar{A})^{\otimes k} \otimes M \quad (3.5)$$

equipped with the differential:

if  $k = 0$ ,

$$D(p[\ ]m) = dp[\ ]n + (-1)^{|p|}p[\ ]dm$$

if  $k > 0$

$$\begin{aligned} D(p[a_1, \dots, a_k]m) &= d_0(p[a_1, \dots, a_k]m) + d_1(p[a_1, \dots, a_k]m) \\ &= dp[a_1, \dots, a_k]m + \sum_{i=1}^k (-1)^{\epsilon_i} p[a_1, \dots, da_i, \dots, a_k]m + (-1)^{\epsilon_{k+1}} p[a_1, \dots, a_k]dm \\ &\quad + (-1)^{|p|} pa_1[a_2, \dots, a_k]m + \sum_{i=2}^k (-1)^{\epsilon_i} p[a_1, \dots, a_{i-1}a_i, \dots, a_k]m + (-1)^{\epsilon_k} p[a_1, \dots, a_{k-1}]a_k m \end{aligned}$$

where

$$\epsilon_i = |p| + |a_1| + \dots + |a_{i-1}| - i - 1$$

Let  $P = A$  and  $\epsilon_M : B(A, A, M) \rightarrow M$  be defined by

$$\epsilon_M(a[a_1, \dots, a_k]m) = \begin{cases} 0 & \text{if } k \geq 1 \\ am & \text{if } k = 0 \end{cases} \quad (3.6)$$

It is clear that  $\epsilon_M$  is a map of left  $A$ -modules if  $M$ .

**Lemma 3.26.** *In the category of left  $A$ -modules,  $\epsilon_M : B(A, A, M) \rightarrow M$  is a semi-free resolution.*

*Proof.* Let us first prove that this is a resolution. Let  $h : B(A, A, M) \rightarrow B(A, A, M)$  be defined

$$h(a[a_1, a_2, \dots, a_k]m) = \begin{cases} [a, a_1, \dots, a_k]m, & \text{if } k \geq 1, \\ [a]m, & \text{if } k = 0. \end{cases} \quad (3.7)$$

One can easily check that for  $[D, h] = id$  on  $\ker \epsilon_M$ , which implies  $H_*(\ker(\epsilon_M)) = 0$ . Since  $\epsilon_M$  is surjective,  $\epsilon_M$  is a quasi-isomorphism. Now we prove that  $B(A, A, M)$  is a semifree  $A$ -module. Let  $F_k = \bigoplus_{i \leq k} A \otimes T(s\bar{A})^{\otimes i} \otimes M$ . Since  $d_1(F_{k+1}) \subset F_k$ , then  $F_{k+1}/F_k$  as a differential graded  $A$ -module is isomorphic to  $(A \otimes (sA)^{\otimes k} \otimes M, d_0) = (A, d) \otimes_{\mathbf{k}} ((sA)^{\otimes k}, d) \otimes (M, d)$ . The latter is a semifree  $(A, d)$ -module since  $((sA)^{\otimes k}, d) \otimes_{\mathbf{k}} (M, d)$  is a semifree  $\mathbf{k}$ -module via the filtration

$$0 \hookrightarrow \ker(d \otimes 1 + 1 \otimes d) \hookrightarrow ((sA)^{\otimes k}, d) \otimes_{\mathbf{k}} (M, d).$$

Therefore  $B(A, A, M)$  is semi-free by Lemma 3.22.  $\square$

**Corollary 3.27.** *The map  $\epsilon_A : B(A, A) := B(A, A, \mathbf{k}) \rightarrow \mathbf{k}$  given by*

$$\epsilon_k(a[a_1, a_2, \dots, a_n]) = \begin{cases} \epsilon(a) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

*is a resolution. Here  $\epsilon : A \rightarrow \mathbf{k}$  is the augmentation of  $A$ . In other words  $B(A, A)$  is acyclic.*

*Proof.* In the previous lemma, let  $M = \mathbf{k}$  be the differential  $A$ -module with trivial differential and the module structure  $a.k := \epsilon(a)k$   $\square$

**Lemma 3.28.** *In the category  $\text{Mod}(A^e)$ ,  $\epsilon_A : B(A, A, A) \rightarrow A$  is a semifree resolution.*

*Proof.* The proof is similar to the previous lemma. First of all, it is obvious that this is a map of  $A^e$ -modules. Let  $F_k = \bigoplus_{i \leq k} A \otimes T(s\bar{A})^{\otimes i} \otimes A$ . Then  $F_{k+1}/F_k$  as a differential graded  $A$ -module is isomorphic to  $(A \otimes (sA)^{\otimes k} \otimes A, d_0) = (A, d) \otimes_{\mathbf{k}} ((sA)^{\otimes k}, d) \otimes (A, d)$ . The later is semi-free as  $A^e$ -module since  $((sA)^{\otimes k}, d)$  is a semi-free  $\mathbf{k}$ -module via the filtration  $\ker d \hookrightarrow (sA)^{\otimes k}$ .  $\square$

Since the two-sided bar construction  $B(A, A, A)$  provides us with a semi-free resolution of  $A$  we have that

$$HH_*(A, M) = H_*(B(A, A, A) \otimes_{A^e} M) = \text{Tor}_*^{A^e}(A, M)$$

and

$$HH^*(A, M) = H^*(\text{Hom}_{A^e}(B(A, A, A), M)) = \text{Ext}_{A^e}^*(A, M).$$

In some special situations, for instance that of Calabi-Yau algebras, one can choose smaller resolutions to compute Hochschild homology or cohomology.

The following will be useful.

**Lemma 3.29.** *If  $H^*(A)$  is finite dimensional then for all finitely generated semi-free  $A$ -bimodules  $P$  and  $Q$ ,  $H^*(P)$ ,  $H^*(Q)$  and  $H^*(\text{Hom}_{A^e}(P, Q))$  are also finite dimensional.*

*Proof.* Since  $A$  has finite dimensional cohomology, we see that  $H^*(A \otimes A^{op})$  is finite dimensional. Similarly  $P$  (or  $Q$ ) has finite cohomological dimension since it is obtained via a finite sequence of extensions of free bimodules of the form  $(A \otimes A^{op})[n]$ . Note that since  $\text{Hom}_{A^e}(A \otimes A^{op}, A \otimes A^{op}) \simeq A \otimes A^{op}$ ,  $A \otimes A^{op}$  is a free  $A$  bimodule of finite cohomological dimension, therefore  $\text{Hom}_{A^e}(P, Q)$  is obtained through a finite sequence of extensions of shifted free  $A$ -bimodules, proving that it has finite cohomological dimension.  $\square$

## 4 Calabi-Yau DG algebras

Throughout this section  $(A, d)$  is a differential graded algebra, and by an  $A$ -bimodule we mean a differential graded  $(A, d)$ -bimodule.

In this section we essentially explain how an isomorphism

$$HH^*(A, A) \simeq HH_*(A, A) \tag{4.1}$$

(of  $HH^*(A, A)$ -modules) gives rise to a BV structure on  $HH^*(A, A)$  extending its canonical Gerstenhaber structure. For Calabi-Yau DG algebra one does have such an isomorphism (4.1) and this is a special case of a more general statement due to Van den Bergh [vdB98]. The main idea is due to V. Ginzburg [Gin] who proved that for a Calabi-Yau algebra  $A$ ,  $HH^*(A, A)$  is a BV-algebra. However he works with ordinary algebras rather than DG algebras. But here we have adapted his result to the case of Calabi-Yau DG algebras. For this purpose one has to work in the correct derived category of  $A$ -bimodules, and this is the derived category of perfect  $A$ -bimodules as it is formulated below. All this can be extended to the case of  $A_\infty$  but for simplicity we refrain from doing so.

## 4.1 Calabi-Yau algebras

We first give the definition of Calabi-Yau algebra which were introduced by Ginzburg in [Gin] for algebras with no differential and then generalized by Kontsevich-Soibelman [KS09] to the differential graded algebras.

**Definition 4.1.** (Kontsevich-Soibelman [KS09])

- (1) An  $A$ -bimodule is *perfect* if it is quasi-isomorphic to a direct summand of a finitely generated semifree  $A$ -bimodules.
- (2)  $A$  is said to be *homologically smooth* if it is perfect as an  $A$ -bimodule.

**Remark 4.2.** In [KS09], the definition of perfectness uses the notion of *extension* [Kel94] and it is essentially the same as ours.

We define *DG-projective  $A$ -modules* to be the direct summands of semifree  $A$ -modules. As a consequence, an  $A$ -bimodule is perfect iff it is quasi-isomorphic to a finitely generated DG-projective  $A$ -bimodule. We call the latter a finitely generated *DG-projective  $A$ -module resolution*. This is analogous to having a bounded projective resolution in the case of ordinary modules (without differential). By Proposition 3.25, DG-projectives have all the nice homotopy theoretic properties that one expects.

The content of the next lemma is that  $A^! := \mathrm{RHom}_{A^e}(A, A^e)$ , called the *derived dual* of  $A$ , is also a perfect  $A$ -bimodule. The  $A$ -bimodule structure of  $A^!$  is induced by the right action of  $A^e$  on itself. Recall that for an  $A$ -bimodule  $M$ ,  $\mathrm{RHom}_{A^e}(-, M)$  is the right derived functor of  $\mathrm{Hom}_{A^e}(-, M)$  *i.e.* for an  $A$ -bimodule  $N$ ,  $\mathrm{RHom}_{A^e}(N, M)$  is the complex  $\mathrm{Hom}_{A^e}(P, M)$  where  $P$  is a DG-projective  $A$ -bimodule quasi-isomorphic to  $N$ . In general,  $M^! = \mathrm{RHom}_{A^e}^*(M, A^e)$  is different from the usual dual

$$M^\vee = \mathrm{Hom}_{A^e}(M, A^e).$$

**Lemma 4.3.** [KS09] *If  $A$  is homologically smooth then  $A^!$  is a perfect  $A$ -bimodule.*

*Proof.* Let  $P = P_i \rightarrow A$  be a finitely generated DG-projective resolution. Note that  $A^! = \mathrm{RHom}_{A^e}(A, A^e)$  is quasi-isomorphic to the complex  $\mathrm{Hom}_{A^e}(P_i, A^e)$ . Each  $P_i$  being a direct summand of a semi-free module  $Q_i$ . Since  $Q_i$  is obtained through a finite sequence of extensions of a free  $A^e$ -modules,  $\mathrm{Hom}(Q_i, A^e)$  is also a semi-free module. Clearly  $\mathrm{Hom}_{A^e}(P_i, A^e)$  is a direct summand of the semi-free module  $\mathrm{Hom}(Q_i, A^e)$ , therefore DG-projective. This proves the lemma. □

We say that a DG algebra  $A$  is *compact* if the cohomology  $H^*(A)$  is finite dimensional.

**Lemma 4.4.** *A compact homologically smooth DG algebra  $A$  has finite dimensional Hochschild cohomology  $HH^*(A, A)$ .*

*Proof.* By assumption  $A$  has finite dimensional cohomology and so does  $A^e = A \otimes A^{op}$ . Now let  $P \rightarrow A$  be a finitely generated DG-projective resolution of  $A$ -bimodules. We have a quasi-isomorphism of complexes  $C^*(A, A) \simeq \mathrm{RHom}_{A^e}(A, A) \simeq \mathrm{Hom}_{A^e}(P, P)$ , which by Lemma 3.29 has finite dimensional cohomology. □

**Definition 4.5.** (Ginzburg, Kontsevich-Soibelman [Gin],[KS09]) A  $d$ -dimensional *Calabi-Yau differential graded algebra* is a homologically smooth DG-algebra endowed with an  $A$ -bimodule quasi-isomorphism

$$\psi : A \xrightarrow{\simeq} A^1[d] \quad (4.2)$$

such that

$$\psi^1 = \psi[d]. \quad (4.3)$$

The main reason to call such algebras Calabi-Yau is that a tilting generator  $\mathcal{E} \in D^b(\text{Coh}(X))$  of the bounded derived category of coherent sheaves on a smooth algebraic variety  $X$  is a Calabi-Yau algebra iff  $X$  is a Calabi-Yau (see [Gin] Proposition 3.3.1 for more details).

There are many other examples provided by representation theory. For instance most of the three dimensional Calabi-Yau algebras are obtained as a quotient of the free associative algebras  $F = \mathbb{C}\langle x_1, \dots, x_n \rangle$  on  $n$  generators. An element  $\Phi$  of  $F_{cyc} := F/[F, F]$  is called a cyclic potential. One can define the partial derivatives  $\frac{\partial}{\partial x_i} : F_{cyc} \rightarrow F$  in this setting. Many of 3-dimensional Calabi-Yau algebras are obtained as a quotient  $\mathcal{U}(F, \phi) = F/\{\frac{\partial \Phi}{\partial x_i=0}_{i=0, \dots, n}\}$ . For instance for  $\Phi(x, y, z) = xyz - yzx$ , we obtain  $\mathcal{U}(F, \phi) = \mathbb{C}[x, y, z]$ , the polynomial algebra in 3 variables. The details of this discussion is irrelevant to the context of this chapter which is the algebraic models of free loop spaces. We, therefore, refer the reader to [Gin] for further details.

Here  $A^1 = \text{RHom}_{A^e}(A, A^e)$  is called the *dualizing the bi-module*, which is also a  $A$ -bimodule using outer multiplication. Condition (4.2) amounts to the following.

**Proposition 4.6.** (Van den Bergh Isomorphism [vdB98]) *Let  $A$  be a Calabi-Yau DG algebra of dimension  $d$ . Then for all  $A$ -bimodules  $A$  we have*

$$HH_{d-*}(A, N) \simeq HH^*(A, N). \quad (4.4)$$

*Proof.* We compute

$$\begin{aligned} HH^*(A, N) &\simeq \text{Ext}_{A^e}^*(A, N) \simeq H^*(\text{RHom}_{A^e}(A, N)) \simeq H^*(\text{RHom}_{A^e}(A, A^e) \otimes_{A^e}^L N) \\ &\simeq H^*(A^1 \otimes_{A^e}^L N) \simeq H_*(A[-d] \otimes_{A^e}^L N) \simeq \text{Tor}_*^{A^e}(A, N) \simeq HH_{d-*}(A, N) \end{aligned} \quad (4.5)$$

□

Note that the choice of the  $A$ -bimodule isomorphism  $\psi$  is important and it is characterized by the image of the unit  $\pi = \psi(1_A) \in A^1$ . By definition,  $\pi$  is the *volume* of the Calabi-Yau algebra  $A$ . For a Calabi-Yau algebra  $A$  with a volume  $\pi$  and  $N = A$ , we obtain an isomorphism

$$D = D_\pi : HH_{d-*}(A, A) \rightarrow HH^*(A, A). \quad (4.6)$$

One can use  $D$  to transfer the Connes operator  $B$  from  $HH_*(A, A)$  to  $HH^*(A, A)$ ,

$$\Delta = \Delta_\pi := D \circ B \circ D^{-1}$$

In the following lemmas  $A$  is a Calabi-Yau algebra with a fixed volume  $\pi$  and the associated operator  $\Delta$ .

**Lemma 4.7.**  $f \in HH^*(A, A)$  and  $a \in HH_*(A, A)$

$$D(i_f a) = f \cup Da.$$

*Proof.* To prove the lemma we use the derived description of Hochschild (co)homology, cap and cup product. Let  $(P, d)$  be a projective resolution of  $A$ . Then  $HH_*(A, A)$  is computed by the complex  $(P \otimes_{A^e} P, d)$ , and similarly  $HH^*(A, A)$  is computed by the complex  $(\text{End}(P), \text{ad}(d) = [-, d])$ . Here  $\text{End}(P) = \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{A^e}(P, P[r])$ . Then the cap product corresponds to the natural pairing

$$ev : (P \otimes_{A^e} P) \otimes_{\mathbf{k}} \text{End}(P) \rightarrow P \otimes_{A^e} P \quad (4.7)$$

given by  $ev : (p_1 \otimes p_2) \otimes f \mapsto p_1 \otimes f(p_2)$ . The quasi-isomorphism  $\psi : A \rightarrow A^1[d]$  yields a morphism  $\phi : P \rightarrow P^\vee$ . Let us explain this in detail.

Using the natural identification  $P^\vee \otimes_{A^e} P = \text{End}(P)$ , we have a commutative diagram

$$\begin{array}{ccc} \text{End}(P) \otimes_{\mathbf{k}} (P \otimes_{A^e} P) & \xrightarrow{ev} & P \otimes_{A^e} P \\ \downarrow id \otimes (\phi \otimes id) & & \downarrow \phi \otimes id \\ \text{End}(P) \otimes_{\mathbf{k}} \text{End}(P) & \xrightarrow{\text{composition}} & \text{End}(P) = P^\vee \otimes_{A^e} P \end{array} \quad (4.8)$$

The evaluation map is defined by  $ev(\psi \boxtimes (x \otimes y)) := x \otimes \psi(y)$ . After passing to (co)homology,  $\phi \otimes id$  becomes  $D$ , the composition induces the cup product and  $ev$  is the contraction (cap product), hence

$$D(i_f a) = f \cup D a$$

which proves the lemma. □

**Lemma 4.8.** *For  $f, g \in HH^*(A, A)$  and  $a \in HH_*(A, A)$  we have*

$$\begin{aligned} [f, g] \cdot a &= (-1)^{|f|} B((f \cup g)a) - f \cdot B(g \cdot a) + (-1)^{(|f|+1)(|g|+1)} g \cdot B(f \cdot a) \\ &\quad + (-1)^{|g|} (f \cup g) \cdot B(a). \end{aligned}$$

*Proof.* We compute

$$\begin{aligned} i_{[f, g]} &= L_f i_g - i_g L_f \\ &= i_f B i_g - B i_f i_g - i_g B i_f - i_g i_f B \\ &= i_f B i_g - i_g B i_f - B i_{f \cup g} - i_g i_f B \end{aligned} \quad (4.9)$$

□

**Corollary 4.9.** *For all  $f, g \in HH^*(A, A)$  and  $a \in HH_*(A, A)$ , we have*

$$\begin{aligned} [f, g] \cup D(a) &= (-1)^{|f|} \Delta(f \cup g \cup D a) - f \cup \Delta(g \cup D(a)) \\ &\quad + (-1)^{(|f|+1)(|g|+1)} g \cup \Delta(f \cup D(a)) - f \cup g \cup D B(a). \end{aligned}$$

*Proof.* After taking  $D$  of the identity in the previous lemma, we obtain

$$\begin{aligned} D(i_{[f, g]} a) &= (-1)^{|f|} D(B i_{f \cup g} a) - D(i_f B i_g(a)) + (-1)^{(|f|+1)(|g|+1)} D(i_g B i_f(a)) \\ &\quad + (-1)^{|g|} D(i_f i_g B(a)) \end{aligned}$$

By Lemma 4.7 and  $DB = \Delta D$ , this reads

$$[f, g] \cup D(a) = (-1)^{|f|} \Delta D(i_{f \cup g} a) - f \cup \Delta D(i_g(a)) + (-1)^{(|f|+1)(|g|+1)} g \cup \Delta D(i_f(a)) - g \cup f \cup DB(a).$$

Once again by Lemma 4.7 we get

$$[f, g] \cup D(a) = (-1)^{|f|} \Delta(f \cup g \cup Da) - f \cup \Delta(g \cup D(a)) + (-1)^{(|f|+1)(|g|+1)} g \cup \Delta(f \cup D(a)) + (-1)^{|g|} f \cup g \cup DB(a).$$

□

**Theorem 4.10.** *For a Calabi-Yau algebra  $A$  with a volume  $\pi \in A^1$ ,  $(HH^*(A, A), \cup, \Delta)$  is a BV-algebra i.e.*

$$[f, g] = (-1)^{|f|} \Delta(f \cup g) - (-1)^{|f|} \Delta(f) \cup g - f \cup \Delta(g). \quad (4.10)$$

*Proof.* In the statement of the previous lemma choose  $a \in HH_d(A, A)$  such that  $D(a) = 1 \in HH^0(A, A)$ . The identity (5.4) will follow since  $B(a) = 0$  for obvious degree reason. □

## 4.2 Poincaré duality groups and chains of Moore based loop space

Let us finish this section with some interesting examples of DG Calabi-Yau algebras. We will also discuss chains of the Moore based loop space. This example plays an important role in symplectic geometry where it appears as the generator of a particular type of Fukaya category called the wrapped Fukaya category (see [Abo11] for more details). One can then compute the Hochschild homology of wrapped Fukaya categories using Burghlea-Fiedorowicz-Goodwillie theorem. This theorem implies that the Hochschild homology of the Fukaya category of a closed oriented manifold is isomorphic to the homology of the free loop space of the manifold.

We start with a more elementary example *i.e.* the Poincaré duality groups. Among the examples, we have the fundamental group of closed oriented aspherical manifolds. Closed oriented irreducible 3-manifolds are aspherical, therefore they provide us an interesting large class of examples.

**Proposition 4.11.** *Let  $G$  be finitely generated oriented Poincaré duality group of dimension  $d$ . Then  $\mathbf{k}[G]$  is a Calabi-Yau algebra of dimension  $d$ , therefore  $HH^*(\mathbf{k}[G], \mathbf{k}[G])$  is a BV algebra*

*Proof.* First note that  $\mathbf{k}[G]$  is only an ordinary algebra without grading and differential. The hypothesis that  $G$  is a finitely generated oriented Poincaré duality group of dimension  $d$  means that  $\mathbf{k}$  has a bounded finite projective resolution  $P = \{P_d \rightarrow \cdots \rightarrow P_1 \rightarrow P_0\} \rightarrow \mathbf{k}$  as a left  $\mathbf{k}[G]$ -module, and  $H^d(G, \mathbf{k}[G]) \simeq \mathbf{k}$  as a  $\mathbf{k}[G]$ -module. Here  $\mathbf{k}$  is equipped with the trivial action and  $\mathbf{k}[G]$  acts on  $H^d(G, \mathbf{k}[G])$  from left via the coefficient module. In particular, we

$$\mathrm{Ext}_{\mathbf{k}[G]}^i(\mathbf{k}, \mathbf{k}[G]) \simeq \begin{cases} \mathbf{k}, & i = d \\ 0, & \text{otherwise.} \end{cases} \quad (4.11)$$

In other words the resolution  $P$  has the property that  $P^\vee := \text{Hom}_{\mathbf{k}[G]}(P, \mathbf{k}[G])$ , after a shift in degree by  $d$ , is also a resolution of  $\mathbf{k}$  as a right  $\mathbf{k}[G]$ -module (See [Bro82] for more details).

Note that using the map  $g \rightarrow (g, g^{-1})$ , we can turn  $\mathbf{k}[G]^e$  into a right  $\mathbf{k}[G]$ -module. More precisely  $(g_1 \otimes g_2)g := g_1 g \otimes g^{-1} g_2$ . The tensor product  $\mathbf{k}[G]^e \otimes_{\mathbf{k}[G]} \mathbf{k}$  is isomorphic to  $\mathbf{k}[G]$  as a left  $\mathbf{k}[G]^e$ -module. The isomorphism is given by  $(g_1 \otimes g_2) \boxtimes 1 \mapsto g_1 g_2$ . Similarly  $\mathbf{k}[G]^e$  can be considered as a left  $\mathbf{k}[G]$ -module using the action  $g(g_1 \otimes g_2) := g g_1 \otimes g_2 g^{-1}$  and once again  $\mathbf{k} \otimes_{\mathbf{k}[G]} \mathbf{k}[G]^e \simeq \mathbf{k}[G]$  as a right  $\mathbf{k}[G]^e$ -module using the isomorphism  $1 \boxtimes (g_1 \otimes g_2) \mapsto g_2 g_1$ .

It is now clear that  $\mathbf{k}[G]^e \otimes_{\mathbf{k}[G]} P_d \rightarrow \cdots \rightarrow \mathbf{k}[G]^e \otimes_{\mathbf{k}[G]} P_1 \rightarrow \mathbf{k}[G]^e \otimes_{\mathbf{k}[G]} P_0 \rightarrow \mathbf{k}[G] \otimes_{\mathbf{k}[G]} \mathbf{k} \simeq \mathbf{k}[G]$  is a projective resolution of  $\mathbf{k}[G]$  as a  $\mathbf{k}[G]$ -bimodule, proving that  $\mathbf{k}[G]$  is homologically smooth. Similarly  $\text{Hom}_{\mathbf{k}[G]}(P, A) \otimes_{\mathbf{k}[G]} \mathbf{k}[G]^e$  is a projective resolution of  $\mathbf{k}[G]$  as  $\mathbf{k}[G]$ -bimodule. Therefore we have a homotopy equivalence

$$\phi : \mathbf{k}[G]^e \otimes_{\mathbf{k}[G]} P \rightarrow \text{Hom}_{\mathbf{k}[G]}(P, A) \otimes_{\mathbf{k}[G]} \mathbf{k}[G]^e.$$

Now if we take  $Q = \mathbf{k}[G]^e \otimes_{\mathbf{k}[G]} P$  as a DG-projective resolution of  $\mathbf{k}[G]$  as  $\mathbf{k}[G]$ -bimodule, then  $\mathbf{k}[G]^! = \text{Hom}_{\mathbf{k}[G]^e}(\mathbf{k}[G]^e \otimes_{\mathbf{k}[G]} P, \mathbf{k}[G]^e) \simeq \text{Hom}_{\mathbf{k}[G]}(P, \mathbf{k}[G]^e)$  where the right  $\mathbf{k}[G]$ -module structure of  $\mathbf{k}[G]^e$  was described above. Since the  $P$  is  $\mathbf{k}[G]$ -projective, the natural map  $\text{Hom}_{\mathbf{k}[G]}(P, \mathbf{k}[G]) \otimes_{\mathbf{k}[G]} \mathbf{k}[G]^e \xrightarrow{\simeq} \text{Hom}_{\mathbf{k}[G]}(P, \mathbf{k}[G]^e)$  is an isomorphism.

Therefore  $\phi$  is nothing but an equivalence  $\mathbf{k}[G] \xrightarrow{\phi} \mathbf{k}[G]^!$  in the derived category of the  $\mathbf{k}[G]$ -bimodules. It remains to prove that after  $\phi^! \simeq \phi[d]$  in the derived category of  $\mathbf{k}[G]$ -bimodules. We have

$$\begin{aligned} \phi^! = \phi^\vee : \text{Hom}_{\mathbf{k}[G]^e}(\text{Hom}_{\mathbf{k}[G]}(P, \mathbf{k}[G]) \otimes_{\mathbf{k}[G]} \mathbf{k}[G]^e, \mathbf{k}[G]^e) &\rightarrow \text{Hom}_{\mathbf{k}[G]^e}(\mathbf{k}[G]^e \otimes_{\mathbf{k}[G]} P, \mathbf{k}[G]^e) \\ &\simeq \mathbf{k}[G]^!. \end{aligned} \quad (4.12)$$

On the other hand we have the natural inclusion map

$$\begin{aligned} i : \mathbf{k}[G]^e \otimes_{\mathbf{k}[G]} P &\rightarrow \text{Hom}_{\mathbf{k}[G]^e}(\text{Hom}_{\mathbf{k}[G]}(P, \mathbf{k}[G]^e) \otimes_{\mathbf{k}[G]} P, \mathbf{k}[G]^e) \\ &\simeq \text{Hom}_{\mathbf{k}[G]^e}(\text{Hom}_{\mathbf{k}[G]}(P, \mathbf{k}[G]) \otimes_{\mathbf{k}[G]} \mathbf{k}[G]^e, \mathbf{k}[G]^e) \end{aligned} \quad (4.13)$$

and one can easily check that  $\phi^! \circ i = \phi$  after a shift in degree by  $d$ . This proves that  $\phi^! \simeq \phi[d]$  in the derived category.  $\square$

**Remark 4.12.** In the case of  $G = \pi_1(M)$  the fundamental group of an aspherical manifold  $M$ , Vaintrob [Vai] has proved that the BV structure on  $HH^{*+d}(\mathbf{k}[G], \mathbf{k}[G]) \simeq HH_*(\mathbf{k}[G], \mathbf{k}[G])$  corresponds to the Chas-Sullivan BV structure on  $H_*(LM, \mathbf{k})$ .

Let  $(X, *)$  be a finite CW complex with a basepoint and Poincaré duality. The Moore loop space of  $X$ ,  $\Omega X = \{\gamma : [0, s] \mid \gamma(0) = \gamma(s) = *, s \in \mathbb{R}^{>0}\}$  is equipped with the standard concatenation which is strictly associative. Therefore the cubic chains  $C_*(\Omega X)$  can be made into a strictly associative algebra using the Eilenberg-Zilber map and the concatenation. In [Gin] there is a sketch of the proof that  $C_*(\Omega X)$  is homologically smooth. Here we prove more using a totally different method.

**Proposition 4.13.** *For a Poincaré duality finite CW-complex  $X$ ,  $C_*(\Omega X)$  is Calabi-Yau DG algebra.*



*Proof.* The main idea of the proof is essentially taken from [FHT95]. Let  $A = C_*(\Omega X)$  be the cubic singular chains complex of the Moore loop space. By composing the Eilenberg-Zilber and concatenation maps  $C_*(\Omega X) \otimes C_*(\Omega X) \xrightarrow{EZ} C_*(\Omega X \times \Omega X) \xrightarrow{\text{concaten.}} C_*(\Omega X)$  one can define an associative product on  $A$ . The product is often called the Pontryagin product. One could switch to the simplicial singular chain complex of the standard base loop space  $\{\gamma : [0, 1] \mid \gamma(0) = \gamma(1) = *\}$  but then one has to work with  $A_\infty$ -algebras and  $A_\infty$ -bimodules and their derived category. All these work nicely [KS09, Mal] and the reader may wish to write down the details in this setting.

Note that  $A$  has some additional structures. First of all, the composition of Alexander-Whitney and the diagonal maps  $C_*(\Omega X) \xrightarrow{\text{diagonal}} C_*(\Omega X \times \Omega X) \xrightarrow{A-W} C_*(\Omega X) \otimes C_*(\Omega X)$  provides  $A$  with a coassociative coproduct, which together with the Pontryagin product make  $C_*(\Omega X)$  into a bialgebra. One can consider the inverse map on  $\Omega X$  which makes the bialgebra  $C_*(\Omega X)$  into a differential graded Hopf algebra up to homotopy. In order to get a strict differential graded Hopf algebra, one finds a topological group  $G$  which is homotopy equivalent to  $\Omega X$  (see [Kan56, HT10]). This can be done, and one can even find a simplicial topological group homotopy equivalent to  $\Omega X$ . Therefore from now on, we assume that  $\Omega X = G$  is a topological group and  $C_*(\Omega X)$  is a differential graded Hopf algebra  $(A, \cdot, \delta, S)$  with the coproduct  $\delta$  and antipode map  $S$ .

First we prove that  $A$  has a finitely generated semifree resolution as an  $A$ -bimodule. The proof which is essentially taken from [FHT95] (Proposition 5.3) relies on the cellular structure of  $X$ . Consider the path space  $E = \{\gamma : [0, s] \rightarrow X \mid \gamma(s) = *\}$ . Using the concatenation of paths and loops, one can define an action of  $\Omega X$  on  $E$ , and thus  $C_*(E)$  becomes a  $C_*(\Omega X)$ -module. This action translates to an action of  $A$  on  $E$  which is from now on an  $A$ -module. Let  $G = \Omega X \rightarrow E \rightarrow X$  be the path space fibration of  $X$ . We will construct a finitely generated semifree resolution of  $C_*(E)$  as an  $A$ -module which, since  $E$  is contractible, provides us with a finitely generated semifree resolution of  $\mathbf{k} \simeq C_*(E)$ . Now by tensoring this resolution with  $A^e$  over  $A$  we obtain a finitely generated semifree resolution of  $A$  as  $A$ -bimodule. Here the  $A$ -module structure of  $A^e$  is defined via the composite  $Ad_0 := (A \otimes S)\delta : A \rightarrow A^e$ , similar to the case of the Poincaré duality groups (see the proof Proposition 4.11).

The semifree resolution of  $C_*(E)$  is constructed as follows. Let  $X_1 \subset X_2 \subset \dots \subset X_m$  be the skeleta of  $X$  and  $D_n = \coprod D_\alpha^n$  the disjoint union of  $n$ -cells and  $\Sigma_n = \coprod S_\alpha^{n-1}$ . Let  $V_n = H_*(X_n, X_{n-1})$  be the free  $\mathbf{k}$ -module on the basis  $v_\alpha^n$ . Using the cellular structure of  $X$  we construct an  $A = C_*(G)$ -linear quasi-isomorphism  $\phi : (V \otimes A) \rightarrow C_*(E)$  where  $V = \bigoplus_n V_n$ , inductively from the restrictions  $\phi_n = \phi|_{\bigoplus_{i \leq n} V_i \otimes A} \rightarrow C_*(E_n)$ . The induction step  $n-1$  to  $n$  goes as follows: Let  $f : (D_n, \Sigma_n) \rightarrow (X_n, X_{n-1})$  be the characteristic map. Since the (homotopy)  $G$ -fibration  $E$  can be trivialized over  $D^n$ , one has a homotopy equivalence of pairs

$$\Phi : (D_n, \Sigma_n) \times G \rightarrow (E_n, E_{n-1}),$$

where  $E_i = \pi^{-1}(X_i)$ . We have a commutative diagram

$$\begin{array}{ccc} & & C_*(E_n) \\ & & \downarrow q \\ C_*(D_n, \Sigma_n) \otimes C_*(G) & \xrightarrow{\Phi_* \circ EZ} & C_*(E_n, E_{n-1}) \\ \downarrow & & \downarrow \pi_* \\ C_*(D_n, \Sigma_n) & \xrightarrow{f_*} & C_*(X_n, X_{n-1}) \end{array} \quad (4.14)$$

whose horizontal arrows are quasi-isomorphisms. Here  $q$  is the standard projection map and  $\pi$  is the fibration map. Since  $q$  is surjective there is an element  $w_n^\alpha \in C_*(E_n)$  such that  $q_*(w_n^\alpha) = \Phi_* \circ EZ(v_n^\alpha \otimes 1)$ . Since  $v_n^\alpha \otimes 1$  is a cycle we have that  $dw_n^\alpha \in C_*(E_{n-1})$ . Because we have assumed that  $m_{n-1}$  is a quasi-isomorphism, there is a cycle  $z_{n-1}^\alpha \in \bigoplus_{i \leq n-1} V_i \otimes A$  such that  $\phi_{n-1}(z_{n-1}^\alpha) = dw_n^\alpha$ . First we extend the differential by  $d(v_n^\alpha \otimes 1) = z_\alpha$ . We extend  $\phi_{n-1}$  to  $\phi_n$  by defining  $\phi_n(v_n^\alpha \otimes 1) = w_\alpha$ . The fact that  $\phi_n$  is a quasi-isomorphism follows from an inductive argument and 5 Lemma and the fact that on the quotient  $\phi_n : V_n \otimes C_*(G) \rightarrow C_*(E_n, E_{n-1})$  is a quasi-isomorphism.

Next, we prove that  $A \simeq A^!$  in the derived category of  $A^e$ -bimodules which is a translation of Poincaré duality. Let  $Ad_0 : A \rightarrow A^e$  be defined by  $Ad_0 = (id \otimes S)\delta$ . For an  $A$ -bimodule  $M$  let  $Ad_0^*(M)$  be the  $A$ -module whose  $A$ -module structure is induced using pull-back by  $Ad_0$ . By applying the result of Félix-Halperin-Thomas on describing the chains of the base space of a  $G$ -fibration  $G \rightarrow EG \rightarrow BG \simeq X$ , we get a quasi-isomorphism

$$C_*(X) \simeq B(\mathbf{k}, A, \mathbf{k}),$$

as coalgebras. Note that  $B(\mathbf{k}, A, \mathbf{k}) \simeq B(\mathbf{k}, A, A) \otimes_A B(A, A, \mathbf{k})$ . The Poincaré duality for  $X$  implies that there is a cycle  $z_1 \in C_*(X)$  such that capping with  $z_1$

$$- \cap z_1 : C^*(X) \rightarrow C_{*-d}(X), \quad (4.15)$$

is a quasi-isomorphism. The class  $z_1$  corresponds to a cycle  $z \in B(\mathbf{k}, A, A) \otimes_A B(A, A, \mathbf{k})$  and the quasi-isomorphism (4.15) corresponds to the quasi-isomorphism

$$ev_{z,P} : \text{Hom}_k(B(\mathbf{k}, A, A), P) \rightarrow B(A, A, \mathbf{k}) \otimes P. \quad (4.16)$$

given by  $ev_z(f) = \sum f(z_i)z'_i$ , where  $f \in \text{Hom}_k(B(\mathbf{k}, A, A), \mathbf{k})$  and  $z = \sum z_i \otimes z'_i$ .

Let  $E = Ad_0^*(A^e)$ . Then we have the quasi-isomorphisms of  $A^e$ -modules,

$$A \simeq B(A, A, A) \simeq B(Ad^*(A^e), A, \mathbf{k}) \simeq E \otimes_A B(A, A, \mathbf{k}), \quad (4.17)$$

where  $A^e = A \otimes A$  acts on the latter from the left and on the factor  $E$ . On the other hand

$$\begin{aligned} A^! &\simeq \text{Hom}_{A^e}(B(A, A, A), A^e) \simeq \text{Hom}_{A^e}(B(\mathbf{k}, A, Ad^*(A^e)), A^e) \\ &\simeq \text{Hom}_A(B(\mathbf{k}, A, A), \text{Hom}_{A^e}(Ad_0^*(A^e), A^e)) \\ &\simeq \text{Hom}_A(B(\mathbf{k}, A, A), Ad_0^*(A^e)). \end{aligned} \quad (4.18)$$

Therefore  $ev_{z,E}$  is a quasi-isomorphism of  $A^e$ -modules from  $A^!$  and  $A[-d]$ .  $\square$

**Corollary 4.14.** *For a closed oriented manifold  $M$ ,  $HH^*(C^*(\Omega M), C^*(\Omega M))$  is a BV-algebra.*

*Proof.* Note that in the proof of Theorem 4.10 we don't use the second part of the Calabi-Yau condition. We only use the derived equivalence  $A \simeq A^!$ .  $\square$

**Remark 4.15.** Recently E. Malm [Mal] has proved that the Burghelaea-Fiedorowicz-Goodwillie isomorphism ([BF86, Goo85])

$$HH^*(C_*(\Omega M), C_*(\Omega M)) \simeq HH_*(C_*(\Omega M), C_*(\Omega M)) \stackrel{\text{Burghelaea-Fiedorowicz-Goodwillie}}{\simeq} H_*(LM).$$

is an isomorphism of BV-algebras where  $H_*(LM)$  is equipped with the Chas-Sullivan [CS] BV-structure.

## 5 Derived Poincaré duality algebras

In this section we essentially show how an isomorphism

$$HH^*(A, A) \simeq HH^*(A, A^\vee)$$

of  $HH^*(A, A)$ -modules gives rise to a BV structure on  $HH^*(A, A)$  whose underlying Gerstenhaber structure is the canonical one. The next lemma follows from Lemma 4.8.

**Lemma 5.1.** *For  $a, b \in HH^*(A, A)$  and  $\phi \in HH^*(A, A^\vee)$  we have*

$$[f, g] \cdot \phi = (-1)^{|f|} B^\vee((f \cup g)\phi) - f \cdot B^\vee(g \cdot \phi) + (-1)^{(|f|+1)(|g|+1)} g \cdot B^\vee(f \cdot \phi) + (-1)^{|g|} (f \cup g) \cdot B^\vee(\phi). \quad (5.1)$$

*Proof.* To prove the identity, one evaluates the cochains in  $C^*(A, A^\vee) = \text{Hom}_{\mathbf{k}}(T(s\bar{A}), A^\vee) \simeq \text{Hom}_{\mathbf{k}}(A \otimes T(s\bar{A}), \mathbf{k})$  on both sides on a chain  $x = a_0[a_1, \dots, a_n] \in A \otimes T(s\bar{A})$ . By (2.19) and Lemma 5.1, we have:

$$\begin{aligned} ([f, g] \cdot \phi)(x) &= (i_{[f, g]}\phi)(x) = (-1)^{|[f, g]| \cdot |\phi|} \phi(i_{[f, g]}(x)) \\ &= (-1)^{|[f, g]| \cdot |\phi|} \phi((-1)^{|f|} B((f \cup g) \cdot x) - f \cdot B(g \cdot x) + (-1)^{(|f|+1)(|g|+1)} g \cdot B(f \cdot x) \\ &\quad + (-1)^{|g|} (f \cup g) \cdot B(x)) = (-1)^{|[f, g]| \cdot |\phi| + |f| + |\phi|} B^\vee(\phi)(i_{f \cup g} x) - (-1)^{|[f, g]| \cdot |\phi|} \phi(i_f B(i_g x)) \\ &\quad (-1)^{|[f, g]| \cdot |\phi| + (|f|+1)(|g|+1)} \phi(i_g B(i_f x)) + (-1)^{|[f, g]| \cdot |\phi| + |g|} \phi(i_{f \cup g} B(x)) \\ &= (-1)^{|[f, g]| \cdot |\phi| + |f| + |\phi| + (|f|+1)|f \cup g|} i_{f \cup g}(B^\vee(\phi))(x) \\ &\quad - (-1)^{|[f, g]| \cdot |\phi| + |g| + (|f|+1)|f| + |\phi| + |f| \cdot |\phi|} i_g(B^\vee(i_f \phi))(x) \\ &\quad (-1)^{|[f, g]| \cdot |\phi| + (|f|+1)(|g|+1) + |f| + (|f|+|g|+1) + |g| + |f| \cdot |\phi|} i_f(B^\vee(i_g \phi))(x) \\ &\quad + (-1)^{|[f, g]| \cdot |\phi| + |g| + (|f|+1)|f \cup g| + |\phi|} i_{f \cup g}(B^\vee \phi)(x) \\ &= (-1)^{|g|} i_{f \cup g}(B^\vee(\phi))(x) + (-1)^{(|f|+1)(|g|+1)} i_g B^\vee(i_f \phi)(x) - i_f B^\vee(i_g \phi)(x) \\ &\quad + (-1)^{|f|} B^\vee((f \cup g) \cdot \phi)(x). \end{aligned}$$

This proves the statement.  $\square$

Now let us suppose that we have an equivalence  $A \simeq A^\vee[d]$  in the derived category of  $A$ -bimodules. This property provides us with an isomorphism  $D : HH^*(A, A^\vee) \rightarrow HH^{*+d}(A, A)$  which allows us to transfer the Connes operator on  $HH^*(A, A^\vee)$  to  $HH^*(A, A)$ ,

$$\Delta := D \circ B^\vee \circ D^{-1}.$$

**Lemma 5.2.** *Let  $A$  be a DGA algebra with an equivalence  $A \simeq A^\vee[d]$  in the derived category of  $A$ -bimodules. Then the induced isomorphism  $D : HH^*(A, A^\vee) \rightarrow HH^{*+d}(A, A)$  is an isomorphism of  $HH^*(A, A)$ -modules, i.e. for all  $f \in HH^*(A, A)$  and  $\phi \in HH^*(A, A^\vee)$  we have*

$$D(i_f(\phi)) = f \cup D(\phi) \quad (5.2)$$

*Proof.* The proof is identical to the proof of Lemma 4.7. One uses a resolution by semi-free modules in the category of  $A$ -bimodules and adapts diagram (4.8) to the case of  $C^*(A, A^\vee)$ , the dual theory of  $C_*(A, A)$ .  $\square$

**Corollary 5.3.** *Let  $A$  be a DG algebra with an equivalence  $A \simeq A^\vee[d]$  in the derived category of  $A$ -bimodules. Then for  $f, g \in HH^*(A, A)$  and  $\phi \in HH^*(A, A^\vee)$  we have*

$$\begin{aligned} [f, g] \cup D(\phi) &= (-1)^{|f|} \Delta((f \cup g) \cdot D\phi) - f \cdot \Delta(g \cdot D\phi) + (-1)^{(|f|+1)(|g|+1)} g \cdot \Delta(f \cdot D\phi) \\ &\quad + (-1)^{|g|} (f \cup g) \cdot DB^\vee(\phi). \end{aligned} \quad (5.3)$$

where  $D : HH^*(A, A^\vee) \rightarrow HH^*(A, A)$  is the isomorphism induced by the derived equivalence.

*Proof.* This is a consequence of Lemma 5.1, the proof being similar to that of Corollary 4.9.  $\square$

**Definition 5.4.** Let  $A$  be a differential graded algebra such that  $A$  is equivalent to  $A^\vee[d]$  in the derived category of  $A$ -bimodules. This means that there is a quasi-isomorphism of  $A$ -bimodules  $\psi : P \rightarrow A^\vee[d]$  where  $P$  is a semi-free resolution of  $A$ . Then  $\psi$  is a cocycle in  $\text{Hom}_{A^e}(P, A^\vee)$  for which  $-\cap[\psi] : HH^*(A, A) \rightarrow HH^*(A, A^\vee)$  is an isomorphism. Under this assumption,  $A$  is said to be a *derived Poincaré duality algebra* (DPD for short) of dimension  $d \in \mathbb{Z}$  if  $B^\vee([\psi]) = 0$ .

**Remark 5.5.** For a cocycle  $\psi \in C^d(A, A^\vee)$ , it is rather easy to check when  $-\cap[\psi] : HH^*(A, A) \rightarrow HH^{*+d}(A, A^\vee)$  is an isomorphism<sup>1</sup>: one only has to check that  $-\cap[\phi] : H^*(A) \hookrightarrow HH^*(A, A) \xrightarrow{\cap[\phi]} H^*(A^\vee)$  is an isomorphism (See [Men09], Proposition 11).

Two immediate consequences of the previous lemma are the following theorems.

**Theorem 5.6.** *For a DPD algebra  $A$ ,  $(HH^*(A, A), \cup, \Delta)$  is a BV-algebra i.e.*

$$[f, g] = (-1)^{|f|} \Delta(f \cup g) - (-1)^{|f|} \Delta(f) \cup g - f \cup \Delta(g). \quad (5.4)$$

*Proof.* Suppose that the derived equivalence  $A^\vee[d] \simeq A$  is realized by a quasi-isomorphism  $\psi : P \rightarrow A^\vee$ , where  $\epsilon : P \rightarrow A$  is a semi-free resolution of  $A$ .

$$\begin{array}{ccc} P & \xrightarrow{\psi} & A^\vee \\ & & \uparrow \psi \\ P & \xrightarrow{id} & P \\ & & \downarrow \epsilon \\ P & \xrightarrow{\epsilon} & A \end{array} \quad (5.5)$$

One can then use  $\text{Hom}_{A^e}(P, A^\vee)$  to compute  $HH^*(A, A^\vee)$ , and similarly  $\text{Hom}_{A^e}(P, A)$  or  $\text{Hom}_{A^e}(P, P)$  to compute the cohomology  $HH^*(A, A)$ . Let  $D = (-\cap[\psi])^{-1} : HH^*(A, A^\vee) \rightarrow HH^*(A, A)$  be the isomorphism induced by the derived equivalence.

<sup>1</sup>Intuitively, one should think of  $HH^*(A, A)$  as the homology of the free loop space of some space, which includes a copy of the homology of the underlying space by the inclusion of constant loops. This condition means that one has to check that the restriction of the cap product to the constants loops corresponds to the Poincaré duality of the underlying manifold.

Then the cohomology class represented by  $id \in \text{Hom}_{A^e}(P, P)$  corresponds to  $1 \in HH^*(A, A)$  using  $\epsilon_* : \text{Hom}_{A^e}(P, P) \rightarrow \text{Hom}_{A^e}(P, A)$ , and to  $\psi$  by the map

$$\psi_* : \text{Hom}_{A^e}(P, P) \rightarrow \text{Hom}_{A^e}(P, A^\vee).$$

Therefore,  $D([\psi]) = 1 \in HH^*(A, A)$  where  $D = \epsilon_* \circ \psi_*^{-1}$ . Now take  $\phi = [\psi]$  in the statement of Corollary 5.3.  $\square$

A similar theorem can be proved under a slightly different assumption.

**Theorem 5.7.** (*Menichi [Men09]*) *Let  $A$  be a differential graded algebra equipped with a quasi-isomorphism  $m : A \rightarrow A^\vee[d]$ . Then  $HH^*(A, A)$  has a BV algebra structure extending its natural Gerstenhaber algebra structure. The BV operator is  $\Delta = DB^\vee D^{-1}$  where  $D : HH^*(A, A^\vee) \rightarrow HH^*(A, A)$  is the isomorphism induced by  $m$ .*

*Proof.* The proof is very similar to that of the previous theorem. For simplicity we take the two-sided bar resolution  $\epsilon : B(A, A, A) \rightarrow A$ , where  $\epsilon : A \otimes \mathbf{k} \otimes A \subset B(A, A, A) \rightarrow A$  is given by the multiplication of  $A$ . Then  $\psi = m \circ \epsilon : B(A, A, A) \rightarrow A^\vee$  is a quasi-isomorphism. Let  $[\psi] \in HH^*(A, A^\vee)$  be the class represented by  $\psi$ , and  $D = (-\cap[\psi])^{-1} : HH^*(A, A^\vee) \rightarrow HH^*(A, A)$  be the inverse of the isomorphism induced by  $\psi$ . Similarly to the proof of the previous theorem, we have  $D([\psi]) = 1 \in HH^*(A, A)$ . It only remains to prove that  $DB^\vee([\psi]) = 0$ . For that we compute  $B^\vee([\psi]) = [B^\vee(\psi)]$ . Note that we have  $\mathbf{k}$ -module isomorphism  $C^*(A, A^\vee) = \text{Hom}_{A^e}(B(A, A, A), A^\vee) \simeq \text{Hom}(A \otimes T(s\bar{A}), \mathbf{k}) = (A \otimes T(s\bar{A}))^\vee$ . The image of the Connes operator  $B : A \otimes T(s\bar{A}) \rightarrow A \otimes T(s\bar{A})$  is included in  $A \otimes T(s\bar{A})^+$ . Since  $\epsilon_{|A \otimes T(s\bar{A})^+ \otimes A} = 0$ , we have  $m \circ B = \psi \circ \epsilon \circ B = 0$ .  $\square$

## 6 Symmetric open Frobenius algebras and BV/coBV structure of Hochschild homology

In this section we study the algebraic structure of the Hochschild cohomology of a symmetric open Frobenius algebra. More precisely, we introduce a BV and coBV structure on the (co) Hochschild homology of such algebras. For the special case of closed Frobenius algebras these structures are a consequence of the action of the Sullivan chord diagrams on the Hochschild (co)homology as we'll explain in Section 8. In this section we will present explicitly all the operations and homotopies related to the BV and coBV structures. The product and coproduct of the BV and coBV structures are identical to those made explicit by Wahl-Westerland in [WW] in the case of closed symmetric Frobenius algebras.

In the special case of a closed Frobenius algebra  $A$  the underlying Gerstenhaber algebra structure of our BV structure on  $HH^*(A, A) \simeq HH^*(A, A^\vee)$  (because of the  $A$ -bimodule isomorphism  $A \simeq A^\vee$ ) is the standard one (see Theorem 2.1) therefore we recover Tradler's theorem in [Tra08] as well. For more details see the discussion that precedes Theorem 6.3 .

All over this section, like the rest of the chapter, the signs are determined by Koszul's rule, we won't give them explicitly. Readers interested in a more detailed sign discussion are referred to [WW, Tra08], .

**Definition 6.1.** (DG open Frobenius algebra). A *differential graded open Frobenius  $\mathbf{k}$ -algebra* of degree  $m$  is a triple  $(A, \cdot, \delta)$  such that:

- (1)  $(A, \cdot)$  is a unital differential graded associative algebra whose product has degree zero,
- (2)  $(A, \delta)$  is differential graded coassociative coalgebra whose coproduct has degree  $m$ ,
- (3)  $\delta : A \rightarrow A \otimes A$  is a right and left  $A$ -module map which using (simplified) Sweedler' notation reads

$$\sum_{(x,y)} (x.y)' \otimes (xy)'' = \sum_{(y)} x.y' \otimes y'' = \sum_{(x)} (-1)^{m|x|} x' \otimes x'' .y$$

Here we have simplified Sweedler's notation for the coproduct  $\delta x = \sum_i x'_i \otimes x''_i$ , to  $\delta x = \sum_{(x)} x' \otimes x''$  where  $(x)$  should be thought of as the index set for  $i$ 's. We shall say  $(A, \cdot, \delta)$  is symmetric if  $\sum_{(1)} 1' \otimes 1'' = \sum_{(1)} 1'' \otimes 1'$ .

We recall that an ordinary (DG) Frobenius algebra, sometimes called *closed Frobenius (DG) algebra*, is a unital associative differential graded algebra equipped with a *nondegenerate symmetric inner product*  $\langle -, - \rangle$  which is *invariant i.e*

$$\langle xy, z \rangle = \langle x, yz \rangle.$$

In particular the inner product allows us to identify  $A$  with its dual  $A^\vee = \text{Hom}_{\mathbf{k}}(A, \mathbf{k})$  (as right and left  $A$ -modules) and defines a coproduct on  $A$  by

$$A \simeq A^\vee \xrightarrow{\text{dual of product}} (A \otimes A)^\vee \simeq A^\vee \otimes A^\vee \simeq A \otimes A.$$

The coproduct is coassociative and satisfies condition (3) of the definition above, in other words a closed Frobenius algebra is also an open Frobenius algebra. Moreover,  $\eta : A \rightarrow \mathbf{k}$  defined by  $\eta(x) = \langle x|1 \rangle$  is a counit (trace), that is

$$\sum_{(x)} \eta(x')x'' = x. \quad (6.1)$$

As a consequence we have the following identities

$$\sum_{(x)} \eta(x'y)x'' = \sum_{(x)} (-1)^{|x'|m} \eta(x''y)x' = xy. \quad (6.2)$$

**Example:** An important example of symmetric open Frobenius algebra is the cohomology with compact support  $H_c^*(M)$  of an oriented  $n$ -dimensional manifold  $M$  (not necessarily closed). Note that  $H_c^*(M)$  is naturally equipped with the usual cup product and using the Poincaré duality (see [Hat02] Theorem 3.35),

$$H_c^*(M) \simeq H_{*-n}(M)$$

one can transfer the natural coproduct  $\Delta$  on  $H_*(M)$  to  $H_c^*(M)$ . Then  $(H_c^*(M), \cup, \Delta)$  is a symmetric open Frobenius algebra whose differential is identically zero. The Frobenius compatibility condition is satisfied because the Poincaré duality isomorphism above is a map of  $H_c^*(M)$ -modules.

If  $M$  is a closed then  $H^*(M) = H_c^*(M)$  is indeed a closed Frobenius manifold since  $H^*(M)$  has a counit given by  $\int : H^*(M) \rightarrow \mathbb{Z}$  be evaluation on the fundamental class of

$M$  and the Poincaré duality is given by capping with the fundamental class. The non-degenerate inner product is defined by  $\langle x, y \rangle := \int_{[M]} x \cup y$ . Over the rationals it is possible to lift this Frobenius algebra structure to the cochains level. By a result of Lambrechts and Stanley [LS08], there is a connected finite dimensional commutative DG algebra  $A$  which is quasi-isomorphic to  $C^*(M)$  the cochains of a given  $n$ -dimensional manifold  $M$  and is equipped with a bimodule isomorphism  $A \rightarrow A^\vee$  inducing the Poincaré duality  $H^*(M) \rightarrow H_{*-n}(M)$ .

**Proposition 6.2.** (1) *A closed Frobenius algebra is finite dimensional and symmetric.*  
(2) *An open Frobenius algebra with a counit is indeed a closed Frobenius algebra, in particular it is finite dimensional.*  
(3) *A symmetric and commutative open Frobenius algebra is cocommutative.*  
(4) *For any  $z$  in a symmetric open Frobenius  $A$ ,  $\sum_{(z)} z'' z'$  belongs to the center of  $A$ .*

*Proof.* 1) The finite dimensionality follows from the identity  $x = \sum_{(1)} 1' \langle x | 1'' \rangle$  where  $\delta(1) = \sum_{(1)} 1' \otimes 1''$ . Being symmetric is a consequence of the inner product being symmetric. We have

$$\langle y, z \rangle = \langle 1, yz \rangle = \sum \langle 1', y \rangle \langle 1'', z \rangle, \quad (6.3)$$

left hand side being symmetric in terms of  $y$  and  $z$  implies that  $\sum_{(1)} 1' \otimes 1'' = \sum_{(1)} 1'' \otimes 1'$ .

2) The inner product can be defined using the counit by setting  $\langle x, y \rangle = \eta(xy)$ . It is clearly invariant and the nondegeneracy follows from the identity (6.2) for  $x = 1$ : If  $\langle y, z \rangle = 0$  for all  $z$ , then

$$y = \sum_{(x)} (-1)^{|x'|m} \eta(1''y) 1' = 0.$$

$$3) \sum x' \otimes x'' = \sum x 1' \otimes 1'' = \sum x 1'' \otimes 1' = \sum 1'' x \otimes 1' = \sum x'' \otimes x'.$$

4)

$$\begin{aligned} x \sum_{(z)} z'' z' &= \sum_{(z)} x z'' z' = \sum_{(1)} x 1'' z 1' = \sum_{(1)} x 1' z 1'' = \sum_{(x)} x' z x'' = \sum_{(1)} 1' z 1'' x \\ &= \sum_{(1)} 1'' z 1' x = \sum_{(z)} z'' z' x. \end{aligned}$$

□

As we know  $HH^*(A, A^\vee)$  is already equipped with a BV operator namely the Connes operator  $B^\vee$ , so we just need a product on  $HH^*(A, A^\vee)$  or equivalently a coproduct on the Hochschild chains  $C_*(A, A)$ . This is given by

$$\theta(a_0[a_1, \dots, a_n]) = \sum_{(a_0), 1 \leq i \leq n} \pm (a'_0[a_1, \dots, a_{i-1}, a_i]) \otimes (a''_0[a_{i+1}, \dots, a_n]) \quad (6.4)$$

Then we can define the cup product of  $f, g \in C^*(A, A^\vee) = \text{Hom}(A \otimes T(s\bar{A}), \mathbf{k})$  by

$$(f * g)(x) := \mu(f \otimes g)\theta(x)$$

where  $\mu : \mathbf{k} \otimes \mathbf{k} \rightarrow \mathbf{k}$  is the multiplication. More explicitly

$$(f * g)(a_0[a_1, \dots, a_n]) = \sum_{(a_0), 1 \leq i < n} \pm f(a'_0[a_1, \dots, a_{i-1}, a_i])g(a''_0[a_{i+1}, \dots, a_n]).$$

In the case of a closed Frobenius algebra this product corresponds to the standard cup product on  $HH^*(A, A)$  using the isomorphism

$$HH^*(A, A) \simeq HH^*(A, A^\vee)$$

induced by the inner product on  $A$ . More explicitly we identify  $A$  with  $A^\vee$  using the map  $a \mapsto (a^\vee(x) := \langle a, x \rangle)$ . Therefore to a cochain  $f \in C^*(A, A)$ ,  $f : A^{\otimes n} \rightarrow A$ , corresponds a cochain  $\tilde{f} \in C^*(A) = C^*(A, A^\vee)$  given by  $\tilde{f} \in \text{Hom}(A^{\otimes n}, A^\vee) \simeq \text{Hom}(A^{\otimes(n+1)}, \mathbf{k})$ ,

$$\tilde{f}(a_0, a_1, \dots, a_n) := \langle f(a_1, \dots, a_n), a_0 \rangle.$$

The inverse of this isomorphism is given by

$$f(a_1, \dots, a_n) := \sum_{(1)} \tilde{f}(1', a_1, \dots, a_n)1''.$$

For two cochain  $f : A^{\otimes p} \rightarrow A$  and  $g : A^{\otimes q} \rightarrow A$  in  $C^*(A, A)$  we have

$$\begin{aligned} \widetilde{f \cup g}(a_0, a_1, \dots, a_{p+q}) &= \langle (f \cup g)(a_1, \dots, a_{p+q}), a_0 \rangle = \langle f(a_1, \dots, a_p)g(a_{p+1}, \dots, a_{p+q}), a_0 \rangle \\ &= \langle f(a_1, \dots, a_p), g(a_{p+1}, \dots, a_{p+q})a_0 \rangle = \sum_{(a_0)} \langle f(a_1, \dots, a_p), \langle g(a_{p+1}, \dots, a_{p+q}), a'_0 \rangle a'_0 \rangle \\ &= \sum_{(a_0)} \langle f(a_1, \dots, a_p), \langle g(a_{p+1}, \dots, a_{p+q}), a''_0 \rangle a'_0 \rangle \\ &= \sum_{(a_0)} \langle f(a_1, \dots, a_p), a'_0 \rangle \langle g(a_{p+1}, \dots, a_{p+q}), a''_0 \rangle = (\tilde{f} * \tilde{g})(a_0[a_1, \dots, a_{p+q}]) \end{aligned}$$

**Theorem 6.3.** *For a symmetric open Frobenius algebra  $A$ ,  $(HH_*(A, A), \theta, B)$  is a coBV-algebra. As a consequence  $(HH^*(A, A^\vee), \theta^\vee, B^\vee)$  is a BV-algebra. In particular if  $A$  is closed Frobenius algebra, then using the natural isomorphism*

$$(CC^*(A, A^\vee), \theta^\vee) \simeq (CC^*(A, A), \cup), \quad (6.5)$$

$(HH^*(A, A), \cup)$  is naturally a BV-algebra whose BV-operator is image of the Connes operators  $B$  under the isomorphism (6.5). This recovers Tradler's [Tra08] result for closed Frobenius algebras.

*Proof.* To prove the theorem we show that  $(C_*(A, A), \theta, B)$  is a homotopy coBV coalgebra. It is a direct check that  $\theta$  is co-associative. Just like the computation above, we transfer the homotopy for commutative (in the case of closed Frobenius algebra) of the cup product as given in Theorem 2.1 to  $C^*(A, A^\vee)$  and then dualize it. It turns out that the obtained formula only depends on the product and coproduct, so it makes also sense for open Frobenius algebras. The homotopy for co-commutativity is given by

$$h(a_0[a_1, \dots, a_n]) := \sum_{(1), 0 \leq i < j \leq n+1} (a_0[a_1, \dots, a_i, 1'', a_j, \dots, a_n]) \otimes (1'[a_{i+1}, \dots, a_{j-1}]). \quad (6.6)$$



where for  $j = n + 1$  and  $i = 0$  the correspondings terms are respectively

$$(a_0[a_1, \dots, a_i, 1'']) \otimes (1'[a_{i+1}, \dots, a_n]).$$

and

$$(a_0[1'', a_j, \dots, a_n]) \otimes (1'[a_1, \dots, a_{j-1}]).$$

It is a direct check that  $hd - (d \otimes 1 + 1 \otimes d)h = \theta - \tau \circ \theta$  where  $\tau : C_*(A) \otimes C_*(A) \rightarrow C_*(A) \otimes C_*(A)$  is given by  $\tau(\alpha_1 \otimes \alpha_2) = \pm \alpha_2 \otimes \alpha_1$ .

To prove that the 7-term (coBV) relation holds, we use the Chas-Sullivan [CS] idea (see also [Tra08]) in the case of the free loop space adapted to the combinatorial (simplicial) situation. First we identify the Gerstenhaber co-bracket explicitly. Let

$$S := h - \tau \circ h$$

Once proven  $S$  is, up to homotopy, the deviation of  $B$  from being a coderivation for  $\theta$ , the 7-term homotopy coBV relation is equivalent to the homotopy co-Leibniz identity for  $S$ .

**Co-Leibniz identity:** The idea of the proof is identical to Lemma 4.6 [CS]. We prove that up to some homotopy we have

$$(\theta \otimes id)S = (id \otimes \tau)(S \otimes id)\theta + (id \otimes S)\theta \quad (6.7)$$

It is a direct check that

$$(id \otimes \tau)(h \otimes id)\theta + (id \otimes h)\theta = (\theta \otimes id)h,$$

so to prove (6.7) we should prove that up to some homotopy

$$(id \otimes \tau)(\tau h \otimes id)\theta + (id \otimes \tau h)\theta = (\theta \otimes id)\tau h. \quad (6.8)$$

The homotopy is given by  $H : C^*(A) \rightarrow (C^*(A))^{\otimes 3}$

$$H(a_0[a_1, \dots, a_n]) = \sum_{0 \leq i < j < k} \sum_{(1), (1)} (1''[a_i, \dots, a_{i-1}]) \otimes (1''[a_j, \dots, a_{k-1}]) \otimes a_0[a_1, \dots, a_i, 1', a_i \cdots a_{j-1}, 1', a_k, \dots, a_n].$$

Note that in the sum above the sequence  $a_i \cdots a_{j-1}$  can be empty. The identity

$$(d \otimes id \otimes id + id \otimes d \otimes id + id \otimes id \otimes d)H - Hd = (id \otimes \tau)(\tau h \otimes id)\theta + (id \otimes \tau h)\theta - (\theta \otimes id)\tau h$$

can be checked directly.

**Compatibility of  $B$  and  $S$ :** The final step is to prove that  $S = \theta B \pm (B \otimes id \pm id \otimes B)\theta$  up to homotopy. To that end we prove that  $h$  is homotopic to  $(\theta B)_2 - (B \otimes id)\theta$  and similarly  $\tau h \simeq (\theta B)_1 - (id \otimes B)\theta$  where  $\theta B = (\theta B)_1 + (\theta B)_2$ , with

$$(\theta B)_1(a_0[a_1, \dots, a_n]) = \sum_{0 \leq i \leq j \leq n} \sum_{(1)} (1'[a_i, \dots, a_j]) \otimes (1''[a_{j+1}, \dots, a_n, a_0 \cdots, a_{i-1}]).$$

and

$$(\theta B)_2(a_0[a_1, \dots, a_n]) = \sum_{0 < j < i \leq n} \sum_{(1)} (1'[a_i, \dots, a_n, a_0, a_1 \cdots, a_j]) \otimes (1''[a_{j+1}, \dots, a_{i-1}]).$$

It can be easily checked that

$$H(a_0[a_1, \dots, a_n]) = \sum_{0 \leq k < j \leq i \leq n} \sum_{(a_i)} (1[a_i, \dots, a_n, a_0, a_1, \dots, a_k, 1', a_j, \dots, a_{i-1}]) \otimes (1''[a_{k+1}, \dots, a_{j-1}]),$$

is a homotopy between  $h$  and  $(\theta B)_2 - (B \otimes id)\theta$ . In the formulae describing  $H$ , the sequence  $a_j, \dots, a_{i-1}$  can be empty.

While computing  $dH$  we encounter, the terms corresponding to  $k = 0$  is exactly

$$(B \otimes 1)\theta(a_0[a_1, \dots, a_n]) = \sum (1[a_j, \dots, a_i, a'_0, a_1, \dots, a_{j-1}]) \otimes (a''_0[a_{i+1}, \dots, a_n]).$$

Similarly one proves that  $\tau h \simeq (\theta B)_1 - (id \otimes B)\theta$ .  $\square$

**Remark 6.4.** By a theorem of Félix-Thomas [Fél], this cup product on  $HH^*(A, A^\vee)$  provides an algebraic model for the Chas-Sullivan product on  $H_*(LM)$  the homology of the free loop space of closed oriented manifold  $M$ . Here one must work over a field of characteristic zero and for  $A$  one can take the closed (commutative) Frobenius algebra provided by Lambrechts-Stanley result [LS08] on the existence of an algebraic model with Poincaré duality for the cochains of a closed oriented manifold.

**Theorem 6.5.** For  $A$  a symmetric open Frobenius algebra,  $HH_*(A, A)$  can be naturally equipped with a BV-structure whose BV-operator is Connes' operator and the product at the chain level is given by

$$a_0[a_1, \dots, a_n] \circ b_0[b_1, \dots, b_m] = \begin{cases} 0 & \text{if } n > 0 \\ a''_0 a'_0 b_0[b_1, \dots, b_m] & \end{cases} \quad (6.9)$$

*Proof.* Using Proposition 6.2 (4), it is easily checked that  $\circ$  is a chain map. The product  $\circ$  is strictly associative. We only have to check this for  $x = a[ ]$ ,  $y = b[ ]$  and  $z = c[c_1, \dots, c_n]$ . We have

$$(x \circ y) \circ z = \sum (a''_0 a'_0 b_0)'' (a''_0 a'_0 b_0)' c_0[c_1, \dots, c_n] = \sum b''_0 a''_0 a'_0 b'_0 c_0[c_1, \dots, c_n]$$

which by Proposition 6.2 (4) is equal to

$$\sum a''_0 a'_0 b''_0 b'_0 c_0[c_1, \dots, c_n] = x \circ (y \circ z).$$

Next we prove that the product is commutative up to homotopy. Indeed the homotopy for  $x = a_0[a_1, \dots, a_m]$  and  $y = b_0[b_1, \dots, b_n]$  is given by

$$K(x, y) = \begin{cases} \sum_{(a)} a'_0 b_0[b_1, \dots, b_n, a''_0] & \text{if } m = 0 \quad \& \quad n \geq 0 \\ \sum_{(a)} b''_0 [b'_0 a_0, a_1, \dots, a_m, ] & \text{if } m \geq 0 \quad \& \quad n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.10)$$

The proof of this fact relies on the following identities,

$$\begin{aligned} \sum a'_0 b_0[b_1, \dots, b_n, a''_0] &= \sum a_0 1' b_0[b_1, \dots, b_n, 1''] = \sum a_0 1'' b_0[b_1, \dots, b_n, 1'] = \sum a_0 b''_n b_0[b_1, \dots, b'_n] \\ &= \sum a_0 1'' b_n b_0[b_1, \dots, 1'] = \sum a_0 1' b_n b_0[b_1, \dots, 1''] = \sum a'_0 b_n b_0[b_1, \dots, a''_0]. \end{aligned} \quad (6.11)$$

In the particular case of  $x = a[ ]$  and  $y = b[ ]$  and for the external differential we have

$$d_1 K(x, y) = \sum (a'ba'' - a''a'b) = \sum a1'b1'' - x \circ y = \sum a1''b1' - x \circ y = \sum ab''b' - x \circ y$$

which by Proposition 6.2 (4) is,

$$\sum b''b'a - x \circ y = y \circ x - x \circ y.$$

Since we have also  $d_0 K(x, y) = K(dx, y) + K(x, dy)$ , therefore

$$dK(x, y) = K(dx, y) + K(x, dy) + y \circ x - x \circ y.$$

The Gerstenhaber bracket is naturally defined to be

$$\{x, y\} := K(x, y) - K(y, x). \quad (6.12)$$

Next we prove that the identity

$$\{x, y\} = \Delta(x \circ y) \pm \Delta x \circ y \pm x \circ \Delta y \quad (6.13)$$

holds up to homotopy. If  $n > 0$  and  $m > 0$  then all the involved are homotopically zero and there is nothing to prove. We prove the claim for the case of  $m = 0$  and  $n > 0$ , the other case is similar. Since  $\Delta x \circ y$  and  $x \circ \Delta y$  are respectively zero and homotopique to zero, we only have to show that  $\Delta(x \circ y)$  is homotopic to  $\{x, y\}$ . We have

$$\Delta(x \circ y) = \sum_{i=1}^n \sum_{(a)} 1[b_i, \dots, b_n, a''_0 a'_0 b_0, b_1, \dots, b_{i-1}] + 1[a''_0 a'_0 b_0, b_1, \dots, b_n]$$

and

$$\{x, y\} = \sum_{(a)} a'_0 b_0 [b_1, \dots, b_n, a''_0] + \sum_{(a)} a''_0 [a'_0 b_0, b_1, \dots, b_n].$$

The desired homotopy is given by

$$H(x, y) := \sum_i \sum_{(a)} 1[b_i \dots, a''_0, a'_0 b_0, b_1, \dots, b_{i-1}].$$

Finally we prove the Leibniz identity

$$\{x \circ y, z\} = x \circ \{y, z\} + \{x, z\} \circ z$$

holds up to homotopy. Note that the  $\{y, z\}$  is either zero or it belongs to  $\oplus_{n>1} A^{\otimes n}$  therefore the right hand side is always zero, so we need to prove that the left hand side is always homotopic to zero which is clear if one of  $x$  or  $y$  belongs to  $\oplus_{n>1} A^{\otimes n}$ . Using the identity (up to homotopy) (6.13) we know that  $\{x \circ y, z\}$  is homotopic to  $\Delta(x \circ y \circ z)$  which is homotopic to zero if  $z \in \oplus_{n>1} A^{\otimes n}$ . So we may assume that  $x = a[ ]$  and  $y = b[ ]$  and  $z = c[ ]$ , and we have

$$\Delta(x \circ y \circ z) = \Delta((a''ab)''(a''a'b)'c) = \Delta(b''a''a'b'c).$$

Since  $a''a'$  is central, we get  $\Delta(x \circ y \circ z) = 1[a''a'b''b'c]$  which is homotopic to zero. The homotopy is given by  $H(a, b, c) = 1[a''a', b''b'c]$  whose boundary is

$$a''a'[b''b'c] + 1[a''a'b''b'c] + b''b'c[a''a'].$$

In fact  $a''a'[b''b'c] = b''b'c[a''a']$  (up to signs) because

$$\begin{aligned} \sum a''a'[b''b'c] &= \sum 1''a1'[1''b1''c] = \sum 1'a1'[1''b1''c] = \sum c'ab'[b''c''] = \sum c1'a1'[1''b1''] \\ &= \sum c1'ab'[b''1''] = \sum c1'ab1'[1''1''] = \sum c1''ab1'[1''1'] = \sum ca''b1'[1''a'] \\ &= \sum c1''b1'[1''a1'] = \sum cb''b'[a''a'] = \sum b''b'c[a''a'].. \end{aligned}$$

□

**Corollary 6.6.** *For a cocommutative open Frobenius algebra the Gerstenhaber bracket defined by (6.12) on  $HH_*(A, A)$  is identically zero. In particular for a commutative symmetric open Frobenius algebra the Gerstenhaber bracket defined by (6.12) vanishes.*

*Proof.* We know that at the homology level  $\{x, y\} = \Delta(x \circ y)$ . The latter is zero unless  $x = a[\ ]$  and  $y = b[\ ]$  for some  $a$  and  $b$  in  $A$ , in which case

$$\Delta(x \circ y) = 1[a''a'b] = d(1[a'', a'b]).$$

Note that the two terms  $a''[a'b]$  and  $a'b[a'']$  in  $d(1[a'', a'b])$  cancel each other since

$$a'b[a''] = a''b[a'] = (ab)''[(ab)'] = (ab)'[(ab)'] = a'[a''b] = a''[a'b].$$

□

## 7 Symmetric commutative open Frobenius algebras and BV structure on shifted relative Hochschild homology

In this section we exhibit a BV structure on the relative Hochschild homology of a symmetric commutative open Frobenius algebra. In particular we introduce a product on the shifted relative Hochschild homology of symmetric commutative Frobenius algebras whose dual could be an algebraic model for Chas-Sullivan [CS04] / Goresky-Hingston [GH09] coproduct on  $H_*(LM, M)$ . One should mention that Chas-Sullivan worked with an equivariant version of this product on  $H_*^{S^1}(LM, M)$  which is a generalization of Turavev co-Lie bracket [Tur91].

For a commutative DG-algebra  $A$  the relative Hochschild chains are defined to be

$$\widetilde{CC}_*(A) = \bigoplus_{n \geq 1} A \otimes \bar{A}^{\otimes n}. \quad (7.1)$$

equipped with the Hochschild differential. Since  $A$  is commutative,  $\widetilde{CC}_*(A)$  is stable under the Hochschild differential and fits in the split short exact sequence of complexes,

$$0 \longrightarrow (A, d_A) \longrightarrow CC_*(A, A) \longrightarrow \widetilde{CC}_*(A) \quad (7.2)$$

The homology of  $\widetilde{CC}_*(A)$  is denoted  $\widetilde{HH}_*(A)$  and is called the relative Hochschild homology of  $A$ .

**Theorem 7.1.** *The shifted relative Hochschild homology  $\widetilde{HH}_*(A)[1-m]$  of a degree  $m$  symmetric commutative open Frobenius algebra  $A$  is a BV algebra whose BV-operator is the Connes operator and the product is given by*

$$\begin{aligned} x \cdot y &= \sum_{(a_0 b_0)} \pm (a_0 b_0)' [a_1, \dots, a_m, (a_0 b_0)'', b_1, \dots, b_n] \\ &= \sum_{a_0} \pm a_0' [a_1, \dots, a_m, a_0'' b_0, b_1, \dots, b_n] \\ &= \sum_{b_0} \pm a_0 b_0' [a_1, \dots, a_m, b_0'', b_1, \dots, b_n] \end{aligned} \quad (7.3)$$

for  $x = a_0[a_1, \dots, a_m]$  and  $y = b_0[b_1, \dots, b_n] \in \widetilde{CC}_*(A)$ .

*Proof.* Note that the identities above hold because  $A$  is an open Frobenius algebra. The product defined above is a chain map and strictly associative, but commutative only up to homotopy, and the homotopy being given by

$$\begin{aligned} H_1(x, y) &= \sum_{(a_0 b_0)} \pm 1 [a_1, \dots, a_n, (a_0 b_0)', b_1, \dots, b_m, (a_0 b_0)''] \\ &\quad + \sum_{i=1}^n \sum_{(a_0 b_0)} \pm 1 [a_{i+1}, \dots, a_n, (a_0 b_0)', b_1, \dots, b_m, (a_0 b_0)'', a_1, \dots, a_i]. \end{aligned} \quad (7.4)$$

To prove that the 7-term relation holds, we adapt once again Chas-Sullivan's [CS] idea to a simplicial situation. First we identify the Gerstenhaber bracket directly. Let

$$x \circ y := \sum_{i=0}^m \sum_{(a_0)} b_0 [b_1, \dots, b_i, a_0', a_1, \dots, a_n, a_0'', b_{i+1}, \dots, b_m], \quad (7.5)$$

and then define  $\{x, y\} := x \circ y \pm y \circ x$ . Next we prove that the bracket  $\{-, -\}$  is homotopic to the deviation of the BV-operator from being a derivation. For that we decompose  $\Delta(x \circ y)$  in two pieces:

$$\begin{aligned} B_1(x, y) &:= \sum_{j=1}^m \sum_{(a_0 b_0)} \pm 1 [b_{j+1}, \dots, b_m, (a_0 b_0)', a_1, \dots, a_n, (a_0 b_0)'', b_1, \dots, b_j], \\ B_2(x, y) &:= \sum_{j=1}^m \sum_{(a_0 b_0)} \pm 1 [a_{j+1}, \dots, a_n, (a_0 b_0)', b_1, \dots, b_m, (a_0 b_0)'', a_1, \dots, a_j], \end{aligned}$$

so that  $B = B_1 + B_2$ . Then  $x \circ y$  is homotopic  $B_1(x, y) - x \cdot B_2 y$ . In fact the homotopy is given by

$$H_2(x, y) = \sum_{0 \leq j \leq i \leq m} \sum_{(a_0)} 1 [b_{j+1}, \dots, b_i, (a_0)', a_1, \dots, a_n, (a_0)'', b_{i+1}, \dots, b_m, b_0, \dots, b_j].$$

Similarly for  $y \circ x$  and  $B_2(x \cdot y) - Bx \cdot y$ . Therefore we have proved that on  $HH_*(A, A)$  the following identity holds:

$$\{x, y\} = B(x \cdot y) - Bx \cdot y \pm x \cdot By.$$

Now proving the 7-term relation is equivalent to prove the Leibniz rule for the bracket and the product, *i.e.*

$$\{x, y \cdot z\} = \{x, y\} \cdot z \pm y \cdot \{x, z\}.$$

It is a direct check that  $x \circ (y \cdot z) = (x \circ y) \cdot z + y \cdot (x \circ z)$ . On the other hand  $(y \cdot z) \circ x$  is homotopic to  $(y \circ x) \cdot z - y \cdot (z \circ x)$  using the homotopy

$$H_3(x, y, z) = \sum a_0[a_1, \dots, a_i, b'_0, b_1, \dots, b_n, b''_0, a_{i+1}, \dots, a_j, c'_0, c_1, \dots, c_m, c''_0, a_{j+1}, \dots, a_p].$$

Here  $z = c_0[c_1, \dots, c_p]$ . This proves that the Leibniz rule holds up to homotopy.  $\square$

**Remark 7.2.** The commutativity assumption was only needed for making  $\widetilde{CC}_*(A)$  a subcomplex of  $CC_*(A)$  otherwise for the proof of this theorem cocommutativity suffices (exercise).

**Remark 7.3.** In [CG10] Chen and Gan proved that for an open Frobenius algebra  $A$ , the reduced coHochschild homology of  $A$  seen as a *coalgebra*, is a BV algebra. They also proved that the *reduced* Hochschild homology is a BV and coBV algebra. It is necessary to take the reduced Hochschild homology in order to get the coBV structure.

## 8 Closed Frobenius Algebras: Action of the moduli space of curves via Sullivan chord diagrams

In this section we extend the operations introduced in Section 6 to an action of Sullivan chord diagrams on the Hochschild chains  $C_*(A, A)$  [and cochains  $C^*(A, A^\vee)$ ] of a closed Frobenius algebra. The main theorem of this section recovers that in [TZ06] because the inner product induces an isomorphism  $A \simeq A^\vee$  of  $A$ -bimodules therefore all structures can be transferred from  $HH^*(A, A^\vee)$  to  $HH^*(A, A)$ . Since we are describing the action on the Hochschild chains there is difference in our terminology with that of [TZ06]. Here the incoming cycles of Sullivan chord diagram correspond to outgoing cycles of the same diagram in [TZ06] and [CG04]. Such action has also been described by N. Wahl and C. Westerland [WW] who work with integral coefficients and explain how such an action implies an action of the moduli spaces of the curves (See Section 2.10 in [WW]).

A *Sullivan chord diagram* [CS04, CG04] of type  $(g, m, n)$  is a fat graph <sup>2</sup> which is a union of  $m$  labeled disjoint oriented circles, called *output circles* or *outgoing boundaries*, and some *disjointly immersed* trees whose endpoints land on the outgoing circles. The trees are called *chords*, which have length zero. We assume that each vertex is at least trivalent, therefore there is no vertex on a circle which is not an end of a tree. The graphs don't need to be connected.

The cyclic ordering basically tells us how to draw the graph on the plane. The cyclic ordering should be such that the  $m$  output circles are among the boundary components,

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<sup>2</sup>A fat graph is a graph with a cyclic ordering on the set of edges entering each vertex.

which is best visualized by thickening the graph to a surface whose genus is  $g$ . The cyclic orderings also allow us to identify the remaining  $n$  labeled *input circles* or *incoming boundaries*. Therefore this surface has  $n + m$  labeled boundary components. We also assume that each incoming circle has a marked point, called *input marked point*, and similarly each outgoing boundary has a marked point, called *outgoing marked point*. As in [WW], one may think of the the input marked point as a leaf, connecting a degree vertex to the corresponding input cycle, but we don't. We don't consider marked point as vertices of the graphs but as some *special points* on the graph. The marked points and the endpoints of the chords may correspond. Because of the cyclic ordering at each vertex, there is a well-defined cyclic ordering on the special points attached to a tree (chord).

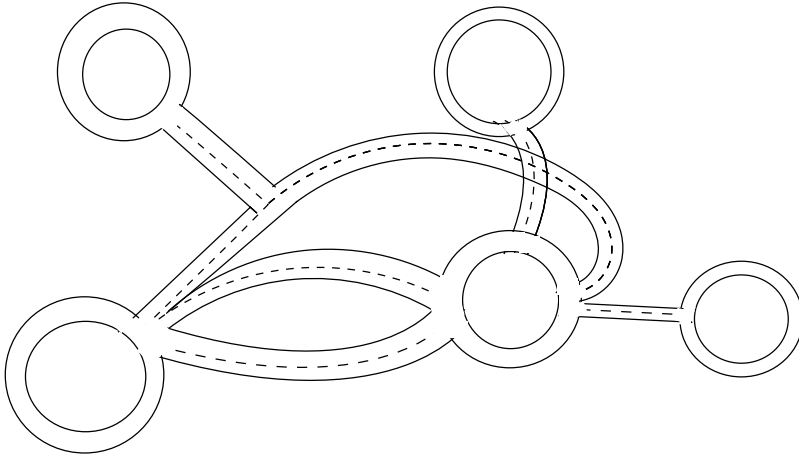


Figure 1.

Figure 1 displays a chord diagram with 5 incoming cycles and 3 outgoing cycles. There is an obvious composition rule for two Sullivan chord diagrams if the number of output circles of the first graph equals the number of input circles of the second one. Of course the labeling matters and marked points get identified. This composition rule makes the space of Sullivan chord diagrams into a PROP (see [WW] and [TZ06] for more details). Here we don't give the definition of a PROP and we refer the interested reader to [May72] and [MSS02] for more detail.

The combinatorial degree of a diagram of type  $(g, m, n)$  is the number of connected components obtained after removing all special points. Let  $CS_k(g, m, n)$  denote the space of  $(g, m, n)$ -diagrams of degree  $k$ . For instance the combinatorial degree of the diagram in Figure 2 is one which corresponds to the BV operator. One makes  $\{CS_k(g, m, n)\}_{k \geq 0}$  into a complex using a boundary map which is defined by collapsing an edge (arc) on input circles and considering the induced cyclic ordering. In what follows we describe the action of chord diagrams on chains in  $C_*(A, A)$  whose degree is exactly the combinatorial degree of the given diagram. In other words we construct a chain map  $(CS_k(g, m, n) \rightarrow (\text{Hom}(C_*(A, A)^{\otimes m}, C_*(A, A)^{\otimes n}), D := [d_{Hoch}, -])$ . Moreover this action is compatible

with the composition rule of the diagrams. Said formally,  $C_*(A, A)$  is a differential graded algebra over the differential PROP  $\{CS_k(g, m, n)\}_{k \geq 0}$ . We won't deal with this last statement.

**The equivalence relation for graphs and essentially trivalent graphs:**

Two graphs are considered equivalent if one is obtained from the other using on of the following moves:

- sliding, one each time, a vertex on the chord over edges of the chord.
- sliding an input marked point over the chord tree.

By doing so one can easily see that each Sullivan chord diagram is equivalent to a Sullivan chord diagram whose each vertex is trivalent or has an input marked point, and no input marked point coincides is on a chord end point.

**The action of the diagrams:** Let  $\gamma$  be a chord diagram with  $m$  input circles and  $n$  output circles. We assume that in  $\gamma$  all vertices are trivalent and no input marked point coincides with a chord end point, otherwise we will replace with an equivalent trivalent graph as explained above.

The aim is to associate to  $\gamma$  a chain map  $(C_*(A, A))^{\otimes m} \rightarrow (C_*(A, A))^{\otimes n}$ . Let  $x_i = a_0^i[a_1^i | \cdots | a_{k_i}^i]$ ,  $1 \leq i \leq m$ , be  $m$  Hochschild chains.

Step 1) Write down  $a_0^i, a_1^i, \dots, a_{k_i}^i$  on the  $i$ th input circle by putting first  $a_0^i$  on the input marked point and then the rest following the orientation of the cycle, on those parts of the  $i$ th input cycle which are not part of the chord tree (at this stage we don't use the output marked point). We consider all the possible ways of placing  $a_1^i, \dots, a_{k_i}^i$  on the  $i$ th cycle following rules specified above.

Step 2) At an output marked point which is not a chord end point or an input marked point we place a 1, otherwise we move to the next step.

Step 3) On the end points of a chord tree with  $r$  end points and no input marked point, we place following orientation of the plane  $1', 1'' \dots 1^{(r)}$  where

$$(\delta \otimes id^{(r-2)}) \otimes \cdots \otimes (\delta \otimes id)\delta(1) = \sum_{(1)} 1' \otimes 1'' \otimes \dots 1^{(r)} \in A^{\otimes r},$$

Step 4) On the end points of a chord tree with  $r$  endpoint which has  $s$  input marked points on its vertices we do as follows: We organize the chord tree as a rooted tree whose roots are input marked point. Now the tree defines a a well-define (because of the Frobenius relations) operation  $A^{\otimes s} \rightarrow A^{\otimes r}$  defined using the product and coproduct of  $A$ . Now by applying this operation on the element placed on the input marked points (the roots of the tree) we obtain a sum  $\sum_i x_i^1 \otimes \cdots \otimes x_i^r$ . We decorate the end points of the chord tree by  $x_i^r, \dots, x_i^r$  following the orientation of the plane.

Step 5) For each output circle, starting from its output marked point and following its orientation, read off all the elements on the outgoing cycle *i.e.*  $a_j^i$  left or created after the previous steps, possibly 1 and the  $a^{(k)}$ , and note them as an element of  $C_*(A, A)$ . Since the output circles are labeled we therefore obtain a well-defined element of  $(C_*(A, A))^{\otimes n}$ . Take the sum over all possible sum appeared in steps 3 and 4. The result is an element of  $C_*(A, A)^{\otimes n}$ .



We clarify this procedure with some examples. The BV operator clearly corresponds to the diagram in Figure 2.

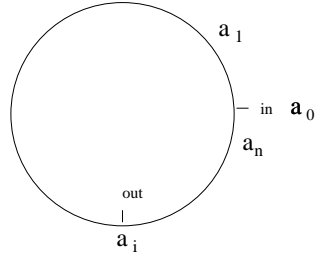


Figure 2. BV operator

The coproduct (6.4)

$$\theta(a_0[a_1, \dots, a_n]) = \sum_{(a_0), 1 \leq i \leq n} \pm(a'_0[a_1, \dots, a_{i-1}, a_i]) \otimes (a''_0[a_{i+1}, \dots, a_n]), \quad (8.1)$$

corresponds to the diagram in Figure 3. The dual of  $\theta$  induces a product on  $HH^*(A, A^\vee)$  which under isomorphism  $HH^*(A, A^\vee) \simeq HH^*(A, A)$  corresponds to the cup product on  $HH^*(A, A)$  (see Section 6). One should think of the latter as the algebraic model of the Chas-Sullivan [CS] product on  $H_*(LM)$ .

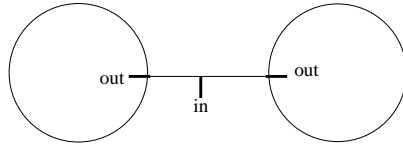
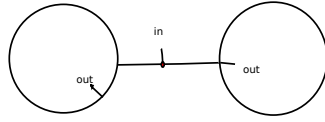


Figure 3. String topology coproduct on  $HH_*(A, A)$  [the dual of the cup product on cohomology  $HH^*(A, A^\vee) \simeq HH^*(A, A)$ ]

The homotopy  $h$  for co-commutativity of the  $\theta$  as defined in (6.6) corresponds to the diagram in Figure 4.

Figure 4. The homotopy for cocommutativity of  $\theta$ 

The degree zero coproduct as defined in Cohen-Godin on  $H_*(LM)$  is the dual of the following product on  $HH_*(A, A)$ :

$$(a_0[a_1, \dots, a_n]) \circ (b_0[b_1, \dots, b_m]) = \begin{cases} 0 & \text{if } n \geq 1 \\ a''a'b_0[b_1, \dots, b_m] & \text{otherwise} \end{cases} \quad (8.2)$$

The product  $*$  corresponds to the diagram in Figure 5 which is equivalent to the essentially trivalent graph in Figure 6.

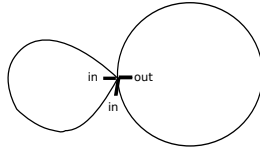


Figure 5. The dual of Cohen-Godin coproduct

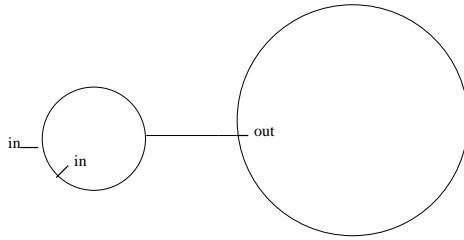


Figure 6. Essentially trivalent graph corresponding to Cohen-Godin coproduct

This is exactly the product introduced in the statement of Theorem 6.5.

Now it remains to deal with differentials. This is quite easy to check since collapsing the arcs on the input circles corresponds to the components of the Hochschild differential. The only nontrivial part concerns collapsing the arcs attached to the special points and this follows from the hypothesis that  $A$  is an open Frobenius algebra with a counit. This shows that to cycles in  $(\{CS_k(g, m, n)\}_{k \geq 0}, \partial)$ , the action associates a chain map and the homotopies between operations correspond to the action of the boundaries of corresponding chains in  $(\{CS_k(g, m, n)\}_{k \geq 0}, \partial)$ . We refer the reader to [TZ06] for more details, or to [WW] for a different approach. Now one can explain all the homotopies in the previous section using this language.

The main the result of this section can be formulated as follows:

**Theorem 8.1.** *For a closed Frobenius (DG) algebra  $A$ ,  $C_*(A, A)$  the Hochschild chain complex of  $A$  is an algebra over the PROP of Sullivan chord diagrams. Similarly the*

Hochschild cochain complex  $C^*(A, A)$  of  $A$  is an algebra over the PROP of Sullivan chord diagrams.

In particular the action of Sullivan chord diagrams provides us the compatibility relations between the product and coproduct in order to obtain an open Frobenius algebra (see [CG04] for more details):

**Corollary 8.2.** *For a closed Frobenius (DG) algebra  $A$ ,  $HH_*(A, A)$  and  $HH^*(A, A^\vee)$  are open Frobenius algebras. The product and coproduct for  $HH_*(A, A)$  are respectively made explicit in Theorem 6.5 and Theorem 6.3.*

Note that we had already identified the product and coproduct of this open Frobenius algebra structure.

**Remark 8.3.** As we saw above, the product and coproduct on  $HH_*(A, A)$  require only an open Frobenius algebra structure on  $A$ . The results of this section do not prove that the product and coproduct are compatible so that  $HH_*(A, A)$  is an open Frobenius algebra. The reason is that our proof of the compatibility identities uses the action of some chord diagrams whose actions are defined only if  $A$  is a closed Frobenius algebra. Still it could be true that  $HH_*(A, A)$  is an open Frobenius algebra if  $A$  is only an open Frobenius algebra, but this needs a direct proof.

## 9 Cyclic cohomology

In this section we briefly describe some of the structures of the cyclic homology and negative cyclic homology, which are induced by those of the Hochschild cohomology via Connes' long exact sequence. We recall that the cyclic and negative cyclic chain complexes of a DG-algebra  $A$  are

$$\begin{aligned} CC_*(A) &= (C_*(A, A)[[u, u^{-1}]/\mathbf{uk}[[u]], d + uB), \\ CC_*^-(A) &= (C_*(A, A)[[u]], d + uB). \end{aligned}$$

Here  $u$  is formal variable of degree 2,  $d = d_{Hoch}$  and  $B$  is the Connes operator, whereas  $\mathbf{uk}[[u]]$  stands for the ideal generated by  $u$  in  $C_*(A, A)[[u, u^{-1}]]$ . Here  $\mathbf{k}[[u, u^{-1}]]$  stands for the Laurent series in  $u$ . The cyclic and negative cyclic cochain complexes are defined to be:

$$\begin{aligned} CC^*(A) &= (C^*(A, A^\vee)[v], d^\vee + vB^\vee), \\ CC_*^-(A) &= (C^*(A, A^\vee)[[v, v^{-1}]/\mathbf{vk}[[v]], d^\vee + vB^\vee). \end{aligned}$$

The cyclic homology of  $A$  is denoted  $CH_*(A)$  and is the homology of the complex  $CC_*(A)$ . The negative cyclic homology  $HC_*^-(A)$  is the homology of  $CC_*^-(A)$ . Here  $v$  is a formal variable of degree  $-2$ .

**Lemma 9.1.** *Let  $(A^*, \cdot, \Delta)$  be a BV-algebra and  $L^*$  a graded vector space with a long exact sequence*

$$\dots \longrightarrow L^{k+2} \longrightarrow L^k \xrightarrow{m} A^{k+1} \xrightarrow{e} L^{k+1} \longrightarrow L^{k-1} \xrightarrow{m} A^k \xrightarrow{e} \dots \quad (9.1)$$

such that  $\Delta = m \circ e$ . Then

$$\{a, b\} := (-1)^{|a|} e(ma \cdot mb)$$

defines a graded Lie bracket on the graded vector space  $L^*$ . Moreover  $m$  sends the Lie bracket to the opposite of the Gerstenhaber bracket i.e.

$$m\{a, b\} = -[ma, mb].$$

*Proof.* We have,

$$\begin{aligned} \{a, \{b, c\}\} &= (-1)^{|a|+|b|} e(ma \cdot \Delta(mb \cdot mc)) \\ \{\{a, b\}, c\} &= (-1)^{|b|} e(\Delta(ma \cdot mb) \cdot mc) \\ \{b, \{a, c\}\} &= (-1)^{|b|+|a|} e(mb \cdot \Delta(ma \cdot mc)). \end{aligned} \quad (9.2)$$

Then

$$\begin{aligned} &\{a, \{b, c\}\} - \{\{a, b\}, c\} - (-1)^{|a| \cdot |b|} \{b, \{a, c\}\} \\ &= (-1)^{|a|+|b|} e[ma \cdot \Delta(mb \cdot mc) + (-1)^{|a|+1} \Delta(ma \cdot mb) \cdot mc + (-1)^{|a||b|+1} mb \cdot \Delta(ma \cdot mc)] \\ &= (-1)^{|b|+1} e[\Delta(ma \cdot mb) \cdot mc + (-1)^{|a|+1} ma \cdot \Delta(mb \cdot mc) + (-1)^{|a|(|b|+1)} mb \cdot \Delta(ma \cdot mc)]. \end{aligned} \quad (9.3)$$

By replacing  $a$ ,  $b$ , and  $c$  in the 7-term relation (2.10) respectively by  $ma$ ,  $mb$  and  $mc$ , we see that the last line in the above identity is equal to  $(-1)^{|b|+1} e\Delta(ma \cdot mb \cdot mc) = -1)^{|b|+1} eme(ma \cdot mb \cdot mc) = 0$  because of the exactness of the long exact sequence. Therefore  $\{a, \{b, c\}\} - \{\{a, b\}, c\} - (-1)^{|a| \cdot |b|} \{b, \{a, c\}\} = 0$ , proving the Jacobi identity.

As for the second statement,

$$\begin{aligned} m\{a, b\} &= (-1)^{|a|} me(ma \cdot mb) = (-1)^{|a|} \Delta(ma \cdot mb) \\ &= (-1)^{|a|} ((-1)^{|a|+1} [ma, mb] - \Delta(ma) \cdot mb + (-1)^{|a|+1} ma \cdot \Delta(mb)] = -[ma, mb]. \end{aligned} \quad (9.4)$$

□

Using this lemma and Connes' exact sequence for the cyclic cohomology (or homology),

$$\dots HC^{k+2}(A) \longrightarrow HC^k(A) \xrightarrow{b} HH^{k+1}(A, A^*) \xrightarrow{e} HC^{k+1}(A) \longrightarrow \dots \quad (9.5)$$

we have:

**Corollary 9.2.** *The cyclic cohomology and negative cyclic cohomology of an algebra whose Hochschild cohomology is a BV algebra, has a natural graded Lie algebra structure given by*

$$\{x, y\} := e(m(x) \cup m(y)).$$

In fact one can prove something slightly better.

**Definition 9.3.** A *gravity algebra* is a graded vector space  $L^*$  equipped with maps

$$\{\cdot, \dots, \cdot\} : L^{\otimes k} \rightarrow L$$

satisfying the following identities:

$$\begin{aligned} \sum_{i,j} (-1)^{\varepsilon_{i,j}} \{\{x_i, x_j\}, x_1 \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_k, y_1, \cdots, y_l\} \\ = \begin{cases} 0, & \text{if } l = 0 \\ \{\{x_1, \cdots, \cdots, x_k\}, y_1, \cdots, y_l\}, & \text{if } l > 0 \end{cases} \end{aligned} \quad (9.6)$$

It is quite easy to prove that

**Proposition 9.4.** *The cyclic and negative cyclic cohomology of an algebra whose Hochschild cohomology is a BV algebra, is naturally a gravity algebra where the brackets are given by*

$$\{x_1, \cdots, x_k\} := e(m(x_1) \cup \cdots \cup m(x_k)).$$

The proof is a consequence of the following identity for BV algebras:

$$\Delta(x_1 \cdots x_n) = \sum \pm \Delta(x_i x_j) \cdot x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n \quad (9.7)$$

This is a generalized form of the 7-term identity which is rather easy to prove. We refer the reader to [Wes08] for a more operadic approach on the gravity algebra structure.

The Lie bracket on cyclic homology is known in the literature under the name *string bracket*. For surfaces it was discovered by W. Goldman [Gol86] who studied the symplectic structure of the representation variety of fundamental groups of surfaces, or equivalently the moduli space of flat connections. His motivation lied in the dynamics of Teichmüller theory and Hamiltonian vector fields of Thurston earthquakes. It was then generalized by Chas-Sullivan using a purely topological construction to manifolds of all dimension. A geometrical description of the string bracket is given in [AZ07] (and [ATZ10]) which generalizes Goldman's computation for surfaces using Chen iterated integrals to arbitrary even dimensions.

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