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Hörmander Functional Calculus for Poisson Estimates

Christoph Kriegler

Abstract. The aim of the article is to show a Hörmander spectral multiplier theorem for an operator A whose kernel of the semigroup $\exp(-zA)$ satisfies certain Poisson estimates for complex times z . Here $\exp(-zA)$ acts on $L^p(\Omega)$, $1 < p < \infty$, where Ω is a space of homogeneous type with the additional conditions that the volume of balls grows polynomially of exponent d and the measure of annuli is controlled by the corresponding euclidean term. In most of the known Hörmander type theorems in the literature, Gaussian bounds and self-adjointness for the semigroup are needed, whereas here the new feature is that the assumptions are the to some extent weaker Poisson bounds, and H^∞ calculus in place of self-adjointness. The order of derivation in our Hörmander multiplier result is typically $\frac{d}{2}$, d being the dimension of the space Ω . Moreover the functional calculus resulting from our Hörmander theorem is shown to be R -bounded. Finally, the result is applied to some examples.

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1. Introduction

Let f be a bounded function on $(0, \infty)$ and $u(f)$ the operator on $L^p(\mathbb{R}^d)$ defined by $[u(f)g]^\wedge(\xi) = f(|\xi|^2)\hat{g}(\xi)$. Hörmander's theorem on Fourier multipliers [9, Theorem 2.5] asserts that $u(f) : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ is bounded for any $p \in (1, \infty)$ provided that for some integer N strictly larger than $\frac{d}{2}$

$$\sup_{R>0} \int_{R/2}^{2R} \left| t^k f^{(k)}(t) \right|^2 \frac{dt}{t} < \infty \quad (k = 0, \dots, N). \quad (1.1)$$

This theorem has many generalisations to similar contexts, for example to elliptic and sub-elliptic differential operators A , including sublaplacians on Lie groups of polynomial growth, Schrödinger operators and elliptic operators on Riemannian manifolds [5]: Note first that the above $u(f)$ equals $f(-\Delta)$,

the functional calculus of the self-adjoint positive operator $-\Delta$. Now for a self-adjoint operator A , a Hörmander theorem states that the operator $f(A)$ extends boundedly to $L^p(\Omega)$, $1 < p < \infty$ for any function f satisfying (1.1) with suitable N . In most of the proofs for a Hörmander theorem in the literature, the assumption of so called Gaussian bounds plays a crucial role. That means the following. Suppose that A acts on $L^p(\Omega)$, $1 < p < \infty$, where (Ω, μ, ρ) is a space of homogeneous type. Then the semigroup $(\exp(-tA))_{t \geq 0}$ generated by A has an integral kernel $k_t(x, y)$ such that

$$|k_t(x, y)| \leq C \mu(B(y, \sqrt{t}))^{-1} \exp\left(-c \frac{\rho(x, y)^2}{t}\right) \quad (t > 0, x, y \in \Omega). \quad (1.2)$$

This hypothesis includes many elliptic differential operators. However there are operators such that the integral kernel of the semigroup satisfies only weaker estimates, see e.g. [8, 15, 19]. Establishing a Hörmander theorem for these operators is the issue of the present article. More precisely, let Ω be a space of homogeneous type with the additional properties that the volume of balls grows polynomially of exponent $d > 0$ and that the measure of annuli is controlled by (a constant times) the corresponding euclidean term, see (2.2) and (2.3) for a precise definition. Let further A act on $L^p(\Omega)$ such that $(\exp(-zA))_{\operatorname{Re} z > 0}$ has an integral kernel $k_z(x, y)$ such that

$$|k_z(x, y)| \leq C (\cos \arg z)^{-\beta} \frac{1}{\mu(B(x, |z|))} \frac{1}{\left|1 + \frac{\rho(x, y)^2}{z^2}\right|^{\frac{d+1}{2}}} \quad (\operatorname{Re} z > 0, x, y \in \Omega) \quad (1.3)$$

holds for some $C, \beta \geq 0$. If $\Omega = \mathbb{R}^d$ and $\beta = 0$, the right hand side of this estimate is (a constant times) the absolute value of the complex Poisson kernel which obviously decays slower as $\rho(x, y) \rightarrow \infty$ than the Gaussian kernel above. Under a further hypothesis on the homogeneous space Ω , see (2.2) and (2.3) below, and the presence of an H^∞ calculus of A on $L^2(\Omega)$, we obtain a Hörmander theorem of the order $N > \frac{d}{2} + \beta$ for operators A satisfying the above estimate. The proof relies on the behaviour of the semigroup $\exp(-zA)$ generated by A when the complex parameter z approaches the imaginary axis. Here simple norm estimates are not sufficient but R -bounds of the semigroup are needed. Our method does not need self-adjointness of A . This is new compared to most of the spectral multiplier results in the literature. In particular, we give a non-self-adjoint example of a Lamé operator to which our main result applies. Note also that Gaussian estimates as in (1.2) and self-adjointness in general yield only a Mihlin calculus [18, Theorem 7.23], i.e. bounded spectral multipliers $f(A)$ for f satisfying

$$\max_{0 \leq k \leq N} |t^k f^{(k)}(t)| < \infty,$$

and $N > \frac{d}{2}$, whereas we show that complex Poisson estimates give a Hörmander functional calculus, i.e. $f(A)$ is bounded for f satisfying (1.1) with $N > \frac{d}{2}$, if $\beta = 0$ in (1.3). This yields better estimates for the Bochner-Riesz means $(1 - A/\lambda)_+^\nu$ on $L^p(\Omega)$, see the discussion in the introduction of [5]. The difficulty in assumption (1.3) is to show an estimate for complex times z ,

something that for Gaussian estimates one gets somehow for free out of an estimate like (1.2) and self-adjointness.

In Section 2 we will introduce the necessary background and cite a theorem which allows to pass from R -bounds on the semigroup to a Hörmander functional calculus. In Section 3 we state and prove the main result, Theorem 3.2 and Corollary 3.6 of this article. In Section 4, an application to a concrete operator is given, for which a Hörmander theorem was previously unknown, and two further examples entering our context are discussed. Finally, in Section 5, two proofs of technical lemmas are annexed.

2. Preliminaries

In this section, we provide the necessary background for the Main Section 3. Let $\omega \in (0, \pi)$. A densely defined and closed operator A on $L^p(\Omega)$, $1 < p < \infty$, is called ω -sectorial if $\sigma(A) \subset \overline{\Sigma_\omega}$ where $\Sigma_\omega = \{z \in \mathbb{C}^* : |\arg z| < \omega\}$, and $\|\lambda(\lambda - A)^{-1}\| \leq C_\theta$ for any $\lambda \in \overline{\Sigma_\theta}^c$ and any $\theta \in (\omega, \pi)$. For an ω -sectorial operator A and a function $f \in H_0^\infty(\Sigma_\theta) = \{g : \Sigma_\theta \rightarrow \mathbb{C} : g \text{ analytic and bounded, } \exists C, \epsilon > 0 : |g(z)| \leq C \min(|z|^\epsilon, |z|^{-\epsilon})\}$ where $0 < \omega < \theta < \pi$, one defines the operator $f(A)$ by

$$f(A)x = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} x d\lambda.$$

Here, Γ is the boundary of $\Sigma_{\frac{\omega+\theta}{2}}$ oriented counterclockwise. This definition coincides with the self-adjoint calculus if applicable. If there is a constant $C > 0$ such that $\|f(A)\| \leq C \sup_{|\arg z| < \theta} |f(z)|$ for any $f \in H_0^\infty(\Sigma_\theta)$, then A is said to have a bounded $H^\infty(\Sigma_\theta)$ calculus, or just bounded H^∞ calculus. Let $\phi_0 \in C_c^\infty(\frac{1}{2}, 2)$ and for $n \in \mathbb{Z}$ put $\phi_n = \phi_0(2^{-n}\cdot)$. We can and do assume that $\sum_{n \in \mathbb{Z}} \phi_n(t) = 1$ for any $t > 0$ [1, Lemma 6.1.7]. Now define

$$\mathcal{H}^\alpha = \{f : [0, \infty) \rightarrow \mathbb{C} : \|f\|_{\mathcal{H}^\alpha} = |f(0)| + \sup_{n \in \mathbb{Z}} \|(\phi_n f) \circ \exp \|_{W_2^\alpha(\mathbb{R})} < \infty\},$$

where $W_2^\alpha(\mathbb{R})$ is the usual Sobolev space. For $\alpha > \frac{1}{2}$, the space \mathcal{H}^α is a Banach algebra endowed with the norm $\|\cdot\|_{\mathcal{H}^\alpha}$. This class refines condition (1.1) in the sense that $f \in \mathcal{H}^\alpha \implies f$ satisfies (1.1) for $\alpha > N$ and the converse holds for $\alpha < N$. If $\|f(A)\| \leq C \|f\|_{\mathcal{H}^\alpha}$ for any $f \in \bigcap_{\omega > 0} H_0^\infty(\Sigma_\omega) \cap \mathcal{H}^\alpha$, then there exists a bounded homomorphism $\mathcal{H}^\alpha \rightarrow B(L^p(\Omega))$, $f \mapsto f(A)$, and A is said to have a bounded \mathcal{H}^α calculus. If A is moreover self-adjoint on $L^2(\Omega)$ then for $f \in \mathcal{H}^\alpha \subseteq L^\infty(\mathbb{R}_+)$ for some $\alpha > \frac{1}{2}$, $f(A)$ is defined twice, but one can show that the definition from the H^∞ calculus plus density in \mathcal{H}^α and the definition from the self-adjoint spectral calculus coincide. In particular, our notion of \mathcal{H}^α calculus is the same as in most of the definitions in the literature.

Let $(\epsilon_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables such that $\text{Prob}(\epsilon_n = 1) = \text{Prob}(\epsilon_n = -1) = \frac{1}{2}$, i.e. a sequence of independent Rademacher variables. Let X be a Banach space. A subset $\tau \subset B(X)$ is called R -bounded if there exists a constant $C > 0$ such that for any choice of finite

families $T_1, \dots, T_n \in \tau$ and $x_1, \dots, x_n \in X$, one has

$$\left(\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k T_k x_k \right\|_X^2 \right)^{\frac{1}{2}} \leq C \left(\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|_X^2 \right)^{\frac{1}{2}}.$$

The least possible constant is denoted by $R(\tau)$, and $R(\tau) = \infty$, if no such constant is admitted. Any R -bounded set τ is norm bounded, i.e. $\sup_{T \in \tau} \|T\| \leq R(\tau)$, but the converse is false in general. If $X = L^p$, $1 \leq p < \infty$, then

$$\left(\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|_X^2 \right)^{\frac{1}{2}} \cong \left\| \left(\sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\|_p \quad (2.1)$$

uniformly in n and x_1, \dots, x_n . A linear mapping $u : Y \rightarrow B(X)$, where Y is a further Banach space is called R -bounded if $R(u(y) : \|y\|_Y \leq 1) < \infty$. The following proposition gives a condition on the semigroup generated by a sectorial operator A so that A has a \mathcal{H}^β calculus.

Proposition 2.1. Let A be an ω -sectorial operator for any $\omega > 0$ defined on an L^p space for some $1 < p < \infty$, and let A have a bounded H^∞ calculus. Suppose that for some $\alpha > 0$ the set $\{\exp(-e^{i\theta} 2^k t A) : k \in \mathbb{Z}\}$ is R -bounded for any $t > 0$ and $|\theta| < \frac{\pi}{2}$, with R -bound $\lesssim (\cos(\theta))^{-\alpha}$. Then for any $\beta > \alpha + \frac{1}{2}$, A has a bounded \mathcal{H}^β calculus. Moreover, this calculus is an R -bounded mapping.

Proof. This is proved in the case that A has dense range in [11, Lemma 4.72 and Proposition 4.79], see also [12]. This proof for which we give a sketch applies also here. First one deduces from the assumption of R -boundedness of the semigroup that

$$\{(1 + |t|)^{-\alpha} (1 + 2^k A)^{-\alpha} \exp(i2^k t A) : t \in \mathbb{R}\}$$

is R -bounded with R -bound independent of $t \in \mathbb{R}$. Then for $g \in C_c^\infty(0, \infty)$ a representation formula of $g(2^k A)(1 + 2^k A)^{-\alpha}$ is available, namely

$$g(2^k A)(1 + 2^k A)^{-\alpha} x = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(t) (1 + |t|)^\beta (1 + |t|)^{-\beta} (1 + 2^k A)^{-\alpha} \exp(i2^k t A) x dt.$$

If $\beta > \alpha + \frac{1}{2}$, then $(1 + |t|)^{-\beta} \|(1 + 2^k A)^{-\alpha} \exp(i2^k t A)\|$ is dominated by a function in $L^2(\mathbb{R})$, and if g belongs to $W_2^\beta(\mathbb{R})$ then also $\hat{g}(t)(1 + |t|)^\beta \in L^2(\mathbb{R})$. In fact, more can be said. By [10, Proposition 4.1, Remark 4.2], the set

$$\{g(2^k A)(1 + 2^k A)^{-\alpha} : g \in C_c^\infty(0, \infty), \|g\|_{W_2^\beta(\mathbb{R})} \leq 1, k \in \mathbb{Z}\}$$

is R -bounded. Next one gets rid of the factor $(1 + 2^k A)^{-\alpha}$ above by using a function $\psi(\lambda) = (1 + \lambda)^\alpha \phi(\lambda)$ where $\phi \in C_c^\infty(0, \infty)$ and $\phi(\lambda) = 1$ for $\lambda \in [\frac{1}{2}, 2]$. The hypotheses of the proposition imply that $\{\psi(2^k A) : k \in \mathbb{Z}\}$ is R -bounded. Then

$$\{g(2^k A) : g \in C_c^\infty(0, \infty), \text{supp } g \subset [\frac{1}{2}, 2], \|g\|_{W_2^\beta(\mathbb{R})} \leq 1, k \in \mathbb{Z}\}$$

is R -bounded. The hypotheses imply moreover that there holds the following equivalences of Paley-Littlewood type:

$$\|f\|_p \cong \left\| \left(\sum_{k \in \mathbb{Z}} |\phi(2^k A) f|^2 \right)^{\frac{1}{2}} \right\|_p \cong \left\| \left(\sum_{k \in \mathbb{Z}} |\tilde{\phi}(2^k A) \phi(2^k A) f|^2 \right)^{\frac{1}{2}} \right\|_p$$

for a function $\phi \in C_c^\infty(0, \infty)$, ϕ not vanishing identically zero, $\text{supp } \phi \subset [\frac{1}{2}, 2]$ and $\tilde{\phi} = \phi(2^{-1}\cdot) + \phi + \phi(2\cdot)$. Then one can show that $g(A)$ is bounded for $\|g\|_{\mathcal{H}^\beta} < \infty$:

$$\begin{aligned} \|g(A)f\|_p &\cong \left\| \left(\sum_{k \in \mathbb{Z}} |\tilde{\phi}(2^k A) g(A) \phi(2^k A) f|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\cong \left\| \left(\sum_k |\tilde{\phi} g(2^{-k}\cdot)(2^k A) \phi(2^k A) f|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\lesssim R(\{\tilde{\phi} g(2^{-k}\cdot) : k \in \mathbb{Z}\}) \left\| \left(\sum_{k \in \mathbb{Z}} |\phi(2^k A) f|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\lesssim \|g\|_{\mathcal{H}^\beta} \|f\|_p. \end{aligned}$$

Thus $\{g(A) : \|g\|_{\mathcal{H}^\beta} \leq 1\}$ is a bounded subset of $B(L^p)$. In a similar manner to the calculation right above, using the fact that L^p has Pisier's property (α) , one shows that this set is moreover R -bounded. \square

The space Ω on which the operator A acts will be a space of homogeneous type. This means that (Ω, ρ) is a metric space endowed with a nonnegative Borel measure μ which satisfies the doubling condition: There exists a constant $C > 0$ such that for all $x \in \Omega$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty,$$

where we set $B(x, r) = \{y \in \Omega : \rho(x, y) < r\}$. Note that the doubling condition implies the following strong homogeneity property: There exists $C > 0$ and a dimension $d > 0$ such that for all $\lambda \geq 1$, for all $x \in \Omega$ and all $r > 0$ we have $\mu(B(x, \lambda r)) \leq C\lambda^d \mu(B(x, r))$. We will assume that the space of homogeneous type (Ω, μ, ρ) has the following two additional properties

$$\begin{aligned} \mu(B(x, r)) &\cong r^d \text{ if } \text{diam}(\Omega) = \infty, \\ \mu(B(x, r)) &\cong \min(r^d, 1) \text{ if } \text{diam}(\Omega) < \infty \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \mu(B(x, r, R)) &\leq C(R^d - r^d) \quad (x \in \Omega, R > r > 0) \text{ if } \text{diam}(\Omega) = \infty, \\ \mu(B(x, r, R)) &\leq C(R - r) \min(R^{d-1}, 1) \quad (x \in \Omega, R > r > \frac{1}{2}R > 0) \text{ if } \text{diam}(\Omega) < \infty \end{aligned} \tag{2.3}$$

where we denote $\text{diam}(\Omega) = \sup\{\rho(x, y) : x, y \in \Omega\}$ and $B(x, r, R) = B(x, R) \setminus B(x, r)$. Note that if (2.2) holds, then $\text{diam}(\Omega) < \infty$ if and only if $\mu(\Omega) < \infty$.

3. The Main Theorem

We let (Ω, μ, ρ) be a space of homogeneous type with the additional properties (2.2) and (2.3). We further let $T_z = \exp(-zA)$ be a semigroup on $L^2(\Omega)$ with the properties: The generator A has a bounded $H^\infty(\Sigma_\omega)$ calculus for some $\omega \in (0, \pi)$ on $L^2(\Omega)$, and T_z has an integral kernel $k_z(x, y)$ for $\text{Re } z > 0$ i.e. $(T_z f)(x) = \int_\Omega k_z(x, y) f(y) d\mu(y)$ for any $f \in L^2(\Omega)$. We assume that

$$|k_z(x, y)| \leq C(\cos \arg z)^{-\beta} \frac{1}{\mu(B(x, |z|))} \frac{1}{\left|1 + \frac{\rho(x, y)^2}{z^2}\right|^{\frac{d+1}{2}}} \quad (z \in \mathbb{C}_+, x, y \in \Omega). \quad (3.1)$$

Proposition 3.1. Let (Ω, μ, ρ) be a space of homogeneous type satisfying (2.2) and (2.3) and $T_t = \exp(-tA)$ a semigroup which acts on all $L^p(\Omega)$, $1 < p < \infty$. Assume that T_z is analytic on $L^2(\Omega)$ on $z \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} : \text{Re } \lambda > 0\}$ and bounded on each subsector Σ_ω for $\omega \in (0, \frac{\pi}{2})$, and that T_z has an integral kernel which satisfies (3.1). Assume moreover that A has a bounded $H^\infty(\Sigma_\omega)$ calculus on $L^2(\Omega)$ for some $\omega \in (0, \pi)$, which is the case e.g. when A is self-adjoint. Then the operator A has an $H^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega)$ for any $\omega \in (0, \pi)$ and $1 < p < \infty$.

Proof. The proposition follows from [6, Theorem 3.1]. Indeed, let $\theta \in (0, \frac{\pi}{2})$. The kernel $k_z(x, y)$ satisfies on $z \in \Sigma_\theta$ the bound

$$\begin{aligned} |k_z(x, y)| &\lesssim \frac{(\cos \arg z)^{-\beta}}{\mu(B(x, |z|))} \frac{1}{\left|1 + \left(\frac{\rho(x, y)}{z}\right)^2\right|^{\frac{d+1}{2}}} \\ &\lesssim_\theta \mu(B(x, |\text{Re } z|))^{-1} \left(1 + \left(\frac{\rho(x, y)}{\text{Re } z}\right)^2\right)^{-\frac{d+1}{2}} \end{aligned}$$

since $|z| \cong \text{Re } z$ for $z \in \Sigma_\theta$ and $\left|1 + \left(\frac{\rho(x, y)}{\text{Re } z}\right)^2\right| \leq 1 + \left|\frac{\rho(x, y)}{\text{Re } z}\right|^2 \lesssim 1 + \left|\frac{\rho(x, y)}{z}\right|^2 \lesssim \left|1 + \left(\frac{\rho(x, y)}{z}\right)^2\right|$. Then with G_t given by [6, (7)] and $g(x) = c(1 + x^2)^{-\frac{d+1}{2}}$, we can deduce from [6, Theorem 3.1] that A has a bounded $H^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega)$ for any $p \in (1, \infty)$ and $\omega > \frac{\pi}{2} - \theta$. \square

The following is the main theorem of this article.

Theorem 3.2. Let (Ω, μ, ρ) be a space of homogeneous type satisfying (2.2) and (2.3) and $T_t = \exp(-tA)$ a semigroup which acts on all $L^p(\Omega)$, $1 < p < \infty$. Assume that T_z is analytic on $L^2(\Omega)$ on $z \in \mathbb{C}_+$ and that T_z has an integral kernel which satisfies (3.1). Assume that for $z \in \mathbb{C}_+$, $\|\exp(-zA)\|_{B(L^2(\Omega))} \lesssim$

$(\cos(\arg z))^{-\frac{d-1}{2}-\beta}(1 + |\log(\cos(\arg z))|)^2$, which is the case e.g. when A is self-adjoint. Then the semigroup $\exp(-zA)$ satisfies on $X = L^p(\Omega)$ for any $1 < p < \infty$ the R -bound estimate

$$R(\exp(-e^{i\theta}2^j t A) : j \in \mathbb{Z}) \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta}(1 + |\log(\cos(\theta))|)^2.$$

For the proof of the theorem, we state two preliminary lemmas.

Lemma 3.3. Let $k_z(x, y)$ be analytic in $z \in \mathbb{C}_+$ and satisfy (3.1). Let $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $|\theta|$ sufficiently close to $\frac{\pi}{2}$ and $0 < t < t_0 < \infty$. Then one has the estimate

$$\begin{aligned} |k_{e^{i\theta}t_0+t}(x, y) - k_{e^{i\theta}t_0}(x, y)| &\lesssim \\ &\frac{(\cos(\theta))^{-\beta} \min(\cos(\theta)t_0, t)}{\mu(B(x, t_0)) \cos(\theta)t_0} \left[\left| 1 - \frac{\rho(x, y)^2}{t_0^2} \right| + \cos(\theta) \right]^{-\frac{d+1}{2}} + \\ &\begin{cases} (5.5) & : t_0^2 \geq \rho(x, y)^2 \text{ and } \frac{1}{\sqrt{2-\rho(x, y)^2/t_0^2-1}} \geq 2t_0/t \\ (5.6) & : t_0^2 \geq \rho(x, y)^2 \text{ and } \frac{1}{\sqrt{2-\rho(x, y)^2/t_0^2-1}} \leq 2t_0/t \\ (5.7) & : t_0^2 \leq \rho(x, y)^2 \leq 2t_0^2 \\ (5.8) & : \rho(x, y)^2 \geq 2t_0^2, \end{cases} \end{aligned}$$

where (5.5), (5.6), (5.7), (5.8) can be found in the proof of this lemma in Section 5.

Lemma 3.4. Let (Ω, μ, ρ) be a space of homogeneous type satisfying (2.2). Let $k_z(x, y)$ be analytic in $z \in \mathbb{C}_+$ and satisfy (3.1). Let $t > 0$, $t_0 \in [1, 2]$, $l \in \mathbb{Z}$ with $l \leq L_{\max} = \max\{j \in \mathbb{Z} : 2^j t_0 \leq \text{diam}(\Omega)\} + 1 \leq \infty$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $|\theta|$ sufficiently close to $\frac{\pi}{2}$ and $x, y \in \Omega$ such that $\rho(x, y) \geq 3t$.

1. If $|\frac{\rho(x, y)^2}{(2^l t_0)^2} - 1| \leq \cos(\theta)$, then

$$\begin{aligned} &\sup_{j \in \mathbb{Z}: 2^j t_0 \geq t} |k_{e^{i\theta}2^j t_0+t}(x, y) - k_{e^{i\theta}2^j t_0}(x, y)| \lesssim \\ &\frac{1}{\mu(B(x, 2^l t_0))} \frac{\min(\cos(\theta)2^l t_0, t)}{\cos(\theta)2^l t_0} (\cos(\theta))^{-\frac{d+1}{2}-\beta}. \end{aligned}$$

2. If $\frac{\rho(x, y)^2}{(2^l t_0)^2} \in [1 + \cos(\theta), 2]$, then with $r = \rho(x, y)^2/(2^l t_0)^2$,

$$\sup_{j \in \mathbb{Z}: 2^j t_0 \geq t} |k_{e^{i\theta}2^j t_0+t}(x, y) - k_{e^{i\theta}2^j t_0}(x, y)| \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \times$$

$$\times \begin{cases} (r-1)^{-\frac{d+1}{2}} \left(1 + \log \frac{1-\sqrt{2-r}}{\cos(\theta)}\right) \\ + (1-\sqrt{2-r})^{-\frac{d+1}{2}} - \left(\frac{t}{2^j t_0}\right)^{-\frac{d+1}{2}}, \\ \text{if } \cos(\theta) \leq 1-\sqrt{2-r} \leq \frac{t}{2^j t_0}; \\ (r-1)^{-\frac{d+1}{2}} \left(1 + \log \frac{t}{2^j t_0 \cos(\theta)}\right) + |\log \cos(\theta)|, \\ \text{if } \cos(\theta) \leq \frac{t}{2^j t_0} \leq 1-\sqrt{2-r}; \\ (\cos(\theta))^{-\frac{d+1}{2}}, \text{ if } 1-\sqrt{2-r} \leq \cos(\theta) \leq \frac{t}{2^j t_0}; \text{ and} \\ \frac{t}{\cos(\theta) 2^j t_0} (r-1)^{-\frac{d+1}{2}}, \text{ if } \cos(\theta) \geq \frac{t}{2^j t_0}. \end{cases}$$

3. If $\frac{\rho(x,y)^2}{(2^j t_0)^2} \in [\frac{1}{2}, 1 - \cos(\theta)]$, then with $r = \rho(x,y)^2 / (2^j t_0)^2$,

$$\sup_{j \in \mathbb{Z}: 2^j t_0 \geq t} |k_{e^{i\theta} 2^j t_0 + t}(x,y) - k_{e^{i\theta} 2^j t_0}(x,y)| \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \times$$

$$\times \begin{cases} (1-r)^{-\frac{d+1}{2}} \left(1 + \log \frac{1-r}{\cos(\theta)}\right) + (1-r)^{-\frac{d+1}{4}-1} \times \\ \times \left[(\sqrt{1-r} - \sqrt{2-r} + 1)^{-\frac{d-1}{2}} - (\sqrt{1-r} - \frac{1}{2}\sqrt{2-r} + \frac{1}{2} - \frac{\cos(\theta)}{2})^{-\frac{d-1}{2}} \right] \\ + |\log(\cos(\theta))|, \text{ if } \cos(\theta) \leq \sqrt{2-r} - 1 \leq \frac{1}{2} \frac{t}{2^j t_0}; \\ (1-r)^{-\frac{d+1}{2}} \left(1 + \log \frac{1-r}{\cos(\theta)}\right) + (1-r)^{-\frac{d+1}{4}-1} \times \\ \times \left[(\sqrt{1-r} - \sqrt{2-r} + 1)^{-\frac{d-1}{2}} - (\sqrt{1-r} - \frac{1}{2}\sqrt{2-r} + \frac{1}{2} - \frac{\cos(\theta)}{2})^{-\frac{d-1}{2}} \right] \\ + (\frac{t}{2^j t_0} - \sqrt{2-r} + 1)(1-r)^{-\frac{d+3}{2}} + |\log(\cos(\theta))|, \\ \text{if } \max(\cos(\theta), \frac{1}{2} \frac{t}{2^j t_0}) \leq \sqrt{2-r} - 1 \leq \frac{t}{2^j t_0}; \\ (1-r)^{-\frac{d+1}{2}} (1 + \log \frac{r-1}{\cos(\theta)}) + (1-r)^{-\frac{d+1}{4}-1} \times \\ \times \left[(\sqrt{1-r} - \sqrt{2-r} + 1)^{-\frac{d+1}{2}} - (\sqrt{1-r} - \frac{1}{2}\sqrt{2-r} + \frac{1}{2} - \frac{\cos(\theta)}{2})^{-\frac{d-1}{2}} \right] \\ + |\log(\cos(\theta))|, \text{ if } \cos(\theta) \leq \frac{t}{2^j t_0} \leq \sqrt{2-r} - 1; \\ (1-r)^{-\frac{d+1}{2}} + (\frac{t}{2^j t_0} - \sqrt{2-r} + 1)(1-r)^{-\frac{d+3}{2}} + |\log(\cos(\theta))|, \\ \text{if } \frac{1}{2} \frac{t}{2^j t_0} \leq \sqrt{2-r} - 1 \leq \cos(\theta) \leq \frac{t}{2^j t_0}; \\ (1-r)^{-\frac{d+1}{2}} + |\log(\cos(\theta))|, \\ \text{if } \sqrt{2-r} - 1 \leq \min(\cos(\theta), \frac{1}{2} \frac{t}{2^j t_0}) \text{ and } \cos(\theta) \leq \frac{t}{2^j t_0}; \\ \frac{t}{\cos(\theta) 2^j t_0} (1-r)^{-\frac{d+1}{2}} \text{ if } \cos(\theta) \geq \frac{t}{2^j t_0}. \end{cases}$$

The proofs of Lemmas 3.3 and 3.4 are technical and deferred to Section 5 due to their length.

Proof of Theorem 3.2. Let $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $j \in \mathbb{Z}$, $t_0 \in [1, 2]$ and $x, y \in \Omega$. Write in short $T_j = \exp(-e^{i\theta} 2^j t_0 A)$. Recall that $\{T_j : j \in \mathbb{Z}\}$ is R -bounded on $L^p(\Omega)$ with R -bound $C < \infty$, if

$$\left\| \left(\sum_{j \in F} |T_j f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \leq C \left\| \left(\sum_{j \in F} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

for any finite index set $F \subseteq \mathbb{Z}$ and $f_j \in L^p(\Omega)$, $j \in F$. To prove the theorem, it thus suffices to show that

$$\left\| T : \begin{cases} L^p(\Omega, \ell^2(F)) & \rightarrow L^p(\Omega, \ell^2(F)) \\ (f_j)_{j \in F} & \mapsto (T_j f_j)_{j \in F} \end{cases} \right\| \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|)^2 \quad (3.2)$$

independently of F . To show this, we apply the boundedness criterion for singular integral operators with non-smooth kernels [4, Theorem 1] in its vector-valued version [16, Theorem 2.3]. Note that the standing assumption in [16] that $\mu(\Omega) = \infty$ is not needed in [16, Theorem 2.3]. First note that $L^2(\Omega, \ell^2(F)) = \ell^2(F, L^2(\Omega))$ isometrically, so that for $p = 2$, $\|T\| \leq \sup_{j \in \mathbb{Z}} \|T_j\|_{B(L^2(\Omega))}$ and that the assumption of the theorem gives that $T : L^2(\Omega, \ell^2(F)) \rightarrow L^2(\Omega, \ell^2(F))$ has the norm bound in (3.2). To conclude, it suffices to show that T is a vector-valued singular integral operator with non-smooth kernel in the sense of [16, Definition 2.1]. We choose the approximation to identity $A_t = A'_t = ((f_j)_{j \in F} \mapsto (\exp(-tA)f_j)_{j \in F})$. Note that (1.4), (1.5) and (1.6) in [16] are satisfied for this choice due to the Poisson estimate (3.1) and the volume control (2.2). Note that $T - A'_t T = T - T A_t : (f_j)_{j \in F} \mapsto ((T_j - T_j \exp(-tA))f_j)_{j \in F}$, that this operator has the $B(\ell^2(F))$ -valued kernel with entry $k_{e^{i\theta}2^j t_0}(x, y) - k_{e^{i\theta}2^j t_0+t}(x, y)$ on the diagonal, so that its $B(\ell^2(F))$ -norm is controlled by $\sup_{j \in \mathbb{Z}} |k_{e^{i\theta}2^j t_0}(x, y) - k_{e^{i\theta}2^j t_0+t}(x, y)|$. By [16, Definition 2.1 (i) and (ii)], we are reduced to show that

$$\int_{\rho(x, y) \geq 3t} \sup_{j \in \mathbb{Z}} |k_{e^{i\theta}2^j t_0+t}(x, y) - k_{e^{i\theta}2^j t_0}(x, y)| d\mu(x) \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|)^2 \quad (y \in \Omega, t > 0). \quad (3.3)$$

At first, we estimate the above integral by

$$\int_{\rho(x, y) \geq 3t} \sup_{j: 2^j t_0 \leq t} |\dots| d\mu(x) + \int_{\rho(x, y) \geq 3t} \sup_{j: 2^j t_0 \geq t} |\dots| d\mu(x)$$

and start by estimating the first integral. Estimate crudely $|k_{e^{i\theta}2^j t_0+t}(x, y) - k_{e^{i\theta}2^j t_0}(x, y)| \leq |k_{e^{i\theta}2^j t_0+t}(x, y)| + |k_{e^{i\theta}2^j t_0}(x, y)|$ and consider both summands separately. For $\rho(x, y) \geq 3t \geq \frac{3}{2}2^j t_0 + \frac{3}{2}t$, we have $|1 + \frac{\rho(x, y)^2}{(e^{i\theta}2^j t_0+t)^2}| \cong \frac{\rho(x, y)^2}{|e^{i\theta}2^j t_0+t|^2}$. Thus,

$$\begin{aligned} & \int_{\rho(x, y) \geq 3t} \sup_{j: 2^j t_0 \leq t} |k_{e^{i\theta}2^j t_0+t}(x, y)| d\mu(x) \\ & \lesssim \int_{\rho(x, y) \geq 3t} \sup_{j: 2^j t_0 \leq t} \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, |e^{i\theta}2^j t_0+t|))} \frac{|e^{i\theta}2^j t_0+t|^{d+1}}{\rho(x, y)^{d+1}} d\mu(x) \\ & \lesssim \int_{\rho(x, y) \geq 3t} \sup_{j: 2^j t_0 \leq t} \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t))} \frac{t^{d+1}}{\rho(x, y)^{d+1}} d\mu(x) \\ & \lesssim \sum_{k=0}^{\infty} \int_{B(y, 3t \cdot 2^k, 3t \cdot 2^{k+1})} \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t))} 2^{-k(d+1)} d\mu(x) \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{k=0}^{\infty} \sum_{n=1}^{M_k} \int_{B(y_n^k, t)} \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t))} 2^{-k(d+1)} d\mu(x) \\
&\lesssim \sum_{k=0}^{\infty} 2^{kd} 2^{-k(d+1)} (\cos(\theta))^{-\beta} \lesssim (\cos(\theta))^{-\beta},
\end{aligned}$$

where $B(y, 3t \cdot 2^0, 3t \cdot 2^{0+1})$ should be replaced by $B(y, 3t \cdot 2)$, and we used the two well-known facts on spaces of homogeneous type that $B(y, 3t \cdot 2^k, 3t \cdot 2^{k+1})$ can be covered by $M_k \cong 2^{kd}$ balls $B(y_n^k, t)$ and that $\sup_{y \in \Omega} \int_{B(y, t)} \frac{1}{\mu(B(x, t))} d\mu(x) \cong 1$. Note that if $\mu(\Omega) < \infty$, then the above sum over k was finite.

Furthermore, for $\rho(x, y) \geq 3t \geq 3 \cdot 2^j t_0$, we have $|1 + \frac{\rho(x, y)^2}{(e^{i\theta} 2^j t_0)^2}| \cong \frac{\rho(x, y)^2}{(2^j t_0)^2}$.

Thus, similarly to the above calculation, with $M_k \lesssim \left(\frac{t \cdot 2^k}{2^j t_0}\right)^d$ this time,

$$\begin{aligned}
&\int_{\rho(x, y) \geq 3t} \sup_{j: 2^j t_0 \leq t} |k_{e^{i\theta} 2^j t_0}(x, y)| d\mu(x) \\
&\lesssim \sum_{j: 2^j t_0 \leq t} \int_{\rho(x, y) \geq 3t} \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{(2^j t_0)^{d+1}}{\rho(x, y)^{d+1}} d\mu(x) \\
&\lesssim \sum_{j: 2^j t_0 \leq t} \sum_{k=0}^{\infty} \int_{B(y, 3t \cdot 2^k, 3t \cdot 2^{k+1})} \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{(2^j t_0)^{d+1}}{(3t \cdot 2^k)^{d+1}} d\mu(x) \\
&\lesssim \sum_{j: 2^j t_0 \leq t} \sum_{k=0}^{\infty} \sum_{n=1}^{M_k} \int_{B(y_n^k, 2^j t_0)} \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{(2^j t_0)^{d+1}}{(3t \cdot 2^k)^{d+1}} d\mu(x) \\
&\lesssim \sum_{j: 2^j t_0 \leq t} \sum_{k=0}^{\infty} (\cos(\theta))^{-\beta} \frac{t^d 2^{kd}}{2^j d t_0^d} \frac{2^j (d+1) t_0^{d+1}}{t^{d+1} 2^{kd} 2^k} \\
&\lesssim \sum_{j: 2^j t_0 \leq t} \sum_{k=0}^{\infty} (\cos(\theta))^{-\beta} t^{-1} 2^{-k} 2^j t_0 \lesssim (\cos(\theta))^{-\beta}.
\end{aligned}$$

To conclude (3.3), it now suffices to show

$$\begin{aligned}
&\int_{\rho(x, y) \geq 3t} \sup_{j: 2^j t_0 \leq t} |k_{e^{i\theta} 2^j t_0 + t}(x, y) - k_{e^{i\theta} 2^j t_0}(x, y)| d\mu(x) \lesssim \\
&\quad (\cos(\theta))^{-\frac{d-1}{2} - \beta} (1 + |\log(\cos(\theta))|)^2 \quad (y \in \Omega, t > 0). \quad (3.4)
\end{aligned}$$

To this end, we use the estimates from Lemma 3.4. Therefore, as a first step, we divide the integral in (3.4) into

$$\begin{aligned}
&\int_{\rho(x, y) \geq 3t} \leq \sum_{l=L_{\min}}^{L_{\max}} \int_{|\rho(x, y)^2 / (2^l t_0)^2 - 1| \leq \cos(\theta)} + \sum_{l=L_{\min}}^{L_{\max}} \int_{\rho(x, y)^2 / (2^l t_0)^2 \in [1 + \cos(\theta), 2]} \\
&\quad + \sum_{l=L_{\min}}^{L_{\max}} \int_{\rho(x, y)^2 / (2^l t_0)^2 \in [\frac{1}{2}, 1 - \cos(\theta)]} =: \sum_{l=L_{\min}}^{L_{\max}} I_1 + I_2 + I_3,
\end{aligned}$$

where $L_{\min} = \min\{l \in \mathbb{Z} : 2^l t_0 \geq \frac{3}{2\sqrt{1-\cos(\theta)}} t\}$ and $L_{\max} = \max\{l \in \mathbb{Z} : 2^l t_0 \leq \text{diam}(\Omega)\} + 1$. For I_1 , we have by Lemma 3.4 1. and the volume control (2.3),

$$\begin{aligned} & \int_{|\rho(x,y)^2/(2^l t_0)^2 - 1| \leq \cos(\theta)} \sup_{j: 2^j t_0 \geq t} |k_{e^{i\theta} 2^j t_0 + t}(x, y) - k_{e^{i\theta} 2^j t_0}(x, y)| d\mu(x) \\ & \lesssim \int_{\dots} \frac{1}{\mu(B(x, 2^l t_0))} \frac{\min(\cos(\theta) 2^l t_0, t)}{\cos(\theta) 2^l t_0} (\cos(\theta))^{-\frac{d+1}{2} - \beta} d\mu(x) \\ & \lesssim 2^{ld} \cos(\theta) 2^{-ld} \frac{\min(\cos(\theta) 2^l t_0, t)}{\cos(\theta) 2^l t_0} (\cos(\theta))^{-\frac{d+1}{2} - \beta} \\ & \cong \frac{\min(\cos(\theta) 2^l t_0, t)}{\cos(\theta) 2^l t_0} (\cos(\theta))^{-\frac{d-1}{2} - \beta}. \end{aligned}$$

Now summing over l , we obtain

$$\begin{aligned} & \sum_{l=L_{\min}}^{L_{\max}} \frac{\min(\cos(\theta) 2^l t_0, t)}{\cos(\theta) 2^l t_0} (\cos(\theta))^{-\frac{d-1}{2} - \beta} \\ & \lesssim (\cos(\theta))^{-\frac{d-1}{2} - \beta} \left(\int_1^{(\cos(\theta))^{-1}} \frac{dx}{x} + \int_{(\cos(\theta))^{-1}}^{\infty} (\cos(\theta))^{-1} x^{-1} \frac{dx}{x} \right) \\ & \cong (\cos(\theta))^{-\frac{d-1}{2} - \beta} (1 + |\log(\cos(\theta))|). \end{aligned}$$

Now for I_2 . We distinguish the cases I) $(\cos(\theta) 2^l t_0 \leq t)$ and II) $(\cos(\theta) 2^l t_0 \geq t)$. In case I), we decompose I_2 into three integrals according to Lemma 3.4 2. We write $r = \rho(x, y)^2 / (2^l t_0)^2$.

$$\begin{aligned} I_2 & \leq \int_{\cos(\theta) \leq 1 - \sqrt{2-r} \leq \frac{t}{2^l t_0}, 1 + \cos(\theta) \leq r \leq 2} + \int_{\cos(\theta) \leq \frac{t}{2^l t_0} \leq 1 - \sqrt{2-r}} + \int_{1 - \sqrt{2-r} \leq \cos(\theta) \leq \frac{t}{2^l t_0}} \\ & =: I_2^1 + I_2^2 + I_2^3. \end{aligned}$$

Then by the volume condition (2.3) and Lemma 3.4 2.,

$$\begin{aligned} I_2^1 & \lesssim \int_{\cos(\theta) \leq 1 - \sqrt{2-r} \leq \frac{t}{2^l t_0}, 1 + \cos(\theta) \leq r \leq 2} (\cos(\theta))^{-\beta} (r-1)^{-\frac{d+1}{2}} \left(1 + \log \frac{r-1}{\cos(\theta)}\right) r^{\frac{d}{2}} \frac{dr}{r} \\ & \lesssim (\cos(\theta))^{-\beta} \int_{1 + \cos(\theta)}^2 (r-1)^{-\frac{d+1}{2}} \left(1 + \log \frac{r-1}{\cos(\theta)}\right) dr \\ & \lesssim (\cos(\theta))^{-\frac{d-1}{2} - \beta} \left(1 + \max_{\cos(\theta) \leq r \leq 1} \log \frac{r}{\cos(\theta)}\right) \\ & \cong (\cos(\theta))^{-\frac{d-1}{2} - \beta} (1 + |\log(\cos(\theta))|). \end{aligned}$$

Further,

$$\begin{aligned} I_2^2 & \lesssim (\cos(\theta))^{-\beta} \int_{1 + \cos(\theta) \leq r \leq 2, \frac{t}{2^l t_0} \leq 1 - \sqrt{2-r}} \left((r-1)^{-\frac{d+1}{2}} \left(1 + \log \frac{t}{2^l t_0 \cos(\theta)}\right) \right. \\ & \quad \left. + |\log(\cos(\theta))| \right) r^{\frac{d}{2}} \frac{dr}{r} \end{aligned}$$

$$\begin{aligned} &\lesssim (\cos(\theta))^{-\beta} \left(\int_{1+\cos(\theta)}^2 (r-1)^{-\frac{d+1}{2}} dr (1 + \log \frac{t}{2^l t_0} + |\log(\cos(\theta))|) + |\log(\cos(\theta))| \right) \\ &\lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|), \end{aligned}$$

due to $\frac{t}{2^l t_0} \leq 1 - \sqrt{2-r} \leq r-1 \leq 1$ in this case. Finally,

$$\begin{aligned} I_2^3 &\lesssim (\cos(\theta))^{-\beta} \int_{1-\sqrt{2-r} \leq \cos(\theta), 1+\cos(\theta) \leq r \leq 2} (\cos(\theta))^{-\frac{d+1}{2}} r^{\frac{d}{2}} \frac{dr}{r} \\ &\lesssim (\cos(\theta))^{-\frac{d+1}{2}-\beta} \int_{1+\cos(\theta)}^{1+2\cos(\theta)} r^{\frac{d}{2}} \frac{dr}{r} \\ &\cong (\cos(\theta))^{-\frac{d-1}{2}-\beta}, \end{aligned}$$

where we have used that $\frac{1}{2}(r-1) \leq 1 - \sqrt{2-r}$, and thus $r-1 \leq 2(1 - \sqrt{2-r}) \leq 2\cos(\theta)$. Now summing up the three estimates, we get $I_2 \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|)$, and thus, as for $\sum_{l=L_{\min}}^{L_{\max}} I_1$, we deduce $\sum_{l=L_{\min}}^{L_{\max}} I_2 \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|)^2$. In case II), we have by the volume condition (2.3) and Lemma 3.4 2.,

$$\begin{aligned} I_2 &\cong (\cos(\theta))^{-\beta} \frac{t}{\cos(\theta) 2^l t_0} \int_{1+\cos(\theta)}^2 (r-1)^{-\frac{d+1}{2}} r^{\frac{d}{2}} \frac{dr}{r} \\ &\cong (\cos(\theta))^{-\beta} \frac{t}{\cos(\theta) 2^l t_0} \left[-(r-1)^{-\frac{d-1}{2}} \right]_{1+\cos(\theta)}^2 \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} \frac{t}{\cos(\theta) 2^l t_0}. \end{aligned}$$

Now summing over those l with $\cos(\theta) 2^l t_0 \geq t$, we obtain $\sum_{l: \cos(\theta) 2^l t_0 \geq t} I_2 \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} \int_0^1 x \frac{dx}{x} = (\cos(\theta))^{-\frac{d-1}{2}-\beta}$.

It remains to estimate I_3 , for which we use part 3. of Lemma 3.4. Again we distinguish the cases I) ($\cos(\theta) 2^l t_0 \leq t$) and II) ($\cos(\theta) 2^l t_0 \geq t$). In case I), we decompose $I_3 \lesssim \int \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|) d\mu(x) + \int \text{remainder } d\mu(x)$. The first term is estimated by

$$\begin{aligned} &\int_{\rho(x,y)^2 / (2^l t_0)^2 \in [\frac{1}{2}, 1-\cos(\theta)]} \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|) d\mu(x) \\ &\lesssim \int_{\frac{1}{2}}^{1-\cos(\theta)} (\cos(\theta))^{-\beta} (1 + |\log(\cos(\theta))|) r^{\frac{d}{2}} \frac{dr}{r} \\ &\cong (\cos(\theta))^{-\beta} (1 + |\log(\cos(\theta))|). \end{aligned}$$

The remainder is decomposed into the following five integrals corresponding to the first five cases in Lemma 3.4 3. We write again $r = \rho(x, y)^2 / (2^l t_0)^2$.

$$\begin{aligned} &\int_{\frac{1}{2} \leq r \leq 1-\cos(\theta), \cos(\theta) \leq \sqrt{2-r}-1 \leq \frac{1}{2} \frac{t}{2^l t_0}} + \int_{\frac{1}{2} \leq r \leq 1-\cos(\theta), \max(\cos(\theta), \frac{1}{2} \frac{t}{2^l t_0}) \leq \sqrt{2-r}-1 \leq \frac{t}{2^l t_0}} \\ &+ \int_{\frac{1}{2} \leq r \leq 1-\cos(\theta), \cos(\theta) \leq \frac{t}{2^l t_0} \leq \sqrt{2-r}-1} + \int_{\frac{1}{2} \leq r \leq 1-\cos(\theta), \frac{1}{2} \frac{t}{2^l t_0} \leq \sqrt{2-r}-1 \leq \cos(\theta) \leq \frac{t}{2^l t_0}} \end{aligned}$$

$$+ \int_{\frac{1}{2} \leq r \leq 1 - \cos(\theta), \sqrt{2-r} - 1 \leq \min(\cos(\theta), \frac{1}{2} \frac{t}{2^l t_0})} =: I_3^1 + I_3^2 + I_3^3 + I_3^4 + I_3^5.$$

We estimate the five integrals separately. For I_3^1 , we have

$$\begin{aligned} (\cos(\theta))^\beta I_3^1 &\lesssim \int_{\frac{1}{2} \leq r \leq 1 - \cos(\theta), \cos(\theta) \leq 1 - r \leq \frac{1}{2} \frac{1}{\sqrt{2-1}} \frac{t}{2^l t_0}} (1-r)^{-\frac{d+1}{2}} (1 + \log \frac{1-r}{\cos(\theta)}) r^{\frac{d}{2}} \frac{dr}{r} \\ &+ \int_{\frac{1}{2} \leq r \leq 1 - \cos(\theta), \cos(\theta) \leq \sqrt{2-r} - 1 \leq \frac{1}{2} \frac{t}{2^l t_0}} (1-r)^{-\frac{d+1}{4} - 1} \times \\ &\times (1-r)(\sqrt{1-r} - (\sqrt{2-r} - 1))^{-\frac{d+1}{2}} r^{\frac{d}{2}} \frac{dr}{r} \\ &\lesssim (\cos(\theta))^{-\frac{d-1}{2}} (1 + |\log(\cos(\theta))|) + \int_{\frac{1}{2}}^{1 - \cos(\theta)} (1-r)^{-\frac{d+1}{4} - 1 + 1 - \frac{d+1}{4}} dr \\ &\lesssim (\cos(\theta))^{-\frac{d-1}{2}} (1 + |\log(\cos(\theta))|) + (\cos(\theta))^{-\frac{d-1}{2}}, \end{aligned}$$

where we have used the mean value theorem to estimate

$$\begin{aligned} (\sqrt{1-r} - (\sqrt{2-r} - 1))^{-\frac{d-1}{2}} - (\sqrt{1-r} - \frac{1}{2}(\sqrt{2-r} - 1) - \frac{\cos(\theta)}{2})^{-\frac{d-1}{2}} &\lesssim \\ (1-r)(\sqrt{1-r} - (\sqrt{2-r} - 1))^{-\frac{d+1}{2}}. \end{aligned}$$

For I_3^2 , we have

$$\begin{aligned} (\cos(\theta))^\beta I_3^2 &\lesssim \int_{\frac{1}{2} \leq r \leq 1 - (\cos(\theta)), \max(\cos(\theta), \frac{1}{2} \frac{t}{2^l t_0}) \leq \sqrt{2-r} - 1 \leq \frac{t}{2^l t_0}} \\ &\left\{ 1^{st} \text{ term} + 2^{nd} \text{ term} + \left(\frac{t}{2^l t_0} - \sqrt{2-r} + 1 \right) (1-r)^{-\frac{d+3}{2}} \right\} r^{\frac{d}{2}} \frac{dr}{r}, \end{aligned}$$

where the 1^{st} and 2^{nd} term are given in Lemma 3.4 3., and can be controlled as in I_3^1 by $(\cos(\theta))^{-\frac{d-1}{2}} (1 + |\log(\cos(\theta))|)$. The third term, we estimate by

$$\begin{aligned} &\lesssim \int_{\frac{1}{2} \leq r \leq 1 - \cos(\theta), \max(\cos(\theta), \frac{1}{2} \frac{t}{2^l t_0}) \leq \sqrt{2-r} - 1 \leq \frac{t}{2^l t_0}} \left(\frac{t}{2^l t_0} - (\sqrt{2-r} - 1) \right) (1-r)^{-\frac{d+3}{2}} dr \\ &\lesssim \int_{\frac{1}{2} \leq r \leq 1 - \cos(\theta)} (1-r)(1-r)^{-\frac{d+3}{2}} dr \lesssim (\cos(\theta))^{-\frac{d-1}{2}}. \end{aligned}$$

For I_3^3 , we apply exactly the same estimate as for I_3^1 , to get $(\cos(\theta))^\beta I_3^3 \lesssim (\cos(\theta))^{-\frac{d-1}{2}} (1 + |\log(\cos(\theta))|)$, too. For I_3^4 , we have again with 1^{st} , 2^{nd} and 3^{rd} term given in the estimate in Lemma 3.4 3.,

$$\begin{aligned} (\cos(\theta))^\beta I_3^4 &\lesssim \int_{\frac{1}{2} \leq r \leq 1 - \cos(\theta), \frac{1}{2} \frac{t}{2^l t_0} \leq \sqrt{2-r} - 1 \leq \cos(\theta) \leq \frac{t}{2^l t_0}} \\ &\left\{ 1^{st} \text{ term} + 2^{nd} \text{ term} + 3^{rd} \text{ term} \right\} r^{\frac{d}{2}} \frac{dr}{r}. \end{aligned}$$

The 1^{st} and 2^{nd} term can be controlled as in I_3^1 , whereas the 3^{rd} term can be controlled as in I_3^2 . We get $(\cos(\theta))^\beta I_3^4 \lesssim (\cos(\theta))^{-\frac{d-1}{2}} (1 + |\log(\cos(\theta))|)$.

Eventually, we estimate I_3^5 by

$$\begin{aligned} (\cos(\theta))^\beta I_3^5 &\lesssim \int_{\frac{1}{2} \leq r \leq 1 - \cos(\theta), \sqrt{2-r} - 1 \leq \min(\cos(\theta), \frac{1}{2} \frac{t}{2^l t_0})} (1-r)^{-\frac{d+1}{2}} r^{\frac{d}{2}} \frac{dr}{r} \\ &\lesssim (\cos(\theta))^{-\frac{d-1}{2}}. \end{aligned}$$

Now summing up we obtain as for I_2 , $\sum_{l=L_{\min}}^{L_{\max}} I_3 \leq \sum_{l=L_{\min}}^{L_{\max}} I_3^1 + I_3^2 + I_3^3 + I_3^4 + I_3^5 \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|)^2$.

In case II), we have

$$\begin{aligned} (\cos(\theta))^\beta I_3 &\lesssim \int_{\frac{1}{2} \leq r \leq 1 - \cos(\theta)} \frac{t}{\cos(\theta) 2^l t_0} (1-r)^{-\frac{d+1}{2}} r^{\frac{d}{2}} \frac{dr}{r} \\ &\lesssim \frac{t}{\cos(\theta) 2^l t_0} (\cos(\theta))^{-\frac{d-1}{2}}. \end{aligned}$$

Summing over l with $\cos(\theta) 2^l t_0 \geq t$, we obtain as for I_2 , $\sum_{l: \cos(\theta) 2^l t_0 \geq t} I_3 \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta}$.

We have shown that

$$\sum_{l=L_{\min}}^{L_{\max}} I_1 + I_2 + I_3 \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|)^2,$$

and thus (3.4) and hence the theorem follows. \square

Remark 3.5. We have proved in Theorem 3.2 that the semigroup is R -bounded when taking dyadic arguments of the form $e^{i\theta} 2^j t_0$, $t_0 \in [1, 2]$ fixed. It is unclear whether one gets also an R -bounded set when this argument is replaced by a continuous variant, i.e. if $R(\exp(-e^{i\theta} t_0 A) : t_0 > 0) \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|)^2$ holds under the same hypotheses as in Theorem 3.2, and at least the above proof does not seem to work. By the method in [14, Last part of 2.16 Example], which passes from dyadic arguments to continuous ones, one only gets a cruder estimate for the continuous variant involving an additional factor $(\cos(\theta))^{-2}$.

Corollary 3.6. Let (Ω, μ, ρ) be a space of homogeneous type of either finite or infinite measure, satisfying (2.2) and (2.3). Let $T_t = \exp(-tA)$ a semigroup which acts on all $L^p(\Omega)$, $1 < p < \infty$. Assume that T_z is analytic on $L^2(\Omega)$ on $z \in \mathbb{C}_+$ and that T_z has an integral kernel $k_z(x, y)$ which satisfies the Poisson estimate

$$|k_{re^{i\theta}}(x, y)| \leq C (\cos(\theta))^{-\beta} \frac{1}{\mu(B(x, r))} \frac{1}{\left|1 + \frac{\rho(x, y)^2}{(re^{i\theta})^2}\right|^{\frac{d+1}{2}}}$$

for any $x, y \in \Omega$, $r > 0$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and some $C, \beta \geq 0$. Assume moreover that A has a bounded $H^\infty(\Sigma_\omega)$ calculus on $L^2(\Omega)$ for some $\omega \in (0, \pi)$, and that for $r > 0$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\|\exp(-re^{i\theta} A)\|_{B(L^2(\Omega))} \leq C (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|)^2,$$

both of which are satisfied e.g. when A is self-adjoint. Then A has a bounded \mathcal{H}^α calculus on $L^p(\Omega)$ for any $1 < p < \infty$ and $\alpha > \frac{d}{2}$. Moreover this calculus is an R -bounded mapping, i.e.

$$R(f(A) : \|f\|_{\mathcal{H}^\alpha} \leq 1) < \infty.$$

Proof. This follows immediately from Propositions 2.1, 3.1 and Theorem 3.2. \square

4. Examples

In this section, we want to give some applications of Theorem 3.2 and Corollary 3.6. Note first that in the most classical example, namely $A = (-\Delta)^{\frac{1}{2}}$ and $\exp(-zA)$ the Poisson semigroup on $L^p(\mathbb{R}^d)$ for some $1 < p < \infty$ and $d \in \mathbb{N}$, Corollary 3.6 gives the sharp order of derivation of the classical Hörmander multiplier theorem and strengthens it in that it includes the R -boundedness of spectral multipliers whose associated functions have bounded \mathcal{H}^α norm.

For a generalization, we consider the situation in [15]. Let M be a positive integer. Consider the constant coefficient second order, $M \times M$ system, differential operator

$$Lu = \sum_{\gamma=1}^M \sum_{r,s=1}^{d+1} (\partial_r (a_{rs}^{\alpha\gamma} \partial_s u_\beta))_{1 \leq \alpha \leq M},$$

where $a_{rs}^{\alpha\beta}$ are real coefficients for $r, s = 1, \dots, d+1$ and $\alpha, \gamma = 1, \dots, M$. Here, u is a function defined on the upper half space $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times [0, \infty)$. Further, we assume as in [15] the ellipticity condition

$$\sum_{\alpha, \gamma=1}^M \sum_{r,s=1}^{d+1} \operatorname{Re} [a_{rs}^{\alpha\gamma} \xi_r \xi_s \overline{\eta_\alpha} \eta_\beta] \geq \kappa_0 |\xi|^2 |\eta|^2$$

for every $(\xi_r)_{1 \leq r \leq d+1} \in \mathbb{R}^{d+1}$, $(\eta_\alpha)_{1 \leq \alpha \leq M} \in \mathbb{C}^M$ and some $\kappa_0 > 0$. Then in [15] the following Dirichlet problem on \mathbb{R}_+^{d+1} is considered:

$$\begin{cases} Lu = 0 & \text{in } \mathbb{R}_+^{d+1} \\ u|_{\partial \mathbb{R}_+^{d+1}}^{n.t.} = f \in L^p(\mathbb{R}^d; \mathbb{C}^M), \end{cases}$$

where $\partial \mathbb{R}_+^{d+1} = \mathbb{R}^d \times \{0\}$, *n.t.* means non-tangential trace of u , and f is a given function in $L^p(\mathbb{R}^d; \mathbb{C}^M)$, $1 < p < \infty$. If $\mathfrak{A}_L^{dis} \neq \emptyset$, a certain condition, see [15, (3.12)], which will be satisfied in our example, then this problem is well-posed in $L^p(\mathbb{R}^d; \mathbb{C}^M)$ [15, Theorem 4.1], so it possesses a unique solution u . As moreover the coefficients defining L are constant, we have $\partial_r [u(\cdot + (0, t))]|_{\cdot=x} = (\partial_r u)(x + (0, t))$, so that the expression $T_t f(x) := u(x, t)$, $x \in \mathbb{R}^d$, $t \geq 0$ defines a semigroup on $L^p(\mathbb{R}^d; \mathbb{C}^M)$. In the sequel, we are interested in the Hörmander functional calculus of the negative generator of that semigroup. Note that for some cases, this semigroup is given by a convolution kernel. We now restrict to the following specific example.

Lamé system of elasticity. Assume that $M = d+1$ above. The so-called Lamé operator in \mathbb{R}^{d+1} has the form

$$Lu = \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u, \quad u = (u_1, \dots, u_{d+1}), \quad (4.1)$$

where the constants $\lambda, \mu \in \mathbb{R}$ (typically called Lamé moduli) are assumed to satisfy $\mu > 0$ and $2\mu + \lambda > 0$. Then according to [15, Theorem 5.2], $T_t f(x) = u(x, t)$ is given by

$$\begin{aligned} (T_t f)_\alpha(x) &= \frac{4\mu}{3\mu + \lambda} \frac{1}{\omega_d} \int_{\mathbb{R}^d} \frac{t}{(|x-y|^2 + t^2)^{\frac{d+1}{2}}} f_\alpha(y) dy \\ &+ \frac{\mu + \lambda}{3\mu + \lambda} \frac{2(d+1)}{\omega_d} \sum_{\gamma=1}^{d+1} \int_{\mathbb{R}^d} \frac{t(x-y, t)_\alpha (x-y, t)_\gamma}{(|x-y|^2 + t^2)^{\frac{d+3}{2}}} f_\gamma(y) dy \quad (\alpha = 1, \dots, d+1), \end{aligned} \quad (4.2)$$

where ω_d is the area of the unit sphere S^d in \mathbb{R}^{d+1} , and $(x-y, t)_\alpha = x_\alpha - y_\alpha$ if $\alpha \in \{1, \dots, d\}$ and $(x-y, t)_{d+1} = t$.

Proposition 4.1. The semigroup in (4.2) is strongly continuous on $L^p(\mathbb{R}^d; \mathbb{C}^{d+1})$ for $1 < p < \infty$, has an analytic extension for $\operatorname{Re} z > 0$ and if $k_z(x, y) = (k_{z; \alpha\gamma})_{\alpha, \gamma=1}^{d+1}(x, y)$ denotes its $(d+1) \times (d+1)$ matrix valued integral kernel, then each of its components $k_{z, \alpha\gamma}(x, y)$ satisfies the Poisson estimate (3.1) with $\beta = 1$. Further, the negative generator A of the semigroup has an H^∞ calculus on $L^2(\mathbb{R}^d; \mathbb{C}^{d+1})$ and

$$\begin{aligned} \|\exp(-e^{i\theta} tA)\|_{B(L^2(\mathbb{R}^d; \mathbb{C}^{d+1}))} &\lesssim (\cos(\theta))^{-1} \max(1 + |\log(\cos(\theta))|, (\cos(\theta))^{-\frac{d-1}{2}}) \\ &\lesssim (\cos(\theta))^{-\frac{d-1}{2} - \beta} (1 + |\log(\cos(\theta))|)^2 \end{aligned}$$

for $t > 0$ and $|\theta| < \frac{\pi}{2}$.

Proof. For the strong continuity of the semigroup, we show first that $\sup_{t>0} \|T_t\|_{L^p(\mathbb{R}^d; \mathbb{C}^{d+1}) \rightarrow L^p(\mathbb{R}^d; \mathbb{C}^{d+1})} < \infty$. As T_t is given by a linear combination of convolutions, we have $\|T_t\|_{p \rightarrow p} \lesssim \int_{\mathbb{R}^d} \frac{t}{(|y|^2 + t^2)^{\frac{d+1}{2}}} dy + \int_{\mathbb{R}^d} t \frac{|y|^2 + t^2}{(|y|^2 + t^2)^{\frac{d+3}{2}}} dy = 2 \int_{\mathbb{R}^d} \frac{1}{(|y/t|^2 + 1)^{\frac{d+1}{2}}} \frac{dy}{t^d}$, which is clearly bounded independently of $t > 0$. Thus it suffices to show the strong continuity $\|T_t f - f\|_p \rightarrow 0$ as $t \rightarrow 0$ for f belonging to the dense subset of continuous functions with compact support. In the following calculation, we use that the kernel $k_t(x, y)$ of T_t satisfies $\int_{\mathbb{R}^d} k_t(x, y) dy = 1_{M_{d+1}}$, since $T_t 1_{\ell_{d+1}^2} = 1_{\ell_{d+1}^2}$ is the unique solution of the Dirichlet problem with constant initial value, and we use Jensen's inequality.

$$\begin{aligned} \|T_t f - f\|_p^p &\cong \sum_{\alpha=1}^{d+1} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} c_1 \frac{t}{(|y|^2 + t^2)^{\frac{d+1}{2}}} f_\alpha(x-y) dy \right. \\ &\left. + \sum_{\gamma=1}^{d+1} \int_{\mathbb{R}^d} c_2 t \frac{(y, t)_\alpha (y, t)_\gamma}{(|y|^2 + t^2)^{\frac{d+3}{2}}} f_\gamma(x-y) dy - f_\alpha(x) \right|^p dx \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha=1}^{d+1} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} c_1 \frac{t}{(|y|^2 + t^2)^{\frac{d+1}{2}}} [f_\alpha(x-y) - f_\alpha(x)] dy \right. \\
&+ \left. \sum_{\gamma=1}^{d+1} \int_{\mathbb{R}^d} c_2 t \frac{(y,t)_\alpha(y,t)_\gamma}{(|y|^2 + t^2)^{\frac{d+3}{2}}} [f_\gamma(x-y) dy - f_\gamma(x)] dy \right|^p dx \\
&\lesssim \sum_{\alpha=1}^{d+1} \int_{\mathbb{R}^d} \left\{ c_1 t \int_{\mathbb{R}^d} \frac{1}{(|y|^2 + t^2)^{\frac{d+1}{2}}} |f_\alpha(x-y) - f_\alpha(x)|^p dy \right. \\
&+ \left. c_2 t \sum_{\gamma=1}^{d+1} \int_{\mathbb{R}^d} \frac{|(y,t)_\alpha(y,t)_\gamma|}{(|y|^2 + t^2)^{\frac{d+3}{2}}} |f_\gamma(x-y) - f_\gamma(x)|^p dy \right\} dx \\
&= \sum_{\alpha=1}^{d+1} \int_{\mathbb{R}^d} \left\{ c_1 \int_{\mathbb{R}^d} \frac{1}{(|y|^2 + 1)^{\frac{d+1}{2}}} |f_\alpha(x-ty) - f_\alpha(x)|^p dy \right. \\
&+ \left. c_2 \sum_{\gamma=1}^{d+1} \int_{\mathbb{R}^d} \frac{|(y,1)_\alpha(y,1)_\gamma|}{(|y|^2 + 1)^{\frac{d+3}{2}}} |f_\gamma(x-ty) - f_\gamma(x)|^p dy \right\} dx \\
&\cong \sum_{\alpha} \int_{|x| \leq R} \int_{\mathbb{R}^d} \dots dy + \sum_{\gamma} \int_{\mathbb{R}^d} \dots dy dx + \int_{|x| \geq R} \int_{\mathbb{R}^d} \dots dy + \sum_{\gamma} \int_{\mathbb{R}^d} \dots dy dx.
\end{aligned}$$

Now choose first $\epsilon > 0$ and $R \gg 1$ sufficiently large. Then there exists $t_0 \ll 1$ such that $|f_\alpha(x-ty) - f_\alpha(x)|^p < \frac{\epsilon}{\mu(B(0,R))}$ for $t \leq t_0$ and $\alpha \in \{1, \dots, d+1\}$. Thus the integral over $|x| \leq R$ is $\lesssim \epsilon$. Let $\text{supp } f \subset B(0, r)$. Then the integral over $|x| \geq R$ is $\lesssim \int_{|y| \geq 1/t_0(R-r)} \frac{1}{(|y|^2+1)^{\frac{d+1}{2}}} 2^p \|f\|_p^p dy \rightarrow 0$ as $R \rightarrow \infty$. We have proved the strong continuity of the semigroup.

For the analyticity, extend the definition of T_t from (4.2) by replacing $t > 0$ by a complex z with $\text{Re } z > 0$ everywhere. Clearly, (4.2) is an analytic expression in t , so that we get an analytic family T_z . The claimed $B(L^2(\mathbb{R}^d, \mathbb{C}^{d+1}))$ estimate of T_z follows from the integral kernel estimate of $\frac{z}{(|\cdot|^2+z^2)^{\frac{d+1}{2}}}$, which is shown in [7, p. 348] and gives the estimate for the first expression in (4.2), and also for the second expression by the pointwise estimate $\frac{|z(y,z)_\alpha(y,z)_\gamma|}{\||y|^2+z^2|^{\frac{d+3}{2}}} \leq (\cos(\arg z))^{-1} \frac{|z|}{\||y|^2+z^2|^{\frac{d+1}{2}}}$, which is shown in the following.

Now for the claimed Poisson estimate (3.1) with $\beta = 1$. Fix some $\alpha, \gamma \in \{1, \dots, d+1\}$. We have by (4.2),

$$k_{z,\alpha\gamma}(x,y) = c_1 \delta_{\alpha=\gamma} \frac{z}{(|x-y|^2+z^2)^{\frac{d+1}{2}}} + c_2 \frac{z(x-y,z)_\alpha(x-y,z)_\gamma}{(|x-y|^2+z^2)^{\frac{d+3}{2}}}. \quad (4.3)$$

Clearly, the first summand admits the complex Poisson estimate (3.1) even with $\beta = 0$. For the second summand, it clearly suffices to show that $|(x,z)_\alpha(x,z)_\gamma| \lesssim (\cos(\theta))^{-1} \||x|^2+z^2\|$, where $\arg z = \theta$. We distinguish the four cases $\alpha, \gamma \leq d$; $\alpha, \gamma = d+1$; $\alpha \leq d, \gamma = d+1$ and $\alpha = d+1, \gamma \leq d$. Suppose first $\alpha, \gamma \leq d$. If $|x|^2 \geq 2|z|^2$, then $|x_\alpha x_\gamma| \leq \frac{1}{2}(x_\alpha^2 + x_\gamma^2) \leq \frac{1}{2}|x|^2 \lesssim \||x|^2+z^2\|$. If

$|x|^2 \leq \frac{1}{2}|z|^2$, then $|x_\alpha x_\gamma| \leq \frac{1}{2}|x|^2 \leq \frac{1}{4}|z|^2 \leq \frac{1}{2}(|z|^2 - |x|^2) \leq \frac{1}{2}|z^2 + |x|^2|$. If $\frac{1}{2}|z|^2 \leq |x|^2 \leq 2|z|^2$, then

$$\begin{aligned} |x_\alpha x_\gamma| \cos(\theta) &\lesssim \cos(\theta)|z|^2 \leq |z|^2 \cos(\theta) + |z|^2(\cos^2(\theta) - \sin^2(\theta) + |x|^2) \\ &\cong |2(\operatorname{Re} z)(\operatorname{Im} z)| + |(\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 + |x|^2| \\ &= |\operatorname{Im}(z^2)| + |\operatorname{Re}(z^2) + |x|^2| \cong |z^2 + |x|^2|. \end{aligned}$$

The other three cases can be treated in the same manner.

We now show that the negative generator A of T_t has a bounded H^∞ calculus on $L^2(\mathbb{R}^d; \mathbb{C}^{d+1})$. According to [3, Theorem 2.4], it suffices to show that both

$$\int_0^\infty t \|Ae^{-tA}f\|_2^2 dt \lesssim \|f\|_2^2 \quad \text{and} \quad \int_0^\infty t \|A'e^{-tA'}f\|_2^2 dt \lesssim \|f\|_2^2 \quad (4.4)$$

hold. We have with F denoting the Fourier transform, using the Plancherel formula and Fubini,

$$\begin{aligned} \int_0^\infty t \|Ae^{-tA}f\|_{L^2(\mathbb{R}^d; \mathbb{C}^{d+1})}^2 dt &= \int_0^\infty \sum_{\alpha=1}^{d+1} t \left\| \frac{\partial}{\partial t} (e^{-tA}f)_\alpha \right\|_{L^2(\mathbb{R}^d)}^2 dt \\ &= \int_0^\infty \sum_{\alpha=1}^{d+1} t \left\| \sum_{\gamma=1}^{d+1} \frac{\partial}{\partial t} k_{t,\alpha\gamma} * f_\gamma \right\|_2^2 dt \\ &= \int_0^\infty \sum_{\alpha=1}^{d+1} t \left\| \sum_{\gamma=1}^{d+1} \frac{\partial}{\partial t} F[k_{t,\alpha\gamma}] \cdot F[f_\gamma] \right\|_2^2 dt \\ &\lesssim \sum_{\alpha,\gamma=1}^{d+1} \int_{\mathbb{R}^d} |F[f_\gamma](\xi)|^2 \int_0^\infty t \left| \frac{\partial}{\partial t} F[k_{t,\alpha\gamma}](\xi) \right|^2 dt d\xi. \end{aligned}$$

For the first estimate in (4.4), it thus suffices to show that

$$\int_0^\infty t \left| \frac{\partial}{\partial t} F[k_{t,\alpha\gamma}](\xi) \right|^2 dt \lesssim 1 \quad (4.5)$$

independently of $\xi \in \mathbb{R}^d$. We decompose $k_{t,\alpha\gamma} = k_{t,\alpha\gamma}^1 + k_{t,\alpha\gamma}^2$ into the two summands according to (4.3). For the first summand, we have $F[k_{t,\alpha\gamma}^1](\xi) = c_1 \delta_{\alpha=\gamma} e^{-t|\xi|}$. Thus,

$$\begin{aligned} \int_0^\infty t \left| \frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^1](\xi) \right|^2 dt &= c_1^2 \delta_{\alpha=\gamma} \int_0^\infty t |\xi|^2 e^{-2t|\xi|} dt \\ &= c_1^2 \delta_{\alpha=\gamma} \int_0^\infty t e^{-2t} dt < \infty. \end{aligned}$$

Next we claim that

$$F \left[x \mapsto \frac{t}{(|x|^2 + t^2)^{\frac{d+3}{2}}} \right] (\xi) = c_d \frac{|\xi|}{t} e^{-t|\xi|} \left(1 + \frac{1}{t|\xi|} \right), \quad (4.6)$$

from which we shall easily deduce $F[k_{t,\alpha\gamma}^2](\xi)$. First note that the function which we Fourier transform in (4.6) is invariant under rotation, so also its Fourier transform is. Thus we can assume that $\xi = (\xi_1, 0, \dots, 0)$ with $\xi_1 \geq 0$ and we get (l.h.s. of (4.6)) $= c_1 \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \frac{t}{(x_1^2 + x_2^2 + \dots + x_d^2 + t^2)^{\frac{d+3}{2}}} e^{-ix_1 \xi_1} dx_2 \dots dx_d dx_1$.

Note that $\int_{\mathbb{R}} \frac{t}{(x^2 + s^2)^a} dx = c_a \frac{t}{(s^2)^{a-\frac{1}{2}}}$ for any $a > \frac{1}{2}$, so that by induction, we get by [17, p. 202], (l.h.s. of (4.6)) $= c'_d \int_{\mathbb{R}} \frac{t}{(x_1^2 + t^2)^2} e^{-ix_1 \xi_1} dx_1 = c'_d 2t \sqrt{\pi} \Gamma(2)^{-1} \left(\frac{\xi_1}{2t}\right)^{\frac{3}{2}} K_{\frac{3}{2}}(t\xi_1)$, where $K_{\frac{3}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} (1 + \frac{1}{x})$ is a modified Bessel function. This immediately gives (4.6). Now we calculate $\frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^2](\xi)$ and distinguish the five cases $\alpha = \gamma = d + 1$; $\alpha \leq d$ and $\gamma = d + 1$; $\alpha = d + 1$ and $\gamma \leq d$; $\alpha, \gamma \leq d$ and $\alpha \neq \gamma$; and finally $\alpha = \gamma \leq d$.

If $\alpha = \gamma = d + 1$, then $\frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^2](\xi) = c_d \frac{\partial}{\partial t} \left(t |\xi| e^{-t|\xi|} (1 + \frac{1}{t|\xi|}) \right) = c_d e^{-t|\xi|} (-t|\xi|^2)$. Thus, $\int_0^\infty t \left| \frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^2](\xi) \right|^2 dt = c_d \int_0^\infty t^2 |\xi|^2 e^{-2t|\xi|} (-t|\xi|)^2 \frac{dt}{t} < \infty$ independently of $|\xi|$. If $\alpha \leq d$ and $\gamma = d + 1$, then $\frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^2](\xi) = c_d i \frac{\partial}{\partial t} \frac{\partial}{\partial \xi_\alpha} \left(|\xi| e^{-t|\xi|} (1 + \frac{1}{t|\xi|}) \right) = c_d i e^{-t|\xi|} (t \xi_\alpha |\xi| - \xi_\alpha)$. Thus, $\int_0^\infty t \left| \frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^2](\xi) \right|^2 dt = c_d \int_0^\infty e^{-t} (t^2 \frac{\xi_\alpha}{|\xi|} - t \frac{\xi_\alpha}{|\xi|})^2 \frac{dt}{t} < \infty$ independently of ξ . If $\alpha = d + 1$ and $\gamma \leq d$, one obtains the same as above, roles of α and γ interchanged. If $\alpha, \gamma \leq d$ and $\alpha \neq \gamma$, we have $\frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^2](\xi) = -c_d \frac{\partial}{\partial t} \frac{\partial}{\partial \xi_\alpha} \frac{\partial}{\partial \xi_\gamma} (e^{-t|\xi|} (\frac{|\xi|}{t} + \frac{1}{t^2})) = -c_d e^{-t|\xi|} (-t \xi_\alpha \xi_\gamma + \frac{\xi_\alpha \xi_\gamma}{|\xi|})$. Thus, $\int_0^\infty t \left| \frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^2](\xi) \right|^2 dt = c_d \int_0^\infty e^{-2t} (-t^2 \frac{\xi_\alpha \xi_\gamma}{|\xi|^2} + t \frac{\xi_\alpha \xi_\gamma}{|\xi|^2})^2 \frac{dt}{t} < \infty$ independently of ξ . If finally $\alpha = \gamma \leq d$, then $\frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^2](\xi) = -c_d e^{-t|\xi|} (-t \xi_\alpha^2 + |\xi| + \frac{\xi_\alpha^2}{\xi})$, and again $\int_0^\infty t \left| \frac{\partial}{\partial t} F[k_{t,\alpha\gamma}^2](\xi) \right|^2 dt < \infty$ independently of ξ .

We have shown the first inequality in (4.4). The second one follows by the same proof, there are only signs without further impact which change. This shows that A has an H^∞ calculus. \square

Corollary 4.2. Let A be the negative generator of the Lamé semigroup given in (4.2) on $L^p(\mathbb{R}^d; \mathbb{C}^{d+1})$ for some $1 < p < \infty$. Then A has an \mathcal{H}^α calculus for any $\alpha > \frac{d}{2} + 1$. Moreover, $\{f(A) : \|f\|_{\mathcal{H}^\alpha} \leq 1\}$ is R -bounded on $L^p(\mathbb{R}^d; \mathbb{C}^{d+1})$ for these α .

Proof. The corollary follows immediately from Propositions 2.1 and 4.1 once we show that we can apply Proposition 3.1 and that $\{T(e^{i\theta} 2^k t_0) : k \in \mathbb{Z}\}$ is R -bounded with

$$R(\{T(e^{i\theta} 2^k t_0) : k \in \mathbb{Z}\}) \lesssim (\cos(\theta))^{-\frac{d-1}{2}-1} (1 + |\log(\cos(\theta))|)^2. \quad (4.7)$$

For the application of Proposition 3.1, note that a careful inspection of the proof of [6, Theorem 3.1] shows that a version in $L^p(\mathbb{R}^d; \mathbb{C}^{d+1})$ also holds, more precisely that: if T_z is an analytic semigroup on $L^2(\mathbb{R}^d; \mathbb{C}^{d+1})$ for $\operatorname{Re} z > 0$ such that its negative generator A has a bounded H^∞ calculus on $L^2(\mathbb{R}^d; \mathbb{C}^{d+1})$ and each of the $(d+1) \times (d+1)$ components of its matrix valued integral kernel $k_{z,\alpha\gamma}(x, y)$ satisfies (3.1), then A has a

bounded H^∞ calculus on $L^p(\mathbb{R}^d; \mathbb{C}^{d+1})$ for $1 < p < \infty$. Note that we have shown the H^∞ calculus on $L^2(\mathbb{R}^d; \mathbb{C}^{d+1})$ and the bounds (3.1) for $k_{z, \alpha\gamma}(x, y)$ in Proposition 4.1. It remains to show (4.7), which would follow immediately from Theorem 3.2, weren't it for the vector valued character. We give the adaption details now. First note that $\left(\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k f_k \right\|_{L^p(\mathbb{R}^d; \mathbb{C}^{d+1})}^2\right)^{\frac{1}{2}} \cong \sum_{\alpha=1}^{d+1} \left\| \left(\sum_{k=1}^n |f_{\alpha, k}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \cong \left\| \left(\sum_{\alpha=1}^{d+1} \sum_{k=1}^n |f_{\alpha, k}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)}$. Thus, $\{T(e^{i\theta} 2^k t_0) : k \in \mathbb{Z}\}$ is R -bounded on $L^p(\mathbb{R}^d; \mathbb{C}^{d+1})$ iff

$$\left\| \left(\sum_{\alpha=1}^{d+1} \sum_k |T(e^{i\theta} 2^k t_0 A) f_k)_\alpha|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \left\| \left(\sum_{\alpha=1}^{d+1} \sum_k |f_{\alpha, k}|^2 \right)^{\frac{1}{2}} \right\|_p,$$

iff $T \in B(L^p(\mathbb{R}^d; \ell_{\{1, \dots, d+1\} \times \mathbb{N}}^2))$, where $T(f_{\alpha, j})_{\alpha, j} = (T(e^{i\theta} 2^j t_0) f_j)_\alpha$. In view of [16, Definition 2.1], choose now the approximation of identity $A_t(f_{\alpha, j})_{\alpha, j} = (T_t f_j)_\alpha$. Then $T A_t(f_{\alpha, j}) = T(T_t f_j) = (T(e^{i\theta} 2^j t_0) T_t f_j) = A_t T(f_{\alpha, j})$. Further, the kernel of A_t , called a_t satisfies

$$A_t(f_{\alpha, j}) = \int_{\mathbb{R}^d} \sum_{\gamma=1}^{d+1} k_{t, \alpha\gamma}(x-y) f_{\gamma, j}(y) dy$$

and

$$\begin{aligned} \|a_t(x-y)\|_{B(\ell_{\{1, \dots, d+1\} \times \mathbb{N}}^2)} &= \sup_{w_{\alpha, j} : \sum_{\alpha, j} |w_{\alpha, j}|^2 \leq 1} \left(\sum_{\alpha, j} \left| \sum_{\gamma=1}^{d+1} k_{t, \alpha\gamma}(x-y) w_{\gamma, j} \right|^2 \right)^{\frac{1}{2}} \\ &\lesssim \sup_{\alpha, \gamma=1}^{d+1} |k_{t, \alpha\gamma}(x-y)| \lesssim \frac{1}{\mu(B(x, t))} \frac{1}{\left| 1 + \frac{|x-y|^2}{t^2} \right|^{\frac{d+1}{2}}}, \end{aligned}$$

so that A_t is indeed an approximation of identity in the sense of [16, Definition 1.1]. If $k_T(x, y)$ and $k_{T A_t}(x, y)$ denote the $B(\ell_{\{1, \dots, d+1\} \times \mathbb{N}}^2)$ valued kernels of T and $T A_t$, then we have in view of the application of [16, Theorem 2.3],

$$\begin{aligned} \int_{|x-y| \geq 3t} \|k_T(x, y) - k_{T A_t}(x, y)\|_{B(\ell_{\{1, \dots, d+1\} \times \mathbb{N}}^2)} &\lesssim \\ \sup_{\alpha, \gamma=1}^{d+1} \int_{|x-y| \geq 3t} \sup_{j \in \mathbb{Z}} |k_{e^{i\theta} 2^j t_0, \alpha\gamma}(x-y) - k_{e^{i\theta} 2^j t_0 + t, \alpha\gamma}(x-y)| d\mu(x). \end{aligned}$$

Apply now the Poisson estimate and analyticity of $k_{z, \alpha\gamma}(x-y)$ in z , for fixed α and γ , exactly as in the proof of Theorem 3.2, to deduce (4.7) as wanted. \square

Remark 4.3. Let us compare Corollary 4.2 with known Hörmander functional calculi for ‘‘Lamé operators’’ in the literature. Let L denote the operator as in (4.1), i.e. $Lu = \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u$, where $u : \Omega \rightarrow \mathbb{R}^{d+1}$ and $\Omega \subset \mathbb{R}^d$ is an open subset satisfying the interior ball condition, i.e. there exists a positive constant c such that for all $x \in \Omega$ and all $r \in (0, \frac{1}{2} \operatorname{diam}(\Omega))$, $\mu(B(x, r)) \geq cr^d$.

Let further A denote the negative generator of the semigroup in (4.2), so that if $\Omega = \mathbb{R}^d$, for functions $u : \mathbb{R}_+^{d+1} \rightarrow \mathbb{C}^{d+1}$ and $f \in L^p(\mathbb{R}^d; \mathbb{C}^{d+1})$, we have $Lu = 0$, $u|_{\partial\mathbb{R}_+^{d+1}} = f$ if and only if $u(x, t) = e^{-tA} f(x)$. If Ω is bounded, the operator $-L$ on $L^p(\Omega; \mathbb{C}^{d+1})$ for $p \in (q'_\Omega, q_\Omega)$ where $q_\Omega > 2$ is some constant, has a \mathcal{H}^α calculus for $\alpha > d|\frac{1}{p} - \frac{1}{2}| + \frac{1}{2}$ [13, Theorem 5.1]. In contrast, for A we get a Hörmander functional calculus on the full range $p \in (1, \infty)$, but with a worse derivation exponent $\alpha > \frac{d}{2} + 1$. Note that A is *not* self-adjoint on $L^2(\mathbb{R}^d; \mathbb{C}^{d+1})$, in contrast to L on $L^2(\Omega; \mathbb{C}^{d+1})$, so that A and L are of a quite different nature.

Dirichlet-to-Neumann operator. Another application of Theorem 3.2 and Corollary 3.6 is the following operator from [19]. There the authors suppose that Ω is the smooth boundary of an open connected subset $\tilde{\Omega}$ of \mathbb{R}^{d+1} and A is the Dirichlet-to-Neumann operator defined as follows: Given $\phi \in L^2(\Omega)$ solve the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \text{ weakly on } \tilde{\Omega} \\ u|_\Omega &= \phi \end{aligned}$$

with $u \in W_2^1(\tilde{\Omega})$. If u has a weak normal derivative $\frac{\partial u}{\partial \nu}$ in $L^2(\Omega)$, then $\phi \in D(A)$ and $A\phi = \frac{\partial u}{\partial \nu}$. This operator is a pseudodifferential operator, self-adjoint on $L^2(\Omega)$. Then in [19] it is shown that the semigroup satisfies the complex Poisson estimate:

$$\begin{aligned} |k_z(x, y)| &\leq C(\cos(\theta))^{-2(d-1)d} \frac{\min(|z|, 1)^{-d}}{\left(1 + \frac{|x-y|}{|z|}\right)^{d+1}} \\ &\lesssim (\cos(\theta))^{-2(d-1)d} \frac{1}{\mu(B(x, |z|))} \frac{1}{\left|1 + \frac{|x-y|^2}{z^2}\right|^{\frac{d+1}{2}}} \end{aligned}$$

for all $x, y \in \Omega$ and $\operatorname{Re} z > 0$, where $\theta = \arg z$. Further a \mathcal{H}^α calculus for A with $\alpha > \frac{d}{2}$ is derived in [19, Section 7]. Suppose that Ω satisfies (2.2) and (2.3). Then since A is self-adjoint on $L^2(\Omega)$, we can apply Corollary 3.6 to deduce that A has an R -bounded \mathcal{H}^α calculus for $\alpha > \frac{d}{2} + 2(d-1)d$. Note that our derivation order in this functional calculus is worse than the one obtained in [19, Section 7], but since it is R -bounded, it contains square function estimates like

$$\left\| \left(\sum_{j=1}^n |g_j(A)f_j|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \max_{j=1}^n \|g_j\|_{\mathcal{H}^\alpha} \left\| \left(\sum_{j=1}^n |f_j|^2 \right)^{\frac{1}{2}} \right\|_p.$$

Pseudodifferential operators on compact manifolds without boundary. Let Ω be a compact closed (i.e. without boundary) d -dimensional Riemannian C^∞ -manifold and A a classical, self-adjoint, strongly elliptic pseudodifferential operator on Ω of order 1 such that $\gamma(A) = \inf\{\operatorname{Re} z : z \in \sigma(A)\} \geq 0$.

Then according to [8, Theorem 3.14], the semigroup generated by $-A$ has an integral kernel satisfying

$$|k_z(x, y)| \lesssim (\cos(\theta))^{-\beta} e^{-\gamma(A) \operatorname{Re} z} \frac{|z|}{\rho(x, y) + |z|} ((\rho(x, y) + |z|)^{-d} + 1) \quad (\operatorname{Re} z > 0)$$

with $\theta = \arg z$ and $\beta = \frac{7}{2}d + 11$. If Ω satisfies (2.2), then this estimate readily gives (3.1). Thus if Ω also satisfies (2.3), we can appeal to Corollary 3.6 and deduce that A has an R -bounded \mathcal{H}^α calculus for $\alpha > \frac{d}{2} + \frac{7}{2}d + 11$ on $L^p(\Omega)$, $1 < p < \infty$. The order of this calculus is worse than what is known in the literature for this kind of operator ($\alpha > \frac{d}{2}$, see [20]), but at least, our result includes the R -boundedness of the calculus.

5. Proofs of Lemmas 3.3 and 3.4

Proof of Lemma 3.3. Since $k_z(x, y)$ is analytic in z , one has

$$\begin{aligned} |k_{e^{i\theta}t_0+t}(x, y) - k_{e^{i\theta}t_0}(x, y)| &\leq \int_0^t \left| \frac{\partial}{\partial s} k_{e^{i\theta}t_0+s}(x, y) \right| ds \\ &\leq \int_0^t \frac{1}{2\pi} \int_{\Gamma_{e^{i\theta}t_0+s}} \left| \frac{k_z(x, y)}{(z - e^{i\theta}t_0 - s)^2} \right| dz ds, \end{aligned}$$

where $\Gamma_{e^{i\theta}t_0+s}$ is the contour of the circle centered at $e^{i\theta}t_0 + s$ with radius $r(\cos(\theta)t_0 + s)$, where $r > 0$ is some small constant determined later on in this proof. The rest of the proof is the rather long task to exploit (3.1) in the above double integral. Let $z \in \Gamma_{e^{i\theta}t_0+s}$. Then $\cos(\arg z) \cong \min(|\frac{\operatorname{Re} z}{\operatorname{Im} z}|, 1) \cong \min(\frac{\cos(\theta)t_0+s}{\sin(\theta)t_0+s}, 1) \cong \cos(\arg(e^{i\theta}t_0 + s)) \geq \cos(\theta)$. Also, $|z| \cong |e^{i\theta}t_0 + s|$. We divide the integral over s in

$$\int_0^t \dots ds = \int_0^{\min(\cos(\theta)t_0, t)} \dots ds + \int_{\min(\cos(\theta)t_0, t)}^t \dots ds.$$

1st case: $\cos(\theta)t_0 \geq s$.

We show that

$$\left| 1 + \frac{\rho(x, y)^2}{[e^{i\theta}t_0 + s + r e^{i\phi}(\cos(\theta)t_0 + s)]^2} \right| \cong \left| 1 + \frac{\rho(x, y)^2}{(e^{i\theta}t_0)^2} \right|. \quad (5.1)$$

First note that (5.1) is equivalent to $|w_l| \cong |w_r|$, where $w_l = [e^{i\theta}t_0 + s + r e^{i\phi}(\cos(\theta)t_0 + s)]^2 + \rho(x, y)^2$ and $w_r = (e^{i\theta}t_0)^2 + \rho(x, y)^2$. We have $\operatorname{Re} w_r = \cos(2\theta)t_0^2 + \rho(x, y)^2 = -t_0^2 + \rho(x, y)^2 + o(\frac{\pi}{2} - |\theta|)t_0^2$ and $|\operatorname{Im} w_r| \cong \cos(\theta)t_0^2$. This gives $|w_r| \cong |\operatorname{Re} w_r| + |\operatorname{Im} w_r| \cong |\rho(x, y)^2 - t_0^2| + \cos(\theta)t_0^2$. On the other hand, $|\operatorname{Im} w_l| = 2(\cos(\theta)t_0 + s)(1 + r \cos(\phi))|\sin(\theta)t_0 + r \sin(\phi)(\cos(\theta)t_0 + s)| \cong \cos(\theta)t_0^2$ and $\operatorname{Re} w_l = \rho(x, y)^2 - t_0^2 + t_0^2(1 - \sin^2(\theta)) - r^2 \sin^2(\phi)(\cos(\theta)t_0 + s)^2 - 2r \sin(\theta)t_0 \sin(\phi)(\cos(\theta)t_0 + s) + (\cos(\theta)t_0 + s)^2(1 + r \cos(\phi))^2$. This gives

$$\begin{aligned} |\operatorname{Re} w_l| &\geq |\rho(x, y)^2 - t_0^2| - \left[\left(\frac{\pi}{2} - |\theta| \right)^2 + o\left(\left(\frac{\pi}{2} - |\theta| \right)^2 \right) \right] t_0^2 \\ &\quad - 4r^2 t_0^2 \left(\frac{\pi}{2} - |\theta| \right)^2 (1 + o(1)) - 2r \cdot 2 \cdot \left(\frac{\pi}{2} - |\theta| \right) t_0^2 \end{aligned}$$

$$\geq |\rho(x, y)^2 - t_0^2| - c_1 \left(\frac{\pi}{2} - |\theta|\right) t_0^2,$$

where $c_1 \ll 1$ is some small constant if r is sufficiently small and $|\theta|$ is sufficiently close to $\frac{\pi}{2}$. Similarly, $|\operatorname{Re} w_l| \leq |\rho(x, y)^2 - t_0^2| + c_1 \left(\frac{\pi}{2} - |\theta|\right) t_0^2$. Thus, $|w_l| \cong |\operatorname{Re} w_l| + |\operatorname{Im} w_l| \cong |\rho(x, y)^2 - t_0^2| + \cos(\theta) t_0^2 \cong |w_r|$, and (5.1) follows. Now we can estimate

$$\begin{aligned} & \int_0^{\min(\cos(\theta)t_0, t)} \left| \frac{\partial}{\partial s} k_{e^{i\theta}t_0+s}(x, y) \right| ds \lesssim \\ & \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \int_0^{\min(\cos(\theta)t_0, t)} \frac{1}{\cos(\theta)t_0} \{|\rho(x, y)^2 - t_0^2| + \cos(\theta)t_0^2\}^{-\frac{d+1}{2}} t_0^{d+1} ds \\ & = \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \frac{\min(\cos(\theta)t_0, t)}{\cos(\theta)t_0} \left\{ \left| 1 - \frac{\rho(x, y)^2}{t_0^2} \right| + \cos(\theta) \right\}^{-\frac{d+1}{2}}. \end{aligned} \quad (5.2)$$

2nd case: $\cos(\theta)t_0 \leq s$.

We show that

$$\left| 1 + \frac{\rho(x, y)^2}{[e^{i\theta}t_0 + s + r e^{i\phi}(\cos(\theta)t_0 + s)]^2} \right| \gtrsim \left| 1 + \frac{\rho(x, y)^2}{(s + it_0)^2} \right|. \quad (5.3)$$

Put $w_l = [e^{i\theta}t_0 + s + r e^{i\phi}(\cos(\theta)t_0 + s)]^2 + \rho(x, y)^2$ and $w_r = [s + it_0]^2 + \rho(x, y)^2$, so that (5.3) $\iff |w_l| \gtrsim |w_r|$. We have $\operatorname{Re} w_r = s^2 - t_0^2 + \rho(x, y)^2$ and $\operatorname{Im} w_r = 2st_0$. On the other hand, $\operatorname{Re} w_l = [\cos(\theta)t_0 + s + r \cos(\phi)(\cos(\theta)t_0 + s)]^2 - [\sin(\theta)t_0 + r \sin(\phi)(\cos(\theta)t_0 + s)]^2 + \rho(x, y)^2$, and

$$\begin{aligned} |\operatorname{Im} w_l| &= 2 [|\cos(\theta)t_0 + s + r \cos(\phi)(\cos(\theta)t_0 + s)| \times \\ & \quad \times |\sin(\theta)t_0 + r \sin(\phi)(\cos(\theta)t_0 + s)|] \cong st_0, \end{aligned}$$

if $r < \frac{1}{4}$. This gives

$$\begin{aligned} \operatorname{Re} w_l &= \cos^2(\theta)t_0^2 + s^2 + r^2 \cos^2(\phi)(\cos(\theta)t_0 + s)^2 + 2s \cos(\theta)t_0 \\ & \quad + 2 \cdot r(\cos(\theta)t_0 + s) \cos(\phi)(\cos(\theta)t_0 + s) - \sin^2(\theta)t_0^2 \\ & \quad - r^2 \sin^2(\phi)(\cos(\theta)t_0 + s) - 2 \cdot r \sin(\theta)t_0 \sin(\phi)(\cos(\theta)t_0 + s) + \rho(x, y)^2. \end{aligned}$$

Now we distinguish the two cases that st_0 is bigger or smaller than $|s^2 - t_0^2 + \rho(x, y)^2|$. In the first case, we have $|w_l| \geq |\operatorname{Im} w_l| \cong st_0 \gtrsim |\operatorname{Re} w_r| + |\operatorname{Im} w_r| \cong |w_r|$. In the second case, we have with constants $c_1 < 1$ arbitrarily close to 1 and $c_2 > 0$ arbitrarily close to 0 if r is sufficiently small, and $c_3 > 0$ such that $(\cos(\theta))^{-1} \leq c_3 \left(\frac{\pi}{2} - |\theta|\right)^{-1}$

$$\begin{aligned} |\operatorname{Re} w_l| &\geq |c_1 s^2 + \cos(2\theta)t_0^2 + \rho(x, y)^2| - c_2 t_0 s \\ &\geq |s^2 - t_0^2 + \rho(x, y)^2| - (1 - c_1)s^2 - (1 + \cos(2\theta))t_0^2 - c_2 t_0 s \\ &\geq |s^2 - t_0^2 + \rho(x, y)^2| - (1 - c_1)s^2 - (2\left(\frac{\pi}{2} - |\theta|\right)^2 + o\left(\left(\frac{\pi}{2} - |\theta|\right)^3\right)) \times \\ & \quad \times (\cos(\theta))^{-1} c_3 st_0 - c_2 t_0 s \\ &\geq |s^2 - t_0^2 + \rho(x, y)^2| - st_0 [(1 - c_1) + 2c_3 \left(\frac{\pi}{2} - |\theta|\right) + o\left(\left(\frac{\pi}{2} - |\theta|\right)^2\right) + c_2] \end{aligned}$$

$$\gtrsim |s^2 - t_0^2 + \rho(x, y)^2|.$$

Thus, $|w_l| \geq |\operatorname{Re} w_l| \gtrsim |s^2 - t_0^2 + \rho(x, y)^2| + st_0 \gtrsim |\operatorname{Re} w_r| + |\operatorname{Im} w_r| \cong |w_r|$. This shows (5.3).

It follows that if $\cos(\theta)t_0 \leq t$, then

$$\int_{\cos(\theta)t_0}^t \left| \frac{\partial}{\partial s} k_{e^{i\theta}t_0+s}(x, y) \right| ds \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \int_{\cos(\theta)t_0}^t \frac{1}{s} \left| 1 + \frac{\rho(x, y)^2}{(s + it_0)^2} \right|^{-\frac{d+1}{2}} ds. \quad (5.4)$$

We have

$$\begin{aligned} \left| 1 + \frac{\rho(x, y)^2}{(s + it_0)^2} \right|^{-\frac{d+1}{2}} &\cong |s + it_0|^{d+1} |(s + it_0)^2 + \rho(x, y)^2|^{-\frac{d+1}{2}} \\ &\cong t_0^{d+1} [\max(|s^2 - t_0^2 + \rho(x, y)^2|, 2st_0)]^{-\frac{d+1}{2}}. \end{aligned}$$

Next we determine the value of the above maximum of two terms. A simple calculation shows that the two terms are the same iff s takes one of the four values $s_{\pm, \pm} = \pm t_0 \pm \sqrt{2t_0^2 - \rho(x, y)^2}$. We distinguish three cases i) - iii).

Case i) $t_0^2 \geq \rho(x, y)^2$.

Then out of $s_{\pm, \pm}$, only $s_{-, +} = -t_0 + \sqrt{2t_0^2 - \rho(x, y)^2}$ lies in $[0, t_0]$. If $s \leq s_{-, +}$, then $|s^2 - t_0^2 + \rho(x, y)^2| \geq 2st_0$ and if $s \geq s_{-, +}$, then $|s^2 - t_0^2 + \rho(x, y)^2| \leq 2st_0$. Now divide the integral in (5.4) accordingly. We get

$$\begin{aligned} &\int_{\cos(\theta)t_0}^t \left| \frac{\partial}{\partial s} k_{e^{i\theta}t_0+s}(x, y) \right| ds \lesssim \\ &\frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \int_{\cos(\theta)t_0}^{t_0(\sqrt{2-\rho(x, y)^2/t_0^2}-1)} \frac{1}{s} t_0^{d+1} |s^2 - t_0^2 + \rho(x, y)^2|^{-\frac{d+1}{2}} ds \\ &+ \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \int_{t_0(\sqrt{2-\rho(x, y)^2/t_0^2}-1)}^t \frac{1}{s} t_0^{d+1} (st_0)^{-\frac{d+1}{2}} ds \\ &\cong \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \left\{ \int_{\cos(\theta)t_0}^{t_0(\sqrt{2-\rho(x, y)^2/t_0^2}-1)} \frac{1}{s} [(\sqrt{t_0^2 - \rho(x, y)^2} - s) \times \right. \\ &\left. \times (\sqrt{t_0^2 - \rho(x, y)^2} + s)]^{-\frac{d+1}{2}} t_0^{d+1} ds + \left(\sqrt{2 - \frac{\rho(x, y)^2}{t_0^2}} - 1 \right)^{-\frac{d+1}{2}} - (t_0/t)^{\frac{d+1}{2}} \right\}. \quad (5.5) \end{aligned}$$

The last integral above, with lower and upper bound abbreviated by a and b , can be further estimated by

$$\int_a^b \frac{1}{s} t_0^{d+1} (t_0^2 - \rho(x, y)^2)^{-\frac{d+1}{4}} \left[\frac{1}{(\sqrt{t_0^2 - \rho(x, y)^2} - s)^{\frac{d+1}{2}}} + \frac{1}{(\sqrt{t_0^2 - \rho(x, y)^2} + s)^{\frac{d+1}{2}}} \right] ds$$

$$\lesssim \int_{a/t_0}^{b/t_0} \left(1 - \frac{\rho(x, y)^2}{t_0^2}\right)^{-\frac{d+1}{4}} \frac{1}{\left(\sqrt{1 - \frac{\rho(x, y)^2}{t_0^2}} - s\right)^{\frac{d+1}{2}} s} ds.$$

Case ia) $\frac{1}{\sqrt{2 - \rho(x, y)^2/t_0^2} - 1} \geq 2t_0/t$.

Then the term after the integral in (5.5) can be simplified to

$$\left(\sqrt{2 - \frac{\rho(x, y)^2}{t_0^2}} - 1\right)^{-\frac{d+1}{2}} \cong \left(1 - \frac{\rho(x, y)^2}{t_0^2}\right)^{-\frac{d+1}{2}} \cong \left(1 - \frac{\rho(x, y)}{t_0}\right)^{-\frac{d+1}{2}}.$$

Case ib) $\frac{1}{\sqrt{2 - \rho(x, y)^2/t_0^2} - 1} \leq 2t_0/t$.

Then we obtain

$$\begin{aligned} & \int_{\cos(\theta)t_0}^t \left| \frac{\partial}{\partial s} k_{e^{i\theta}t_0+s}(x, y) \right| ds \lesssim \\ & \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \left\{ \left(1 - \frac{\rho(x, y)^2}{t_0^2}\right)^{-\frac{d+1}{4}} \left[\left(\sqrt{1 - \frac{\rho(x, y)^2}{t_0^2}}\right)^{-\frac{d+1}{2}} \times \right. \right. \\ & \times \log \left(\frac{\frac{1}{2}[\sqrt{2 - \rho(x, y)^2/t_0^2} - 1 + \cos(\theta)]}{\cos(\theta)} \right) + [\sqrt{2 - \rho(x, y)^2/t_0^2} - 1 + \cos(\theta)]^{-1} \times \\ & \left. \left[\left(\sqrt{1 - \frac{\rho(x, y)^2}{t_0^2}} - \sqrt{2 - \frac{\rho(x, y)^2}{t_0^2}} + 1\right)^{-\frac{d-1}{2}} \right. \right. \\ & \left. \left. - \left(\sqrt{1 - \frac{\rho(x, y)^2}{t_0^2}} - \frac{1}{2}\sqrt{2 - \frac{\rho(x, y)^2}{t_0^2}} + \frac{1}{2} - \frac{\cos(\theta)}{2}\right)^{-\frac{d-1}{2}} \right] \right\} \\ & + \left(\frac{t}{t_0} - \sqrt{2 - \frac{\rho(x, y)^2}{t_0^2}} + 1 \right) \left(1 - \frac{\rho(x, y)^2}{t_0^2}\right)^{-\frac{d+3}{2}} \}. \end{aligned} \quad (5.6)$$

Hereby, the last summand only exists for $\sqrt{2 - \rho(x, y)^2/t_0^2} - 1 \leq t/t_0$ and if $\rho(x, y)/t_0$ is so close to 1 that $\sqrt{2 - \rho(x, y)^2/t_0^2} - 1 \leq \cos(\theta)$ then $\sqrt{2 - \rho(x, y)^2/t_0^2} - 1$ in the last summand has to be replaced by $\cos(\theta)$. Furthermore, everything before the last summand has to be replaced by 0 if $\cos(\theta) \geq \sqrt{2 - \rho(x, y)^2/t_0^2} - 1$.

Case ii) $t_0^2 \leq \rho(x, y)^2 \leq 2t_0^2$.

Then only $s_{+,-}$ lies in $[0, t_0]$. If $s \leq s_{+,-}$, then $|s^2 - t_0^2 + \rho(x, y)^2| \geq 2st_0$ and if $s \geq s_{+,-}$, then $|s^2 - t_0^2 + \rho(x, y)^2| \leq 2st_0$. Then we obtain

$$\int_{\cos(\theta)t_0}^t \left| \frac{\partial}{\partial s} k_{e^{i\theta}t_0+s}(x, y) \right| ds \lesssim$$

$$\begin{aligned}
& \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \left[\int_{\cos(\theta)t_0}^{t_0(1-\sqrt{2-\rho(x,y)^2/t_0^2})} [s^2 - t_0^2 + \rho(x, y)^2]^{-\frac{d+1}{2}} t_0^{d+1} \frac{ds}{s} \right. \\
& \left. + \int_{t_0(1-\sqrt{2-\rho(x,y)^2/t_0^2})}^t t_0^{d+1} (st_0)^{-\frac{d+1}{2}} \frac{ds}{s} \right] \\
& \cong \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \left\{ \left[\log \frac{1 - \sqrt{2 - \rho(x, y)^2/t_0^2}}{\cos(\theta)} \right]_+ \left(\frac{\rho(x, y)^2}{t_0^2} - 1 \right)^{-\frac{d+1}{2}} \right. \\
& \left. + \left[\left(1 - \sqrt{2 - \rho(x, y)^2/t_0^2} \right)^{-\frac{d+1}{2}} - (t/t_0)^{-\frac{d+1}{2}} \right]_+ \right\}, \tag{5.7}
\end{aligned}$$

where $[x]_+ = \max(x, 0)$. Note that in the last expression above, in the first summand, $1 - \sqrt{2 - \rho(x, y)^2/t_0^2}$ has to be replaced by t/t_0 if $t/t_0 \leq 1 - \sqrt{2 - \rho(x, y)^2/t_0^2}$. In the second summand, $1 - \sqrt{2 - \rho(x, y)^2/t_0^2}$ has to be replaced by $\cos(\theta)$ if $1 - \sqrt{2 - \rho(x, y)^2/t_0^2} \leq \cos(\theta)$.

Case iii) $\rho(x, y)^2 \geq 2t_0^2$.

Then none of $s_{\pm, \pm}$ is real. We always have $|s^2 - t_0^2 + \rho(x, y)^2| \geq 2st_0$. Then we obtain

$$\begin{aligned}
& \int_{\cos(\theta)t_0}^t \left| \frac{\partial}{\partial s} k_{e^{i\theta}t_0+s}(x, y) \right| ds \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \int_{\cos(\theta)t_0}^t t_0^{d+1} |s^2 - t_0^2 + \rho(x, y)^2|^{-\frac{d+1}{2}} \frac{ds}{s} \\
& \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, t_0))} \left[\log \left(\frac{t}{t_0 \cos(\theta)} \right) \right]_+ \left(\frac{\rho(x, y)^2}{t_0^2} - 1 \right)^{-\frac{d+1}{2}}. \tag{5.8}
\end{aligned}$$

Summarizing (5.2), (5.5), (5.6), (5.7), (5.8), we finally obtain the claimed estimate of the lemma. \square

Proof of Lemma 3.4. We start with estimating $K := |k_{e^{i\theta}2^j t_0+t}(x, y) - k_{e^{i\theta}2^j t_0}(x, y)|$ for a fixed $j \in \mathbb{Z}$ such that $2^j t_0 \geq t$, hereby using Lemma 3.3. Write in short $R = \rho(x, y)^2/(2^j t_0)^2$. We distinguish the following fifteen cases Iai), Iaii), Iaiii), Iaiiv), Iav), Ibi), Ibi), Ic), Idi), Idii), Idiii), Ie), IIa), IIb+c), IId+e), which depend on the values of $\theta, 2^j t_0, t$ and $\rho(x, y)$. Here, case I stands for $\cos(\theta)2^j t_0 \leq t$, case II stands for $\cos(\theta)2^j t_0 \geq t$, case a stands for $R \leq 1 - \cos(\theta)$, b for $1 - \cos(\theta) \leq R \leq 1$, c for $1 \leq R \leq 1 + \cos(\theta)$, d for $1 + \cos(\theta) \leq R \leq 2$ and e for $2 \leq R$.

Case Iai) $\cos(\theta)2^j t_0 \leq t$, $R \leq 1 - \cos(\theta)$ and $\cos(\theta) \leq \sqrt{2 - R} - 1 \leq \frac{1}{2} \frac{t}{2^j t_0}$.

Then with Lemma 3.3,

$$\begin{aligned}
K & \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \left\{ (1 - R)^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{\frac{1}{2}[\sqrt{2 - R} - 1 + \cos(\theta)]}{\cos(\theta)} \right) \right) \right. \\
& \left. + [\sqrt{2 - R} - 1 + \cos(\theta)]^{-1} [(\sqrt{1 - R} - \sqrt{2 - R} + 1)^{-\frac{d-1}{2}} \right.
\end{aligned}$$

$$- \left(\sqrt{1-R} - \frac{1}{2}\sqrt{2-R} + \frac{1}{2} - \frac{\cos(\theta)}{2} \right)^{-\frac{d-1}{2}} \left. \vphantom{\left(\sqrt{1-R} - \frac{1}{2}\sqrt{2-R} + \frac{1}{2} - \frac{\cos(\theta)}{2} \right)^{-\frac{d-1}{2}}} \right\} (1-R)^{-\frac{d+1}{4}} + (1-R)^{-\frac{d+1}{2}} \left. \vphantom{(1-R)^{-\frac{d+1}{4}}} \right\}.$$

Case Iaii) $\cos(\theta)2^j t_0 \leq t$, $R \leq 1 - \cos(\theta)$ and $\max(\cos(\theta), \frac{1}{2}\frac{t}{2^j t_0}) \leq \sqrt{2-R} - 1 \leq \frac{t}{2^j t_0}$.

$$\begin{aligned} K &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \left\{ (1-R)^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{\frac{1}{2}[\sqrt{2-R} - 1 + \cos(\theta)]}{\cos(\theta)} \right) \right) \right. \\ &\quad + (1-R)^{-\frac{d+1}{4}-1} \left[(\sqrt{1-R} - \sqrt{2-R} + 1)^{-\frac{d-1}{2}} \right. \\ &\quad \left. \left. - \left(\sqrt{1-R} - \frac{1}{2}\sqrt{2-R} + \frac{1}{2} - \frac{\cos(\theta)}{2} \right)^{-\frac{d-1}{2}} \right] \right. \\ &\quad \left. + \left(\frac{t}{2^j t_0} - \sqrt{2-R} + 1 \right) (1-R)^{-\frac{d+3}{2}} \right\} \end{aligned}$$

Case Iaiii) $\cos(\theta)2^j t_0 \leq t$, $R \leq 1 - \cos(\theta)$ and $\max(\cos(\theta), \frac{t}{2^j t_0}) = \frac{t}{2^j t_0} \leq \sqrt{2-R} - 1$.

$$\begin{aligned} K &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \left\{ (1-R)^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{\frac{1}{2}[\sqrt{2-R} - 1 + \cos(\theta)]}{\cos(\theta)} \right) \right) \right. \\ &\quad + (1-R)^{-\frac{d+1}{4}-1} \left[(\sqrt{1-R} - \sqrt{2-R} + 1)^{-\frac{d-1}{2}} \right. \\ &\quad \left. \left. - \left(\sqrt{1-R} - \frac{1}{2}\sqrt{2-R} + \frac{1}{2} - \frac{\cos(\theta)}{2} \right)^{-\frac{d-1}{2}} \right] \right\}. \end{aligned}$$

Case Iaiv) $\cos(\theta)2^j t_0 \leq t$, $R \leq 1 - \cos(\theta)$ and $\frac{1}{2}\frac{t}{2^j t_0} \leq \sqrt{2-R} - 1 \leq \cos(\theta)$.

$$K \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \left\{ (1-R)^{-\frac{d+1}{2}} + \left(\frac{t}{2^j t_0} - \sqrt{2-R} + 1 \right) (1-R)^{-\frac{d+3}{2}} \right\}$$

Case Iav) $\cos(\theta)2^j t_0 \leq t$, $R \leq 1 - \cos(\theta)$ and $\sqrt{2-R} - 1 \leq \min(\cos(\theta), \frac{1}{2}\frac{t}{2^j t_0})$.

$$K \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (1-R)^{-\frac{d+1}{2}}$$

Case Ib) $\cos(\theta)2^j t_0 \leq t$ and $1 - \cos(\theta) \leq R \leq 1$.

In this case, we have $\sqrt{2-R} - 1 \leq \frac{1}{2}\cos(\theta) + o(\cos(\theta))$, so that always $\sqrt{2-R} - 1 \leq \cos(\theta)$ if $|\theta|$ is sufficiently close to $\frac{\pi}{2}$.

Case Ibi) $\cos(\theta)2^j t_0 \leq t$, $1 - \cos(\theta) \leq R \leq 1$ and $\frac{1}{2}\frac{t}{2^j t_0} \leq \sqrt{2-R} - 1$.

$$K \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \left\{ (\cos(\theta))^{-\frac{d+1}{2}} + (\cos(\theta))^{-\frac{d+1}{2}} - \left(\frac{t}{2^j t_0} \right)^{-\frac{d+1}{2}} \right\}$$

$$\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (\cos(\theta))^{-\frac{d+1}{2}}$$

Case Ibii) $\cos(\theta)2^j t_0 \leq t$, $1 - \cos(\theta) \leq R \leq 1$ and $\sqrt{2 - R} - 1 \leq \frac{1}{2} \frac{t}{2^j t_0}$.

$$K \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (\cos(\theta))^{-\frac{d+1}{2}}$$

Case Ic) $\cos(\theta)2^j t_0 \leq t$, $1 \leq R \leq 1 + \cos(\theta)$.

In this case, we have $0 \leq 1 - \sqrt{2 - R} \leq \frac{1}{2} \cos(\theta) + o(\cos(\theta))$. Then

$$\begin{aligned} K &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \left\{ (\cos(\theta))^{-\frac{d+1}{2}} + (\cos(\theta))^{-\frac{d+1}{2}} - \left(\frac{t}{2^j t_0} \right)^{-\frac{d+1}{2}} \right\} \\ &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (\cos(\theta))^{-\frac{d+1}{2}} \end{aligned}$$

Case Id) $\cos(\theta)2^j t_0 \leq t$ and $1 + \cos(\theta) \leq R \leq 2$.

In this case, $-1 \leq 1 - R \leq -\cos(\theta)$, so that $1 - (R - 1) \leq \sqrt{1 + (1 - R)} \leq 1 - \frac{1}{2}(R - 1)$ and thus $\frac{1}{2}(R - 1) \leq 1 - \sqrt{2 - R} \leq R - 1$.

Case Idi) $\cos(\theta)2^j t_0 \leq t$, $1 + \cos(\theta) \leq R \leq 2$ and $\cos(\theta) \leq 1 - \sqrt{2 - R} \leq \frac{t}{2^j t_0}$.

$$\begin{aligned} K &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \left\{ (R - 1)^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{1 - \sqrt{2 - R}}{\cos(\theta)} \right) \right) \right. \\ &\quad \left. + (1 - \sqrt{2 - R})^{-\frac{d+1}{2}} - \left(\frac{t}{2^j t_0} \right)^{-\frac{d+1}{2}} \right\} \\ &\cong \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (R - 1)^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{R - 1}{\cos(\theta)} \right) \right) \end{aligned}$$

Case Idii) $\cos(\theta)2^j t_0 \leq t$, $1 + \cos(\theta) \leq R \leq 2$ and $\max(\cos(\theta), \frac{t}{2^j t_0}) = \frac{t}{2^j t_0} \leq 1 - \sqrt{2 - R}$.

$$K \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (R - 1)^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{t}{2^j t_0 \cos(\theta)} \right) \right)$$

Case Idiii) $\cos(\theta)2^j t_0 \leq t$, $1 + \cos(\theta) \leq R \leq 2$ and $1 - \sqrt{2 - R} \leq \cos(\theta)$.

$$\begin{aligned} K &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \left\{ (R - 1)^{-\frac{d+1}{2}} + (\cos(\theta))^{-\frac{d+1}{2}} - \left(\frac{t}{2^j t_0} \right)^{-\frac{d+1}{2}} \right\} \\ &\cong \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (\cos(\theta))^{-\frac{d+1}{2}} \end{aligned}$$

Case Ie) $\cos(\theta)2^j t_0 \leq t$ and $R \geq 2$.

$$K \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (R-1)^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{t}{2^j t_0 \cos(\theta)} \right) \right)$$

Case IIa) $\cos(\theta)2^j t_0 \geq t$ and $R \leq 1 - \cos(\theta)$.

$$K \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{t}{\cos(\theta)2^j t_0} (1-R)^{-\frac{d+1}{2}}$$

Case IIb+c) $\cos(\theta)2^j t_0 \geq t$ and $1 - \cos(\theta) \leq R \leq 1 + \cos(\theta)$.

$$K \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{t}{\cos(\theta)2^j t_0} (\cos(\theta))^{-\frac{d+1}{2}}$$

Case IId+e) $\cos(\theta)2^j t_0 \geq t$ and $R \geq 1 + \cos(\theta)$.

$$K \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{t}{\cos(\theta)2^j t_0} (R-1)^{-\frac{d+1}{2}}$$

Now we want to prove part 1. of the lemma, i.e. to estimate

$$\sup_{j \in \mathbb{Z}: 2^j t_0 \geq t} |k_{e^{i\theta} 2^j t_0 + t}(x, y) - k_{e^{i\theta} 2^j t_0}(x, y)| \quad (5.9)$$

in the case that $|\rho(x, y)^2 / (2^l t_0)^2 - 1| \leq \cos(\theta)$ for some $l \leq L_{\max}$. We claim that in (5.9), the supremum is essentially attained for $j = l$, more precisely, that (5.9) can be estimated by the above estimate for $j = l$ (which has to be in one of the four cases I or II, b or c).

For $j < l$, we have $\frac{\rho(x, y)^2}{(2^j t_0)^2} = \frac{\rho(x, y)^2}{(2^l t_0)^2} 2^{2l-2j} \geq (1 - \cos(\theta)) \cdot 4$, so that case e applies. We note again $R = \frac{\rho(x, y)^2}{(2^j t_0)^2}$, and moreover $K_j = |k_{e^{i\theta} 2^j t_0 + t}(x, y) - k_{e^{i\theta} 2^j t_0}(x, y)|$, and M_l the right hand side of the estimate obtained for K_l . Note that since $j < l < L_{\max}$, we have $\mu(B(x, 2^j t_0)) \cong (2^j t_0)^d$ by (2.2). For $\cos(\theta)2^j t_0 \leq t$, we have by case Ie),

$$\begin{aligned} K_j &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (R-1)^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{t}{2^j t_0 \cos(\theta)} \right) \right) \\ &\lesssim \frac{(\cos(\theta))^{-\beta}}{(2^j t_0)^d} \frac{(2^j t_0)^{d+1}}{\rho(x, y)^{d+1}} \left(1 + \log \left(\frac{t}{2^j t_0 \cos(\theta)} \right) \right) \\ &\cong (\cos(\theta))^{-\beta} 2^{j-l} (2^l t_0)^{-d} \left(1 + \log \left(\frac{t}{2^j t_0 \cos(\theta)} \right) \right). \end{aligned}$$

If $\cos(\theta)2^j t_0 \leq t \leq \cos(\theta)2^l t_0$, then we have $K_j \lesssim M_l \iff 2^{j-l} (1 + \log \frac{t}{2^j t_0 \cos(\theta)}) \lesssim \frac{t}{t_0} 2^{-l} (\cos(\theta))^{-\frac{d+3}{2}} \iff 2^j t_0 \cos(\theta) / t (1 + \log \frac{t}{2^j t_0} + |\log \cos(\theta)|) \lesssim (\cos(\theta))^{-\frac{d+1}{2}}$, which is true, since $2^j t_0 \cos(\theta) / t \leq 1$ and $\log \frac{t}{2^j t_0} \leq 0$. If $\cos(\theta)2^l t_0 \leq t$, then we have $K_j \lesssim M_l \iff 2^{j-l} (1 + \log \frac{t}{2^j t_0 \cos(\theta)}) \lesssim (\cos(\theta))^{-\frac{d+1}{2}} \iff 2^{j-l} (1 + \log \frac{t}{2^j t_0} + |\log \cos(\theta)|) \lesssim (\cos(\theta))^{-\frac{d+1}{2}}$, which is true, since $2^{j-l} \leq 1$ and $\log \frac{t}{2^j t_0} \leq 0$.

For $\cos(\theta)2^j t_0 \geq t$, we have by case IIe),

$$\begin{aligned} K_j &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{t}{\cos(\theta)2^j t_0} (R-1)^{-\frac{d+1}{2}} \\ &\cong \frac{(\cos(\theta))^{-\beta}}{(2^j t_0)^d} \frac{t}{\cos(\theta)2^j t_0} \frac{(2^j t_0)^{d+1}}{\rho(x, y)^{d+1}} \\ &\cong \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \frac{t}{\cos(\theta)2^l t_0}. \end{aligned}$$

This is indeed majorized by M_l , since $\cos(\theta)2^l t_0 \geq t$, and thus,

$$M_l = \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \frac{t}{\cos(\theta)2^l t_0} (\cos(\theta))^{-\frac{d+1}{2}}.$$

For $j > l$, we have $\frac{\rho(x, y)^2}{(2^j t_0)^2} = \frac{\rho(x, y)^2}{(2^l t_0)^2} \cdot 2^{2l-2j} \leq (1 + \cos(\theta)) \cdot \frac{1}{4}$, so that case a applies. We have $1 - R \cong 1$, $\sqrt{2 - R} - 1 \cong 1$, $\sqrt{1 - R} - (\sqrt{2 - R} - 1) \cong 1$, $\sqrt{1 - R} - \frac{1}{2}\sqrt{2 - R} + \frac{1}{2} - \frac{\cos(\theta)}{2} \cong 1$. Thus, in case Iai),

$$\begin{aligned} K_j &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (1 + |\log \cos(\theta)|) \\ &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (\cos(\theta))^{-\frac{d+1}{2}} \cong M_l. \end{aligned}$$

In case Iaii),

$$\begin{aligned} K_j &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (1 + |\log \cos(\theta)| + \frac{t}{2^j t_0} - (\sqrt{2 - R} - 1)) \\ &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (\cos(\theta))^{-\frac{d+1}{2}} \cong M_l. \end{aligned}$$

In case Iaiii), $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (1 + |\log \cos(\theta)|) \lesssim M_l$.

Cases Iaiv) and Iav) cannot appear here, since $\sqrt{2 - R} - 1 > \cos(\theta)$.

In case IIa),

$$\begin{aligned} K_j &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{t}{\cos(\theta)2^j t_0} \cdot 1 \\ &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \min(1, \frac{t}{\cos(\theta)2^l t_0}) (\cos(\theta))^{-\frac{d+1}{2}} \cong M_l. \end{aligned}$$

Looking up M_l in the four cases I or II, b or c, now yields part 1. of the lemma.

Now for the proof of part 2. Suppose first that $1 + \cos(\theta) \leq \frac{\rho(x, y)^2}{(2^l t_0)^2} \leq 2$. Then M_l is given by one of the four cases Idi), Idii), Idiii) and IId). We will show that $\sup_{j \in \mathbb{Z}: 2^j t_0 \geq t} K_j \lesssim M_l + \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log \cos(\theta)|)$, the logarithmic term only appearing in case Idii).

Suppose first that M_l is given by case Idi). Consider a $j < l$. Then $\frac{\rho(x,y)^2}{(2^j t_0)^2} = \frac{\rho(x,y)^2}{(2^l t_0)^2} \cdot 2^{2l-2j} \geq 4$, so that for K_j , case Ie) or Iie) applies. In fact, one is never in the case Iie), since then $\cos(\theta)2^j t_0 \leq t$, $\cos(\theta)2^j t_0 \geq t$ and $j < l$. In case Ie), we have

$$\begin{aligned} K_j &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} R^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{t}{2^j t_0 \cos(\theta)} \right) \right) \\ &\cong (\cos(\theta))^{-\beta} 2^j t_0 \rho(x, y)^{-(d+1)} \left(1 + \log \left(\frac{t}{2^j t_0 \cos(\theta)} \right) \right) \\ &\cong (\cos(\theta))^{-\beta} 2^{j-l} (2^l t_0)^{-d} \left(1 + \log \left(\frac{t}{2^j t_0 \cos(\theta)} \right) \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} M_l &= \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \left\{ (\rho(x, y)^2 / (2^l t_0)^2 - 1)^{-\frac{d+1}{2}} \left(1 + \log \left(\frac{1 - \sqrt{2 - \rho(x, y)^2 / (2^l t_0)^2}}{\cos(\theta)} \right) \right) \right. \\ &\quad \left. + \left(1 - \sqrt{2 - \rho(x, y)^2 / (2^l t_0)^2} \right)^{-\frac{d+1}{2}} - \left(\frac{t}{2^l t_0} \right)^{-\frac{d+1}{2}} \right\}. \end{aligned}$$

Now we have $K_j \lesssim M_l$ if

$$1 + |\log(\cos(\theta))| \lesssim (\rho(x, y)^2 / (2^l t_0)^2 - 1)^{-\frac{d+1}{2}} \left(1 + \log \frac{\rho(x, y)^2 / (2^l t_0)^2 - 1}{\cos(\theta)} \right).$$

An elementary calculation shows that the minimum of the right hand side for $1 + \cos(\theta) \leq \frac{\rho(x, y)^2}{(2^l t_0)^2} \leq 2$ is equivalent to $1 + |\log(\cos(\theta))|$. Thus $K_j \lesssim M_l$ for $j < l$. Now consider $j > l$. Then $\frac{\rho(x, y)^2}{(2^j t_0)^2} = \frac{\rho(x, y)^2}{(2^l t_0)^2} 2^{2l-2j} \leq \frac{1}{2}$, so that case a) applies for K_j . Now we have $1 - R \cong 1$, $\sqrt{2 - R} - 1 \cong 1$, $\sqrt{1 - R} - (\sqrt{2 - R} - 1) \cong 1$, and $\sqrt{1 - R} - \frac{1}{2}(\sqrt{2 - R} - 1) - \frac{\cos(\theta)}{2} \cong 1$. With this, we obtain easily in case Iai) that $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (1 + |\log(\cos(\theta))|) \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|) \lesssim M_l$. In case Iaii), we also have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (1 + |\log(\cos(\theta))|) \lesssim M_l$, and the cases Iaiii), Iaiiv) and Iav) can be handled in the same way. In case Iia), we have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{t}{\cos(\theta) 2^j t_0} \lesssim M_l$. Thus $\sup_{j: 2^j t_0 \geq t} K_j \lesssim M_l$ if M_l is given by case Idi).

Now suppose that M_l is given by case Idii). Take first a $j < l$. Then $\rho(x, y)^2 / (2^j t_0)^2 \geq 4$, so that for K_j , case e) applies. In case Ie), we have

$$\begin{aligned} K_j &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \left(\frac{\rho(x, y)^2}{(2^j t_0)^2} \right)^{-\frac{d+1}{2}} \left(1 + \log \frac{t}{2^j t_0 \cos(\theta)} \right) \\ &\lesssim (\cos(\theta))^{-\beta} 2^{j-l} (2^l t_0)^{-d} \left(1 + \log \frac{t}{2^j t_0 \cos(\theta)} \right) \\ &\lesssim (\cos(\theta))^{-\beta} (2^l t_0)^{-d} \left(1 + \log \frac{t}{2^l t_0 \cos(\theta)} \right) \cong M_l. \end{aligned}$$

Again, case IIe) cannot appear since then, $\cos(\theta)2^l t_0 \leq t$, $\cos(\theta)2^j t_0 \geq t$ and $j < l$.

Now consider a $j > l$. We have $\rho(x, y)^2 / (2^j t_0)^2 \leq \frac{1}{2}$, so that case a) applies for the estimate of K_j . In case Iai), we have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (1 + |\log(\cos(\theta))|) \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|)$. The same estimate holds in the cases Iaii) and Iaiii). In case Iaiv), we have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \lesssim M_l$, and similarly, also in the cases, Iav) and IIa), $K_j \lesssim M_l$ holds. We thus have $\sup_{j: 2^j t_0 \geq t} K_j \lesssim M_l + \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|)$, if M_l is given by case Idii).

Suppose now that M_l is given by case Idiii). Then $M_l \cong \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (\cos(\theta))^{-\frac{d+1}{2}}$. Consider a $j < l$. Again, for the estimate of K_j , case e) applies, and case IIe) is ruled out by the constraints on j and l . We have as above

$$\begin{aligned} K_j &\lesssim (\cos(\theta))^{-\beta} 2^{j-l} (2^l t_0)^{-d} \left(1 + \log \frac{t}{2^j t_0 \cos(\theta)} \right) \\ &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + \log \frac{t}{2^j t_0} + |\log(\cos(\theta))|) \lesssim M_l. \end{aligned}$$

Consider now a $j > l$. Again $\rho(x, y)^2 / (2^j t_0)^2 \leq \frac{1}{2}$, so that for K_j , case a) applies, and again, several expressions involving R appearing in this case are equivalent to 1. This gives in case Iai), Iaii) and Iaiii), $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|) \lesssim M_l$, and in case Iaiv) and Iav), $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \lesssim M_l$. In case IIa), we have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \frac{t}{\cos(\theta) 2^j t_0} \lesssim M_l$. Thus $\sup_{j: 2^j t_0 \geq t} K_j \lesssim M_l$ if M_l is given by case Idiii).

Suppose now that M_l is given by case IId). Then

$$M_l = \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \frac{t}{\cos(\theta) 2^l t_0} \left(\frac{\rho(x, y)^2}{(2^l t_0)^2} - 1 \right)^{-\frac{d+1}{2}}.$$

Consider $j < l$, so that for K_j , the estimate in case e) applies. In case Ie), we have

$$\begin{aligned} K_j &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} 2^{(l-j)d} \left(\frac{\rho(x, y)^2}{(2^j t_0)^2} \right)^{-\frac{d+1}{2}} \left(1 + \log \frac{t}{2^j t_0 \cos(\theta)} \right) \\ &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} 2^{-l+j} \left(1 + \log \frac{t}{2^j t_0 \cos(\theta)} \right) \\ &\lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \frac{t}{\cos(\theta) 2^l t_0} \lesssim M_l. \end{aligned}$$

In case IIe), we have

$$K_j \lesssim (\cos(\theta))^{-\beta} (2^j t_0)^{-d} \frac{t}{\cos(\theta) 2^j t_0} \left(\frac{2^l}{2^j} \right)^{-(d+1)} \cong (2^l t_0)^{-d} 2^{j-l} \frac{t}{\cos(\theta) 2^j t_0} \lesssim M_l.$$

Consider now $j > l$, so that for K_j , the estimate in case a) applies. The cases Iai) - Iav) cannot appear due to restrictions on j, l and $\cos(\theta)t_0/t$ similar as before. In case IIa), we have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{t}{\cos(\theta)2^j t_0} \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \lesssim M_l$. Thus $\sup_{j: 2^j t_0 \geq t} K_j \lesssim M_l$ if M_l is given by case II d). We have proved part 2. of the lemma.

Now for the proof of part 3. of the lemma. We proceed similarly as before. The expression M_l is given by one of the six cases Iai) - Iav) or IIa). Suppose first that case Iai) applies. Consider $j < l$. Then $R \geq 2$, so that for K_j , case e) applies. In case Ie), we have

$$\begin{aligned} K_j &\lesssim (\cos(\theta))^{-\beta} (2^j t_0)^{-d} \frac{(2^l t_0)^{-(d+1)}}{(2^j t_0)^{-(d+1)}} \left(1 + \log \frac{t}{2^j t_0 \cos(\theta)} \right) \\ &\lesssim (\cos(\theta))^{-\beta} (2^l t_0)^{-d} (1 + |\log(\cos(\theta))|). \end{aligned}$$

Case IIe) cannot appear. Now consider $j > l$. Then for K_j , case a) applies. As several times before, diverse terms in R are equivalent to 1. In cases Iai), Iaii) and Iaiii), we have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} (1 + |\log(\cos(\theta))|) \leq \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|)$. In cases Iai v) and Iav), we have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \lesssim M_l$, and in case IIa), we have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{t}{\cos(\theta)2^j t_0} \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))}$. Thus, if M_l is given by Iai), we have $\sup_{j: 2^j t_0 \geq t} K_j \lesssim M_l + \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|)$.

Suppose that M_l is given by case Iaii). Consider a $j < l$. Then K_j is estimated by case Ie), and we have $K_j \lesssim (\cos(\theta))^{-\beta} (2^l t_0)^{-d} (1 + |\log(\cos(\theta))|)$. Consider a $j > l$. Then K_j is estimated by case a), and we have $K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|)$. Thus, if M_l is given by Iaii), we have again $\sup_{j: 2^j t_0 \geq t} K_j \lesssim M_l + \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|)$.

Similarly, we obtain in the cases that M_l is given by Iaiiii), Iai v) or Iav) that $\sup_{j: 2^j t_0 \geq t} K_j \lesssim M_l + \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} (1 + |\log(\cos(\theta))|)$.

If M_l is given by IIa), we have $M_l = \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \frac{t}{\cos(\theta)2^l t_0} \left(1 - \frac{\rho(x, y)^2}{(2^l t_0)^2} \right)^{-\frac{d+1}{2}}$. Consider a $j < l$. Then $R \geq 2$, so that case Ie) or IIe) applies for K_j . In case Ie), we have

$$\begin{aligned} K_j &\lesssim (\cos(\theta))^{-\beta} (2^j t_0)^{-d} \frac{(2^j t_0)^{d+1}}{(2^l t_0)^{d+1}} \left(1 + \log \frac{t}{2^j t_0 \cos(\theta)} \right) \\ &\lesssim (\cos(\theta))^{-\beta} (2^l t_0)^{-d} 2^{j-l} \left(1 + \log \frac{t}{2^j t_0 \cos(\theta)} \right) \lesssim M_l. \end{aligned}$$

In case IIe), we have $K_j \lesssim (2^l t_0)^{-d} 2^{j-l} \frac{t}{\cos(\theta)2^j t_0} \lesssim M_l$. Consider now a $j > l$. Then for K_j , case a) applies. Cases Iai) - Iav) cannot occur due to restrictions on j, l and $\cos(\theta)t_0/t$ as several times before. In case IIa), we have

$K_j \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^j t_0))} \frac{t}{\cos(\theta) 2^j t_0} \lesssim \frac{(\cos(\theta))^{-\beta}}{\mu(B(x, 2^l t_0))} \frac{t}{\cos(\theta) 2^l t_0} \lesssim M_l$. Thus if M_l is given by IIa), we have $\sup_{j: 2^j t_0 \geq t} K_j \lesssim M_l$. \square

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