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# Hörmander Functional Calculus for Poisson Estimates

Christoph Kriegler

**Abstract.** The aim of the article is to show a Hörmander spectral multiplier theorem for an operator  $A$  whose kernel of the semigroup  $\exp(-zA)$  satisfies certain Poisson estimates for complex times  $z$ . Here  $\exp(-zA)$  acts on  $L^p(\Omega)$ ,  $1 < p < \infty$ , where  $\Omega$  is a space of homogeneous type with the additional condition that the measure of annuli is controlled. In most of the known Hörmander type theorems in the literature, Gaussian bounds for the semigroup are needed, whereas here the new feature is that the assumption are the to some extent weaker Poisson bounds. The order of derivation in our Hörmander multiplier result is  $\frac{d}{2} + 1$ ,  $d$  being the dimension of the space  $\Omega$ . Moreover the functional calculus resulting from our Hörmander theorem is shown to be  $R$ -bounded.

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## 1. Introduction

Let  $f$  be a bounded function on  $(0, \infty)$  and  $u(f)$  the operator on  $L^p(\mathbb{R}^d)$  defined by  $[u(f)g]^\wedge(\xi) = f(|\xi|^2)\hat{g}(\xi)$ . Hörmander's theorem on Fourier multipliers [5, Theorem 2.5] asserts that  $u(f) : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$  is bounded for any  $p \in (1, \infty)$  provided that for some integer  $N$  strictly larger than  $\frac{d}{2}$

$$\sup_{R>0} \int_{R/2}^{2R} \left| t^k f^{(k)}(t) \right|^2 \frac{dt}{t} < \infty \quad (k = 0, \dots, N). \quad (1.1)$$

This theorem has many generalisations to similar contexts, for example to elliptic and sub-elliptic differential operators  $A$ , including sublaplacians on Lie groups of polynomial growth, Schrödinger operators and elliptic operators on Riemannian manifolds [3]: Note first that the above  $u(f)$  equals  $f(-\Delta)$ , the functional calculus of the self-adjoint positive operator  $-\Delta$ . Now for a self-adjoint operator  $A$ , a Hörmander theorem states that the operator

$f(A)$  extends boundedly to  $L^p(\Omega)$ ,  $1 < p < \infty$  for any function  $f$  satisfying (1.1) with suitable  $N$ . In most of the proofs for a Hörmander theorem in the literature, the assumption of so called Gaussian bounds plays a crucial role. That means the following. Suppose that  $A$  acts on  $L^p(\Omega)$ ,  $1 < p < \infty$ , where  $(\Omega, \mu, \rho)$  is a space of homogeneous type. Then the semigroup  $(\exp(-tA))_{t \geq 0}$  generated by  $A$  has an integral kernel  $k_t(x, y)$  such that

$$|k_t(x, y)| \leq C \mu(B(y, \sqrt{t}))^{-1} \exp\left(-c \frac{\rho(x, y)^2}{t}\right) \quad (t > 0, x, y \in \Omega).$$

This hypothesis includes many elliptic differential operators. However there are operators such that the integral kernel of the semigroup satisfies only weaker estimates, see e.g. [10]. Establishing a Hörmander theorem for these operators is the issue of the present article. More precisely, let  $A$  act as above on  $L^p(\Omega)$  such that  $(\exp(-zA))_{\operatorname{Re} z > 0}$  has an integral kernel  $k_z(x, y)$  such that

$$|k_z(x, y)| \leq C \frac{|z|}{|z^2 + \rho(x, y)^2|^{\frac{d+1}{2}}} \quad (\operatorname{Re} z > 0, x, y \in \Omega)$$

and

$$|k_z(x, y) - k_z(x, \bar{y})| \leq C \left| \frac{|z|}{|z^2 + \rho(x, y)^2|^{\frac{d+1}{2}}} - \frac{|z|}{|z^2 + \rho(x, \bar{y})^2|^{\frac{d+1}{2}}} \right|$$

hold for  $\operatorname{Re} z > 0$ ,  $x, y, \bar{y} \in \Omega$ . The right hand side of the first estimate is (a constant times) the absolute value of the complex Poisson kernel which obviously decays slower as  $\rho(x, y) \rightarrow \infty$  than the Gaussian kernel above. Under a further hypothesis on the homogeneous space  $\Omega$  we obtain a Hörmander theorem of the order  $N > \frac{d+2}{2}$  for operators  $A$  satisfying the above two estimates. The proof relies on the behaviour of the semigroup  $\exp(-zA)$  generated by  $A$  when the complex parameter  $z$  approaches the imaginary axis. Here simple norm estimates are not sufficient but  $R$ -bounds of the semigroup are needed.

In Section 2 we will introduce the necessary background and cite a theorem which allows to pass from  $R$ -bounds on the semigroup to a Hörmander functional calculus. In Section 3 we state and prove the result of this article.

## 2. Preliminaries

In this section, we provide the necessary background for the Main Section 3. Let  $A$  be a self-adjoint positive operator on  $L^2(\Omega)$ , where  $\Omega$  is some  $\sigma$ -finite measure space. Then for any bounded measurable function  $f : [0, \infty) \rightarrow \mathbb{C}$ , the operator  $f(A) \in B(L^2(\Omega))$  is defined via the self-adjoint functional calculus of  $A$ . In several situations, this functional calculus extends partially to  $L^p(\Omega)$  for  $1 < p < \infty$ . Let  $\phi_0 \in C_c^\infty(\frac{1}{2}, 2)$  and for  $n \in \mathbb{Z}$  put  $\phi_n = \phi_0(2^{-n}\cdot)$ . We can and do assume that  $\sum_{n \in \mathbb{Z}} \phi_n(t) = 1$  for any  $t > 0$  [1, Lemma 6.1.7]. Now define

$$\mathcal{H}^\alpha = \{f : [0, \infty) \rightarrow \mathbb{C} : \|f\|_{\mathcal{H}^\alpha} = |f(0)| + \sup_{n \in \mathbb{Z}} \|(\phi_n f) \circ \exp\|_{W_2^\alpha(\mathbb{R})} < \infty\},$$

where  $W_2^\alpha(\mathbb{R})$  is the usual Sobolev space. For  $\alpha > \frac{1}{2}$ , the space  $\mathcal{H}^\alpha$  is a Banach algebra endowed with the norm  $\|\cdot\|_{\mathcal{H}^\alpha}$ . This class refines condition (1.1) in the sense that  $f \in \mathcal{H}^\alpha \implies f$  satisfies (1.1) for  $\alpha > N$  and the converse holds for  $\alpha < N$ . Then  $A$  is said to have a bounded  $\mathcal{H}^\alpha$  calculus on  $L^p(\Omega)$  if for any  $f \in \mathcal{H}^\alpha$ , the operator  $f(A)$  extends boundedly from  $L^p(\Omega) \cap L^2(\Omega) \rightarrow L^2(\Omega)$  to an operator in  $B(L^p(\Omega))$ .

Let  $\omega \in (0, \pi)$ . A densely defined and closed operator  $A$  on  $L^p(\Omega)$ ,  $1 < p < \infty$ , is called  $\omega$ -sectorial if  $\sigma(A) \subset \overline{\Sigma_\omega}$  where  $\Sigma_\omega = \{z \in \mathbb{C}^* : |\arg z| < \omega\}$ , and  $\|\lambda(\lambda - A)^{-1}\| \leq C_\theta$  for any  $\lambda \in \overline{\Sigma_\theta}^c$  and any  $\theta \in (\omega, \pi)$ . For an  $\omega$ -sectorial operator  $A$  and a function  $f \in H_0^\infty(\Sigma_\theta) = \{g : \Sigma_\theta \rightarrow \mathbb{C} : g \text{ analytic and bounded, } \exists C, \epsilon > 0 : |g(z)| \leq C \min(|z|^\epsilon, |z|^{-\epsilon})\}$  where  $0 < \omega < \theta < \pi$ , one defines the operator  $f(A)$  by

$$f(A)x = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} x d\lambda.$$

Here,  $\Gamma$  is the boundary of  $\Sigma_{\frac{\omega+\theta}{2}}$  oriented counterclockwise. This definition coincides with the self-adjoint calculus if applicable. If there is a constant  $C > 0$  such that  $\|f(A)\| \leq C \sup_{|\arg z| < \theta} |f(z)|$  for any  $f \in H_0^\infty(\Sigma_\theta)$ , then  $A$  is said to have a bounded  $H^\infty(\Sigma_\theta)$  calculus, or just bounded  $H^\infty$  calculus. If  $\|f(A)\| \leq C \|f\|_{\mathcal{H}^\alpha}$  for any  $f \in \bigcap_{\omega > 0} H_0^\infty(\Sigma_\omega) \cap \mathcal{H}^\alpha$ , then there exists a bounded homomorphism  $\mathcal{H}^\alpha \rightarrow B(L^p(\Omega))$ ,  $f \mapsto f(A)$ , and  $A$  is said to have a bounded  $\mathcal{H}^\alpha$  calculus. If  $A$  is moreover self-adjoint on  $L^2(\Omega)$  then the notion of a bounded  $\mathcal{H}^\alpha$  calculus coincides with the one from the preceding paragraph.

Let  $(\epsilon_n)_{n \in \mathbb{N}}$  be a sequence of independent random variables such that  $\text{Prob}(\epsilon_n = 1) = \text{Prob}(\epsilon_n = -1) = \frac{1}{2}$ , i.e. a sequence of independent Rademacher variables. Let  $X$  be a Banach space. A subset  $\tau \subset B(X)$  is called  $R$ -bounded if there exists a constant  $C > 0$  such that for any choice of finite families  $T_1, \dots, T_n \in \tau$  and  $x_1, \dots, x_n \in X$ , one has

$$\left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k T_k x_k \right\|_X^2 \right)^{\frac{1}{2}} \leq C \left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|_X^2 \right)^{\frac{1}{2}}.$$

The least possible constant is denoted by  $R(\tau)$ , and  $R(\tau) = \infty$ , if no such constant is admitted. Any  $R$ -bounded set  $\tau$  is norm bounded, i.e.  $\sup_{T \in \tau} \|T\| \leq R(\tau)$ , but the converse is false in general. If  $X = L^p$ ,  $1 \leq p < \infty$ , then

$$\left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|_X^2 \right)^{\frac{1}{2}} \cong \left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\|_p \quad (2.1)$$

uniformly in  $n$  and  $x_1, \dots, x_n$ . A linear mapping  $u : Y \rightarrow B(X)$ , where  $Y$  is a further Banach space is called  $R$ -bounded if  $R(u(y) : \|y\|_Y \leq 1) < \infty$ . The following proposition gives a condition on the semigroup generated by a sectorial operator  $A$  so that  $A$  has a  $\mathcal{H}^\beta$  calculus.

**Proposition 2.1.** Let  $A$  be an  $\omega$ -sectorial operator for any  $\omega > 0$  defined on an  $L^p$  space for some  $1 < p < \infty$ , and let  $A$  have a bounded  $H^\infty$  calculus. Suppose that for some  $\alpha > 0$  the set  $\{\exp(-e^{i\theta}2^k t A) : k \in \mathbb{Z}\}$  is  $R$ -bounded for any  $t > 0$  and  $|\theta| < \frac{\pi}{2}$ , with  $R$ -bound  $\lesssim (\frac{\pi}{2} - |\theta|)^{-\alpha}$ . Then for any  $\beta > \alpha + \frac{1}{2}$ ,  $A$  has a bounded  $\mathcal{H}^\beta$  calculus. Moreover, this calculus is an  $R$ -bounded mapping.

*Proof.* This is proved in the case that  $A$  has dense range in [7, Lemma 4.72 and Proposition 4.79] and also in [9, Proposition 3.1 and Proposition 6.7]. This proof for which we give a sketch applies also here. First one deduces from the assumption of  $R$ -boundedness of the semigroup that

$$\{(1 + |t|)^{-\alpha}(1 + 2^k A)^{-\alpha} \exp(i2^k t A) : t \in \mathbb{R}\}$$

is  $R$ -bounded with  $R$ -bound independent of  $t \in \mathbb{R}$ . Then for  $g \in C_c^\infty(0, \infty)$  a representation formula of  $g(2^k A)(1 + 2^k A)^{-\alpha}$  is available, namely

$$g(2^k A)(1 + 2^k A)^{-\alpha} x = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(t)(1 + |t|)^\beta (1 + |t|)^{-\beta} (1 + 2^k A)^{-\alpha} \exp(i2^k t A) x dt.$$

If  $\beta > \alpha + \frac{1}{2}$ , then  $(1 + |t|)^{-\beta} \|(1 + 2^k A)^{-\alpha} \exp(i2^k t A)\|$  is dominated by a function in  $L^2(\mathbb{R})$ , and if  $g$  belongs to  $W_2^\beta(\mathbb{R})$  then also  $\hat{g}(t)(1 + |t|)^\beta \in L^2(\mathbb{R})$ . In fact, more can be said. By [6, Proposition 4.1, Remark 4.2], the set

$$\{g(2^k A)(1 + 2^k A)^{-\alpha} : g \in C_c^\infty(0, \infty), \|g\|_{W_2^\beta(\mathbb{R})} \leq 1, k \in \mathbb{Z}\}$$

is  $R$ -bounded. Next one gets rid of the factor  $(1 + 2^k A)^{-\alpha}$  above by using a function  $\psi(\lambda) = (1 + \lambda)^\alpha \phi(\lambda)$  where  $\phi \in C_c^\infty(0, \infty)$  and  $\phi(\lambda) = 1$  for  $\lambda \in [\frac{1}{2}, 2]$ . The hypotheses of the proposition imply that  $\{\psi(2^k A) : k \in \mathbb{Z}\}$  is  $R$ -bounded. Then

$$\{g(2^k A) : g \in C_c^\infty(0, \infty), \text{supp } g \subset [\frac{1}{2}, 2], \|g\|_{W_2^\beta(\mathbb{R})} \leq 1, k \in \mathbb{Z}\}$$

is  $R$ -bounded. The hypotheses imply moreover that there holds the following equivalences of Paley-Littlewood type:

$$\|f\|_p \cong \left\| \left( \sum_{k \in \mathbb{Z}} |\phi(2^k A) f|^2 \right)^{\frac{1}{2}} \right\|_p \cong \left\| \left( \sum_{k \in \mathbb{Z}} |\tilde{\phi}(2^k A) \phi(2^k A) f|^2 \right)^{\frac{1}{2}} \right\|_p$$

for a function  $\phi \in C_c^\infty(0, \infty)$ ,  $\phi$  not vanishing identically zero,  $\text{supp } \phi \subset [\frac{1}{2}, 2]$  and  $\tilde{\phi} = \phi(2^{-1}\cdot) + \phi + \phi(2\cdot)$ . Then one can show that  $g(A)$  is bounded for  $\|g\|_{\mathcal{H}^\beta} < \infty$ :

$$\begin{aligned} \|g(A)f\|_p &\cong \left\| \left( \sum_{k \in \mathbb{Z}} |\tilde{\phi}(2^k A) g(A) \phi(2^k A) f|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\cong \left\| \left( \sum_k |\tilde{\phi} g(2^{-k}\cdot)(2^k A) \phi(2^k A) f|^2 \right)^{\frac{1}{2}} \right\|_p \end{aligned}$$

$$\lesssim R(\{\tilde{\phi}g(2^{-k}\cdot) : k \in \mathbb{Z}\}) \left\| \left( \sum_{k \in \mathbb{Z}} |\phi(2^k A)f|^2 \right)^{\frac{1}{2}} \right\|_p$$

$$\lesssim \|g\|_{\mathcal{H}^\beta} \|f\|_p.$$

Thus  $\{g(A) : \|g\|_{\mathcal{H}^\beta} \leq 1\}$  is a bounded subset of  $B(L^p)$ . In a similar manner to the calculation right above, using the fact that  $L^p$  has Pisier's property  $(\alpha)$ , one shows that this set is moreover  $R$ -bounded.  $\square$

The space  $\Omega$  on which the operator  $A$  acts will be a space of homogeneous type. This means that  $(\Omega, \rho)$  is a metric space endowed with a nonnegative Borel measure  $\mu$  which satisfies the doubling condition: There exists a constant  $C > 0$  such that for all  $x \in \Omega$  and  $r > 0$ ,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty,$$

where we set  $B(x, r) = \{y \in \Omega : \rho(x, y) < r\}$ . Note that the doubling condition implies the following strong homogeneity property: There exists  $C > 0$  and a dimension  $d > 0$  such that for all  $\lambda \geq 1$ , for all  $x \in \Omega$  and all  $r > 0$  we have  $\mu(B(x, \lambda r)) \leq C\lambda^d \mu(B(x, r))$ . We will assume that the space of homogeneous type  $(\Omega, \mu, \rho)$  has the additional property

$$\mu(B(x, r, R)) \leq C(R^d - r^d) \quad (x \in \Omega, R > r > 0), \quad (2.2)$$

where we denote  $B(x, r, R) = B(x, R) \setminus B(x, r)$ .

### 3. The Main Theorem

We let  $(\Omega, \mu, \rho)$  be a space of homogeneous type with the additional property (2.2). We further let  $T_z = \exp(-zA)$  be a semigroup on  $L^2(\Omega)$  with the properties: The generator  $A$  is selfadjoint, and  $T_z$  has an integral kernel  $k_z(x, y)$  for  $\operatorname{Re} z \geq 0$  i.e.  $(T_z f)(x) = \int_\Omega k_z(x, y)f(y)d\mu(y)$  for any  $f \in L^2(\Omega)$ . We assume that

$$|k_t(x, y)| \leq C \frac{t}{(t^2 + \rho(x, y)^2)^{\frac{d+1}{2}}} \quad (t > 0, x, y \in \Omega), \quad (3.1)$$

$$|k_{it}(x, y)| \leq C \frac{|t|}{|t^2 - \rho(x, y)^2|^{\frac{d+1}{2}}} \quad (t \in \mathbb{R}, x, y \in \Omega), \quad (3.2)$$

and

$$|k_z(x, y)| \leq C \frac{|z|}{|z^2 + \rho(x, y)^2|^{\frac{d+1}{2}}} \exp(\max(|z|^a, |z|^{-a})) \quad (3.3)$$

for some  $a < \pi$ . Then (3.1), (3.2) and (3.3) imply

$$|k_z(x, y)| \leq C \frac{|z|}{|z^2 + \rho(x, y)^2|^{\frac{d+1}{2}}} \quad (z \in \mathbb{C}_+, x, y \in \Omega). \quad (3.4)$$

Indeed, define  $f(z) = k_z(x, y) \frac{(z^2 + \rho(x, y)^2)^{\frac{d+1}{2}}}{z}$  for  $x, y \in \Omega$  fixed and  $\operatorname{Re} z \geq 0$ ,  $z \neq 0$ . Then  $|f(z)| \leq C$  for  $z \in \mathbb{R}_+$  and  $z \in i\mathbb{R} \setminus \{0\}$ . Moreover,  $f$  is analytic

and of admissible growth in the sectors  $\{z \in \mathbb{C}^* : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$  and  $\{z \in \mathbb{C}^* : \operatorname{Re} z > 0, \operatorname{Im} z < 0\}$  in the sense of [11]. By [11] it follows from a variant of the three lines lemma that  $|f(z)| \leq C$  for  $\operatorname{Re} z > 0$ , which shows (3.4). We further assume in the sequel that

$$|k_z(x, y) - k_z(x, \bar{y})| \leq C \left| \frac{|z|}{|z^2 + \rho(x, y)^2|^{\frac{d+1}{2}}} - \frac{|z|}{|z^2 + \rho(x, \bar{y})^2|^{\frac{d+1}{2}}} \right| \quad (3.5)$$

holds for  $\operatorname{Re} z > 0$ ,  $x, y, \bar{y} \in \Omega$ .

**Proposition 3.1.** Let  $(\Omega, \mu, \rho)$  be a space of homogeneous type satisfying (2.2) and  $T_z = \exp(-zA)$  a semigroup satisfying (3.4) and (3.5) with self-adjoint  $A$ . Then the operator  $A$  has an  $H^\infty$  calculus on  $L^p(\Omega)$  for  $1 < p < \infty$ .

*Proof.* The proposition follows from [4, Theorem 3.1]. Indeed, let  $\theta \in (0, \frac{\pi}{2})$ . Then  $T_z$  is analytic on  $L^2(\Omega)$  in the sector  $\Sigma_\theta$ , and since  $A$  is self-adjoint, it has a bounded  $H^\infty(\Sigma_\mu)$  calculus for any  $\mu > \frac{\pi}{2} - \theta$ . The kernel  $k_z(x, y)$  satisfies on  $z \in \Sigma_\theta$  the bound

$$\begin{aligned} |k_z(x, y)| &\lesssim \frac{|z|^{-d}}{\left|1 + \left(\frac{\rho(x, y)}{z}\right)^2\right|^{\frac{d+1}{2}}} \lesssim \frac{|\operatorname{Re} z|^{-d}}{\left(1 + \left(\frac{\rho(x, y)}{\operatorname{Re} z}\right)^2\right)^{\frac{d+1}{2}}} \\ &\lesssim \mu(B(x, |\operatorname{Re} z|))^{-1} \left(1 + \left(\frac{\rho(x, y)}{\operatorname{Re} z}\right)^2\right)^{-\frac{d+1}{2}} \end{aligned}$$

since  $|z| \cong \operatorname{Re} z$  for  $z \in \Sigma_\theta$ ,  $|1 + \left(\frac{\rho(x, y)}{\operatorname{Re} z}\right)^2| \leq 1 + \left|\frac{\rho(x, y)}{\operatorname{Re} z}\right|^2 \lesssim 1 + \left|\frac{\rho(x, y)}{z}\right|^2 \lesssim \left|1 + \left(\frac{\rho(x, y)}{z}\right)^2\right|$ , and  $\mu(B(x, t)) \lesssim t^d$  by (2.2). Then with  $G_t$  given by [4, (7)] and  $g(x) = c(1 + x^2)^{-\frac{d+1}{2}}$ , we can deduce from [4, Theorem 3.1] that  $A$  has a bounded  $H^\infty(\Sigma_\mu)$  calculus on  $L^p(\Omega)$  for any  $p \in (1, \infty)$  and  $\mu > \frac{\pi}{2} - \theta$ .  $\square$

For later use we state the following lemma.

**Lemma 3.2.** Let  $(\Omega, \mu, \rho)$  be a space of homogeneous type satisfying (2.2). Let  $|\theta| < \frac{\pi}{2}$ ,  $a > 0$ ,  $b > \frac{d}{2}$  and  $y \in \Omega$ . Then

$$\int_{\Omega} |e^{2i\theta} + (a\rho(x, y))^2|^{-b} a^d d\mu(x) \lesssim \left(\frac{\pi}{2} - |\theta|\right)^{-b+1}$$

*Proof.* We split the integral over  $\Omega$  into four parts

$$\begin{aligned} \int_{\Omega} &= \int_{B(y, a^{-1}\sqrt{1 - (\frac{\pi}{2} - |\theta|)})} + \int_{B(y, a^{-1}\sqrt{1 - (\frac{\pi}{2} - |\theta|)}, a^{-1}\sqrt{1 + \frac{\pi}{2} - |\theta|})} \\ &+ \int_{B(y, a^{-1}\sqrt{1 + \frac{\pi}{2} - |\theta|}, 2a^{-1})} + \int_{B(y, 2a^{-1}, \infty)}. \end{aligned}$$

For the first integral, we have  $|e^{2i\theta} + (a\rho(x, y))^2| \geq |\cos(2\theta) + (a\rho(x, y))^2|$  and thus,

$$\begin{aligned}
& \int_{B(y, a^{-1}\sqrt{1-(\frac{\pi}{2}-|\theta|)})} |e^{2i\theta} + (a\rho(x, y))^2|^{-b} a^d d\mu(x) \\
& \leq \int_{B(y, a^{-1}\sqrt{1-(\frac{\pi}{2}-|\theta|)})} |\cos(2\theta) + (a\rho(x, y))^2|^{-b} a^d d\mu(x) \\
& \lesssim \int_0^{a^{-1}\sqrt{1-(\frac{\pi}{2}-|\theta|)}} |\cos(2\theta) + (ar)^2|^{-b} a^d r^d \frac{dr}{r} \\
& = \int_0^{\sqrt{1-(\frac{\pi}{2}-|\theta|)}} |\cos(2\theta) + r^2|^{-b} r^d \frac{dr}{r} \\
& \lesssim (\frac{\pi}{2} - |\theta|)^{-b+1}.
\end{aligned}$$

Note that after the second inequality, the factor  $r^{d-1}$  follows from assumption (2.2). For the second integral, we have  $|e^{2i\theta} + (a\rho(x, y))^2| \geq |\sin(2\theta)| \cong \frac{\pi}{2} - |\theta|$  and thus,

$$\begin{aligned}
& \int_{B(y, a^{-1}\sqrt{1-(\frac{\pi}{2}-|\theta|)}, a^{-1}\sqrt{1+\frac{\pi}{2}-|\theta|})} |e^{2i\theta} + (a\rho(x, y))^2|^{-b} a^d d\mu(x) \\
& \lesssim (\frac{\pi}{2} - |\theta|)^{-b} a^d \mu(B(y, a^{-1}\sqrt{1-(\frac{\pi}{2}-|\theta|)}, a^{-1}\sqrt{1+\frac{\pi}{2}-|\theta|})) \\
& \lesssim (\frac{\pi}{2} - |\theta|)^{-b} ((1 + (\frac{\pi}{2} - |\theta|))^{\frac{d}{2}} - (1 - (\frac{\pi}{2} - |\theta|))^{\frac{d}{2}}) \\
& \lesssim (\frac{\pi}{2} - |\theta|)^{-b+1}.
\end{aligned}$$

For the third integral, we have  $|e^{2i\theta} + (a\rho(x, y))^2| \geq |\cos(2\theta) + (a\rho(x, y))^2|$  and therefore,

$$\begin{aligned}
& \int_{B(y, a^{-1}\sqrt{1+\frac{\pi}{2}-|\theta|}, 2a^{-1})} |e^{2i\theta} + (a\rho(x, y))^2|^{-b} a^d d\mu(x) \\
& \lesssim \int_{a^{-1}\sqrt{1+\frac{\pi}{2}-|\theta|}}^{2a^{-1}} |\cos(2\theta) + (ar)^2|^{-b} a^d r^d \frac{dr}{r} \\
& \lesssim \int_{\sqrt{1+\frac{\pi}{2}-|\theta|}}^2 |\cos(2\theta) + r^2|^{-b} r^d \frac{dr}{r} \\
& \lesssim (\frac{\pi}{2} - |\theta|)^{-b+1}.
\end{aligned}$$

Finally, for the fourth integral, we have  $|e^{2i\theta} + (a\rho(x, y))^2| \gtrsim (a\rho(x, y))^2$  and therefore,

$$\begin{aligned}
\int_{B(y, 2a^{-1}, \infty)} |e^{2i\theta} + (a\rho(x, y))^2|^{-b} a^d d\mu(x) & \lesssim \int_{2a^{-1}}^{\infty} (ar)^{-2b+d} \frac{dr}{r} \\
& \lesssim 1
\end{aligned}$$

since  $b > \frac{d}{2}$ . Now summing up the four estimates proves the lemma.  $\square$



The following is the main proposition of this section.

**Proposition 3.3.** Let  $A$  be self-adjoint positive such that (3.4) and (3.5) are satisfied. Then the semigroup  $\exp(-zA)$  satisfies on  $X = L^p(\Omega)$  for any  $1 < p < \infty$  the  $R$ -bound estimate

$$R(\exp(-e^{i\theta}2^j t A) : j \in \mathbb{Z}) \lesssim \left(\frac{\pi}{2} - |\theta|\right)^{-(d+1)/2}.$$

*Proof.* Write in short  $k_j(x, y) = k_{e^{i\theta}2^j t}(x, y)$ . According to [12, p. 19, Theorem 3 and p. 28, Section 6.4], see also [2, p. 74, Théorème 2.4], it suffices to show that

$$\int_{\rho(x, y) \geq 3\rho(y, \bar{y})} \sup_{j \in \mathbb{Z}} |k_j(x, y) - k_j(x, \bar{y})| d\mu(x) \leq C \left(\frac{\pi}{2} - |\theta|\right)^{-(d+1)/2}. \quad (3.6)$$

Indeed, then it follows from the self-adjointness of  $A$  and, with  $B_1 = B_2 = \ell^2$  in [12, p. 28, Section 6.4], that for any  $1 < p \leq 2$  and  $f_j \in L^p(\Omega)$ ,  $\|(\sum_j |k_j * f_j|^2)^{\frac{1}{2}}\|_p \leq C_p (\frac{\pi}{2} - |\theta|)^{-(d+1)/2} \|(\sum_j |f_j|^2)^{\frac{1}{2}}\|_p$ . The same inequality holds then for  $2 \leq p < \infty$  by duality which shows the claimed  $R$ -boundedness by (2.1). At first, we have

$$\begin{aligned} & \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} \sup_j |k_j(x, y) - k_j(x, \bar{y})| d\mu(x) \\ & \leq \sum_{j: 2^j < \rho(y, \bar{y})} \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} |k_j(x, y) - k_j(x, \bar{y})| d\mu(x) \end{aligned} \quad (3.7)$$

$$+ \sum_{j: 2^j \geq \rho(y, \bar{y})} \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} |k_j(x, y) - k_j(x, \bar{y})| d\mu(x). \quad (3.8)$$

We estimate (3.7) and (3.8) separately. For (3.7), we assume without loss of generality that  $t \in [\frac{1}{2}, 1]$ . Then

$$\begin{aligned} & \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} |k_j(x, y) - k_j(x, \bar{y})| d\mu(x) \\ & \leq \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} |k_j(x, y)| + |k_j(x, \bar{y})| d\mu(x) \\ & \lesssim \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} \frac{2^j t}{|(2^j t e^{i\theta})^2 + \rho(x, y)^2|^{(d+1)/2}} + \frac{2^j t}{|(2^j t e^{i\theta})^2 + \rho(x, \bar{y})^2|^{(d+1)/2}} d\mu(x) \\ & \lesssim \int_{2\rho(y, \bar{y})}^{\infty} 2^j t r^{-d-1} r^{d-1} dr + \int_{\rho(y, \bar{y})}^{\infty} 2^j t r^{-d-1} r^{d-1} dr \\ & \lesssim 2^j t \rho(y, \bar{y})^{-1} + 2^j t \rho(y, \bar{y})^{-1}. \end{aligned}$$

Here, in the second inequality, we have used assumption (3.4). In the third inequality we have used that  $|2^j t e^{i\theta}| \leq \rho(y, \bar{y})$ , so that  $|(2^j t e^{i\theta})^2 + \rho(x, y)^2| \cong \rho(x, y)^2$  and  $|(2^j t e^{i\theta})^2 + \rho(x, \bar{y})^2| \cong \rho(x, \bar{y})^2$ . Then the two integrals after the third inequality are concentric around  $y$  and  $\bar{y}$ , with  $r = \rho(x, y)$  and

$r = \rho(x, \bar{y})$ , and the factor  $r^{d-1}$  follows from assumption (2.2). Replacing this in (3.7) and summing up we obtain

$$\sum_{j: 2^j < \rho(y, \bar{y})} \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} |k_j(x, y) - k_j(x, \bar{y})| d\mu(x) \lesssim \sum_{j: 2^j < \rho(y, \bar{y})} 2^j t \rho(y, \bar{y})^{-1} \lesssim 1.$$

Let us turn to (3.8) and pick a  $j \in \mathbb{Z}$  such that  $2^j > \rho(y, \bar{y})$ , and a  $t \in [\frac{1}{2}, 1]$ . Using (3.5), we get

$$\begin{aligned} & \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} |k_j(x, y) - k_j(x, \bar{y})| d\mu(x) \\ & \lesssim \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} 2^j t \left| \frac{1}{|(2^j t e^{i\theta})^2 + \rho(x, y)|^{\frac{d+1}{2}}} - \frac{1}{|(2^j t e^{i\theta})^2 + \rho(x, \bar{y})|^{\frac{d+1}{2}}} \right| d\mu(x) \\ & = \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} 2^j t \left| |(2^j t e^{i\theta})^2 + \rho(x, \bar{y})|^{\frac{d+1}{2}} - |(2^j t e^{i\theta})^2 + \rho(x, y)|^{\frac{d+1}{2}} \right| \times \\ & \times |(2^j t e^{i\theta})^2 + \rho(x, y)|^{-\frac{d+1}{2}} |(2^j t e^{i\theta})^2 + \rho(x, \bar{y})|^{\frac{d+1}{2}} d\mu(x) \\ & \lesssim \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} 2^j t \left| |(2^j t e^{i\theta})^2 + \rho(x, \bar{y})|^{\frac{1}{2}} - |(2^j t e^{i\theta})^2 + \rho(x, y)|^{\frac{1}{2}} \right| \times \\ & \times \sum_{l=0}^d |(2^j t e^{i\theta})^2 + \rho(x, \bar{y})|^{\frac{d-l}{2}} |(2^j t e^{i\theta})^2 + \rho(x, y)|^{\frac{l}{2}} \times \\ & \times |(2^j t e^{i\theta})^2 + \rho(x, y)|^{-\frac{d+1}{2}} |(2^j t e^{i\theta})^2 + \rho(x, \bar{y})|^{\frac{d+1}{2}} d\mu(x) \\ & \lesssim \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} 2^j t |\rho(x, \bar{y})^2 - \rho(x, y)^2| \cdot |(2^j t e^{i\theta})^2 + \rho(x, y)|^{-\frac{1}{2}} \times \\ & \times \sum_{l=0}^d |(2^j t e^{i\theta})^2 + \rho(x, \bar{y})|^{\frac{d-l}{2}} |(2^j t e^{i\theta})^2 + \rho(x, y)|^{\frac{l}{2}} \times \\ & \times |(2^j t e^{i\theta})^2 + \rho(x, y)|^{-\frac{d+1}{2}} |(2^j t e^{i\theta})^2 + \rho(x, \bar{y})|^{\frac{d+1}{2}} d\mu(x) \\ & \lesssim \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} \rho(y, \bar{y}) (2^j t)^{-1} \left( \frac{\rho(x, y)}{2^j t} + \frac{\rho(x, \bar{y})}{2^j t} \right) \left| e^{2i\theta} + \left( \frac{\rho(x, y)}{2^j t} \right)^2 \right|^{-\frac{1}{2}} \times \\ & \times \sum_{l=0}^d \left| e^{2i\theta} + \left( \frac{\rho(x, \bar{y})}{2^j t} \right)^2 \right|^{\frac{d-l}{2}} \left| e^{2i\theta} + \left( \frac{\rho(x, y)}{2^j t} \right)^2 \right|^{\frac{l}{2}} \times \\ & \times \left| e^{2i\theta} + \left( \frac{\rho(x, y)}{2^j t} \right)^2 \right|^{-\frac{d+1}{2}} \left| e^{2i\theta} + \left( \frac{\rho(x, \bar{y})}{2^j t} \right)^2 \right|^{\frac{d+1}{2}} (2^j t)^{-d} d\mu(x). \end{aligned}$$

We divide this integral into the two terms

$$\int \left\{ \left| \left( \frac{\rho(x, y)}{2^j t} \right)^2 - 1 \right| \leq \frac{\pi}{2} - |\theta| \right\} + \int \left\{ \left| \left( \frac{\rho(x, y)}{2^j t} \right)^2 - 1 \right| > \frac{\pi}{2} - |\theta| \right\}.$$

For the first integral, we have  $\frac{\rho(x,y)}{2^j t} + \frac{\rho(x,\bar{y})}{2^j t} \lesssim 1$ ,  $\left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right| \geq |\sin(2\theta)| \gtrsim \frac{\pi}{2} - |\theta|$  and  $\left| e^{2i\theta} + \left( \frac{\rho(x,\bar{y})}{2^j t} \right)^2 \right| \geq |\sin(2\theta)| \gtrsim \frac{\pi}{2} - |\theta|$ . Therefore,

$$\begin{aligned}
& \int_{\left\{ \left| \left( \frac{\rho(x,y)}{2^j t} \right)^2 - 1 \right| \leq \frac{\pi}{2} - |\theta| \right\}} \rho(y,\bar{y}) (2^j t)^{-1} \left( \frac{\rho(x,y)}{2^j t} + \frac{\rho(x,\bar{y})}{2^j t} \right) \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{-\frac{1}{2}} \times \\
& \times \sum_{l=0}^d \left| e^{2i\theta} + \left( \frac{\rho(x,\bar{y})}{2^j t} \right)^2 \right|^{\frac{d-l}{2}} \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{\frac{l}{2}} \times \\
& \times \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{-\frac{d+1}{2}} \left| e^{2i\theta} + \left( \frac{\rho(x,\bar{y})}{2^j t} \right)^2 \right|^{-\frac{d+1}{2}} (2^j t)^{-d} d\mu(x) \\
& \lesssim \rho(y,\bar{y}) (2^j t)^{-1} \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{d+3}{2}} (2^j t)^{-d} \mu \left( \left\{ \left| \left( \frac{\rho(x,y)}{2^j t} \right)^2 - 1 \right| \leq \frac{\pi}{2} - |\theta| \right\} \right) \\
& \lesssim \rho(y,\bar{y}) (2^j t)^{-1} \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{d+3}{2}} (2^j t)^{-d} \mu \left( B(y, 2^j t - \frac{1}{2} 2^j t \left( \frac{\pi}{2} - |\theta| \right), 2^j t + \frac{1}{2} 2^j t \left( \frac{\pi}{2} - |\theta| \right)) \right) \\
& \lesssim \rho(y,\bar{y}) (2^j t)^{-1} \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{d+3}{2}} (2^j t)^{-d} (2^j t)^d \left( \left( 1 + \frac{1}{2} \left( \frac{\pi}{2} - |\theta| \right) \right)^d - \left( 1 - \frac{1}{2} \left( \frac{\pi}{2} - |\theta| \right) \right)^d \right) \\
& \lesssim \rho(y,\bar{y}) (2^j t)^{-1} \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{d+1}{2}}.
\end{aligned}$$

For the second integral, if  $\left| \left( \frac{\rho(x,y)}{2^j t} \right)^2 - 1 \right| > \frac{\pi}{2} - |\theta|$ , then  $\frac{\rho(x,y)}{2^j t} \leq 1 + \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{\frac{1}{2}} \leq \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{1}{2}} \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{\frac{1}{2}} + \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{\frac{1}{2}} \lesssim \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{1}{2}} \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{\frac{1}{2}}$ . Since  $\rho(x,y) \geq 3\rho(y,\bar{y})$ , we also have  $\frac{\rho(x,\bar{y})}{2^j t} \cong \frac{\rho(x,y)}{2^j t} \lesssim \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{1}{2}} \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{\frac{1}{2}}$ . Note also that  $r^{-a} s^{-b} \leq r^{-a-b} + s^{-a-b}$  for  $r, s, a, b > 0$ . Then

$$\begin{aligned}
& \int_{\left\{ \left| \left( \frac{\rho(x,y)}{2^j t} \right)^2 - 1 \right| > \frac{\pi}{2} - |\theta| \right\}} \rho(y,\bar{y}) (2^j t)^{-1} \left( \frac{\rho(x,y)}{2^j t} + \frac{\rho(x,\bar{y})}{2^j t} \right) \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{-\frac{1}{2}} \times \\
& \times \sum_{l=0}^d \left| e^{2i\theta} + \left( \frac{\rho(x,\bar{y})}{2^j t} \right)^2 \right|^{\frac{d-l}{2}} \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{\frac{l}{2}} \times \\
& \times \left| e^{2i\theta} + \left( \frac{\rho(x,y)}{2^j t} \right)^2 \right|^{-\frac{d+1}{2}} \left| e^{2i\theta} + \left( \frac{\rho(x,\bar{y})}{2^j t} \right)^2 \right|^{-\frac{d+1}{2}} (2^j t)^{-d} d\mu(x) \\
& \lesssim \int_{\left\{ \left| \left( \frac{\rho(x,y)}{2^j t} \right)^2 - 1 \right| > \frac{\pi}{2} - |\theta| \right\}} \rho(y,\bar{y}) (2^j t)^{-1} \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{1}{2}} \times
\end{aligned}$$

$$\times \left( \left| e^{2i\theta} + \left( \frac{\rho(x, y)}{2^j t} \right)^2 \right|^{-\frac{d+2}{2}} + \left| e^{2i\theta} + \left( \frac{\rho(x, \bar{y})}{2^j t} \right)^2 \right|^{-\frac{d+2}{2}} \right) (2^j t)^{-d} d\mu(x).$$

Now apply Lemma 3.2 with  $a = (2^j t)^{-1}$ ,  $b = \frac{d+2}{2}$ , and for  $y$  and  $\bar{y}$  in place of  $y$ . This gives the estimate

$$\lesssim \rho(y, \bar{y}) (2^j t)^{-1} \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{1}{2} - \frac{d+2}{2} + 1} = \rho(y, \bar{y}) (2^j t)^{-1} \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{d+1}{2}}.$$

Now summing up the estimates for  $j$  gives the desired estimate of (3.8):

$$\begin{aligned} & \sum_{j: 2^j \geq \rho(y, \bar{y})} \int_{\rho(x, y) \geq 3\rho(y, \bar{y})} |k_j(x, y) - k_j(x, \bar{y})| d\mu(x) \\ & \lesssim \sum_{j: 2^j \geq \rho(y, \bar{y})} \rho(y, \bar{y}) (2^j t)^{-1} \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{d+1}{2}} \lesssim \left( \frac{\pi}{2} - |\theta| \right)^{-\frac{d+1}{2}}. \end{aligned}$$

□

**Remark 3.4.** For the Poisson semigroup  $T_z = \exp(-(-\Delta)^{\frac{1}{2}} z)$  on  $X = L^p(\mathbb{R}^d)$ , a better estimate than Proposition 3.3 has been obtained in [8, Theorem 5.1], where it is shown that  $R(\{\exp(-e^{i\theta} 2^k t (-\Delta)^{\frac{1}{2}}) : k \in \mathbb{Z}\}) \lesssim (\frac{\pi}{2} - |\theta|)^{-\alpha}$  for any  $\alpha > \frac{d-1}{2}$ . In the above proof, if in place of (3.5) one has the better estimate  $|k_j(x, y) - k_j(x, \bar{y})| \lesssim \rho(y, \bar{y}) (2^j t)^{-d-1} |e^{2i\theta} + (\frac{\rho(x, \bar{y})}{2^j t})^2|^{-\frac{d+1}{2}}$ , then with Lemma 3.2, one can show the better bound  $R(\{\exp(-e^{i\theta} 2^k t A) : k \in \mathbb{Z}\}) \lesssim (\frac{\pi}{2} - |\theta|)^{-(d-1)/2}$ .

**Corollary 3.5.** Let  $A$  be self-adjoint positive on  $L^2(\Omega)$  with  $\Omega$  a space of homogeneous type satisfying (2.2) such that the Poisson estimates (3.4) and (3.5) are satisfied. Then  $A$  has a bounded  $\mathcal{H}^\alpha$  calculus on  $L^p(\Omega)$  for any  $1 < p < \infty$  and  $\alpha > \frac{d+2}{2}$ . Moreover this calculus is an  $R$ -bounded mapping, i.e.

$$R(f(A) : \|f\|_{\mathcal{H}^\alpha} \leq 1) < \infty.$$

*Proof.* This follows immediately from Propositions 2.1, 3.1 and 3.3. □

**Remark 3.6.** Let us compare our corollary to the results in [10]. There the authors suppose that  $\Omega$  is the smooth boundary of an open connected subset  $\tilde{\Omega}$  of  $\mathbb{R}^{d+1}$  and  $A$  is the Dirichlet-to-Neumann operator defined as follows: Given  $\phi \in L^2(\Omega)$  solve the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \text{ weakly on } \tilde{\Omega} \\ u|_{\Omega} &= \phi \end{aligned}$$

with  $u \in W_2^1(\tilde{\Omega})$ . If  $u$  has a weak normal derivative  $\frac{\partial u}{\partial \nu}$  in  $L^2(\Omega)$ , then  $\phi \in D(A)$  and  $A\phi = \frac{\partial u}{\partial \nu}$ . This operator is a pseudodifferential operator, selfadjoint

on  $L^2(\Omega)$ . Then in [10] it is shown that the semigroup satisfies the following variant of the complex Poisson estimate:

$$|k_z(x, y)| \leq C(\cos \theta)^{-2(d-1)d} \frac{\min(|z|, 1)^{-d}}{\left(1 + \frac{|x-y|}{|z|}\right)^{d+1}}$$

for all  $x, y \in \Omega$  and  $\operatorname{Re} z > 0$ , where  $\theta = \arg z$ . Further a  $\mathcal{H}^\alpha$  calculus for  $A$  with  $\alpha > \frac{d}{2}$  is derived from the complex Poisson bounds in [10, Section 7].

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