



HAL
open science

A non-linear problem involving a critical Sobolev exponent.

Soohyun Bae, Rejeb Hadiji, Francois Vigneron, Habib Yazidi

► **To cite this version:**

Soohyun Bae, Rejeb Hadiji, Francois Vigneron, Habib Yazidi. A non-linear problem involving a critical Sobolev exponent.. *Journal of Mathematical Analysis and Applications*, 2012, 396 (1), pp.98-107. hal-00794199

HAL Id: hal-00794199

<https://hal.science/hal-00794199>

Submitted on 25 Feb 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Non-Linear problem involving critical Sobolev exponent

Soohyun Bae^{*} – Rejeb Hadiji[†] – François Vigneron[†] – Habib Yazidi^{□ †}

^{*} *Hanbat National University
Daejeon 305719 – Republic of Korea.*

[†] *Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050 du CNRS
61, avenue du Général de Gaulle, F-94010 Créteil – France.*

[□] *E.S.S.T.T. Département de Mathématiques
5, Avenue Taha Hssine, Bab Mnar 1008 Tunis – Tunisie.*

Abstract

We study the non-linear minimization problem on $H_0^1(\Omega) \subset L^q$ with $q = \frac{2n}{n-2}$:

$$\inf_{\|u\|_{L^q}=1} \int_{\Omega} (1 + |x|^\beta |u|^k) |\nabla u|^2.$$

We show that minimizers exist only in the range $\beta < kn/q$ which corresponds to a dominant non-linear term. On the contrary, the linear influence for $\beta \geq kn/q$ prevents their existence.

Keywords : *Critical Sobolev exponent, Minimization problem, Non-linear effects.*

AMS classification : 35A01, 35A15, 35J57, 35J62.

1 Introduction

Given a smooth bounded open subset $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, let us consider the minimizing problem

$$S_{\Omega}(\beta, k) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^q(\Omega)}=1}} I_{\Omega; \beta, k}(u) \quad \text{with} \quad I_{\Omega; \beta, k}(u) = \int_{\Omega} p(x, u(x)) |\nabla u(x)|^2 dx \quad (1)$$

and $p(x, y) = 1 + |x|^\beta |y|^k$. Here $q = \frac{2n}{n-2}$ denotes the critical exponent of the Sobolev injection $H_0^1(\Omega) \subset L^q(\Omega)$. We restrict ourselves to the case $\beta \geq 0$ and $0 \leq k \leq q$. The Sobolev injection $H^{s+1}(\Omega)$ into $H^s(\Omega)$ gives :

$$I_{\Omega; \beta, k}(u) \leq \|u\|_{H_0^1(\Omega)}^2 + C_s \left(\sup_{x \in \Omega} |x|^\beta \right) \|u\|_{H^{s+1}(\Omega)}^2 \quad \text{for} \quad s \geq \frac{kn}{q(k+2)}$$

so $I_{\Omega; \beta, k}(u) < \infty$ on a dense subset of $H_0^1(\Omega)$. Note in particular that one can have $I_{\Omega; \beta, k}(u) < \infty$ without having $u \in L_{\text{loc}}^\infty(\Omega)$. If $0 \notin \bar{\Omega}$, the problem is essentially equivalent to the case $\beta = 0$ thus one will also assume from now on that $0 \in \Omega$. The case $0 \in \partial\Omega$ is interesting but will not be addressed in this paper.

For any $u \in H_0^1(\Omega)$, one has

$$I_{\Omega; \beta, k}(u) = I_{\Omega; \beta, k}(|u|) \quad (2)$$

thus, when dealing with (1), one can assume without loss of generality that $u \geq 0$.

The Euler-Lagrange equation formally associated to (1) is

$$\begin{cases} -\operatorname{div}(p(x, u(x))\nabla u) + Q(x, u(x))|\nabla u(x)|^2 = \mu|u(x)|^{q-2}u(x) & \text{in } \Omega \\ u \geq 0 & \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

with $Q(x, y) = \frac{k}{2}|x|^\beta|y|^{k-2}y$ and μ is a Lagrange multiplier. However, the logical relation between (1) and (3) is subtle : $I_{\Omega; \beta, k}$ is not Gateaux differentiable on $H_0^1(\Omega)$ because one can only expect $I_{\Omega; \beta, k}(u) = +\infty$ for a general function $u \in H_0^1(\Omega)$. However, if a minimizer u of (1) belongs to $H_0^1 \cap L^\infty(\Omega)$ then, without restriction, one can assume $u \geq 0$ and for any test-function $\phi \in H_0^1 \cap L^\infty(\Omega)$, one has

$$\forall t \in \mathbb{R}, \quad I_{\Omega; \beta, k} \left(\frac{u + t\phi}{\|u + t\phi\|_{L^q}} \right) < \infty.$$

A finite expansion around $t = 0$ then gives (3) in the weak sense, with the test-function ϕ .

The following generalization of (1) will be adressed in a subsequent paper :

$$S_\Omega(\lambda; \beta, k) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^q(\Omega)}=1}} J_{\Omega; \beta, k}(\lambda, u) \quad \text{with} \quad J_{\Omega; \beta, k}(\lambda, u) = I_{\Omega; \beta, k}(u) - \lambda \int_\Omega |u|^2. \quad (4)$$

for $\lambda > 0$, which is a compact perturbation of the case $\lambda = 0$.

A first motivation can be found in the line of [4] for the study of sharp Sobolev and Gagliardo-Nirenberg inequalities. For example, among other striking results it is shown that, for an arbitrary norm $\|\cdot\|$ on \mathbb{R}^n :

$$\inf_{\|u\|_{L^q}=1} \int_{\mathbb{R}^n} \|\nabla u(x)\|^2 dx = \|\nabla h\|_{L^2} \quad \text{with} \quad h(x) = \frac{1}{(c + \|x\|^2)^{\frac{n-2}{2}}}$$

and a constant c such that $\|h\|_{L^q} = 1$. The problem (1) can be seen as a quasi-linear generalisation where the norm $\|\cdot\|$ measuring ∇u is allowed to depend on u itself.

This type of problem is also a toy model for the Yamabe problem which has been the source of a large literature. The Yamabe invariant of a compact Riemannian manifold (M, g) is :

$$\mathcal{Y}(M) = \inf_{\substack{\phi \in C^\infty(M; \mathbb{R}_+) \\ \|\phi\|_{L^q(M)}=1}} \int_M \left(4 \frac{n-1}{n-2} |\nabla \phi|^2 + \sigma \phi^2 \right) dV_g$$

where ∇ denotes the covariant derivative with respect to g and σ is the scalar curvature of g ; $\mathcal{Y}(M)$ is an invariant of the conformal class \mathcal{C} of (M, g) . One can check easily that $\mathcal{Y}(M) \leq \mathcal{Y}(\mathbb{S}^n)$. The so called *Yamabe problem* which is the question of finding a manifold in \mathcal{C} with constant scalar curvature can be solved if $\mathcal{Y}(M) < \mathcal{Y}(\mathbb{S}^n)$. (see for example [11] for an in-depth review of this historical problem and more precise statements).

Even though problems (1) and (4) seem of much less geometric nature, they should be considered as a toy model of the Yamabe problem that can be played with in \mathbb{R}^n . As it will be shown in this paper, those toy models retain some interesting properties from their geometrical counterpart : the functions u_ε that realise the infimum $\mathcal{Y}(\mathbb{S}^n)$ still play a crucial role in (1) and (4) and the existence of a solution is an exclusively non-linear effect.

Problems that resemble to (1) have an extensive literature and we refer to papers [1], [2], [5], [9] and references therein.

1.1 Bibliographical notes

The case $\beta = k = 0$ *i.e.* a constant weight $p(x, y) = 1$ has been addressed in the celebrated [2] where it is shown in particular that the equation

$$-\Delta u = u^{q-1} + \lambda u, \quad u > 0 \quad (5)$$

has a solution $u \in H_0^1(\Omega)$ if $n \geq 4$ and $0 < \lambda < \lambda_1(\Omega) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{I_{\Omega;0,0}(u)}{\int_{\Omega} |u|^2 dx}$.

On the contrary, for $\lambda = 0$, the problem (5) has no solution if Ω is star-shaped around the origin. In dimension $n = 3$, the situation is more subtle. For example, if $\Omega = \{x \in \mathbb{R}^3; |x| < 1\}$, then (5) admits solutions for $\lambda \in]\frac{\pi^2}{4}, \pi^2[$ but has none if $\lambda \in]0, \frac{\pi^2}{4}[$. See also [6] for the behavior of solutions when $\lambda \rightarrow (\pi^2/4)_+$ and for generalizations to general domains.

A first attempt to the case $\beta \neq 0$ but with $k = 0$ (*i.e.* a weight that does not depend on u , which is the semi-linear case) was achieved in [10]. More precisely, [10] deals with a weight $p \in H^1(\Omega) \cap C(\bar{\Omega})$ that admits a global minimum of the form

$$p(x) = p_0 + c|x - a|^\beta + o(|x - a|^\beta), \quad c > 0.$$

They show that for $n \geq 3$ and $\beta > 0$, there exists $\lambda_0 \geq 0$ such that (4) admits a solution for any $\lambda \in]\lambda_0, \lambda_1[$ where λ_1 is the first eigenvalue of the operator $-\operatorname{div}(p(x)\nabla \cdot)$ in Ω , with Dirichlet boundary conditions (and for $n \geq 4$ and $\beta > 2$, one can check that $\lambda_0 = 0$). On the contrary, the problem (4) admits no solution if $\lambda \leq \lambda'_0$ for some $\lambda'_0 \in [0; \lambda_0]$ or for $\lambda \geq \lambda_1$. See [10] for more precise statements.

Similarly, the semi-linear case in which the minimum value of the weight is achieved in more than one point was studied in [9]; namely in dimension $n \geq 4$ if

$$p^{-1}\left(\inf_{x \in \Omega} p(x)\right) = \{a_0, a_1, \dots, a_N\}$$

then multiple solutions that concentrate around each of the a_j can be found for $\lambda > 0$ small enough.

For $\lambda = 0$ and a star-shaped domain, it is well known (see [2]) that the linear problem $\beta = k = 0$ has no solution. However, when the topology of the domain is not trivial, the problem (3) has at least one solution (see [5] for $\beta = k = 0$; [9] and [10] for $k = 0, \beta \neq 0$).

1.2 Ideas and main results

In this article, the introduction of the fully non-linear term $|x|^\beta |u|^k$ in (1) provides a more unified approach and generates a sharp contrast between sub- and super-critical cases. Moreover, the existence of minimizers will be shown to occur exactly in the sub-cases where the nonlinearity is dominant.

The critical value for (1) can be found by the following scaling argument. As $0 \in \Omega$, the non-linear term tends to concentrate minimizing sequences around $x = 0$. Let us therefore consider the blow-up of $u \in H_0^1(\Omega)$ around $x = 0$. This means one looks at the function v_ε defined by :

$$\forall \varepsilon > 0, \quad u(x) = \varepsilon^{-n/q} v_\varepsilon(x/\varepsilon). \quad (6)$$

One has $v_\varepsilon \in H_0^1(\Omega_\varepsilon)$ with $\Omega_\varepsilon = \{\varepsilon^{-1}y; y \in \Omega\}$ and $\|v_\varepsilon\|_{L^q(\Omega_\varepsilon)} = \|u\|_{L^q(\Omega)}$. Moreover, the definition of q ensures that $2 - n + \frac{2n}{q} = 0$, thus :

$$I_{\Omega;\beta,k}(u) = \int_{\Omega_\varepsilon} \left(1 + \varepsilon^{\beta - \frac{kn}{q}} |y|^\beta |v_\varepsilon(y)|^k\right) |\nabla v_\varepsilon(y)|^2 dy. \quad (7)$$

Depending on the ratio β/k , different situations occur.

- If $\frac{\beta}{k} < \frac{n}{q}$ leading term of the blow-up around $x = 0$ is

$$I_{\Omega;\beta,k}(u) \underset{\varepsilon \rightarrow 0}{\sim} \varepsilon^{-\left(\frac{kn}{q}-\beta\right)} \int_{\Omega_\varepsilon} |y|^\beta |v_\varepsilon(y)|^k |\nabla v_\varepsilon(y)|^2 dy.$$

One can expect the effect of the non-linearity to be dominant and one will show in this paper that (1) admits indeed minimizers in this case.

- If $\frac{\beta}{k} = \frac{n}{q}$ both terms have the same weight and

$$\forall \varepsilon > 0, \quad I_{\Omega;\beta,k}(u) = I_{\Omega_\varepsilon;\beta,k}(v_\varepsilon).$$

One will show that, similarly to the classical case $\beta = k = 0$, the corresponding infimum $S(\beta, k)$ does not depend on Ω and that (1) admits no smooth minimizer.

- If $\frac{\beta}{k} > \frac{n}{q}$, the blow-up around 0 gives

$$I_{\Omega;\beta,k}(u) \underset{\varepsilon \rightarrow 0}{\sim} \int_{\Omega_\varepsilon} |\nabla v_\varepsilon(y)|^2 dy.$$

In this case, one can show that the linear behavior is dominant and that (1) admits no minimizer. Moreover, one can find a common minimizing sequences for both the linear and the non-linear problem. A cheap way to justify this is as follows. The problem (1) tends to concentrate u as a radial decreasing function around the origin. Thus, when $\beta/k > n/q$, one can expect $|u(x)|^q \ll 1/|x|^{\beta q/k}$ because the right-hand side would not be locally integrable while the left-hand side is required to. In turn, this inequality reads $|x|^\beta |u(x)|^k \ll 1$ which eliminates the non-linear contribution in the minimizing problem (1).

The infimum for the classical problem with $\beta = k = 0$ is (see *e.g.* [2]) :

$$S = \inf_{\substack{w \in H_0^1(\Omega) \\ \|w\|_{L^q} = 1}} \int_{\Omega} |\nabla w|^2 \tag{8}$$

which does not depend on Ω . Let us now state the main Theorem concerning (1).

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded domain with $n \geq 3$ and $q = \frac{2n}{n-2}$ the critical exponent for the Sobolev injection $H_0^1(\Omega) \subset L^q(\Omega)$.*

1. *If $0 \leq \beta < kn/q$ then $S_\Omega(\beta, k) > S$ and the infimum for $S_\Omega(\beta, k)$ is achieved.*
2. *If $\beta = kn/q$ then $S_\Omega(\beta, k)$ does not depend on Ω and $S_\Omega(\beta, k) \geq S$. Moreover, if Ω is star-shaped around $x = 0$, then the minimizing problem (1) admits no minimizers in the class :*

$$H_0^1 \cap H^{3/2} \cap L^\infty(\Omega).$$

If $k < 1$, the negative result holds, provided additionally $u^{k-1} \in L^n(\Omega)$.

3. *If $\beta > kn/q$ then $S_\Omega(\beta, k) = S$ and the infimum for $S_\Omega(\beta, k)$ is not achieved in $H_0^1(\Omega)$.*

Remarks.

1. In the first case, one has $k > 0$, thus results concerning $k = 0$ (such as those of *e.g.* [9] and [10]) are included either in our second or third case.
2. If the minimizing problem (1) is achieved for $u \in H_0^1(\Omega)$, then $|u|$ is a positive minimizer. In particular, if $\beta < kn/q$, the problem always admits positive minimizers.
3. In the critical case $\beta = kn/q$, it is not known whether a non-smooth minimizer could exist in $H_0^1 \setminus (H^{3/2} \cap L^\infty)$. Such a minimizer could have a non-constant sign.

This paper is organized as follows. In section 2 we establish existence of minimizers of (1) for the subcritical case. Section 3 and Section 4 are devoted to study respectively the case $\beta > kn/q$ and the critical case.

2 Subcritical case ($0 \leq \beta < kn/q$) : existence of minimizers

The case $\beta < kn/q$ is especially interesting because it reveals that the non-linear weight $|u|^k$ helps for the existence of a minimizer. Note that $k > 0$ throughout this section.

Proposition 2 *If $0 \leq \frac{\beta}{k} < \frac{n}{q}$, the minimization problem (1) has at least one solution $u \in H_0^1(\Omega)$. Moreover, one has*

$$S_\Omega(\beta, k) > S \tag{9}$$

where S is defined by (8).

Proof. Let us prove first that the existence of a solution implies the strict inequality in (9). By contradiction, if $S_\Omega(\beta, k) = S$ and if u is a minimizer for (1) thus $u \not\equiv 0$, one has

$$S = \int_\Omega (1 + |x|^\beta |u(x)|^k) |\nabla u(x)|^2 dx > \int_\Omega |\nabla u(x)|^2 dx$$

which contradicts the definition of S . Thus, if the minimization problem has a solution, the strict inequality (9) must hold.

Let us prove now that (1) has at least one solution $u \in H_0^1(\Omega)$. Let $(u_j)_{j \in \mathbb{N}} \in H_0^1(\Omega)$ be a minimizing sequence for (1), *i.e.* :

$$I_{\Omega; \beta, k}(u_j) = S_\Omega(\beta, k) + o(1), \quad \text{and} \quad \|u_j\|_{L^q} = 1.$$

As noticed in the introduction, one can assume without restriction that $u_j \geq 0$. Up to a subsequence, still denoted by u_j , there exists $u \in H_0^1(\Omega)$ such that $u_j(x) \rightarrow u(x)$ for almost every $x \in \Omega$ and such that :

$$\begin{aligned} u_j &\rightharpoonup u \quad \text{weakly in } H_0^1 \cap L^q(\Omega), \\ u_j &\rightarrow u \quad \text{strongly in } L^\ell(\Omega) \text{ for any } \ell < q. \end{aligned}$$

The idea of the proof is to introduce $v_j = u_j^{\frac{k}{2}+1}$ and prove that v_j is a bounded sequence in $W_0^{1,r} \subset L^p$ for indices r and p such that

$$p \left(\frac{k}{2} + 1 \right) \geq q.$$

The key point is the formula :

$$I_{\Omega; \beta, k}(u_j) = \int_\Omega |\nabla u_j|^2 + \left(\frac{k}{2} + 1 \right)^{-2} \int_\Omega |x|^\beta |\nabla v_j|^2 \tag{10}$$

which gives “almost” an H_0^1 bound on v_j (and does indeed if $\beta = 0$). For $r \in [1, 2[$, one has :

$$\int_{\Omega} |\nabla v_j|^r \leq \left(\int_{\Omega} |x|^\beta |\nabla v_j|^2 dx \right)^{r/2} \left(\int_{\Omega} |x|^{-\frac{\beta r}{2-r}} dx \right)^{1-r/2}$$

The integral in the right-hand side is bounded provided $\frac{\beta r}{2-r} < n$. All the previous conditions are satisfied if one can find r such that :

$$1 \leq r < 2, \quad \beta < n \left(\frac{2}{r} - 1 \right), \quad \frac{k}{2} + 1 > \frac{q}{p_0} = q \left(\frac{1}{r} - \frac{1}{n} \right).$$

This system of inequalities boils down to :

$$1 \leq r < 2, \quad \frac{\beta}{n} < \frac{2}{r} - 1 < \frac{2}{q} \left(\frac{k}{2} + 1 + \frac{q}{n} \right) - 1$$

which is finally equivalent to $\beta < kn/q$ provided $k \leq q$. Using the compactity of the inclusion $W_0^{1,r} \subset L^p$ for $p < p_0$ and up to a subsequence, one has $v_j \rightarrow v = u^{\frac{k}{2}+1}$ strongly in L^p (in particular for $p = \frac{q}{k/2+1}$). Finally, as $u_j \geq 0$ and $u \geq 0$, one has :

$$|u_j - u|^q \leq C \left| u_j^q - u^q \right| = C \left| v_j^{q/(k/2+1)} - v^{q/(k/2+1)} \right|$$

and thus $u_j \rightarrow u$ strongly in L^q . One gets $\|u\|_{L^q} = 1$. The following compactity result then implies that u is a minimizer for (1). \blacksquare

Proposition 3 *If $u_j \in H_0^1(\Omega)$ is a minimizing sequence for (1) with $\|u_j\|_{L^q(\Omega)} = 1$ and such that*

$$u_j \rightarrow u \quad \text{in } L^2(\Omega), \quad \text{and} \quad \nabla u_j \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega),$$

the weak limit $u \in H_0^1(\Omega)$ is a minimizer of the problem (1) if and only if $\|u\|_{L^q(\Omega)} = 1$.

Proof. It is an consequence of the main Theorem of [7, p. 77] (see also [14]) applied to the function :

$$f(x, z, p) = (1 + |x|^\beta |z|^k) |p|^2$$

which is positive, measurable on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, continuous with respect to z , convex with respect to p . Then

$$I(u) = \int_{\Omega} f(x, u, \nabla u) \leq \liminf_{j \rightarrow \infty} \int_{\Omega} f(x, u_j, \nabla u_j) = \liminf_{j \rightarrow \infty} I(u_j).$$

If u_j is a minimizing sequence, then $I(u) = S_{\Omega}(\beta, k)$ and u is a minimizer if and only if $\|u\|_{L^q} = 1$. \blacksquare

Remarks

- The sequence u_j converges strongly in $H_0^1(\Omega)$ towards u because $\nabla u_j \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$ and :

$$\int_{\Omega} |\nabla u_j|^2 - \int_{\Omega} |\nabla u|^2 = I(u_j) - I(u) + \int_{\Omega} |x|^\beta u^k |\nabla u|^2 - \int_{\Omega} |x|^\beta u_j^k |\nabla u_j|^2.$$

Using again Theorem of [7, p. 77] with $\tilde{f}(x, z, p) = |x|^\beta |z|^k |p|^2$ provides

$$\forall j \in \mathbb{N}, \quad \int_{\Omega} |\nabla u_j|^2 \leq \int_{\Omega} |\nabla u|^2 + o(1)$$

and Fatou’s lemma provides the converse inequality.

- This proof implies also that $S_{\Omega}(\beta, k)$ is continuous with respect to (β, k) in the range $0 \leq \beta < kn/q$ and that the corresponding minimizer depends continuously on (β, k) in $L^q(\Omega)$ and $H_0^1(\Omega)$.

3 Semi-linear case ($\beta > kn/q$) : non-compact minimizing sequence

When $\beta > kn/q$, the problem (1) is under the total influence of the linear problem (8). Let us recall that its minimizer S is independent of the smooth bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) and that this minimizing problem has no solution. According to [2], a minimizing sequence of (8) is given by $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$ where :

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{n-2}{4}} \zeta(x)}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}} \quad (11)$$

with $\zeta \in C^\infty(\bar{\Omega}; [0, 1])$ is a smooth compactly supported cutoff function that satisfy $\zeta(x) = 1$ in a small neighborhood of the origin in Ω . Recall that $\frac{n-2}{2} = n/q$. Recall that $(k+1)(n-2) > kn/q$ for any $k \geq 0$. We know from [2] that

$$\|\nabla u_\varepsilon\|_{L^2}^2 = K_1 + O(\varepsilon^{\frac{n-2}{2}}), \quad \|u_\varepsilon\|_{L^q}^2 = K_2 + o(\varepsilon^{\frac{n-2}{2}})$$

and that $S = K_1/K_2$.

The goal of this section is the proof of the following Proposition.

Proposition 4 *If $\frac{\beta}{k} > \frac{n}{q}$, one has*

$$S_\Omega(\beta, k) = S \quad (12)$$

and the problem (1) admits no minimizer in $H_0^1(\Omega)$. Moreover, the sequence $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$ defined by (11) is a minimizing sequence for both (1) and the linear problem (8).

Proof. Suppose by contradiction that (1) is achieved by $u \in H_0^1(\Omega)$. Then $u \neq 0$ and therefore the following strict inequality holds :

$$S \leq \int_{\Omega} |\nabla u|^2 < I_{\Omega; \beta, k}(u) = S_\Omega(\beta, k).$$

Therefore the identity (12) implies that (1) has no minimizer. To prove (12) and the rest of the statement, it is sufficient to show that

$$I_{\Omega; \beta, k} \left(\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon \right) = S + o(1) \quad (13)$$

in the limit $\varepsilon \rightarrow 0$, because one obviously has $S \leq S_\Omega(\beta, k) \leq I_{\Omega; \beta, k}(\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon)$. The limit (13) will follow immediately from the next result. \blacksquare

Proposition 5 *With the previous notations, (13) holds and more precisely, as $\varepsilon \rightarrow 0$, one has :*

$$\int_{\Omega} |x|^\beta |u_\varepsilon|^k |\nabla u_\varepsilon|^2 dx = \begin{cases} C \varepsilon^{\frac{2\beta - k(n-2)}{4}} + o\left(\varepsilon^{\frac{2\beta - k(n-2)}{4}}\right) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log \varepsilon|\right) & \text{if } \beta = (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}}\right) & \text{if } \beta > (k+1)(n-2) \end{cases} \quad (14)$$

with $C = \int_{\mathbb{R}^n} \frac{|x|^{\beta+2}}{(1+|x|^2)^{\frac{kn-2}{2}+n}} dx$ and thus :

$$I_{\Omega; \beta, k} \left(\frac{u_\varepsilon}{\|u_\varepsilon\|_{L^q}} \right) = S + \begin{cases} \frac{C}{K_2^{k/2+1}} \varepsilon^{\frac{2\beta - k(n-2)}{4}} + o\left(\varepsilon^{\frac{2\beta - k(n-2)}{4}}\right) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log \varepsilon|\right) & \text{if } \beta = (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}}\right) & \text{if } \beta > (k+1)(n-2). \end{cases} \quad (15)$$

Proof. The only verification is that of (14).

$$\begin{aligned}
\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&+ \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^k |\nabla \zeta(x)|^2 |x|^{\beta}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&- 2(n-2) \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+1} |x|^{\beta} \nabla \zeta(x) \cdot x}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n - 1}} dx.
\end{aligned}$$

Since $\zeta \equiv 1$ on a neighborhood of 0 and using the Dominated Convergence Theorem, a direct computation gives

$$\begin{aligned}
\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&+ O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).
\end{aligned}$$

Here we will consider the following three subcases.

1. Case $\beta < (k+1)(n-2)$

$$\begin{aligned}
\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx &= \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx - \int_{\mathbb{R}^n \setminus \Omega} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&+ \int_{\Omega} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} (|\zeta(x)|^{k+2} - 1) |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx.
\end{aligned}$$

Using the Dominated Convergence Theorem, and the fact that $\zeta \equiv 1$ on a neighborhood of 0, one obtains

$$\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx = \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).$$

By a simple change of variable, one gets

$$\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx = \varepsilon^{\frac{2\beta - k(n-2)}{4}} \int_{\mathbb{R}^n} \frac{|y|^{\beta+2}}{(1 + |y|^2)^{\frac{k(n-2)}{2} + n}} dy + o(\varepsilon^{\frac{2\beta - k(n-2)}{4}})$$

which gives (14) in this case.

2. Case $\beta = (k+1)(n-2)$

$$\begin{aligned}
\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}) \\
&= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{(|\zeta(x)|^{k+2} - 1) |x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx \\
&\quad + (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}) \\
&= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}})
\end{aligned}$$

One has, for some constants $R_1 < R_2$:

$$\int_{B(0,R_1)} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx \leq \int_{\Omega} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx \leq \int_{B(0,R_2)} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx$$

with

$$\begin{aligned}
\int_{B(0,R)} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx &= \omega_n \int_0^R \frac{r^{k(n-2)+2n-1}}{(\varepsilon + r^2)^{k\frac{(n-2)}{2}+n}} dr \\
&= \frac{1}{2} \omega_n |\log \varepsilon| + O(1).
\end{aligned}$$

Consequently, one has :

$$\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx = O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log \varepsilon|\right).$$

3. Case $\beta > (k+1)(n-2)$

$$\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx = (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).$$

One can apply the Dominated Convergence Theorem :

$$\frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} \longrightarrow |\zeta(x)|^{k+2} |x|^{\beta-(k(n-2)+2n-2)} \quad \text{when } \varepsilon \rightarrow 0$$

and

$$\frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} \leq |\zeta(x)|^{k+2} |x|^{\beta-(k(n-2)+2n-2)} \in L^1(\Omega).$$

So, it follows that

$$\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx = O(\varepsilon^{\frac{(k+2)(n-2)}{4}})$$

which again is (14). ■

4 The critical case ($\beta = kn/q$) : non-existence of smooth minimizers

The critical case is a natural generalization of the well known problem with $\beta = k = 0$. In this section, the following result will be established.

Proposition 6 *If $\beta = kn/q$, one has*

$$S_{\Omega}(\beta, k) = S_{\tilde{\Omega}}(\beta, k) \quad (16)$$

for any two smooth neighborhoods $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ of the origin. Moreover, if Ω is star-shaped around $x = 0$, the minimization problem (1) admits no solution in the class :

$$H_0^1 \cap H^{3/2} \cap L^\infty(\Omega).$$

If $k < 1$, the negative result holds, provided additionally $u^{k-1} \in L^n(\Omega)$.

The rest of this section is devoted to the proof of this statement. Note that if the minimization problem (1) had a minimizer u with non constant sign in this class of regularity, then $|u|$ would be a positive minimizer in the same class, thus it is sufficient to show that there are no positive minimizers.

4.1 $S_{\Omega}(\beta, k)$ does not depend on the domain

If $\Omega \subset \Omega'$, there is a natural injection $i : H_0^1(\Omega) \hookrightarrow H_0^1(\Omega')$ that corresponds to the process of extension by zero. Let $u_j \in H_0^1(\Omega)$ be a minimizing sequence for $S_{\Omega}(\beta, k)$. Then $\|i(u_j)\|_{L^q(\Omega')} = 1$ thus

$$S_{\Omega'}(\beta, k) \leq I_{\Omega';\beta,k}(i(u_j)) = I_{\Omega;\beta,k}(u_j)$$

and therefore $S_{\Omega'}(\beta, k) \leq S_{\Omega}(\beta, k)$.

Conversely, let us now consider the scaling transformation (6) which, in the case of $\frac{\beta}{k} = \frac{n}{q}$, leaves both $\|u\|_{L^q(\Omega)}$ and $I_{\Omega;\beta,k}(u)$ invariant. If u_j is a minimizing sequence on Ω then $v_j = u_{j,\lambda^{-1}}$ is an admissible sequence on Ω_λ thus :

$$S_{\Omega_\lambda}(\beta, k) \leq I_{\Omega_\lambda;\beta,k}(v_j) = I_{\Omega;\beta,k}(u_j) \rightarrow S_{\Omega}(\beta, k).$$

Conversely, if v_j is a minimizing sequence on Ω_λ then $u_j = v_{j,\lambda}$ is an admissible sequence on Ω and :

$$S_{\Omega}(\beta, k) \leq I_{\Omega;\beta,k}(u_j) = I_{\Omega_\lambda;\beta,k}(v_j) \rightarrow S_{\Omega_\lambda}(\beta, k).$$

This ensures that $S_{\Omega_\lambda}(\beta, k) = S_{\Omega}(\beta, k)$ for any $\lambda > 0$.

Finally, given two smooth bounded open subsets Ω and $\tilde{\Omega}$ of \mathbb{R}^n that both contain 0, one can find $\lambda, \mu > 0$ such that $\Omega_\lambda \subset \tilde{\Omega} \subset \Omega_\mu$ and the previous inequalities read

$$S_{\Omega_\mu}(\beta, k) \leq S_{\tilde{\Omega}}(\beta, k) \leq S_{\Omega_\lambda}(\beta, k) \quad \text{and} \quad S_{\Omega}(\beta, k) = S_{\Omega_\lambda}(\beta, k) = S_{\Omega_\mu}(\beta, k)$$

thus ensuring $S_{\Omega}(\beta, k) = S_{\tilde{\Omega}}(\beta, k)$.

4.2 Pohozaev identity and the non-existence of smooth minimizers

Suppose by contradiction that a bounded minimizer u of (1) exists for some star-shaped domain Ω with $\beta = kn/q$, i.e. $u \in H_0^1 \cap L^\infty(\Omega)$. As mentioned in the introduction $|u|$ is also a minimizer thus, without loss of generality, one can also assume that $u \geq 0$. Moreover, u will satisfy the Euler-Lagrange equation (3) in the weak sense, for any test-function in $H_0^1 \cap L^\infty(\Omega)$.

In the following argument, inspired by [13], one will use $(x \cdot \nabla)u$ and u as test functions. The later is fine but the former must be checked out carefully. A brutal assumption like $(x \cdot \nabla)u \in H_0^1 \cap L^\infty(\Omega)$ is much too restrictive. Let us assume instead that

$$u \in H_0^1 \cap H^{3/2} \cap L^\infty \quad \text{and (if } k < 1) \quad u^{k-1} \in L^n(\Omega). \quad (17)$$

Note that if $v \in H^{3/2}$ then $|v| \in H^{3/2}$ thus the assumption $u \geq 0$ still holds without loss of generality. Then one can find a sequence $\phi_n \in H_0^1 \cap L^\infty(\Omega)$ such that $\phi_n \rightarrow \phi = (x \cdot \nabla)u$ in $H^{1/2}(\Omega)$ and almost everywhere and such that each sequence of integrals converges to the expected limit :

$$\begin{aligned} (-\Delta u|\phi_n) &\rightarrow (-\Delta u|\phi), & (u^k|\phi_n) &\rightarrow (u^k|\phi) \\ (u^{k-1}\nabla u|\phi_n) &\rightarrow (u^{k-1}\nabla u|\phi) & \text{and} & \quad (u^{q-1}|\phi_n) \rightarrow (u^{q-1}|\phi). \end{aligned}$$

Indeed, each integral satisfies a domination assumption :

$$\begin{aligned} |(-\Delta u|\phi_n - \phi)| &\leq \|u\|_{H^{3/2}} \|\phi_n - \phi\|_{H^{1/2}}, \\ |(u^k|\phi_n - \phi)| &\leq \|u^k\|_{L^{2n/(n+1)}} \|\phi_n - \phi\|_{L^{2n/(n-1)}} \leq C_\Omega \|u\|_{L^\infty}^k \|\phi_n - \phi\|_{H^{1/2}}, \\ |(u^{k-1}\nabla u|\phi_n - \phi)| &\leq \begin{cases} \|u\|_{L^\infty}^{k-1} \|\nabla u\|_{L^2} \|\phi_n - \phi\|_{L^2} & \text{if } k \geq 1, \\ \|u^{k-1}\|_{L^n} \|\nabla u\|_{L^{2n/(n-1)}} \|\phi_n - \phi\|_{L^{2n/(n-1)}} \\ \leq C_\Omega \|u^{k-1}\|_{L^n} \|u\|_{H^{3/2}} \|\phi_n - \phi\|_{H^{1/2}} & \text{if } k < 1, \end{cases} \\ |(u^{q-1}|\phi_n - \phi)| &\leq \|u^{q-1}\|_{L^{2n/(n+1)}} \|\phi_n - \phi\|_{L^{2n/(n-1)}} \leq C_\Omega \|u\|_{L^\infty}^{q-1} \|\phi_n - \phi\|_{H^{1/2}}. \end{aligned}$$

Thus, the Euler-Lagrange is also satisfied in the weak sense for the test-function $\phi = (x \cdot \nabla)u$.

Let us multiply by $(x \cdot \nabla)u$ and integrate by parts :

$$-\int_\Omega \operatorname{div}(p(x, u)\nabla u) \times (x \cdot \nabla)u + \frac{k}{2} \int_\Omega |x|^\beta |u|^{k-2} |\nabla u|^2 u (x \cdot \nabla)u = \mu \int_\Omega |u|^{q-2} u (x \cdot \nabla)u.$$

An integration by part in the right-hand side and the condition $u \in H_0^1(\Omega)$ provide :

$$\mu \int_\Omega |u|^{q-2} u (x \cdot \nabla)u = -\mu \frac{n-2}{2} \int_\Omega |u|^q = -\frac{n}{q} \mu.$$

The first term of the left-hand side is :

$$-\int_\Omega \operatorname{div}(p(x, u)\nabla u) \times (x \cdot \nabla)u = B(u) + \int_\Omega p(x, u) |\nabla u|^2 - \int_{\partial\Omega} p(x, u) (x \cdot \nabla)u \frac{\partial u}{\partial \nu}$$

with $B(u)$ define as follows and dealt with by a second integration by part

$$\begin{aligned} B(u) &= \sum_{i,j} \int_\Omega x_j \left(1 + |x|^\beta |u|^k\right) (\partial_i u)(\partial_i \partial_j u) \\ &= -B(u) - n \int_\Omega p(x, u) |\nabla u|^2 - \beta \int_\Omega |x|^\beta |u|^k |\nabla u|^2 \\ &\quad - k \int_\Omega |x|^\beta |u|^{k-2} |\nabla u|^2 u (x \cdot \nabla)u + \int_{\partial\Omega} p(x, u) |\nabla u|^2 (x \cdot \mathbf{n}). \end{aligned}$$

On the boundary, $p(x, u) = 1$ and as $u \in H_0^1(\Omega)$, one has also $\nabla u = \frac{\partial u}{\partial \nu} \mathbf{n}$ where \mathbf{n} denotes the normal unit vector to $\partial\Omega$ and in particular $|\nabla u| = \left|\frac{\partial u}{\partial \nu}\right|$, thus

$$B(u) = -\frac{n}{2} \int_\Omega p(x, u) |\nabla u|^2 - \frac{\beta}{2} \int_\Omega |x|^\beta |u|^k |\nabla u|^2 - \frac{k}{2} \int_\Omega |x|^\beta |u|^{k-2} |\nabla u|^2 u (x \cdot \nabla)u + \frac{1}{2} \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 (x \cdot \mathbf{n}).$$

The whole energy estimate with $(x \cdot \nabla)u$ boils down to :

$$\frac{n-2}{2} \int_{\Omega} p(x, u) |\nabla u|^2 + \frac{\beta}{2} \int_{\Omega} |x|^{\beta} |u|^k |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 (x \cdot \mathbf{n}) = \frac{n}{q} \mu.$$

Finally, to deal with the first term, let us multiply (3) by u and integrate by parts ; one gets :

$$\int_{\Omega} p(x, u) |\nabla u|^2 = \int_{\Omega} (1 + |x|^{\beta} |u|^k) |\nabla u|^2 = -\frac{k}{2} \int_{\Omega} |x|^{\beta} |u|^k |\nabla u|^2 + \mu.$$

Combining both estimates provides :

$$\frac{1}{2} \left(\beta - \frac{kn}{q} \right) \int_{\Omega} |x|^{\beta} |u|^k |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 (x \cdot \mathbf{n}) = 0. \quad (18)$$

As $\beta = kn/q$ and $x \cdot \mathbf{n} > 0$ (Ω is star-shaped), one gets $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

The Euler-Lagrange equation (3) now reads :

$$-p(x, u) \Delta u = \frac{k}{2} |x|^{\beta} |u|^{k-2} u |\nabla u|^2 + \beta |x|^{\beta-2} |u|^k (x \cdot \nabla) u + \mu |u|^{q-2} u$$

which for $u \geq 0$ boils down to

$$\begin{aligned} -p(x, u) \Delta u &= |x|^{\beta-2} u^{k-1} \left(\frac{k}{2} |x|^2 |\nabla u|^2 + u (x \cdot \nabla) u \right) + \mu u^{q-1} \\ &= |x|^{\beta-2} u^{k-1} \left(\sqrt{\frac{k}{2}} |x| |\nabla u + Cux \right)^2 - C^2 |x|^{\beta} u^{k+1} + \mu u^{q-1} \end{aligned}$$

with $2\sqrt{k/2}C = \beta$. For any $t \in \mathbb{R}$, one has therefore :

$$-\Delta u + tu = \frac{|x|^{\beta-2} u^{k-1}}{p(x, u)} \left(\sqrt{\frac{k}{2}} |x| |\nabla u + Cux \right)^2 + \frac{\mu u^{q-1}}{p(x, u)} + tu - \frac{C^2 |x|^{\beta} u^{k+1}}{p(x, u)} = f(t, x).$$

As $u \in L^{\infty}$, one can chose $t > C^2 |x|^{\beta} \|u\|_{L^{\infty}}^k$. Then $f(t, x) \geq 0$ and the maximum principle implies that either $u = 0$ or $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$. In particular, only the solution $u = 0$ satisfies simultaneously Dirichlet and Neumann boundary conditions, which leads to a contradiction because $\|u\|_{L^q} = 1$.

Remarks

1. Note that Pohozaev identity (18) prevents the existence of minimizers when $\beta \geq kn/q$. However, the technique we used in §3 (when $\beta > kn/q$) enlightens the leading term of the problem and avoids dealing with artificial regularity assumptions.
2. Similarly, one could check that the computation is also correct if

$$u \in H_0^1 \cap H^2 \cap L^{\infty}(\Omega) \quad \text{and (if } k < 1) \quad u^{k-1} \in L^{n/2}. \quad (19)$$

Assumption (19) is only preferable over (17) for $k < 1$. But it requires additional regularity in the interior of Ω and would not allow to assume $u \geq 0$ without loss of generality because in general, $v \in H^2 \not\Rightarrow |v| \in H^2$.

Corollary 7 (Thanks to the referee)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a smooth bounded open set containing 0.

If $\beta = kn/q$ then the minimization problem (1) admits no solution in the class

$$H^1 \cap H_0^{3/2} \cap L^\infty(\Omega).$$

Proof. Take $R > 0$ such that $\Omega \subset B(0, R)$.

Suppose by contradiction that u is a minimizing solution of (1) such that $u \in H^1(\Omega) \cap H_0^{3/2}(\Omega) \cap L^\infty(\Omega)$. Extend u by 0 to $B(0, R)$, we obtain a minimizing solution of (1) such that $u \in H^1(B(0, R)) \cap H_0^{3/2}(B(0, R)) \cap L^\infty(B(0, R))$. Now, arguing as in the proof of Proposition 6, we obtain a contradiction. ■

References

- [1] Aubin, T., Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. J. Math. Pures Appl. (9) 55 (1976), no. 3, 269-296.
- [2] Brezis, H.; Nirenberg, L., Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math. 36 (1983), no. 4, 437-477.
- [3] Brezis, H.; Lieb, E., A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc. 88 (1983), no. 3, 486-490.
- [4] Cordero-Erausquin, D.; Nazaret, B.; Villani, C., A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. Adv. Math. 182 (2004), no. 2, 307-332.
- [5] Coron, J.-M., Topologie et cas limite des injections de Sobolev. C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), no. 7, 209-212.
- [6] Crouau, R.; Hadiji, R.; Lewandowski, R., Critical Sobolev exponent and the dimension three. Houston J. Math. 18 (1992), no. 2, 189-204.
- [7] Eisen, G., A selection lemma for sequences of measurable sets, and lower semicontinuity of multiple integrals. Manuscripta Math. 27 (1979), no. 1, 73-79.
- [8] Hadiji, R., Solutions positives de l'équation $-\Delta u = u^p + \mu u^q$ dans un domaine à trou. Ann. Fac. Sci. Toulouse Math. (5) 11 (1990), no. 3, 55-71.
- [9] Hadiji, R.; Molle, R.; Passaseo, D.; Yazidi, H., Localization of solutions for nonlinear elliptic problems with critical growth. C. R. Math. Acad. Sci. Paris 343 (2006), no. 11-12, 725-730.
- [10] Hadiji, R.; Yazidi, H., Problem with critical Sobolev exponent and with weight. Chin. Ann. Math. Ser. B 28 (2007), no. 3, 327-352.
- [11] Lee, J.M.; Parker, T.H., The Yamabe problem. Bull. Amer. Math. Soc. 17 (1987), no. 1, 37-91.
- [12] Lions, P.-L., The concentration compactness principle in the calculus of variations. The limit case. I. Rev. Mat. Iberoamericana 1 (1985), no. 1, 145-201.
- [13] Pohozaev, S. I., On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. Dokl. Akad. Nauk SSSR 165 (1965), 36-39.
- [14] Struwe, M., Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. Fourth edition. A Series of Modern Surveys in Mathematics, 34. Springer-Verlag, Berlin, 2008.