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► **To cite this version:**

Soohyun Bae, Rejeb Hadiji, Francois Vigneron, Habib Yazidi. A non-linear problem involving a critical Sobolev exponent.. *Journal of Mathematical Analysis and Applications*, 2012, 396 (1), pp.98-107. hal-00794199

**HAL Id: hal-00794199**

**<https://hal.science/hal-00794199v1>**

Submitted on 25 Feb 2013

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# A Non-Linear problem involving critical Sobolev exponent

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## Abstract

We study the non-linear minimization problem on  $H_0^1(\Omega) \subset L^q$  with  $q = \frac{2n}{n-2}$  :

$$\inf_{\|u\|_{L^q}=1} \int_{\Omega} (1 + |x|^\beta |u|^k) |\nabla u|^2.$$

We show that minimizers exist only in the range  $\beta < kn/q$  which corresponds to a dominant non-linear term. On the contrary, the linear influence for  $\beta \geq kn/q$  prevents their existence.

**Keywords :** *Critical Sobolev exponent, Minimization problem, Non-linear effects.*

**AMS classification :** 35A01, 35A15, 35J57, 35J62.

## 1 Introduction

Given a smooth bounded open subset  $\Omega \subset \mathbb{R}^n$  with  $n \geq 3$ , let us consider the minimizing problem

$$S_{\Omega}(\beta, k) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^q(\Omega)}=1}} I_{\Omega; \beta, k}(u) \quad \text{with} \quad I_{\Omega; \beta, k}(u) = \int_{\Omega} p(x, u(x)) |\nabla u(x)|^2 dx \quad (1)$$

and  $p(x, y) = 1 + |x|^\beta |y|^k$ . Here  $q = \frac{2n}{n-2}$  denotes the critical exponent of the Sobolev injection  $H_0^1(\Omega) \subset L^q(\Omega)$ . We restrict ourselves to the case  $\beta \geq 0$  and  $0 \leq k \leq q$ . The Sobolev injection  $H^{s+1}(\Omega)$  into  $H^s(\Omega)$  gives :

$$I_{\Omega; \beta, k}(u) \leq \|u\|_{H_0^1(\Omega)}^2 + C_s \left( \sup_{x \in \Omega} |x|^\beta \right) \|u\|_{H^{s+1}(\Omega)}^2 \quad \text{for} \quad s \geq \frac{kn}{q(k+2)}$$

so  $I_{\Omega; \beta, k}(u) < \infty$  on a dense subset of  $H_0^1(\Omega)$ . Note in particular that one can have  $I_{\Omega; \beta, k}(u) < \infty$  without having  $u \in L_{\text{loc}}^\infty(\Omega)$ . If  $0 \notin \bar{\Omega}$ , the problem is essentially equivalent to the case  $\beta = 0$  thus one will also assume from now on that  $0 \in \Omega$ . The case  $0 \in \partial\Omega$  is interesting but will not be addressed in this paper.

For any  $u \in H_0^1(\Omega)$ , one has

$$I_{\Omega; \beta, k}(u) = I_{\Omega; \beta, k}(|u|) \quad (2)$$

thus, when dealing with (1), one can assume without loss of generality that  $u \geq 0$ .

The Euler-Lagrange equation formally associated to (1) is

$$\begin{cases} -\operatorname{div}(p(x, u(x))\nabla u) + Q(x, u(x))|\nabla u(x)|^2 = \mu|u(x)|^{q-2}u(x) & \text{in } \Omega \\ u \geq 0 \\ u = 0 \end{cases} \quad \text{on } \partial\Omega \quad (3)$$

with  $Q(x, y) = \frac{k}{2}|x|^\beta|y|^{k-2}y$  and  $\mu$  is a Lagrange multiplier. However, the logical relation between (1) and (3) is subtle :  $I_{\Omega; \beta, k}$  is not Gateaux differentiable on  $H_0^1(\Omega)$  because one can only expect  $I_{\Omega; \beta, k}(u) = +\infty$  for a general function  $u \in H_0^1(\Omega)$ . However, if a minimizer  $u$  of (1) belongs to  $H_0^1 \cap L^\infty(\Omega)$  then, without restriction, one can assume  $u \geq 0$  and for any test-function  $\phi \in H_0^1 \cap L^\infty(\Omega)$ , one has

$$\forall t \in \mathbb{R}, \quad I_{\Omega; \beta, k} \left( \frac{u + t\phi}{\|u + t\phi\|_{L^q}} \right) < \infty.$$

A finite expansion around  $t = 0$  then gives (3) in the weak sense, with the test-function  $\phi$ .

The following generalization of (1) will be adressed in a subsequent paper :

$$S_\Omega(\lambda; \beta, k) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^q(\Omega)}=1}} J_{\Omega; \beta, k}(\lambda, u) \quad \text{with} \quad J_{\Omega; \beta, k}(\lambda, u) = I_{\Omega; \beta, k}(u) - \lambda \int_\Omega |u|^2. \quad (4)$$

for  $\lambda > 0$ , which is a compact perturbation of the case  $\lambda = 0$ .

A first motivation can be found in the line of [4] for the study of sharp Sobolev and Gagliardo-Nirenberg inequalities. For example, among other striking results it is shown that, for an arbitrary norm  $\|\cdot\|$  on  $\mathbb{R}^n$  :

$$\inf_{\|u\|_{L^q}=1} \int_{\mathbb{R}^n} \|\nabla u(x)\|^2 dx = \|\nabla h\|_{L^2} \quad \text{with} \quad h(x) = \frac{1}{(c + \|x\|^2)^{\frac{n-2}{2}}}$$

and a constant  $c$  such that  $\|h\|_{L^q} = 1$ . The problem (1) can be seen as a quasi-linear generalisation where the norm  $\|\cdot\|$  measuring  $\nabla u$  is allowed to depend on  $u$  itself.

This type of problem is also a toy model for the Yamabe problem which has been the source of a large literature. The Yamabe invariant of a compact Riemannian manifold  $(M, g)$  is :

$$\mathcal{Y}(M) = \inf_{\substack{\phi \in C^\infty(M; \mathbb{R}_+) \\ \|\phi\|_{L^q(M)}=1}} \int_M \left( 4 \frac{n-1}{n-2} |\nabla \phi|^2 + \sigma \phi^2 \right) dV_g$$

where  $\nabla$  denotes the covariant derivative with respect to  $g$  and  $\sigma$  is the scalar curvature of  $g$  ;  $\mathcal{Y}(M)$  is an invariant of the conformal class  $\mathcal{C}$  of  $(M, g)$ . One can check easily that  $\mathcal{Y}(M) \leq \mathcal{Y}(\mathbb{S}^n)$ . The so called *Yamabe problem* which is the question of finding a manifold in  $\mathcal{C}$  with constant scalar curvature can be solved if  $\mathcal{Y}(M) < \mathcal{Y}(\mathbb{S}^n)$ . (see for example [11] for an in-depth review of this historical problem and more precise statements).

Even though problems (1) and (4) seem of much less geometric nature, they should be considered as a toy model of the Yamabe problem that can be played with in  $\mathbb{R}^n$ . As it will be shown in this paper, those toy models retain some interesting properties from their geometrical counterpart : the functions  $u_\varepsilon$  that realise the infimum  $\mathcal{Y}(\mathbb{S}^n)$  still play a crucial role in (1) and (4) and the existence of a solution is an exclusively non-linear effect.

Problems that resemble to (1) have an extensive literature and we refer to papers [1], [2], [5], [9] and references therein.

## 1.1 Bibliographical notes

The case  $\beta = k = 0$  *i.e.* a constant weight  $p(x, y) = 1$  has been addressed in the celebrated [2] where it is shown in particular that the equation

$$-\Delta u = u^{q-1} + \lambda u, \quad u > 0 \quad (5)$$

has a solution  $u \in H_0^1(\Omega)$  if  $n \geq 4$  and  $0 < \lambda < \lambda_1(\Omega) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{I_{\Omega;0,0}(u)}{\int_{\Omega} |u|^2 dx}$ .

On the contrary, for  $\lambda = 0$ , the problem (5) has no solution if  $\Omega$  is star-shaped around the origin. In dimension  $n = 3$ , the situation is more subtle. For example, if  $\Omega = \{x \in \mathbb{R}^3; |x| < 1\}$ , then (5) admits solutions for  $\lambda \in ]\frac{\pi^2}{4}, \pi^2[$  but has none if  $\lambda \in ]0, \frac{\pi^2}{4}[$ . See also [6] for the behavior of solutions when  $\lambda \rightarrow (\pi^2/4)_+$  and for generalizations to general domains.

A first attempt to the case  $\beta \neq 0$  but with  $k = 0$  (*i.e.* a weight that does not depend on  $u$ , which is the semi-linear case) was achieved in [10]. More precisely, [10] deals with a weight  $p \in H^1(\Omega) \cap C(\bar{\Omega})$  that admits a global minimum of the form

$$p(x) = p_0 + c|x - a|^\beta + o(|x - a|^\beta), \quad c > 0.$$

They show that for  $n \geq 3$  and  $\beta > 0$ , there exists  $\lambda_0 \geq 0$  such that (4) admits a solution for any  $\lambda \in ]\lambda_0, \lambda_1[$  where  $\lambda_1$  is the first eigenvalue of the operator  $-\operatorname{div}(p(x)\nabla \cdot)$  in  $\Omega$ , with Dirichlet boundary conditions (and for  $n \geq 4$  and  $\beta > 2$ , one can check that  $\lambda_0 = 0$ ). On the contrary, the problem (4) admits no solution if  $\lambda \leq \lambda'_0$  for some  $\lambda'_0 \in [0; \lambda_0]$  or for  $\lambda \geq \lambda_1$ . See [10] for more precise statements.

Similarly, the semi-linear case in which the minimum value of the weight is achieved in more than one point was studied in [9]; namely in dimension  $n \geq 4$  if

$$p^{-1}\left(\inf_{x \in \Omega} p(x)\right) = \{a_0, a_1, \dots, a_N\}$$

then multiple solutions that concentrate around each of the  $a_j$  can be found for  $\lambda > 0$  small enough.

For  $\lambda = 0$  and a star-shaped domain, it is well known (see [2]) that the linear problem  $\beta = k = 0$  has no solution. However, when the topology of the domain is not trivial, the problem (3) has at least one solution (see [5] for  $\beta = k = 0$ ; [9] and [10] for  $k = 0, \beta \neq 0$ ).

## 1.2 Ideas and main results

In this article, the introduction of the fully non-linear term  $|x|^\beta |u|^k$  in (1) provides a more unified approach and generates a sharp contrast between sub- and super-critical cases. Moreover, the existence of minimizers will be shown to occur exactly in the sub-cases where the nonlinearity is dominant.

The critical value for (1) can be found by the following scaling argument. As  $0 \in \Omega$ , the non-linear term tends to concentrate minimizing sequences around  $x = 0$ . Let us therefore consider the blow-up of  $u \in H_0^1(\Omega)$  around  $x = 0$ . This means one looks at the function  $v_\varepsilon$  defined by :

$$\forall \varepsilon > 0, \quad u(x) = \varepsilon^{-n/q} v_\varepsilon(x/\varepsilon). \quad (6)$$

One has  $v_\varepsilon \in H_0^1(\Omega_\varepsilon)$  with  $\Omega_\varepsilon = \{\varepsilon^{-1}y; y \in \Omega\}$  and  $\|v_\varepsilon\|_{L^q(\Omega_\varepsilon)} = \|u\|_{L^q(\Omega)}$ . Moreover, the definition of  $q$  ensures that  $2 - n + \frac{2n}{q} = 0$ , thus :

$$I_{\Omega;\beta,k}(u) = \int_{\Omega_\varepsilon} \left(1 + \varepsilon^{\beta - \frac{kn}{q}} |y|^\beta |v_\varepsilon(y)|^k\right) |\nabla v_\varepsilon(y)|^2 dy. \quad (7)$$

Depending on the ratio  $\beta/k$ , different situations occur.

- If  $\frac{\beta}{k} < \frac{n}{q}$  leading term of the blow-up around  $x = 0$  is

$$I_{\Omega;\beta,k}(u) \underset{\varepsilon \rightarrow 0}{\sim} \varepsilon^{-\left(\frac{kn}{q}-\beta\right)} \int_{\Omega_\varepsilon} |y|^\beta |v_\varepsilon(y)|^k |\nabla v_\varepsilon(y)|^2 dy.$$

One can expect the effect of the non-linearity to be dominant and one will show in this paper that (1) admits indeed minimizers in this case.

- If  $\frac{\beta}{k} = \frac{n}{q}$  both terms have the same weight and

$$\forall \varepsilon > 0, \quad I_{\Omega;\beta,k}(u) = I_{\Omega_\varepsilon;\beta,k}(v_\varepsilon).$$

One will show that, similarly to the classical case  $\beta = k = 0$ , the corresponding infimum  $S(\beta, k)$  does not depend on  $\Omega$  and that (1) admits no smooth minimizer.

- If  $\frac{\beta}{k} > \frac{n}{q}$ , the blow-up around 0 gives

$$I_{\Omega;\beta,k}(u) \underset{\varepsilon \rightarrow 0}{\sim} \int_{\Omega_\varepsilon} |\nabla v_\varepsilon(y)|^2 dy.$$

In this case, one can show that the linear behavior is dominant and that (1) admits no minimizer. Moreover, one can find a common minimizing sequences for both the linear and the non-linear problem. A cheap way to justify this is as follows. The problem (1) tends to concentrate  $u$  as a radial decreasing function around the origin. Thus, when  $\beta/k > n/q$ , one can expect  $|u(x)|^q \ll 1/|x|^{\beta q/k}$  because the right-hand side would not be locally integrable while the left-hand side is required to. In turn, this inequality reads  $|x|^\beta |u(x)|^k \ll 1$  which eliminates the non-linear contribution in the minimizing problem (1).

The infimum for the classical problem with  $\beta = k = 0$  is (see *e.g.* [2]) :

$$S = \inf_{\substack{w \in H_0^1(\Omega) \\ \|w\|_{L^q} = 1}} \int_{\Omega} |\nabla w|^2 \tag{8}$$

which does not depend on  $\Omega$ . Let us now state the main Theorem concerning (1).

**Theorem 1** *Let  $\Omega \subset \mathbb{R}^n$  a smooth bounded domain with  $n \geq 3$  and  $q = \frac{2n}{n-2}$  the critical exponent for the Sobolev injection  $H_0^1(\Omega) \subset L^q(\Omega)$ .*

1. *If  $0 \leq \beta < kn/q$  then  $S_\Omega(\beta, k) > S$  and the infimum for  $S_\Omega(\beta, k)$  is achieved.*
2. *If  $\beta = kn/q$  then  $S_\Omega(\beta, k)$  does not depend on  $\Omega$  and  $S_\Omega(\beta, k) \geq S$ . Moreover, if  $\Omega$  is star-shaped around  $x = 0$ , then the minimizing problem (1) admits no minimizers in the class :*

$$H_0^1 \cap H^{3/2} \cap L^\infty(\Omega).$$

*If  $k < 1$ , the negative result holds, provided additionally  $u^{k-1} \in L^n(\Omega)$ .*

3. *If  $\beta > kn/q$  then  $S_\Omega(\beta, k) = S$  and the infimum for  $S_\Omega(\beta, k)$  is not achieved in  $H_0^1(\Omega)$ .*

**Remarks.**

1. In the first case, one has  $k > 0$ , thus results concerning  $k = 0$  (such as those of *e.g.* [9] and [10]) are included either in our second or third case.
2. If the minimizing problem (1) is achieved for  $u \in H_0^1(\Omega)$ , then  $|u|$  is a positive minimizer. In particular, if  $\beta < kn/q$ , the problem always admits positive minimizers.
3. In the critical case  $\beta = kn/q$ , it is not known whether a non-smooth minimizer could exist in  $H_0^1 \setminus (H^{3/2} \cap L^\infty)$ . Such a minimizer could have a non-constant sign.

This paper is organized as follows. In section 2 we establish existence of minimizers of (1) for the subcritical case. Section 3 and Section 4 are devoted to study respectively the case  $\beta > kn/q$  and the critical case.

## 2 Subcritical case ( $0 \leq \beta < kn/q$ ) : existence of minimizers

The case  $\beta < kn/q$  is especially interesting because it reveals that the non-linear weight  $|u|^k$  helps for the existence of a minimizer. Note that  $k > 0$  throughout this section.

**Proposition 2** *If  $0 \leq \frac{\beta}{k} < \frac{n}{q}$ , the minimization problem (1) has at least one solution  $u \in H_0^1(\Omega)$ . Moreover, one has*

$$S_\Omega(\beta, k) > S \tag{9}$$

where  $S$  is defined by (8).

**Proof.** Let us prove first that the existence of a solution implies the strict inequality in (9). By contradiction, if  $S_\Omega(\beta, k) = S$  and if  $u$  is a minimizer for (1) thus  $u \not\equiv 0$ , one has

$$S = \int_\Omega (1 + |x|^\beta |u(x)|^k) |\nabla u(x)|^2 dx > \int_\Omega |\nabla u(x)|^2 dx$$

which contradicts the definition of  $S$ . Thus, if the minimization problem has a solution, the strict inequality (9) must hold.

Let us prove now that (1) has at least one solution  $u \in H_0^1(\Omega)$ . Let  $(u_j)_{j \in \mathbb{N}} \in H_0^1(\Omega)$  be a minimizing sequence for (1), *i.e.* :

$$I_{\Omega; \beta, k}(u_j) = S_\Omega(\beta, k) + o(1), \quad \text{and} \quad \|u_j\|_{L^q} = 1.$$

As noticed in the introduction, one can assume without restriction that  $u_j \geq 0$ . Up to a subsequence, still denoted by  $u_j$ , there exists  $u \in H_0^1(\Omega)$  such that  $u_j(x) \rightarrow u(x)$  for almost every  $x \in \Omega$  and such that :

$$\begin{aligned} u_j &\rightharpoonup u \quad \text{weakly in } H_0^1 \cap L^q(\Omega), \\ u_j &\rightarrow u \quad \text{strongly in } L^\ell(\Omega) \text{ for any } \ell < q. \end{aligned}$$

The idea of the proof is to introduce  $v_j = u_j^{\frac{k}{2}+1}$  and prove that  $v_j$  is a bounded sequence in  $W_0^{1,r} \subset L^p$  for indices  $r$  and  $p$  such that

$$p \left( \frac{k}{2} + 1 \right) \geq q.$$

The key point is the formula :

$$I_{\Omega; \beta, k}(u_j) = \int_\Omega |\nabla u_j|^2 + \left( \frac{k}{2} + 1 \right)^{-2} \int_\Omega |x|^\beta |\nabla v_j|^2 \tag{10}$$

which gives “almost” an  $H_0^1$  bound on  $v_j$  (and does indeed if  $\beta = 0$ ). For  $r \in [1, 2[$ , one has :

$$\int_{\Omega} |\nabla v_j|^r \leq \left( \int_{\Omega} |x|^\beta |\nabla v_j|^2 dx \right)^{r/2} \left( \int_{\Omega} |x|^{-\frac{\beta r}{2-r}} dx \right)^{1-r/2}$$

The integral in the right-hand side is bounded provided  $\frac{\beta r}{2-r} < n$ . All the previous conditions are satisfied if one can find  $r$  such that :

$$1 \leq r < 2, \quad \beta < n \left( \frac{2}{r} - 1 \right), \quad \frac{k}{2} + 1 > \frac{q}{p_0} = q \left( \frac{1}{r} - \frac{1}{n} \right).$$

This system of inequalities boils down to :

$$1 \leq r < 2, \quad \frac{\beta}{n} < \frac{2}{r} - 1 < \frac{2}{q} \left( \frac{k}{2} + 1 + \frac{q}{n} \right) - 1$$

which is finally equivalent to  $\beta < kn/q$  provided  $k \leq q$ . Using the compacity of the inclusion  $W_0^{1,r} \subset L^p$  for  $p < p_0$  and up to a subsequence, one has  $v_j \rightarrow v = u^{\frac{k}{2}+1}$  strongly in  $L^p$  (in particular for  $p = \frac{q}{k/2+1}$ ). Finally, as  $u_j \geq 0$  and  $u \geq 0$ , one has :

$$|u_j - u|^q \leq C \left| u_j^q - u^q \right| = C \left| v_j^{q/(k/2+1)} - v^{q/(k/2+1)} \right|$$

and thus  $u_j \rightarrow u$  strongly in  $L^q$ . One gets  $\|u\|_{L^q} = 1$ . The following compacity result then implies that  $u$  is a minimizer for (1).  $\blacksquare$

**Proposition 3** *If  $u_j \in H_0^1(\Omega)$  is a minimizing sequence for (1) with  $\|u_j\|_{L^q(\Omega)} = 1$  and such that*

$$u_j \rightarrow u \quad \text{in } L^2(\Omega), \quad \text{and} \quad \nabla u_j \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega),$$

*the weak limit  $u \in H_0^1(\Omega)$  is a minimizer of the problem (1) if and only if  $\|u\|_{L^q(\Omega)} = 1$ .*

**Proof.** It is an consequence of the main Theorem of [7, p. 77] (see also [14]) applied to the function :

$$f(x, z, p) = (1 + |x|^\beta |z|^k) |p|^2$$

which is positive, measurable on  $\Omega \times \mathbb{R} \times \mathbb{R}^n$ , continuous with respect to  $z$ , convex with respect to  $p$ . Then

$$I(u) = \int_{\Omega} f(x, u, \nabla u) \leq \liminf_{j \rightarrow \infty} \int_{\Omega} f(x, u_j, \nabla u_j) = \liminf_{j \rightarrow \infty} I(u_j).$$

If  $u_j$  is a minimizing sequence, then  $I(u) = S_{\Omega}(\beta, k)$  and  $u$  is a minimizer if and only if  $\|u\|_{L^q} = 1$ .  $\blacksquare$

## Remarks

- The sequence  $u_j$  converges strongly in  $H_0^1(\Omega)$  towards  $u$  because  $\nabla u_j \rightharpoonup \nabla u$  weakly in  $L^2(\Omega)$  and :

$$\int_{\Omega} |\nabla u_j|^2 - \int_{\Omega} |\nabla u|^2 = I(u_j) - I(u) + \int_{\Omega} |x|^\beta u^k |\nabla u|^2 - \int_{\Omega} |x|^\beta u_j^k |\nabla u_j|^2.$$

Using again Theorem of [7, p. 77] with  $\tilde{f}(x, z, p) = |x|^\beta |z|^k |p|^2$  provides

$$\forall j \in \mathbb{N}, \quad \int_{\Omega} |\nabla u_j|^2 \leq \int_{\Omega} |\nabla u|^2 + o(1)$$

and Fatou’s lemma provides the converse inequality.

- This proof implies also that  $S_{\Omega}(\beta, k)$  is continuous with respect to  $(\beta, k)$  in the range  $0 \leq \beta < kn/q$  and that the corresponding minimizer depends continuously on  $(\beta, k)$  in  $L^q(\Omega)$  and  $H_0^1(\Omega)$ .

### 3 Semi-linear case ( $\beta > kn/q$ ) : non-compact minimizing sequence

When  $\beta > kn/q$ , the problem (1) is under the total influence of the linear problem (8). Let us recall that its minimizer  $S$  is independent of the smooth bounded domain  $\Omega \subset \mathbb{R}^n$  ( $n \geq 3$ ) and that this minimizing problem has no solution. According to [2], a minimizing sequence of (8) is given by  $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$  where :

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{n-2}{4}} \zeta(x)}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}} \quad (11)$$

with  $\zeta \in C^\infty(\bar{\Omega}; [0, 1])$  is a smooth compactly supported cutoff function that satisfy  $\zeta(x) = 1$  in a small neighborhood of the origin in  $\Omega$ . Recall that  $\frac{n-2}{2} = n/q$ . Recall that  $(k+1)(n-2) > kn/q$  for any  $k \geq 0$ . We know from [2] that

$$\|\nabla u_\varepsilon\|_{L^2}^2 = K_1 + O(\varepsilon^{\frac{n-2}{2}}), \quad \|u_\varepsilon\|_{L^q}^2 = K_2 + o(\varepsilon^{\frac{n-2}{2}})$$

and that  $S = K_1/K_2$ .

The goal of this section is the proof of the following Proposition.

**Proposition 4** *If  $\frac{\beta}{k} > \frac{n}{q}$ , one has*

$$S_\Omega(\beta, k) = S \quad (12)$$

*and the problem (1) admits no minimizer in  $H_0^1(\Omega)$ . Moreover, the sequence  $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$  defined by (11) is a minimizing sequence for both (1) and the linear problem (8).*

**Proof.** Suppose by contradiction that (1) is achieved by  $u \in H_0^1(\Omega)$ . Then  $u \neq 0$  and therefore the following strict inequality holds :

$$S \leq \int_{\Omega} |\nabla u|^2 < I_{\Omega; \beta, k}(u) = S_\Omega(\beta, k).$$

Therefore the identity (12) implies that (1) has no minimizer. To prove (12) and the rest of the statement, it is sufficient to show that

$$I_{\Omega; \beta, k} \left( \|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon \right) = S + o(1) \quad (13)$$

in the limit  $\varepsilon \rightarrow 0$ , because one obviously has  $S \leq S_\Omega(\beta, k) \leq I_{\Omega; \beta, k}(\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon)$ . The limit (13) will follow immediately from the next result.  $\blacksquare$

**Proposition 5** *With the previous notations, (13) holds and more precisely, as  $\varepsilon \rightarrow 0$ , one has :*

$$\int_{\Omega} |x|^\beta |u_\varepsilon|^k |\nabla u_\varepsilon|^2 dx = \begin{cases} C \varepsilon^{\frac{2\beta - k(n-2)}{4}} + o\left(\varepsilon^{\frac{2\beta - k(n-2)}{4}}\right) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log \varepsilon|\right) & \text{if } \beta = (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}}\right) & \text{if } \beta > (k+1)(n-2) \end{cases} \quad (14)$$

with  $C = \int_{\mathbb{R}^n} \frac{|x|^{\beta+2}}{(1 + |x|^2)^{\frac{kn-2}{2} + n}} dx$  and thus :

$$I_{\Omega; \beta, k} \left( \frac{u_\varepsilon}{\|u_\varepsilon\|_{L^q}} \right) = S + \begin{cases} \frac{C}{K_2^{k/2+1}} \varepsilon^{\frac{2\beta - k(n-2)}{4}} + o\left(\varepsilon^{\frac{2\beta - k(n-2)}{4}}\right) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log \varepsilon|\right) & \text{if } \beta = (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}}\right) & \text{if } \beta > (k+1)(n-2). \end{cases} \quad (15)$$



**Proof.** The only verification is that of (14).

$$\begin{aligned}
\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&+ \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^k |\nabla \zeta(x)|^2 |x|^{\beta}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&- 2(n-2) \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+1} |x|^{\beta} \nabla \zeta(x) \cdot x}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n - 1}} dx.
\end{aligned}$$

Since  $\zeta \equiv 1$  on a neighborhood of 0 and using the Dominated Convergence Theorem, a direct computation gives

$$\begin{aligned}
\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&+ O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).
\end{aligned}$$

Here we will consider the following three subcases.

**1. Case  $\beta < (k+1)(n-2)$**

$$\begin{aligned}
\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx &= \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx - \int_{\mathbb{R}^n \setminus \Omega} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&+ \int_{\Omega} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} (|\zeta(x)|^{k+2} - 1) |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx.
\end{aligned}$$

Using the Dominated Convergence Theorem, and the fact that  $\zeta \equiv 1$  on a neighborhood of 0, one obtains

$$\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx = \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).$$

By a simple change of variable, one gets

$$\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx = \varepsilon^{\frac{2\beta - k(n-2)}{4}} \int_{\mathbb{R}^n} \frac{|y|^{\beta+2}}{(1 + |y|^2)^{\frac{k(n-2)}{2} + n}} dy + o(\varepsilon^{\frac{2\beta - k(n-2)}{4}})$$

which gives (14) in this case.

**2. Case  $\beta = (k+1)(n-2)$**

$$\begin{aligned}
\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}) \\
&= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{(|\zeta(x)|^{k+2} - 1) |x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx \\
&\quad + (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}) \\
&= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}})
\end{aligned}$$

One has, for some constants  $R_1 < R_2$  :

$$\int_{B(0,R_1)} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx \leq \int_{\Omega} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx \leq \int_{B(0,R_2)} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx$$

with

$$\begin{aligned}
\int_{B(0,R)} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx &= \omega_n \int_0^R \frac{r^{k(n-2)+2n-1}}{(\varepsilon + r^2)^{k\frac{(n-2)}{2}+n}} dr \\
&= \frac{1}{2} \omega_n |\log \varepsilon| + O(1).
\end{aligned}$$

Consequently, one has :

$$\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx = O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log \varepsilon|\right).$$

**3. Case  $\beta > (k+1)(n-2)$**

$$\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx = (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).$$

One can apply the Dominated Convergence Theorem :

$$\frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} \longrightarrow |\zeta(x)|^{k+2} |x|^{\beta-(k(n-2)+2n-2)} \quad \text{when } \varepsilon \rightarrow 0$$

and

$$\frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} \leq |\zeta(x)|^{k+2} |x|^{\beta-(k(n-2)+2n-2)} \in L^1(\Omega).$$

So, it follows that

$$\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx = O(\varepsilon^{\frac{(k+2)(n-2)}{4}})$$

which again is (14). ■

## 4 The critical case ( $\beta = kn/q$ ) : non-existence of smooth minimizers

The critical case is a natural generalization of the well known problem with  $\beta = k = 0$ . In this section, the following result will be established.

**Proposition 6** *If  $\beta = kn/q$ , one has*

$$S_{\Omega}(\beta, k) = S_{\tilde{\Omega}}(\beta, k) \quad (16)$$

for any two smooth neighborhoods  $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$  of the origin. Moreover, if  $\Omega$  is star-shaped around  $x = 0$ , the minimization problem (1) admits no solution in the class :

$$H_0^1 \cap H^{3/2} \cap L^\infty(\Omega).$$

If  $k < 1$ , the negative result holds, provided additionally  $u^{k-1} \in L^n(\Omega)$ .

The rest of this section is devoted to the proof of this statement. Note that if the minimization problem (1) had a minimizer  $u$  with non constant sign in this class of regularity, then  $|u|$  would be a positive minimizer in the same class, thus it is sufficient to show that there are no positive minimizers.

### 4.1 $S_{\Omega}(\beta, k)$ does not depend on the domain

If  $\Omega \subset \Omega'$ , there is a natural injection  $i : H_0^1(\Omega) \hookrightarrow H_0^1(\Omega')$  that corresponds to the process of extension by zero. Let  $u_j \in H_0^1(\Omega)$  be a minimizing sequence for  $S_{\Omega}(\beta, k)$ . Then  $\|i(u_j)\|_{L^q(\Omega')} = 1$  thus

$$S_{\Omega'}(\beta, k) \leq I_{\Omega';\beta,k}(i(u_j)) = I_{\Omega;\beta,k}(u_j)$$

and therefore  $S_{\Omega'}(\beta, k) \leq S_{\Omega}(\beta, k)$ .

Conversely, let us now consider the scaling transformation (6) which, in the case of  $\frac{\beta}{k} = \frac{n}{q}$ , leaves both  $\|u\|_{L^q(\Omega)}$  and  $I_{\Omega;\beta,k}(u)$  invariant. If  $u_j$  is a minimizing sequence on  $\Omega$  then  $v_j = u_{j,\lambda^{-1}}$  is an admissible sequence on  $\Omega_\lambda$  thus :

$$S_{\Omega_\lambda}(\beta, k) \leq I_{\Omega_\lambda;\beta,k}(v_j) = I_{\Omega;\beta,k}(u_j) \rightarrow S_{\Omega}(\beta, k).$$

Conversely, if  $v_j$  is a minimizing sequence on  $\Omega_\lambda$  then  $u_j = v_{j,\lambda}$  is an admissible sequence on  $\Omega$  and :

$$S_{\Omega}(\beta, k) \leq I_{\Omega;\beta,k}(u_j) = I_{\Omega_\lambda;\beta,k}(v_j) \rightarrow S_{\Omega_\lambda}(\beta, k).$$

This ensures that  $S_{\Omega_\lambda}(\beta, k) = S_{\Omega}(\beta, k)$  for any  $\lambda > 0$ .

Finally, given two smooth bounded open subsets  $\Omega$  and  $\tilde{\Omega}$  of  $\mathbb{R}^n$  that both contain 0, one can find  $\lambda, \mu > 0$  such that  $\Omega_\lambda \subset \tilde{\Omega} \subset \Omega_\mu$  and the previous inequalities read

$$S_{\Omega_\mu}(\beta, k) \leq S_{\tilde{\Omega}}(\beta, k) \leq S_{\Omega_\lambda}(\beta, k) \quad \text{and} \quad S_{\Omega}(\beta, k) = S_{\Omega_\lambda}(\beta, k) = S_{\Omega_\mu}(\beta, k)$$

thus ensuring  $S_{\Omega}(\beta, k) = S_{\tilde{\Omega}}(\beta, k)$ .

### 4.2 Pohozaev identity and the non-existence of smooth minimizers

Suppose by contradiction that a bounded minimizer  $u$  of (1) exists for some star-shaped domain  $\Omega$  with  $\beta = kn/q$ , i.e.  $u \in H_0^1 \cap L^\infty(\Omega)$ . As mentioned in the introduction  $|u|$  is also a minimizer thus, without loss of generality, one can also assume that  $u \geq 0$ . Moreover,  $u$  will satisfy the Euler-Lagrange equation (3) in the weak sense, for any test-function in  $H_0^1 \cap L^\infty(\Omega)$ .

In the following argument, inspired by [13], one will use  $(x \cdot \nabla)u$  and  $u$  as test functions. The later is fine but the former must be checked out carefully. A brutal assumption like  $(x \cdot \nabla)u \in H_0^1 \cap L^\infty(\Omega)$  is much too restrictive. Let us assume instead that

$$u \in H_0^1 \cap H^{3/2} \cap L^\infty \quad \text{and (if } k < 1) \quad u^{k-1} \in L^n(\Omega). \quad (17)$$

Note that if  $v \in H^{3/2}$  then  $|v| \in H^{3/2}$  thus the assumption  $u \geq 0$  still holds without loss of generality. Then one can find a sequence  $\phi_n \in H_0^1 \cap L^\infty(\Omega)$  such that  $\phi_n \rightarrow \phi = (x \cdot \nabla)u$  in  $H^{1/2}(\Omega)$  and almost everywhere and such that each sequence of integrals converges to the expected limit :

$$\begin{aligned} (-\Delta u|\phi_n) &\rightarrow (-\Delta u|\phi), & (u^k|\phi_n) &\rightarrow (u^k|\phi) \\ (u^{k-1}\nabla u|\phi_n) &\rightarrow (u^{k-1}\nabla u|\phi) & \text{and} & \quad (u^{q-1}|\phi_n) \rightarrow (u^{q-1}|\phi). \end{aligned}$$

Indeed, each integral satisfies a domination assumption :

$$\begin{aligned} |(-\Delta u|\phi_n - \phi)| &\leq \|u\|_{H^{3/2}} \|\phi_n - \phi\|_{H^{1/2}}, \\ |(u^k|\phi_n - \phi)| &\leq \|u^k\|_{L^{2n/(n+1)}} \|\phi_n - \phi\|_{L^{2n/(n-1)}} \leq C_\Omega \|u\|_{L^\infty}^k \|\phi_n - \phi\|_{H^{1/2}}, \\ |(u^{k-1}\nabla u|\phi_n - \phi)| &\leq \begin{cases} \|u\|_{L^\infty}^{k-1} \|\nabla u\|_{L^2} \|\phi_n - \phi\|_{L^2} & \text{if } k \geq 1, \\ \|u^{k-1}\|_{L^n} \|\nabla u\|_{L^{2n/(n-1)}} \|\phi_n - \phi\|_{L^{2n/(n-1)}} \\ \leq C_\Omega \|u^{k-1}\|_{L^n} \|u\|_{H^{3/2}} \|\phi_n - \phi\|_{H^{1/2}} & \text{if } k < 1, \end{cases} \\ |(u^{q-1}|\phi_n - \phi)| &\leq \|u^{q-1}\|_{L^{2n/(n+1)}} \|\phi_n - \phi\|_{L^{2n/(n-1)}} \leq C_\Omega \|u\|_{L^\infty}^{q-1} \|\phi_n - \phi\|_{H^{1/2}}. \end{aligned}$$

Thus, the Euler-Lagrange is also satisfied in the weak sense for the test-function  $\phi = (x \cdot \nabla)u$ .

Let us multiply by  $(x \cdot \nabla)u$  and integrate by parts :

$$-\int_\Omega \operatorname{div}(p(x, u)\nabla u) \times (x \cdot \nabla)u + \frac{k}{2} \int_\Omega |x|^\beta |u|^{k-2} |\nabla u|^2 u (x \cdot \nabla)u = \mu \int_\Omega |u|^{q-2} u (x \cdot \nabla)u.$$

An integration by part in the right-hand side and the condition  $u \in H_0^1(\Omega)$  provide :

$$\mu \int_\Omega |u|^{q-2} u (x \cdot \nabla)u = -\mu \frac{n-2}{2} \int_\Omega |u|^q = -\frac{n}{q} \mu.$$

The first term of the left-hand side is :

$$-\int_\Omega \operatorname{div}(p(x, u)\nabla u) \times (x \cdot \nabla)u = B(u) + \int_\Omega p(x, u) |\nabla u|^2 - \int_{\partial\Omega} p(x, u) (x \cdot \nabla)u \frac{\partial u}{\partial \nu}$$

with  $B(u)$  define as follows and dealt with by a second integration by part

$$\begin{aligned} B(u) &= \sum_{i,j} \int_\Omega x_j \left(1 + |x|^\beta |u|^k\right) (\partial_i u)(\partial_i \partial_j u) \\ &= -B(u) - n \int_\Omega p(x, u) |\nabla u|^2 - \beta \int_\Omega |x|^\beta |u|^k |\nabla u|^2 \\ &\quad - k \int_\Omega |x|^\beta |u|^{k-2} |\nabla u|^2 u (x \cdot \nabla)u + \int_{\partial\Omega} p(x, u) |\nabla u|^2 (x \cdot \mathbf{n}). \end{aligned}$$

On the boundary,  $p(x, u) = 1$  and as  $u \in H_0^1(\Omega)$ , one has also  $\nabla u = \frac{\partial u}{\partial \nu} \mathbf{n}$  where  $\mathbf{n}$  denotes the normal unit vector to  $\partial\Omega$  and in particular  $|\nabla u| = \left|\frac{\partial u}{\partial \nu}\right|$ , thus

$$B(u) = -\frac{n}{2} \int_\Omega p(x, u) |\nabla u|^2 - \frac{\beta}{2} \int_\Omega |x|^\beta |u|^k |\nabla u|^2 - \frac{k}{2} \int_\Omega |x|^\beta |u|^{k-2} |\nabla u|^2 u (x \cdot \nabla)u + \frac{1}{2} \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 (x \cdot \mathbf{n}).$$

The whole energy estimate with  $(x \cdot \nabla)u$  boils down to :

$$\frac{n-2}{2} \int_{\Omega} p(x, u) |\nabla u|^2 + \frac{\beta}{2} \int_{\Omega} |x|^{\beta} |u|^k |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 (x \cdot \mathbf{n}) = \frac{n}{q} \mu.$$

Finally, to deal with the first term, let us multiply (3) by  $u$  and integrate by parts ; one gets :

$$\int_{\Omega} p(x, u) |\nabla u|^2 = \int_{\Omega} (1 + |x|^{\beta} |u|^k) |\nabla u|^2 = -\frac{k}{2} \int_{\Omega} |x|^{\beta} |u|^k |\nabla u|^2 + \mu.$$

Combining both estimates provides :

$$\frac{1}{2} \left( \beta - \frac{kn}{q} \right) \int_{\Omega} |x|^{\beta} |u|^k |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 (x \cdot \mathbf{n}) = 0. \quad (18)$$

As  $\beta = kn/q$  and  $x \cdot \mathbf{n} > 0$  ( $\Omega$  is star-shaped), one gets  $\frac{\partial u}{\partial \nu} = 0$  on  $\partial\Omega$ .

The Euler-Lagrange equation (3) now reads :

$$-p(x, u) \Delta u = \frac{k}{2} |x|^{\beta} |u|^{k-2} u |\nabla u|^2 + \beta |x|^{\beta-2} |u|^k (x \cdot \nabla) u + \mu |u|^{q-2} u$$

which for  $u \geq 0$  boils down to

$$\begin{aligned} -p(x, u) \Delta u &= |x|^{\beta-2} u^{k-1} \left( \frac{k}{2} |x|^2 |\nabla u|^2 + u (x \cdot \nabla) u \right) + \mu u^{q-1} \\ &= |x|^{\beta-2} u^{k-1} \left( \sqrt{\frac{k}{2}} |x| |\nabla u + Cux \right)^2 - C^2 |x|^{\beta} u^{k+1} + \mu u^{q-1} \end{aligned}$$

with  $2\sqrt{k/2}C = \beta$ . For any  $t \in \mathbb{R}$ , one has therefore :

$$-\Delta u + tu = \frac{|x|^{\beta-2} u^{k-1}}{p(x, u)} \left( \sqrt{\frac{k}{2}} |x| |\nabla u + Cux \right)^2 + \frac{\mu u^{q-1}}{p(x, u)} + tu - \frac{C^2 |x|^{\beta} u^{k+1}}{p(x, u)} = f(t, x).$$

As  $u \in L^{\infty}$ , one can chose  $t > C^2 |x|^{\beta} \|u\|_{L^{\infty}}^k$ . Then  $f(t, x) \geq 0$  and the maximum principle implies that either  $u = 0$  or  $\frac{\partial u}{\partial n} < 0$  on  $\partial\Omega$ . In particular, only the solution  $u = 0$  satisfies simultaneously Dirichlet and Neumann boundary conditions, which leads to a contradiction because  $\|u\|_{L^q} = 1$ .

## Remarks

1. Note that Pohozaev identity (18) prevents the existence of minimizers when  $\beta \geq kn/q$ . However, the technique we used in §3 (when  $\beta > kn/q$ ) enlightens the leading term of the problem and avoids dealing with artificial regularity assumptions.
2. Similarly, one could check that the computation is also correct if

$$u \in H_0^1 \cap H^2 \cap L^{\infty}(\Omega) \quad \text{and (if } k < 1) \quad u^{k-1} \in L^{n/2}. \quad (19)$$

Assumption (19) is only preferable over (17) for  $k < 1$ . But it requires additional regularity in the interior of  $\Omega$  and would not allow to assume  $u \geq 0$  without loss of generality because in general,  $v \in H^2 \not\Rightarrow |v| \in H^2$ .

**Corollary 7** (Thanks to the referee)

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a smooth bounded open set containing 0.

If  $\beta = kn/q$  then the minimization problem (1) admits no solution in the class

$$H^1 \cap H_0^{3/2} \cap L^\infty(\Omega).$$

**Proof.** Take  $R > 0$  such that  $\Omega \subset B(0, R)$ .

Suppose by contradiction that  $u$  is a minimizing solution of (1) such that  $u \in H^1(\Omega) \cap H_0^{3/2}(\Omega) \cap L^\infty(\Omega)$ . Extend  $u$  by 0 to  $B(0, R)$ , we obtain a minimizing solution of (1) such that  $u \in H^1(B(0, R)) \cap H_0^{3/2}(B(0, R)) \cap L^\infty(B(0, R))$ . Now, arguing as in the proof of Proposition 6, we obtain a contradiction. ■

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