

A Non-Linear problem involving critical Sobolev exponent

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Abstract

We study the non-linear minimization problem on $H_0^1(\Omega) \subset L^q$ with $q = \frac{2n}{n-2}$:

$$\inf_{\|u\|_{L^q}=1} \int_{\Omega} (1 + |x|^\beta |u|^k) |\nabla u|^2.$$

We show that minimizers exist only in the range $\beta < kn/q$ which corresponds to a dominant non-linear term. On the contrary, the linear influence for $\beta \geq kn/q$ prevents their existence.

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1 Introduction

Given a smooth bounded open subset $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, let us consider the minimizing problem

$$S_{\Omega}(\beta, k) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^q(\Omega)}=1}} I_{\Omega; \beta, k}(u) \quad \text{with} \quad I_{\Omega; \beta, k}(u) = \int_{\Omega} p(x, u(x)) |\nabla u(x)|^2 dx \quad (1)$$

and $p(x, y) = 1 + |x|^\beta |y|^k$. Here $q = \frac{2n}{n-2}$ denotes the critical exponent of the Sobolev injection $H_0^1(\Omega) \subset L^q(\Omega)$. We restrict ourselves to the case $\beta \geq 0$ and $0 \leq k \leq q$. The Sobolev injection $H^{s+1}(\Omega)$ into $H^s(\Omega)$ gives :

$$I_{\Omega; \beta, k}(u) \leq \|u\|_{H_0^1(\Omega)}^2 + C_s \left(\sup_{x \in \Omega} |x|^\beta \right) \|u\|_{H^{s+1}(\Omega)}^2 \quad \text{for} \quad s \geq \frac{kn}{q(k+2)}$$

so $I_{\Omega; \beta, k}(u) < \infty$ on a dense subset of $H_0^1(\Omega)$. Note in particular that one can have $I_{\Omega; \beta, k}(u) < \infty$ without having $u \in L_{\text{loc}}^\infty(\Omega)$. If $0 \notin \bar{\Omega}$, the problem is essentially equivalent to the case $\beta = 0$ thus one will also assume from now on that $0 \in \Omega$. The case $0 \in \partial\Omega$ is interesting but will not be addressed in this paper.

For any $u \in H_0^1(\Omega)$, one has

$$I_{\Omega; \beta, k}(u) = I_{\Omega; \beta, k}(|u|) \quad (2)$$

thus, when dealing with (1), one can assume without loss of generality that $u \geq 0$.

The Euler-Lagrange equation formally associated to (1) is

$$\begin{cases} -\operatorname{div}(p(x, u(x))\nabla u) + Q(x, u(x))|\nabla u(x)|^2 = \mu|u(x)|^{q-2}u(x) & \text{in } \Omega \\ u \geq 0 & \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

with $Q(x, y) = \frac{k}{2}|x|^\beta|y|^{k-2}y$ and μ is a Lagrange multiplier. However, the logical relation between (1) and (3) is subtle : $I_{\Omega; \beta, k}$ is not Gateaux differentiable on $H_0^1(\Omega)$ because one can only expect $I_{\Omega; \beta, k}(u) = +\infty$ for a general function $u \in H_0^1(\Omega)$. However, if a minimizer u of (1) belongs to $H_0^1 \cap L^\infty(\Omega)$ then, without restriction, one can assume $u \geq 0$ and for any test-function $\phi \in H_0^1 \cap L^\infty(\Omega)$, one has

$$\forall t \in \mathbb{R}, \quad I_{\Omega; \beta, k} \left(\frac{u + t\phi}{\|u + t\phi\|_{L^q}} \right) < \infty.$$

A finite expansion around $t = 0$ then gives (3) in the weak sense, with the test-function ϕ .

The following generalization of (1) will be adressed in a subsequent paper :

$$S_\Omega(\lambda; \beta, k) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^q(\Omega)}=1}} J_{\Omega; \beta, k}(\lambda, u) \quad \text{with} \quad J_{\Omega; \beta, k}(\lambda, u) = I_{\Omega; \beta, k}(u) - \lambda \int_\Omega |u|^2. \quad (4)$$

for $\lambda > 0$, which is a compact perturbation of the case $\lambda = 0$.

A first motivation can be found in the line of [4] for the study of sharp Sobolev and Gagliardo-Nirenberg inequalities. For example, among other striking results it is shown that, for an arbitrary norm $\|\cdot\|$ on \mathbb{R}^n :

$$\inf_{\|u\|_{L^q}=1} \int_{\mathbb{R}^n} \|\nabla u(x)\|^2 dx = \|\nabla h\|_{L^2} \quad \text{with} \quad h(x) = \frac{1}{(c + \|x\|^2)^{\frac{n-2}{2}}}$$

and a constant c such that $\|h\|_{L^q} = 1$. The problem (1) can be seen as a quasi-linear generalisation where the norm $\|\cdot\|$ measuring ∇u is allowed to depend on u itself.

This type of problem is also a toy model for the Yamabe problem which has been the source of a large literature. The Yamabe invariant of a compact Riemannian manifold (M, g) is :

$$\mathcal{Y}(M) = \inf_{\substack{\phi \in C^\infty(M; \mathbb{R}_+) \\ \|\phi\|_{L^q(M)}=1}} \int_M \left(4\frac{n-1}{n-2} |\nabla \phi|^2 + \sigma \phi^2 \right) dV_g$$

where ∇ denotes the covariant derivative with respect to g and σ is the scalar curvature of g ; $\mathcal{Y}(M)$ is an invariant of the conformal class \mathcal{C} of (M, g) . One can check easily that $\mathcal{Y}(M) \leq \mathcal{Y}(\mathbb{S}^n)$. The so called *Yamabe problem* which is the question of finding a manifold in \mathcal{C} with constant scalar curvature can be solved if $\mathcal{Y}(M) < \mathcal{Y}(\mathbb{S}^n)$. (see for example [11] for an in-depth review of this historical problem and more precise statements).

Even though problems (1) and (4) seem of much less geometric nature, they should be considered as a toy model of the Yamabe problem that can be played with in \mathbb{R}^n . As it will be shown in this paper, those toy models retain some interesting properties from their geometrical counterpart : the functions u_ε that realise the infimum $\mathcal{Y}(\mathbb{S}^n)$ still play a crucial role in (1) and (4) and the existence of a solution is an exclusively non-linear effect.

Problems that resemble to (1) have an extensive literature and we refer to papers [1], [2], [5], [9] and references therein.

1.1 Bibliographical notes

The case $\beta = k = 0$ *i.e.* a constant weight $p(x, y) = 1$ has been addressed in the celebrated [2] where it is shown in particular that the equation

$$-\Delta u = u^{q-1} + \lambda u, \quad u > 0 \quad (5)$$

has a solution $u \in H_0^1(\Omega)$ if $n \geq 4$ and $0 < \lambda < \lambda_1(\Omega) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{I_{\Omega;0,0}(u)}{\int_{\Omega} |u|^2 dx}$.

On the contrary, for $\lambda = 0$, the problem (5) has no solution if Ω is star-shaped around the origin. In dimension $n = 3$, the situation is more subtle. For example, if $\Omega = \{x \in \mathbb{R}^3; |x| < 1\}$, then (5) admits solutions for $\lambda \in]\frac{\pi^2}{4}, \pi^2[$ but has none if $\lambda \in]0, \frac{\pi^2}{4}[$. See also [6] for the behavior of solutions when $\lambda \rightarrow (\pi^2/4)_+$ and for generalizations to general domains.

A first attempt to the case $\beta \neq 0$ but with $k = 0$ (*i.e.* a weight that does not depend on u , which is the semi-linear case) was achieved in [10]. More precisely, [10] deals with a weight $p \in H^1(\Omega) \cap C(\bar{\Omega})$ that admits a global minimum of the form

$$p(x) = p_0 + c|x - a|^\beta + o(|x - a|^\beta), \quad c > 0.$$

They show that for $n \geq 3$ and $\beta > 0$, there exists $\lambda_0 \geq 0$ such that (4) admits a solution for any $\lambda \in]\lambda_0, \lambda_1[$ where λ_1 is the first eigenvalue of the operator $-\operatorname{div}(p(x)\nabla \cdot)$ in Ω , with Dirichlet boundary conditions (and for $n \geq 4$ and $\beta > 2$, one can check that $\lambda_0 = 0$). On the contrary, the problem (4) admits no solution if $\lambda \leq \lambda'_0$ for some $\lambda'_0 \in [0; \lambda_0]$ or for $\lambda \geq \lambda_1$. See [10] for more precise statements.

Similarly, the semi-linear case in which the minimum value of the weight is achieved in more than one point was studied in [9]; namely in dimension $n \geq 4$ if

$$p^{-1}\left(\inf_{x \in \Omega} p(x)\right) = \{a_0, a_1, \dots, a_N\}$$

then multiple solutions that concentrate around each of the a_j can be found for $\lambda > 0$ small enough.

For $\lambda = 0$ and a star-shaped domain, it is well known (see [2]) that the linear problem $\beta = k = 0$ has no solution. However, when the topology of the domain is not trivial, the problem (3) has at least one solution (see [5] for $\beta = k = 0$; [9] and [10] for $k = 0, \beta \neq 0$).

1.2 Ideas and main results

In this article, the introduction of the fully non-linear term $|x|^\beta |u|^k$ in (1) provides a more unified approach and generates a sharp contrast between sub- and super-critical cases. Moreover, the existence of minimizers will be shown to occur exactly in the sub-cases where the nonlinearity is dominant.

The critical value for (1) can be found by the following scaling argument. As $0 \in \Omega$, the non-linear term tends to concentrate minimizing sequences around $x = 0$. Let us therefore consider the blow-up of $u \in H_0^1(\Omega)$ around $x = 0$. This means one looks at the function v_ε defined by :

$$\forall \varepsilon > 0, \quad u(x) = \varepsilon^{-n/q} v_\varepsilon(x/\varepsilon). \quad (6)$$

One has $v_\varepsilon \in H_0^1(\Omega_\varepsilon)$ with $\Omega_\varepsilon = \{\varepsilon^{-1}y; y \in \Omega\}$ and $\|v_\varepsilon\|_{L^q(\Omega_\varepsilon)} = \|u\|_{L^q(\Omega)}$. Moreover, the definition of q ensures that $2 - n + \frac{2n}{q} = 0$, thus :

$$I_{\Omega;\beta,k}(u) = \int_{\Omega_\varepsilon} \left(1 + \varepsilon^{\beta - \frac{kn}{q}} |y|^\beta |v_\varepsilon(y)|^k\right) |\nabla v_\varepsilon(y)|^2 dy. \quad (7)$$

Depending on the ratio β/k , different situations occur.

- If $\frac{\beta}{k} < \frac{n}{q}$ leading term of the blow-up around $x = 0$ is

$$I_{\Omega;\beta,k}(u) \underset{\varepsilon \rightarrow 0}{\sim} \varepsilon^{-\left(\frac{kn}{q}-\beta\right)} \int_{\Omega_\varepsilon} |y|^\beta |v_\varepsilon(y)|^k |\nabla v_\varepsilon(y)|^2 dy.$$

One can expect the effect of the non-linearity to be dominant and one will show in this paper that (1) admits indeed minimizers in this case.

- If $\frac{\beta}{k} = \frac{n}{q}$ both terms have the same weight and

$$\forall \varepsilon > 0, \quad I_{\Omega;\beta,k}(u) = I_{\Omega_\varepsilon;\beta,k}(v_\varepsilon).$$

One will show that, similarly to the classical case $\beta = k = 0$, the corresponding infimum $S(\beta, k)$ does not depend on Ω and that (1) admits no smooth minimizer.

- If $\frac{\beta}{k} > \frac{n}{q}$, the blow-up around 0 gives

$$I_{\Omega;\beta,k}(u) \underset{\varepsilon \rightarrow 0}{\sim} \int_{\Omega_\varepsilon} |\nabla v_\varepsilon(y)|^2 dy.$$

In this case, one can show that the linear behavior is dominant and that (1) admits no minimizer. Moreover, one can find a common minimizing sequences for both the linear and the non-linear problem. A cheap way to justify this is as follows. The problem (1) tends to concentrate u as a radial decreasing function around the origin. Thus, when $\beta/k > n/q$, one can expect $|u(x)|^q \ll 1/|x|^{\beta q/k}$ because the right-hand side would not be locally integrable while the left-hand side is required to. In turn, this inequality reads $|x|^\beta |u(x)|^k \ll 1$ which eliminates the non-linear contribution in the minimizing problem (1).

The infimum for the classical problem with $\beta = k = 0$ is (see *e.g.* [2]) :

$$S = \inf_{\substack{w \in H_0^1(\Omega) \\ \|w\|_{L^q} = 1}} \int_{\Omega} |\nabla w|^2 \tag{8}$$

which does not depend on Ω . Let us now state the main Theorem concerning (1).

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded domain with $n \geq 3$ and $q = \frac{2n}{n-2}$ the critical exponent for the Sobolev injection $H_0^1(\Omega) \subset L^q(\Omega)$.*

1. *If $0 \leq \beta < kn/q$ then $S_\Omega(\beta, k) > S$ and the infimum for $S_\Omega(\beta, k)$ is achieved.*
2. *If $\beta = kn/q$ then $S_\Omega(\beta, k)$ does not depend on Ω and $S_\Omega(\beta, k) \geq S$. Moreover, if Ω is star-shaped around $x = 0$, then the minimizing problem (1) admits no minimizers in the class :*

$$H_0^1 \cap H^{3/2} \cap L^\infty(\Omega).$$

If $k < 1$, the negative result holds, provided additionally $u^{k-1} \in L^n(\Omega)$.

3. *If $\beta > kn/q$ then $S_\Omega(\beta, k) = S$ and the infimum for $S_\Omega(\beta, k)$ is not achieved in $H_0^1(\Omega)$.*

Remarks.

1. In the first case, one has $k > 0$, thus results concerning $k = 0$ (such as those of *e.g.* [9] and [10]) are included either in our second or third case.
2. If the minimizing problem (1) is achieved for $u \in H_0^1(\Omega)$, then $|u|$ is a positive minimizer. In particular, if $\beta < kn/q$, the problem always admits positive minimizers.
3. In the critical case $\beta = kn/q$, it is not known whether a non-smooth minimizer could exist in $H_0^1 \setminus (H^{3/2} \cap L^\infty)$. Such a minimizer could have a non-constant sign.

This paper is organized as follows. In section 2 we establish existence of minimizers of (1) for the subcritical case. Section 3 and Section 4 are devoted to study respectively the case $\beta > kn/q$ and the critical case.

2 Subcritical case ($0 \leq \beta < kn/q$) : existence of minimizers

The case $\beta < kn/q$ is especially interesting because it reveals that the non-linear weight $|u|^k$ helps for the existence of a minimizer. Note that $k > 0$ throughout this section.

Proposition 2 *If $0 \leq \frac{\beta}{k} < \frac{n}{q}$, the minimization problem (1) has at least one solution $u \in H_0^1(\Omega)$. Moreover, one has*

$$S_\Omega(\beta, k) > S \tag{9}$$

where S is defined by (8).

Proof. Let us prove first that the existence of a solution implies the strict inequality in (9). By contradiction, if $S_\Omega(\beta, k) = S$ and if u is a minimizer for (1) thus $u \not\equiv 0$, one has

$$S = \int_\Omega (1 + |x|^\beta |u(x)|^k) |\nabla u(x)|^2 dx > \int_\Omega |\nabla u(x)|^2 dx$$

which contradicts the definition of S . Thus, if the minimization problem has a solution, the strict inequality (9) must hold.

Let us prove now that (1) has at least one solution $u \in H_0^1(\Omega)$. Let $(u_j)_{j \in \mathbb{N}} \in H_0^1(\Omega)$ be a minimizing sequence for (1), *i.e.* :

$$I_{\Omega; \beta, k}(u_j) = S_\Omega(\beta, k) + o(1), \quad \text{and} \quad \|u_j\|_{L^q} = 1.$$

As noticed in the introduction, one can assume without restriction that $u_j \geq 0$. Up to a subsequence, still denoted by u_j , there exists $u \in H_0^1(\Omega)$ such that $u_j(x) \rightarrow u(x)$ for almost every $x \in \Omega$ and such that :

$$\begin{aligned} u_j &\rightharpoonup u \quad \text{weakly in } H_0^1 \cap L^q(\Omega), \\ u_j &\rightarrow u \quad \text{strongly in } L^\ell(\Omega) \text{ for any } \ell < q. \end{aligned}$$

The idea of the proof is to introduce $v_j = u_j^{\frac{k}{2}+1}$ and prove that v_j is a bounded sequence in $W_0^{1,r} \subset L^p$ for indices r and p such that

$$p \left(\frac{k}{2} + 1 \right) \geq q.$$

The key point is the formula :

$$I_{\Omega; \beta, k}(u_j) = \int_\Omega |\nabla u_j|^2 + \left(\frac{k}{2} + 1 \right)^{-2} \int_\Omega |x|^\beta |\nabla v_j|^2 \tag{10}$$

which gives “almost” an H_0^1 bound on v_j (and does indeed if $\beta = 0$). For $r \in [1, 2[$, one has :

$$\int_{\Omega} |\nabla v_j|^r \leq \left(\int_{\Omega} |x|^\beta |\nabla v_j|^2 dx \right)^{r/2} \left(\int_{\Omega} |x|^{-\frac{\beta r}{2-r}} dx \right)^{1-r/2}$$

The integral in the right-hand side is bounded provided $\frac{\beta r}{2-r} < n$. All the previous conditions are satisfied if one can find r such that :

$$1 \leq r < 2, \quad \beta < n \left(\frac{2}{r} - 1 \right), \quad \frac{k}{2} + 1 > \frac{q}{p_0} = q \left(\frac{1}{r} - \frac{1}{n} \right).$$

This system of inequalities boils down to :

$$1 \leq r < 2, \quad \frac{\beta}{n} < \frac{2}{r} - 1 < \frac{2}{q} \left(\frac{k}{2} + 1 + \frac{q}{n} \right) - 1$$

which is finally equivalent to $\beta < kn/q$ provided $k \leq q$. Using the compacty of the inclusion $W_0^{1,r} \subset L^p$ for $p < p_0$ and up to a subsequence, one has $v_j \rightarrow v = u^{\frac{k}{2}+1}$ strongly in L^p (in particular for $p = \frac{q}{k/2+1}$). Finally, as $u_j \geq 0$ and $u \geq 0$, one has :

$$|u_j - u|^q \leq C \left| u_j^q - u^q \right| = C \left| v_j^{q/(k/2+1)} - v^{q/(k/2+1)} \right|$$

and thus $u_j \rightarrow u$ strongly in L^q . One gets $\|u\|_{L^q} = 1$. The following compacty result then implies that u is a minimizer for (1). \blacksquare

Proposition 3 *If $u_j \in H_0^1(\Omega)$ is a minimizing sequence for (1) with $\|u_j\|_{L^q(\Omega)} = 1$ and such that*

$$u_j \rightarrow u \quad \text{in } L^2(\Omega), \quad \text{and} \quad \nabla u_j \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega),$$

the weak limit $u \in H_0^1(\Omega)$ is a minimizer of the problem (1) if and only if $\|u\|_{L^q(\Omega)} = 1$.

Proof. It is an consequence of the main Theorem of [7, p. 77] (see also [14]) applied to the function :

$$f(x, z, p) = (1 + |x|^\beta |z|^k) |p|^2$$

which is positive, measurable on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, continuous with respect to z , convex with respect to p . Then

$$I(u) = \int_{\Omega} f(x, u, \nabla u) \leq \liminf_{j \rightarrow \infty} \int_{\Omega} f(x, u_j, \nabla u_j) = \liminf_{j \rightarrow \infty} I(u_j).$$

If u_j is a minimizing sequence, then $I(u) = S_{\Omega}(\beta, k)$ and u is a minimizer if and only if $\|u\|_{L^q} = 1$. \blacksquare

Remarks

- The sequence u_j converges strongly in $H_0^1(\Omega)$ towards u because $\nabla u_j \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$ and :

$$\int_{\Omega} |\nabla u_j|^2 - \int_{\Omega} |\nabla u|^2 = I(u_j) - I(u) + \int_{\Omega} |x|^\beta u^k |\nabla u|^2 - \int_{\Omega} |x|^\beta u_j^k |\nabla u_j|^2.$$

Using again Theorem of [7, p. 77] with $\tilde{f}(x, z, p) = |x|^\beta |z|^k |p|^2$ provides

$$\forall j \in \mathbb{N}, \quad \int_{\Omega} |\nabla u_j|^2 \leq \int_{\Omega} |\nabla u|^2 + o(1)$$

and Fatou’s lemma provides the converse inequality.

- This proof implies also that $S_{\Omega}(\beta, k)$ is continuous with respect to (β, k) in the range $0 \leq \beta < kn/q$ and that the corresponding minimizer depends continuously on (β, k) in $L^q(\Omega)$ and $H_0^1(\Omega)$.

3 Semi-linear case ($\beta > kn/q$) : non-compact minimizing sequence

When $\beta > kn/q$, the problem (1) is under the total influence of the linear problem (8). Let us recall that its minimizer S is independent of the smooth bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) and that this minimizing problem has no solution. According to [2], a minimizing sequence of (8) is given by $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$ where :

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{n-2}{4}} \zeta(x)}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}} \quad (11)$$

with $\zeta \in C^\infty(\bar{\Omega}; [0, 1])$ is a smooth compactly supported cutoff function that satisfy $\zeta(x) = 1$ in a small neighborhood of the origin in Ω . Recall that $\frac{n-2}{2} = n/q$. Recall that $(k+1)(n-2) > kn/q$ for any $k \geq 0$. We know from [2] that

$$\|\nabla u_\varepsilon\|_{L^2}^2 = K_1 + O(\varepsilon^{\frac{n-2}{2}}), \quad \|u_\varepsilon\|_{L^q}^2 = K_2 + o(\varepsilon^{\frac{n-2}{2}})$$

and that $S = K_1/K_2$.

The goal of this section is the proof of the following Proposition.

Proposition 4 *If $\frac{\beta}{k} > \frac{n}{q}$, one has*

$$S_\Omega(\beta, k) = S \quad (12)$$

and the problem (1) admits no minimizer in $H_0^1(\Omega)$. Moreover, the sequence $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$ defined by (11) is a minimizing sequence for both (1) and the linear problem (8).

Proof. Suppose by contradiction that (1) is achieved by $u \in H_0^1(\Omega)$. Then $u \neq 0$ and therefore the following strict inequality holds :

$$S \leq \int_{\Omega} |\nabla u|^2 < I_{\Omega; \beta, k}(u) = S_\Omega(\beta, k).$$

Therefore the identity (12) implies that (1) has no minimizer. To prove (12) and the rest of the statement, it is sufficient to show that

$$I_{\Omega; \beta, k} \left(\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon \right) = S + o(1) \quad (13)$$

in the limit $\varepsilon \rightarrow 0$, because one obviously has $S \leq S_\Omega(\beta, k) \leq I_{\Omega; \beta, k}(\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon)$. The limit (13) will follow immediately from the next result. \blacksquare

Proposition 5 *With the previous notations, (13) holds and more precisely, as $\varepsilon \rightarrow 0$, one has :*

$$\int_{\Omega} |x|^\beta |u_\varepsilon|^k |\nabla u_\varepsilon|^2 dx = \begin{cases} C \varepsilon^{\frac{2\beta - k(n-2)}{4}} + o\left(\varepsilon^{\frac{2\beta - k(n-2)}{4}}\right) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log \varepsilon|\right) & \text{if } \beta = (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}}\right) & \text{if } \beta > (k+1)(n-2) \end{cases} \quad (14)$$

with $C = \int_{\mathbb{R}^n} \frac{|x|^{\beta+2}}{(1+|x|^2)^{\frac{kn-2}{2}+n}} dx$ and thus :

$$I_{\Omega; \beta, k} \left(\frac{u_\varepsilon}{\|u_\varepsilon\|_{L^q}} \right) = S + \begin{cases} \frac{C}{K_2^{k/2+1}} \varepsilon^{\frac{2\beta - k(n-2)}{4}} + o\left(\varepsilon^{\frac{2\beta - k(n-2)}{4}}\right) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log \varepsilon|\right) & \text{if } \beta = (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}}\right) & \text{if } \beta > (k+1)(n-2). \end{cases} \quad (15)$$

Proof. The only verification is that of (14).

$$\begin{aligned}
\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&+ \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^k |\nabla \zeta(x)|^2 |x|^{\beta}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&- 2(n-2) \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+1} |x|^{\beta} \nabla \zeta(x) \cdot x}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n - 1}} dx.
\end{aligned}$$

Since $\zeta \equiv 1$ on a neighborhood of 0 and using the Dominated Convergence Theorem, a direct computation gives

$$\begin{aligned}
\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&+ O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).
\end{aligned}$$

Here we will consider the following three subcases.

1. Case $\beta < (k+1)(n-2)$

$$\begin{aligned}
\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx &= \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx - \int_{\mathbb{R}^n \setminus \Omega} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx \\
&+ \int_{\Omega} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} (|\zeta(x)|^{k+2} - 1) |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx.
\end{aligned}$$

Using the Dominated Convergence Theorem, and the fact that $\zeta \equiv 1$ on a neighborhood of 0, one obtains

$$\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx = \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).$$

By a simple change of variable, one gets

$$\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2} + n}} dx = \varepsilon^{\frac{2\beta - k(n-2)}{4}} \int_{\mathbb{R}^n} \frac{|y|^{\beta+2}}{(1 + |y|^2)^{\frac{k(n-2)}{2} + n}} dy + o(\varepsilon^{\frac{2\beta - k(n-2)}{4}})$$

which gives (14) in this case.

2. Case $\beta = (k+1)(n-2)$

$$\begin{aligned}
\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}) \\
&= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{(|\zeta(x)|^{k+2} - 1) |x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx \\
&\quad + (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}) \\
&= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}})
\end{aligned}$$

One has, for some constants $R_1 < R_2$:

$$\int_{B(0,R_1)} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx \leq \int_{\Omega} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx \leq \int_{B(0,R_2)} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx$$

with

$$\begin{aligned}
\int_{B(0,R)} \frac{|x|^{k(n-2)+n}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx &= \omega_n \int_0^R \frac{r^{k(n-2)+2n-1}}{(\varepsilon + r^2)^{k\frac{(n-2)}{2}+n}} dr \\
&= \frac{1}{2} \omega_n |\log \varepsilon| + O(1).
\end{aligned}$$

Consequently, one has :

$$\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx = O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log \varepsilon|\right).$$

3. Case $\beta > (k+1)(n-2)$

$$\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx = (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_{\Omega} \frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).$$

One can apply the Dominated Convergence Theorem :

$$\frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} \longrightarrow |\zeta(x)|^{k+2} |x|^{\beta-(k(n-2)+2n-2)} \quad \text{when } \varepsilon \rightarrow 0$$

and

$$\frac{|\zeta(x)|^{k+2} |x|^{\beta+2}}{(\varepsilon + |x|^2)^{\frac{k(n-2)}{2}+n}} \leq |\zeta(x)|^{k+2} |x|^{\beta-(k(n-2)+2n-2)} \in L^1(\Omega).$$

So, it follows that

$$\int_{\Omega} |x|^{\beta} |u_{\varepsilon}|^k |\nabla u_{\varepsilon}|^2 dx = O(\varepsilon^{\frac{(k+2)(n-2)}{4}})$$

which again is (14). ■

4 The critical case ($\beta = kn/q$) : non-existence of smooth minimizers

The critical case is a natural generalization of the well known problem with $\beta = k = 0$. In this section, the following result will be established.

Proposition 6 *If $\beta = kn/q$, one has*

$$S_{\Omega}(\beta, k) = S_{\tilde{\Omega}}(\beta, k) \quad (16)$$

for any two smooth neighborhoods $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ of the origin. Moreover, if Ω is star-shaped around $x = 0$, the minimization problem (1) admits no solution in the class :

$$H_0^1 \cap H^{3/2} \cap L^\infty(\Omega).$$

If $k < 1$, the negative result holds, provided additionally $u^{k-1} \in L^n(\Omega)$.

The rest of this section is devoted to the proof of this statement. Note that if the minimization problem (1) had a minimizer u with non constant sign in this class of regularity, then $|u|$ would be a positive minimizer in the same class, thus it is sufficient to show that there are no positive minimizers.

4.1 $S_{\Omega}(\beta, k)$ does not depend on the domain

If $\Omega \subset \Omega'$, there is a natural injection $i : H_0^1(\Omega) \hookrightarrow H_0^1(\Omega')$ that corresponds to the process of extension by zero. Let $u_j \in H_0^1(\Omega)$ be a minimizing sequence for $S_{\Omega}(\beta, k)$. Then $\|i(u_j)\|_{L^q(\Omega')} = 1$ thus

$$S_{\Omega'}(\beta, k) \leq I_{\Omega';\beta,k}(i(u_j)) = I_{\Omega;\beta,k}(u_j)$$

and therefore $S_{\Omega'}(\beta, k) \leq S_{\Omega}(\beta, k)$.

Conversely, let us now consider the scaling transformation (6) which, in the case of $\frac{\beta}{k} = \frac{n}{q}$, leaves both $\|u\|_{L^q(\Omega)}$ and $I_{\Omega;\beta,k}(u)$ invariant. If u_j is a minimizing sequence on Ω then $v_j = u_{j,\lambda^{-1}}$ is an admissible sequence on Ω_λ thus :

$$S_{\Omega_\lambda}(\beta, k) \leq I_{\Omega_\lambda;\beta,k}(v_j) = I_{\Omega;\beta,k}(u_j) \rightarrow S_{\Omega}(\beta, k).$$

Conversely, if v_j is a minimizing sequence on Ω_λ then $u_j = v_{j,\lambda}$ is an admissible sequence on Ω and :

$$S_{\Omega}(\beta, k) \leq I_{\Omega;\beta,k}(u_j) = I_{\Omega_\lambda;\beta,k}(v_j) \rightarrow S_{\Omega_\lambda}(\beta, k).$$

This ensures that $S_{\Omega_\lambda}(\beta, k) = S_{\Omega}(\beta, k)$ for any $\lambda > 0$.

Finally, given two smooth bounded open subsets Ω and $\tilde{\Omega}$ of \mathbb{R}^n that both contain 0, one can find $\lambda, \mu > 0$ such that $\Omega_\lambda \subset \tilde{\Omega} \subset \Omega_\mu$ and the previous inequalities read

$$S_{\Omega_\mu}(\beta, k) \leq S_{\tilde{\Omega}}(\beta, k) \leq S_{\Omega_\lambda}(\beta, k) \quad \text{and} \quad S_{\Omega}(\beta, k) = S_{\Omega_\lambda}(\beta, k) = S_{\Omega_\mu}(\beta, k)$$

thus ensuring $S_{\Omega}(\beta, k) = S_{\tilde{\Omega}}(\beta, k)$.

4.2 Pohozaev identity and the non-existence of smooth minimizers

Suppose by contradiction that a bounded minimizer u of (1) exists for some star-shaped domain Ω with $\beta = kn/q$, i.e. $u \in H_0^1 \cap L^\infty(\Omega)$. As mentioned in the introduction $|u|$ is also a minimizer thus, without loss of generality, one can also assume that $u \geq 0$. Moreover, u will satisfy the Euler-Lagrange equation (3) in the weak sense, for any test-function in $H_0^1 \cap L^\infty(\Omega)$.

In the following argument, inspired by [13], one will use $(x \cdot \nabla)u$ and u as test functions. The later is fine but the former must be checked out carefully. A brutal assumption like $(x \cdot \nabla)u \in H_0^1 \cap L^\infty(\Omega)$ is much too restrictive. Let us assume instead that

$$u \in H_0^1 \cap H^{3/2} \cap L^\infty \quad \text{and (if } k < 1) \quad u^{k-1} \in L^n(\Omega). \quad (17)$$

Note that if $v \in H^{3/2}$ then $|v| \in H^{3/2}$ thus the assumption $u \geq 0$ still holds without loss of generality. Then one can find a sequence $\phi_n \in H_0^1 \cap L^\infty(\Omega)$ such that $\phi_n \rightarrow \phi = (x \cdot \nabla)u$ in $H^{1/2}(\Omega)$ and almost everywhere and such that each sequence of integrals converges to the expected limit :

$$\begin{aligned} (-\Delta u|\phi_n) &\rightarrow (-\Delta u|\phi), & (u^k|\phi_n) &\rightarrow (u^k|\phi) \\ (u^{k-1}\nabla u|\phi_n) &\rightarrow (u^{k-1}\nabla u|\phi) & \text{and} & \quad (u^{q-1}|\phi_n) \rightarrow (u^{q-1}|\phi). \end{aligned}$$

Indeed, each integral satisfies a domination assumption :

$$\begin{aligned} |(-\Delta u|\phi_n - \phi)| &\leq \|u\|_{H^{3/2}} \|\phi_n - \phi\|_{H^{1/2}}, \\ |(u^k|\phi_n - \phi)| &\leq \|u^k\|_{L^{2n/(n+1)}} \|\phi_n - \phi\|_{L^{2n/(n-1)}} \leq C_\Omega \|u\|_{L^\infty}^k \|\phi_n - \phi\|_{H^{1/2}}, \\ |(u^{k-1}\nabla u|\phi_n - \phi)| &\leq \begin{cases} \|u\|_{L^\infty}^{k-1} \|\nabla u\|_{L^2} \|\phi_n - \phi\|_{L^2} & \text{if } k \geq 1, \\ \|u^{k-1}\|_{L^n} \|\nabla u\|_{L^{2n/(n-1)}} \|\phi_n - \phi\|_{L^{2n/(n-1)}} \\ \leq C_\Omega \|u^{k-1}\|_{L^n} \|u\|_{H^{3/2}} \|\phi_n - \phi\|_{H^{1/2}} & \text{if } k < 1, \end{cases} \\ |(u^{q-1}|\phi_n - \phi)| &\leq \|u^{q-1}\|_{L^{2n/(n+1)}} \|\phi_n - \phi\|_{L^{2n/(n-1)}} \leq C_\Omega \|u\|_{L^\infty}^{q-1} \|\phi_n - \phi\|_{H^{1/2}}. \end{aligned}$$

Thus, the Euler-Lagrange is also satisfied in the weak sense for the test-function $\phi = (x \cdot \nabla)u$.

Let us multiply by $(x \cdot \nabla)u$ and integrate by parts :

$$-\int_\Omega \operatorname{div}(p(x, u)\nabla u) \times (x \cdot \nabla)u + \frac{k}{2} \int_\Omega |x|^\beta |u|^{k-2} |\nabla u|^2 u (x \cdot \nabla)u = \mu \int_\Omega |u|^{q-2} u (x \cdot \nabla)u.$$

An integration by part in the right-hand side and the condition $u \in H_0^1(\Omega)$ provide :

$$\mu \int_\Omega |u|^{q-2} u (x \cdot \nabla)u = -\mu \frac{n-2}{2} \int_\Omega |u|^q = -\frac{n}{q} \mu.$$

The first term of the left-hand side is :

$$-\int_\Omega \operatorname{div}(p(x, u)\nabla u) \times (x \cdot \nabla)u = B(u) + \int_\Omega p(x, u) |\nabla u|^2 - \int_{\partial\Omega} p(x, u) (x \cdot \nabla)u \frac{\partial u}{\partial \nu}$$

with $B(u)$ define as follows and dealt with by a second integration by part

$$\begin{aligned} B(u) &= \sum_{i,j} \int_\Omega x_j \left(1 + |x|^\beta |u|^k\right) (\partial_i u)(\partial_i \partial_j u) \\ &= -B(u) - n \int_\Omega p(x, u) |\nabla u|^2 - \beta \int_\Omega |x|^\beta |u|^k |\nabla u|^2 \\ &\quad - k \int_\Omega |x|^\beta |u|^{k-2} |\nabla u|^2 u (x \cdot \nabla)u + \int_{\partial\Omega} p(x, u) |\nabla u|^2 (x \cdot \mathbf{n}). \end{aligned}$$

On the boundary, $p(x, u) = 1$ and as $u \in H_0^1(\Omega)$, one has also $\nabla u = \frac{\partial u}{\partial \nu} \mathbf{n}$ where \mathbf{n} denotes the normal unit vector to $\partial\Omega$ and in particular $|\nabla u| = \left|\frac{\partial u}{\partial \nu}\right|$, thus

$$B(u) = -\frac{n}{2} \int_\Omega p(x, u) |\nabla u|^2 - \frac{\beta}{2} \int_\Omega |x|^\beta |u|^k |\nabla u|^2 - \frac{k}{2} \int_\Omega |x|^\beta |u|^{k-2} |\nabla u|^2 u (x \cdot \nabla)u + \frac{1}{2} \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 (x \cdot \mathbf{n}).$$

The whole energy estimate with $(x \cdot \nabla)u$ boils down to :

$$\frac{n-2}{2} \int_{\Omega} p(x, u) |\nabla u|^2 + \frac{\beta}{2} \int_{\Omega} |x|^{\beta} |u|^k |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 (x \cdot \mathbf{n}) = \frac{n}{q} \mu.$$

Finally, to deal with the first term, let us multiply (3) by u and integrate by parts ; one gets :

$$\int_{\Omega} p(x, u) |\nabla u|^2 = \int_{\Omega} (1 + |x|^{\beta} |u|^k) |\nabla u|^2 = -\frac{k}{2} \int_{\Omega} |x|^{\beta} |u|^k |\nabla u|^2 + \mu.$$

Combining both estimates provides :

$$\frac{1}{2} \left(\beta - \frac{kn}{q} \right) \int_{\Omega} |x|^{\beta} |u|^k |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 (x \cdot \mathbf{n}) = 0. \quad (18)$$

As $\beta = kn/q$ and $x \cdot \mathbf{n} > 0$ (Ω is star-shaped), one gets $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

The Euler-Lagrange equation (3) now reads :

$$-p(x, u) \Delta u = \frac{k}{2} |x|^{\beta} |u|^{k-2} u |\nabla u|^2 + \beta |x|^{\beta-2} |u|^k (x \cdot \nabla) u + \mu |u|^{q-2} u$$

which for $u \geq 0$ boils down to

$$\begin{aligned} -p(x, u) \Delta u &= |x|^{\beta-2} u^{k-1} \left(\frac{k}{2} |x|^2 |\nabla u|^2 + u (x \cdot \nabla) u \right) + \mu u^{q-1} \\ &= |x|^{\beta-2} u^{k-1} \left(\sqrt{\frac{k}{2}} |x| \nabla u + C u x \right)^2 - C^2 |x|^{\beta} u^{k+1} + \mu u^{q-1} \end{aligned}$$

with $2\sqrt{k/2}C = \beta$. For any $t \in \mathbb{R}$, one has therefore :

$$-\Delta u + tu = \frac{|x|^{\beta-2} u^{k-1}}{p(x, u)} \left(\sqrt{\frac{k}{2}} |x| \nabla u + C u x \right)^2 + \frac{\mu u^{q-1}}{p(x, u)} + tu - \frac{C^2 |x|^{\beta} u^{k+1}}{p(x, u)} = f(t, x).$$

As $u \in L^{\infty}$, one can chose $t > C^2 |x|^{\beta} \|u\|_{L^{\infty}}^k$. Then $f(t, x) \geq 0$ and the maximum principle implies that either $u = 0$ or $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$. In particular, only the solution $u = 0$ satisfies simultaneously Dirichlet and Neumann boundary conditions, which leads to a contradiction because $\|u\|_{L^q} = 1$.

Remarks

1. Note that Pohozaev identity (18) prevents the existence of minimizers when $\beta \geq kn/q$. However, the technique we used in §3 (when $\beta > kn/q$) enlightens the leading term of the problem and avoids dealing with artificial regularity assumptions.
2. Similarly, one could check that the computation is also correct if

$$u \in H_0^1 \cap H^2 \cap L^{\infty}(\Omega) \quad \text{and (if } k < 1) \quad u^{k-1} \in L^{n/2}. \quad (19)$$

Assumption (19) is only preferable over (17) for $k < 1$. But it requires additional regularity in the interior of Ω and would not allow to assume $u \geq 0$ without loss of generality because in general, $v \in H^2 \not\Rightarrow |v| \in H^2$.

Corollary 7 (Thanks to the referee)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a smooth bounded open set containing 0.

If $\beta = kn/q$ then the minimization problem (1) admits no solution in the class

$$H^1 \cap H_0^{3/2} \cap L^\infty(\Omega).$$

Proof. Take $R > 0$ such that $\Omega \subset B(0, R)$.

Suppose by contradiction that u is a minimizing solution of (1) such that $u \in H^1(\Omega) \cap H_0^{3/2}(\Omega) \cap L^\infty(\Omega)$. Extend u by 0 to $B(0, R)$, we obtain a minimizing solution of (1) such that $u \in H^1(B(0, R)) \cap H_0^{3/2}(B(0, R)) \cap L^\infty(B(0, R))$. Now, arguing as in the proof of Proposition 6, we obtain a contradiction. ■

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