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► **To cite this version:**

Julien Brémont. Davenport series and almost-sure convergence. Quarterly Journal of Mathematics, 2011, 62 (1), pp.825-843. <hal-00794135>

HAL Id: hal-00794135

<https://hal.science/hal-00794135v1>

Submitted on 25 Feb 2013

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DAVENPORT SERIES AND ALMOST-SURE CONVERGENCE

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Abstract

We consider Davenport-like series with coefficients in l^2 and discuss L^2 -convergence as well as almost-everywhere convergence. We give an example where both fail to hold. We next improve former sufficient conditions under which these convergences are true.

1 Introduction

Let $\mathbb{R}\backslash\mathbb{Z}$ be the Circle and L^2 be the restriction of the space $L^2(\mathbb{R}\backslash\mathbb{Z} \rightarrow \mathbb{R})$ to odd functions. For a real parameter $\lambda > 1/2$, we introduce the map :

$$g_\lambda(x) = \sum_{m \geq 1} \frac{\sin(2\pi mx)}{m^\lambda}.$$

This function is defined everywhere on $\mathbb{R}\backslash\mathbb{Z}$. It is continuous, except at 0 when $1/2 < \lambda \leq 1$, and belongs to L^2 . For real sequences $(a_n) \in l^2$, we consider expansions based on the dilated functions system $\{g_\lambda(nx)\}_{n \geq 1}$ of the following form :

$$\sum a_n g_\lambda(nx), \tag{1}$$

where we write \sum for the summation $\sum_{n=1}^{+\infty}$. We are interested in the questions of L^2 -convergence and Lebesgue almost-everywhere (a.e) convergence of such series.

This is a natural problem which can be formulated with g_λ replaced by a general $g \in L^2(\mathbb{R}\backslash\mathbb{Z})$. A reason for focusing on odd functions is that sin series in general better converge than cos series. When $g(x) = \sin(2\pi x)$, the L^2 -convergence of $\sum a_n g(nx)$ follows from the fact that the $\{g(nx)\}_{n \geq 1}$ are orthonormal. A.e-convergence in this case is the difficult theorem of Carleson [4]. For a different g such that the $\{g(nx)\}_{n \geq 1}$ are complete in L^2 , the $\{g(nx)\}_{n \geq 1}$ are not orthogonal, see Bourgin and Mendel [2], and the question of L^2 -convergence is already not clear. The case of $g = g_\lambda$ was introduced by Wintner in [17]. A special motivation comes Arithmetics and the case $\lambda = 1$, corresponding to the first Bernoulli polynomial or “sawtooth function” $\{x\} := x - [x] - 1/2$, where $[x]$ is the integer part of $x \in \mathbb{R}$. Indeed :

$$\{x\} = -\frac{1}{\pi} \sum_{m \geq 1} \frac{\sin(2\pi mx)}{m}.$$

Series of the form $\sum a_n \{nx\}$ appear since long ago in the litterature, at the interface of Number Theory and Analysis. We recommend the very detailed presentation of such series by Jaffard in [12], where they are called Davenport series, due to Davenport’s initial systematic study [5, 6]. In

AMS 2000 subject classifications : 11A25, 26A30, 42A16, 42A61, 60F05

Key words and phrases : Davenport series, Carlitz’s lemma, L^2 -convergence, almost-everywhere convergence.

this article and for simplicity we call D_λ -series a series of the form (1), the case of Davenport series corresponding to $\lambda = 1$.

Beginning with a discussion in L^2 , Wintner [17] established that the family $\{g_\lambda(nx)\}_{n \geq 1}$ is complete in L^2 for any $\lambda > 1/2$. We now consider the L^2 -convergence of D_λ -series with $(a_n) \in l^2$. According to work by Wintner [17] and next Hedenmalm, Lindqvist and Seip [11], the $\{g_\lambda(nx)\}_{n \geq 1}$ form a Riesz basis when $\lambda > 1$. By a ‘‘Riesz basis’’ we mean a complete sequence (ξ_n) in L^2 such that for some constant $C > 0$:

$$C^{-1} \sum a_n^2 \leq \left\| \sum a_n \xi_n \right\|^2 \leq C \sum a_n^2, \forall (a_n) \in l^2.$$

Here and in the whole article we denote by $\| \cdot \|$ the usual L^2 -norm, with scalar product $\langle \cdot, \cdot \rangle$. Lindqvist and Seip [15] provide the inequalities :

$$\frac{\zeta(2\lambda)}{\zeta(\lambda)^2} \sum a_n^2 \leq \left\| \sum a_n g_\lambda(nx) \right\|^2 \leq \frac{\zeta(\lambda)^2}{\zeta(2\lambda)} \sum a_n^2,$$

where $\zeta(s) = \sum_{n \geq 1} n^{-s}$, for $s > 1$, is the Riemann Zeta function. Constants are optimal. This settles the question of L^2 -convergence when $\lambda > 1$. In this case, the a.e-convergence of D_λ -series for $(a_n) \in l^2$ follows from Carleson’s theorem [4] on the a.e-convergence of Fourier series. Indeed :

$$\sum_{n=1}^N a_n g_\lambda(nx) = \sum_{m \geq 1} m^{-\lambda} \sum_{n=1}^N a_n \sin(2\pi mn x). \quad (2)$$

For each $m \geq 1$, $\sum_{n=1}^N a_n \sin(2\pi mn x)$ converges a.e, as $N \rightarrow +\infty$, by Carleson’s theorem. Next :

$$\left| \sum_{n=1}^N a_n \sin(2\pi mn x) \right| \leq \sup_{K \geq 1} \left| \sum_{n=1}^K a_n \sin(2\pi mn x) \right| =: M(mx).$$

By classical work on the maximal operator, $\|M\|_{L^1(\mathbb{R} \setminus \mathbb{Z})} \leq C \sum a_n^2$ (cf for instance Fefferman [7]). Thus $\sum_{m \geq 1} m^{-\lambda} M(mx)$ is integrable and in particular a.e finite. One can now a.e apply the Lebesgue dominated convergence theorem in (2) to conclude. Of course this argument does not work when $1/2 < \lambda \leq 1$. Mention also that Carleson’s theorem uses properties of the Fourier basis. There exists orthonormal bases of L^2 for which L^2 -convergence does not imply a.e-convergence, see Rademacher [16].

For the rest of the article we suppose that $1/2 < \lambda \leq 1$. As a consequence of an analysis by Wintner [17] of some Dirichlet series associated to D_λ -series, for any $1/2 < \lambda \leq 1$ there exists $(a_n) \in l^2$ such that $\sum a_n g_\lambda(nx)$ is L^2 -divergent. In particular for $1/2 < \lambda \leq 1$, the $\{g_\lambda(nx)\}_{n \geq 1}$ do not form a Riesz basis of L^2 . We now detail known sufficient conditions for L^2 and a.e-convergence. We are essentially aware of results concerning Davenport series. Wintner [17] proved that $\sum a_n \{nx\}$ converges in L^2 whenever $a_n = O(n^{-\kappa})$, with $\kappa > 1/2$. Extending this result, Jaffard [12] showed that for $(a_n) \in l^2$ the sum $\sum a_n \{nx\}$ converges in a Sobolev space very close to L^2 . About a.e-convergence, Davenport in his fundamental papers [5, 6] gave non-trivial arithmetical examples where a.e-convergence is true, such as :

$$\sum_{n=1}^{+\infty} \frac{\lambda(n)}{n} \{nx\}, \sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n} \{nx\}, \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^2} \{n^2 x\}, \quad (3)$$

where $\lambda(n)$, $\Lambda(n)$ and $\mu(n)$ are respectively Liouville’s function, Von Mangolt’s function and Mo-bius’ function. When the a_n are slowly varying, the a.e-convergence of $\sum a_n \{nx\}$ follows, via an Abel transform, from estimates on $\sum_{n < N} \{nx\}$, cf Lang [14]. Jaffard [12] deduced the a.e-convergence of $\sum a_n \{nx\}$, whenever $a_n = O(\log n)^{-\alpha}$ and $a_{n+1} - a_n = O(n^{-1}(\log n)^{-(1+\alpha)})$ for some $\alpha > 2$. In particular, Hecke series :

$$\mathcal{H}_s(x) = \sum_{n=1}^{+\infty} \frac{\{nx\}}{n^s} \quad (4)$$

converge a.e for $\operatorname{Re}(s) > 0$, a result already shown by Hardy and Littlewood [8]. For general sequences, Hartman proved in [9] the a.e-convergence of $\sum a_n \{nx\}$ when $a_n = O(n^{-\kappa})$, with $\kappa > 2/3$. Mention finally some results going further than a.e-convergence. Using P-summation techniques, de la Bretèche and Tenenbaum [3] proved that (4) is convergent when $s = 1$ outside a set of Hausdorff dimension zero that they describe. Also the second series in (3) is everywhere convergent.

We now detail the content of the article. We discuss the L^2 and a.e-convergence of D_λ -series of the form (1) for general $(a_n) \in l^2$. We first complete the L^2 -divergence result of Wintner [17] by an a.e-divergence result. We next improve former sufficient conditions for L^2 and a.e-convergence.

Theorem 1.1

Assume that $1/2 < \lambda \leq 1$.

i) There exists $(a_n) \in l^2$ such that $\sum a_n g_\lambda(nx)$ is simultaneously L^2 -divergent and a.e-divergent.

ii) Suppose that for some $\varepsilon > 0$:

$$\begin{cases} \sum a_n^2 n^{\frac{(1+\varepsilon)(\log n) - (2\lambda-1)}{2(1-\lambda) \log \log n}} < \infty, \text{ when } 1/2 < \lambda < 1, \\ \sum a_n^2 (\log n)^3 (\log \log n)^{2+\varepsilon} < \infty, \text{ when } \lambda = 1. \end{cases}$$

Then $\sum a_n g_\lambda(nx)$ converges in L^2 and a.e.

In particular, the latter conditions are verified if $\sum a_n^2 n^\varepsilon < \infty$, for some $\varepsilon > 0$. For example, the following series converge in L^2 and a.e when $s > 1/2$:

$$\sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s} \{nx\}, \quad \sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n^s} \{nx\} \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{2s}} \{n^2x\}.$$

We note that whereas Wintner's approach is abstract we build here an explicit example. The fact that $(a_n) \in l^2$ does not imply the a.e-convergence of D_λ -series is not that surprising, since this condition is not the correct one for L^2 -convergence. One can make the second moment explode and develop a probabilistic argument based on the Central Limit Theorem. The true question, more difficult, is whether L^2 -convergence implies a.e-convergence. A weak formulation is as follows :

Question. If $1/2 < \lambda \leq 1$ and $\sum_{k \geq 1} \left(\sum_{n \geq 1} n^{-\lambda} |a_{kn}| \right)^2 < +\infty$, does $\sum a_n g_\lambda(nx)$ converge a.e ?

The above condition is strictly stronger than $(a_n) \in l^2$, when $1/2 < \lambda \leq 1$. As detailed below, it ensures the L^2 -convergence of $\sum a_n g_\lambda(nx)$ and is necessary when the a_n have constant sign.

We next consider three classical situations, for instance Hadamard lacunarity, where we can prove L^2 -convergence, but a.e-convergence only under stronger conditions. We define the support $\operatorname{supp}(n)$ of an integer n as its set of prime divisors and write $|\operatorname{supp}(n)|$ for the cardinal of this set.

Theorem 1.2

Suppose that $1/2 < \lambda \leq 1$.

i) Let (n_k) check $n_{k+1}/n_k \geq c$, where $c > 1$. Then $\sum a_k g_\lambda(n_k x)$ converges in L^2 whenever $(a_k) \in l^2$ and more precisely :

$$C_1 \sum a_k^2 \leq \left\| \sum a_k g_\lambda(n_k x) \right\|^2 \leq C_2 \sum a_k^2,$$

where :

$$C_1 = (1 - 1/e) \frac{\zeta(4\lambda)}{2} \left(\frac{2\lambda - 1}{2\lambda} \right) \left(\frac{\ln(c^{2\lambda-1})}{1 + \ln(c^{2\lambda-1})} \right)^2 \quad \text{and} \quad C_2 = \frac{\zeta(2\lambda)}{2} \left(\frac{c^\lambda + 1}{c^\lambda - 1} \right).$$

If the stronger condition $\sum a_k^2 (\log k)^2 < \infty$ is verified, then $\sum a_k g_\lambda(n_k x)$ is also a.e-convergent.

ii) If $(a_n) \in l^2$ and $\{|\text{supp}(n)|, a_n \neq 0\}$ is finite, then $\sum a_n g_\lambda(nx)$ converges in L^2 . In fact, for $N \geq 1$ there exists $C(\lambda, N) > 0$ such that for $(a_n) \in l^2$ with $a_n = 0$ if $|\text{supp}(n)| > N$, then :

$$C^{-1}(\lambda, N) \sum a_n^2 \leq \left\| \sum a_n g_\lambda(nx) \right\|^2 \leq C(\lambda, N) \sum a_n^2.$$

If moreover $\sum a_n^2 (\log n)^2 < \infty$, then $\sum a_n g_\lambda(nx)$ is also a.e-convergent.

iii) Let $a_n = O(b_n)$, where $(b_n)_{n \geq 1} \in l^2 \cap (\mathbb{R}_+)^{\mathbb{N}}$ satisfies $b_{nm} = b_n b_m$ whenever n and m are relatively prime. Then $\sum a_n g_\lambda(nx)$ converges in L^2 .

A word on the method. The study of the convergence of Davenport series often starts with trying to write $\sum a_n \{nx\}$ as a Fourier series $\sum c_m \sin 2\pi m x$. It was indeed remarked by Davenport [5] that formally the (c_m) are explicitly given in terms of the (a_n) and vice-versa. An alternative approach, developed here, when considering L^2 -convergence is to orthonormalize the $\{g_\lambda(nx)\}_{n \geq 1}$. The Gram-Schmidt orthonormalisation procedure is explicit and a consequence of Carlitz's lemma on the reduction of quadratic forms. We provide details for simplicity. This furnishes a rather simple characterization of L^2 -convergence. Theorem 1.2 follows via more or less standard computations. Concerning a.e-convergence, the orthonormalisation approach allows to adapt a technique of Rademacher [16] initially developed for the pointwise convergence of series built with general orthonormal systems.

A few notations. We write $i \wedge j$ and $i \vee j$ respectively for the greatest common divisor and the smallest common multiple of integers i and j . The set of primes is $\mathcal{P} = \{p_n, n \geq 1\}$.

2 Orthonormalization

Recall that $1/2 < \lambda \leq 1$. We first study correlations. The following computation is already contained in Lindqvist and Seip [15].

Lemma 2.1

Let $i, j \geq 1$. Then $\langle g_\lambda(i \cdot), g_\lambda(j \cdot) \rangle = \frac{\zeta(2\lambda)}{2} \left(\frac{i \wedge j}{i \vee j} \right)^\lambda$.

Proof of the lemma :

Let $i' = i/i \wedge j$, $j' = j/i \wedge j$. Since Lebesgue measure on $\mathbb{R} \setminus \mathbb{Z}$ is invariant by $x \mapsto px$ for any integer p , we have $\langle g_\lambda(i \cdot), g_\lambda(j \cdot) \rangle = \langle g_\lambda(i' \cdot), g_\lambda(j' \cdot) \rangle$. Using the Fourier expansion of g_λ :

$$\langle g_\lambda(i' \cdot), g_\lambda(j' \cdot) \rangle = \sum_{k, l \geq 1} \int_0^1 \frac{\sin(2\pi k i' x)}{k^\lambda} \frac{\sin(2\pi l j' x)}{l^\lambda} dx = \frac{1}{2} \sum_{m \geq 1} \frac{1}{(m^2 i' j')^\lambda} = \frac{\zeta(2\lambda)}{2} (i' j')^{-\lambda},$$

since a relation $ki' = lj'$ reduces to $k = j'm$ and $l = i'm$ for some integer m . This concludes the proof of the lemma. \square

Remark. — The correlations being positive, if $\sum b_n g_\lambda(nx)$ is L^2 -convergent with $(b_n) \in (\mathbb{R}_+)^{\mathbb{N}}$ and if $a_n = O(b_n)$, then $\sum a_n g_\lambda(nx)$ also converges in L^2 .

We turn to the orthonormalization of the $\{g_\lambda(nx)\}_{n \geq 1}$. The following proposition is an application of Carlitz's lemma, cf for instance Haukkanen, Wang and Sillanp [10].

Recall that the Möbius function μ on the integers is defined by $\mu(1) = 1$, $\mu(p_{i_1} \cdots p_{i_k}) = (-1)^k$ and $\mu(n) = 0$ if n is not square-free. If f and g are real maps defined on the integers related by $f(n) = \sum_{k|n} g(k)$, then (Möbius inversion formula) we have $g(n) = \sum_{k|n} \mu(n/k) f(k)$.

Proposition 2.2

i) Let $f_{n,\lambda}(x) = n^{-\lambda} \sum_{k|n} k^\lambda \mu(n/k) g_\lambda(kx)$, $n \geq 1$. Then $\{f_{n,\lambda}\}_{n \geq 1}$ is an orthogonal family with :

$$\|f_{n,\lambda}\|^2 = \frac{\zeta(2\lambda)}{2} \prod_{p|n, p \in \mathcal{P}} (1 - p^{-2\lambda}) \in \left(\frac{1}{2}, \frac{\zeta(2\lambda)}{2}\right).$$

The $\{f_{n,\lambda}\}_{n \geq 1}$ form an orthogonal Riesz basis of L^2 , with :

$$2\zeta(2\lambda)^{-1} \sum_{n \geq 1} \langle f_{n,\lambda}, h \rangle^2 \leq \|h\|^2 \leq 2 \sum_{n \geq 1} \langle f_{n,\lambda}, h \rangle^2, \forall h \in L^2.$$

ii) An equality $\sum_{i=1}^n a_i g_\lambda(i.) = \sum_{i=1}^n b_i f_{i,\lambda}$ holds if and only if $b_i = \sum_{k=1}^{[n/i]} k^{-\lambda} a_{ki}$, $1 \leq i \leq n$. These equalities are reversed into $a_i = \sum_{k=1}^{[n/i]} k^{-\lambda} \mu(k) b_{ki}$, $1 \leq i \leq n$.

Proof of the proposition :

Let $n \geq 1$. We introduce n -square matrices D and T , where D is diagonal and T is upper-triangular. Set $T = (t_{ij})$ with $t_{ij} = j^{-\lambda} 1_{i|j}$ and write $D = \text{diag}(d_i)$, where the (d_i) are defined below. First :

$$({}^t T D T)_{ij} = \sum_{1 \leq k \leq n} t_{ki} d_k t_{kj} = (ij)^{-\lambda} \sum_{k|i \wedge j} d_k.$$

We choose D so that $\sum_{k|m} d_k = (\zeta(2\lambda)/2) m^{2\lambda}$, which by the Möbius inversion formula corresponds to setting $d_k = (\zeta(2\lambda)/2) \sum_{l|k} \mu(k/l) l^{2\lambda}$. Lemma 2.1 gives $({}^t T D T)_{ij} = \langle g_\lambda(i.), g_\lambda(j.) \rangle$.

Next, the inverse of T is given by $T_{ij}^{-1} = 1_{i|j} i^\lambda \mu(j/i)$, since for any $i \leq j$:

$$\sum_{k=1}^n 1_{i|k} i^\lambda \mu(k/i) j^{-\lambda} 1_{k|j} = 1_{i|j} i^\lambda j^{-\lambda} \sum_{l|j/i} \mu(l) = 1_{i=j},$$

using that $\sum_{k|m} \mu(k) = 0$, if $m \geq 2$. Observe that $f_{i,\lambda} = i^{-\lambda} \sum_{1 \leq k \leq n} ({}^t T^{-1})_{ik} g_\lambda(k.)$, for $1 \leq i \leq n$. The $\{f_{i,\lambda}\}_{i \geq 1}$ are therefore orthogonal in L^2 , with $\|f_{i,\lambda}\|^2 = d_i i^{-2\lambda}$. They also form a complete family, since it is the case for the $\{g_\lambda(nx)\}$, cf Wintner [17]. Decomposing $i = p_{i_1}^{\alpha_1} \cdots p_{i_k}^{\alpha_k}$ in prime factors, we have :

$$\|f_{i,\lambda}\|^2 = \frac{\zeta(2\lambda)}{2} \sum_{d|i} d^{-2\lambda} \mu(d) = \frac{\zeta(2\lambda)}{2} \prod_{j=1}^k \sum_{d|p_{i_j}^{\alpha_j}} d^{-2\lambda} \mu(d) = \frac{\zeta(2\lambda)}{2} \prod_{p|i, p \in \mathcal{P}} (1 - p^{-2\lambda}).$$

Finally :

$$\sum_{i=1}^n a_i g_\lambda(i.) = \sum_{i=1}^n a_i \sum_{k=1}^n ({}^t T)_{ik} k^\lambda f_{k,\lambda} = \sum_{i=1}^n a_i \sum_{k=1}^n i^{-\lambda} 1_{k|i} k^\lambda f_{k,\lambda} = \sum_{k=1}^n f_{k,\lambda} \sum_{i=1}^{[n/k]} i^{-\lambda} a_{ki}.$$

The reversed formula is proved in a similar way. □

We deduce the following characterization of L^2 -convergence.

Corollary 2.3

i) The series $\sum a_n g_\lambda(nx)$ converges in L^2 if and only if the numerical series $\sum_{k \geq 1} k^{-\lambda} a_{ki}$ converge for all $i \geq 1$, together with the uniformity condition :

$$\sum_{i \geq 1} \left(\sum_{k > [n/i]} k^{-\lambda} a_{ki} \right)^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

ii) A sufficient condition for $\sum a_n g_\lambda(nx)$ to be L^2 -convergent is :

$$\sum_{i \geq 1} \left(\sum_{k \geq 1} k^{-\lambda} |a_{ki}| \right)^2 < +\infty.$$

This condition is necessary when $(a_n) \in (\mathbb{R}_+)^{\mathbb{N}}$.

Proof of the corollary :

If $\sum a_n g_\lambda(nx)$ converges in L^2 , by proposition 2.2 the component $\sum_{k=1}^{[n/i]} k^{-\lambda} a_{ki}$ with respect to each $f_{i,\lambda}$ converges as $n \rightarrow +\infty$. The L^2 -limit then has to be $\sum_{i \geq 1} f_{i,\lambda}(\sum_{k \geq 1} k^{-\lambda} a_{ki})$. The uniformity condition is a consequence from the fact that the norm of $f_{i,\lambda}$ belongs to $(1/2, \zeta(2\lambda)/2)$. The first assertion of the second item is an application of the Lebesgue Dominated Convergence Theorem, whereas the second one follows from the first item. \square

Remark. — Corollary 2.3 can also be obtained when considering directly the Fourier expansion of $\sum a_n g_\lambda(nx)$ given by g_λ . In the sequel, the orthonormalization point of view has the practical advantage to keep finite all partial sums.

3 A l^2 -example of a L^2 -divergent and a.e-divergent series

We prove theorem 1.1 i), using that for $1/2 < \lambda \leq 1$ the series $\sum_{p \in \mathcal{P}} p^{-\lambda}$ is divergent. For each integer $K \geq 1$, we choose a finite set $\mathcal{P}_K = \{p_{j,K}\}_{1 \leq j \leq l_K}$ of consecutive primes satisfying :

$$\sum_{j=1}^{l_K} (p_{j,K})^{-\lambda} \geq K. \tag{5}$$

We fix $m_K \geq 2$ so that $\left(\frac{m_K - 1}{m_K}\right)^{l_K} \geq 1/2$. Introduce sets :

$$\left\{ \begin{array}{l} F_{1,K} = \{p_{1,K}^{u_1} \cdots p_{l_K,K}^{u_{l_K}}, 1 \leq u_1, \dots, u_{l_K} \leq m_K\} \\ F'_{1,K} = \{p_{1,K}^{u_1} \cdots p_{l_K,K}^{u_{l_K}}, 1 \leq u_1, \dots, u_{l_K} \leq m_K - 1\}. \end{array} \right.$$

We have $|F_{1,K}| = (m_K)^{l_K}$ and $|F'_{1,K}| = (m_K - 1)^{l_K}$. Let $q_{1,K} = 1$ and take next infinitely many primes $q_{2,K} < \dots < q_{n,K} < \dots$, larger than $p_{l_K,K}$ and subject to the condition :

$$(p_{1,K} \cdots p_{l_K,K})^{m_K} \left(1 + \frac{(m_K)^{l_K/2}}{K}\right) \sum_{r=2}^{+\infty} \frac{(q_{r-1,K})^{r-1}}{q_{r,K}} \leq \frac{1}{K}. \tag{6}$$

Define the random variable :

$$X_{1,K} = \frac{1}{K |F_{1,K}|^{1/2}} \sum_{n \in F_{1,K}} g_\lambda(nx).$$

It has zero mean and belongs to L^2 . We write $\sigma_K^2 = \int_0^1 (X_{1,K})^2(x) dx$ for its variance and choose an integer $T_K \geq K$ so that :

$$\int_0^1 (X_{1,K})^2(x) 1_{\{|X_{1,K}| > (T_K)^{1/12} \sigma_K\}} dx \leq \frac{1}{K}. \quad (7)$$

We define another collection of sets :

$$\left\{ \begin{array}{l} F_{2,K} = q_{2,K} F_{1,K} \\ F'_{2,K} = q_{2,K} F'_{1,K} \end{array} \right. \cdots \left\{ \begin{array}{l} F_{T_K,K} = q_{T_K,K} F_{1,K} \\ F'_{T_K,K} = q_{T_K,K} F'_{1,K} \end{array} \right.$$

Grouping sets, we define :

$$E_K = \bigcup_{r=1}^{T_K} F_{r,K} \text{ and } E'_K = \bigcup_{r=1}^{T_K} F'_{r,K}.$$

We have $|E_K| = T_K |F_{1,K}|$ and $|E'_K| = T_K |F'_{1,K}|$. In particular :

$$\frac{|E'_K|}{|E_K|} = \frac{|F'_{1,K}|}{|F_{1,K}|} = \left(\frac{m_K - 1}{m_K} \right)^{l_K} \geq \frac{1}{2}. \quad (8)$$

When considering the next integer (ie $K+1$) we start with $p_{1,K+1} > q_{T_K,K} (p_{1,K} \cdots p_{l_K,K})^{m_K}$. Observe that all the $(F_{r,K})_{K \geq 1, 1 \leq r \leq T_K}$, are pairwise disjoint and in particular the $(E_K)_{K \geq 1}$, which furthermore are consecutive. We finally set :

$$a_n = \begin{cases} \frac{1}{K|E_K|^{1/2}} & , \text{ when } n \in E_K, \text{ for some } K \geq 1, \\ 0 & , \text{ otherwise.} \end{cases}$$

This completes the definition of the sequence (a_n) . Formally $\sum a_n g_\lambda(nx) = \sum_{K \geq 1} Z_K$, with :

$$Z_K = \frac{1}{\sqrt{T_K}} \sum_{r=1}^{T_K} X_{r,K}(x) \text{ and } X_{r,K}(x) = \frac{1}{K|F_{1,K}|^{1/2}} \sum_{n \in F_{r,K}} g_\lambda(nx). \quad (9)$$

From the previous construction, observe that a partial sum $\sum_{K=1}^N Z_K(x)$ corresponds to a partial sum of $\sum a_n g_\lambda(nx)$. We now proceed to verifications.

i) The sequence (a_n) belongs to l^2 . Indeed :

$$\sum_{n \geq 1} a_n^2 = \sum_{K \geq 1} \sum_{n \in E_K} \frac{1}{K^2 |E_K|} = \sum_{K \geq 1} \frac{1}{K^2} < \infty.$$

ii) The series $\sum a_n g_\lambda(nx)$ is L^2 -divergent. Indeed, using (5) and (8) :

$$\begin{aligned} \sum_{n \geq 1} \left(\sum_{k \geq 1} k^{-\lambda} a_{kn} \right)^2 &\geq \sum_{K \geq 1} \sum_{n \in E'_K} \left(\sum_{k \geq 1} k^{-\lambda} a_{kn} \right)^2 \geq \sum_{K \geq 1} \sum_{n \in E'_K} \left(\sum_{k \in \mathcal{P}_K} k^{-\lambda} a_{kn} \right)^2 \\ &\geq \sum_{K \geq 1} \sum_{n \in E'_K} \frac{1}{K^2 |E_K|} \left(\sum_{k \in \mathcal{P}_K} k^{-\lambda} \right)^2 \geq \sum_{K \geq 1} |E'_K| \frac{K^2}{K^2 |E_K|} \geq \sum_{K \geq 1} \frac{1}{2} = +\infty. \end{aligned}$$

Since the a_n are positive, the conclusion comes from corollary 2.3.

iii) The series $\sum a_n g_\lambda(nx)$ is a.e.-divergent. This requires longer computations. For a fixed $K \geq 1$, all $(X_{r,K})_{1 \leq r \leq T_K}$ have the same law, due to the invariance of Lebesgue measure on $\mathbb{R} \setminus \mathbb{Z}$ under multiplication by an integer. They do not form a stationary process, but are nearly independent. Under our hypotheses, it is routine to check that the law of $(\sigma_K^2 T_K)^{-1/2} \sum_{r=1}^{T_K} X_{r,K}$ is asymptotically normal.

In a first step, we compute the variance σ_K^2 and verify that it grows rapidly to infinity, as suggested by ii). Via lemma 2.1, we have :

$$\begin{aligned} \sigma_K^2 = \text{Var}(X_{1,K}) &= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \sum_{n, m \in F_{1,K}} \left(\frac{n \wedge m}{n \vee m} \right)^\lambda \\ &= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \sum_{1 \leq a_j, b_j \leq m_K, 1 \leq j \leq l_K} \prod_{j=1}^{l_K} (p_{j,K})^{-\lambda |a_j - b_j|} \\ &= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \prod_{j=1}^{l_K} \left(\sum_{a,b=1}^{m_K} (p_{j,K})^{-\lambda |a-b|} \right) \\ &= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \prod_{j=1}^{l_K} \left(m_K + 2(p_{j,K})^{-\lambda(m_K-1)} \sum_{k=1}^{m_K-1} k (p_{j,K})^{\lambda(k-1)} \right). \end{aligned}$$

For $x > 1$, we have $\sum_{k=1}^{n-1} kx^{k-1} = ((n-1)x^n - nx^{n-1} + 1)/(x-1)^2 = nx^{n-2}(1 + o(1))$, when x and n are large. Inserting this in the previous calculations, we obtain, with a uniform $o(1)$:

$$\begin{aligned} \sigma_K^2 &= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \prod_{j=1}^{l_K} (m_K + 2m_K (p_{j,K})^{-\lambda} (1 + o(1))) \\ &= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2} e^{\sum_{j=1}^{l_K} \log(1 + 2(p_{j,K})^{-\lambda} (1 + o(1)))} \\ &= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2} e^{2(\sum_{j=1}^{l_K} (p_{j,K})^{-\lambda}) (1 + o(1))} \geq e^K, \end{aligned} \tag{10}$$

for large K , using (5).

We now establish the convergence :

$$(\sigma_K^2 T_K)^{-1/2} \sum_{r=1}^{T_K} X_{r,K} \rightarrow \mathcal{N}(0, 1), \text{ in law.} \tag{11}$$

We write E for the expectation under Lebesgue measure on $\mathbb{R} \setminus \mathbb{Z}$. Set $Y_{r,K} = (\sigma_K^2 T_K)^{-1/2} X_{r,K}$ and $S_N = \sum_{r=1}^N Y_{r,K}$. For $1 \leq r \leq T_K$, introduce the finite partitions :

$$\mathcal{F}_r = \{[k/q_{r,K}, (k+1)/q_{r,K}), 0 \leq k < q_{r,K}\}.$$

Each $Y_{r,K}$ being $(1/q_{r,K})$ -periodic, for a bounded measurable f :

$$E(f(Y_{r,K})) = E(f(Y_{r,K}) | \mathcal{F}_r). \tag{12}$$

For $t \in \mathbb{R}$ and $2 \leq N \leq T_K$, we have :

$$\begin{aligned} E(e^{itS_N}) &= E(e^{itS_{N-1}} e^{itY_{N,K}}) \\ &= E(E(e^{itS_{N-1}} | \mathcal{F}_N) e^{itY_{N,K}}) + E((e^{itS_{N-1}} - E(e^{itS_{N-1}} | \mathcal{F}_N)) e^{itY_{N,K}}) \\ &= A + B. \end{aligned}$$

First of all, taking conditional expectation and using (12) :

$$\begin{aligned} A &= E(E(e^{itS_{N-1}}|\mathcal{F}_N)E(e^{itY_{N,K}}|\mathcal{F}_N)) = E(E(e^{itS_{N-1}}|\mathcal{F}_N)E(e^{itY_{N,K}})) \\ &= E(e^{itS_{N-1}})E(e^{itY_{N,K}}). \end{aligned} \quad (13)$$

Next :

$$|B| \leq E(|e^{itS_{N-1}} - E(e^{itS_{N-1}}|\mathcal{F}_N)|). \quad (14)$$

The map $x \mapsto e^{itx}$ is $|t|$ -Lipschitz. On each piece of \mathcal{F}_N which contains no discontinuity of S_{N-1} , when counting the oscillation we have :

$$\begin{aligned} |e^{itS_{N-1}} - E(e^{itS_{N-1}}|\mathcal{F}_N)| &\leq \frac{|t|}{q_{N,K}} \frac{(\sigma_K^2 T_K)^{-1/2}}{(K(m_K)^{l_K})^{1/2}} \sum_{r=1}^{N-1} (m_K)^{l_K} (p_{1,K} \cdots p_{l_K,K})^{m_K} q_{r,K} \\ &\leq |t| \left(\frac{(m_K)^{l_K/2}}{K} (p_{1,K} \cdots p_{l_K,K})^{m_K} \right) \frac{(N-1)q_{N-1,K}}{q_{N,K}}, \end{aligned} \quad (15)$$

since $T_K \geq K$ and $\sigma_K \geq 1$ for large K , by (10). Next, S_{N-1} is continuous on the interior of each segment of the partition whose step⁻¹ is $q_{1,K} \cdots q_{N-1,K} (p_{1,K} \cdots p_{l_K,K})^{m_K}$. The total measure of the pieces of \mathcal{F}_N which may contain a discontinuity of S_{N-1} is bounded from above by :

$$q_{1,K} \cdots q_{N-1,K} (p_{1,K} \cdots p_{l_K,K})^{m_K} \frac{1}{q_{N,K}} \leq (p_{1,K} \cdots p_{l_K,K})^{m_K} \frac{(q_{N-1,K})^{N-1}}{q_{N,K}}. \quad (16)$$

From (14), (15) and (16), we deduce that :

$$|B| \leq 2(1+|t|)(p_{1,K} \cdots p_{l_K,K})^{m_K} \left(1 + \frac{(m_K)^{l_K/2}}{K} \right) \frac{(q_{N-1,K})^{N-1}}{q_{N,K}}. \quad (17)$$

Using that for all $1 \leq N \leq T_K$, we have $|E(e^{itY_{N,K}})| \leq 1$, when iterating the procedure with (13) and (14), we obtain via (6) :

$$\begin{aligned} |E(e^{itS_{T_K}}) - \prod_{r=1}^{T_K} E(e^{itY_{r,K}})| &\leq 2(1+|t|)(p_{1,K} \cdots p_{l_K,K})^{m_K} \left(1 + \frac{(m_K)^{l_K/2}}{K} \right) \sum_{r=2}^{T_K} \frac{(q_{r-1,K})^{r-1}}{q_{r,K}} \\ &\leq 2(1+|t|) \frac{1}{K}. \end{aligned} \quad (18)$$

As a consequence of (18), in order to show (11) we only need to focus on :

$$\prod_{r=1}^{T_K} E(e^{itY_{r,K}}) = E(e^{itY_{1,K}})^{T_K}. \quad (19)$$

We now use the fact that for all $t \in \mathbb{R}$:

$$|e^{it} - (1 + it - t^2/2)| \leq \min\{|t|^3/6, |t|^2\}, \quad (20)$$

which comes from $e^{it} - (1 + it - t^2/2) = i^3/2 \int_0^t (t-s)^2 e^{is} ds = i^2 \int_0^t (t-s)(e^{is} - 1) ds$. Via (20) and the property that $X_{1,K}$ has zero mean, we now deduce the following inequalities :

$$\begin{aligned}
\left| E \left(e^{itY_{1,K}} \right) - \left(1 - \frac{t^2}{2T_K} \right) \right| &\leq \left| E \left(e^{itY_{1,K}} - \left(1 + itY_{1,K} - \frac{t^2}{2} Y_{1,K}^2 \right) \right) \right| \\
&\leq E \left| \left(e^{itY_{1,K}} - \left(1 + itY_{1,K} - \frac{t^2}{2} Y_{1,K}^2 \right) \right) \right| \\
&\leq E \left(\min\{|tY_{1,K}|^3/6, |tY_{1,K}|^2\} \right).
\end{aligned}$$

With $\varepsilon = (T_K)^{-5/12}$ and using (7), as well as $T_K \geq K$ and $\sigma_K \geq 1$ for large K :

$$\begin{aligned}
\left| E \left(e^{itY_{1,K}} \right) - \left(1 - \frac{t^2}{2T_K} \right) \right| &\leq \frac{|t|^3}{6} E(|Y_{1,K}|^3 1_{|Y_{1,K}| \leq \varepsilon}) + |t|^2 E(|Y_{1,K}|^2 1_{|Y_{1,K}| > \varepsilon}) \\
&\leq \frac{|t|^3}{6(T_K)^{5/4}} + \frac{|t|^2}{\sigma_K^2 T_K} E(|X_{1,K}|^2 1_{|X_{1,K}| > (T_K)^{1/12} \sigma_K}) \\
&\leq \frac{1}{T_K} \left(\frac{|t|^3}{6(T_K)^{1/4}} + \frac{|t|^2}{K \sigma_K^2} \right) \leq \frac{1}{T_K} \left(\frac{|t|^3}{6K^{1/4}} + \frac{|t|^2}{K} \right). \quad (21)
\end{aligned}$$

Since $T_K \rightarrow +\infty$, as $K \rightarrow +\infty$, we deduce from (18), (19) and (21) that $E(e^{itS_{T_K}}) \rightarrow e^{-t^2/2}$, as $K \rightarrow +\infty$, for all $t \in \mathbb{R}$. This proves (11).

To conclude, for all $L \geq 1$ we choose $K_L \geq L$ so that :

$$P(|S_{T_{K_L}}| \leq 1/L^2) \leq 2 \int_{|t| \leq 1/L^2} d\mathcal{N}(0,1)(t) =: \delta_L.$$

Clearly $\sum_{L \geq 1} \delta_L < \infty$, so by Borel-Cantelli's lemma, for a.e x when L is large enough we have $|S_{T_{K_L}}| \geq 1/L^2$. For such a x , using (10) and when L is large enough :

$$|Z_{K_L}| = \left| \frac{1}{\sqrt{T_{K_L}}} \sum_{r=1}^{T_{K_L}} X_{r,K_L}(x) \right| = \sigma_{K_L} |S_{T_{K_L}}| \geq \frac{e^{K_L/2}}{L^2} \geq \frac{e^{L/2}}{L^2}.$$

Since partial sums $\sum_{K=1}^N Z_K(x)$ are partial sums of $\sum a_n g_\lambda(nx)$, this prevents $\sum a_n g_\lambda(nx)$ from converging at x . This completes the proof of item *i*) of theorem 1.1.

4 Sufficient conditions for L^2 and a.e-convergence

We take a finite sequence (a_n) and write $\sum a_n g_\lambda(nx) = \sum b_n f_{n,\lambda}(x)$, where (b_n) is also finite. By proposition 2.2 :

$$\left\| \sum a_n g_\lambda(nx) \right\|^2 = \left\| \sum b_n f_{n,\lambda} \right\|^2 \leq \frac{\zeta(2\lambda)}{2} \sum b_n^2 \leq \frac{\zeta(2\lambda)}{2} \sum_{n \geq 1} \left(\sum_{k \geq 1} k^{-\lambda} |a_{kn}| \right)^2.$$

Set $\psi_\lambda(k) = k^{1-\lambda}(\log k)^2$ if $1/2 < \lambda < 1$ and $\psi_1(k) = \log k(\log \log k)^{1+\varepsilon}$, for some $\varepsilon > 0$. For simplicity we write $\log(x)$ for $\max\{1, \log(x)\}$. Using Cauchy-Schwarz's inequality :

$$\begin{aligned}
\sum_{n \geq 1} \left(\sum_{k \geq 1} k^{-\lambda} |a_{kn}| \right)^2 &= \sum_{k, k' \geq 1} (kk')^{-\lambda} \sum_{n \geq 1} |a_{nk} a_{nk'}| \leq \sum_{k, k' \geq 1} (kk')^{-\lambda} \left(\sum_{n \geq 1} a_{nk}^2 \right)^{1/2} \left(\sum_{n \geq 1} a_{nk'}^2 \right)^{1/2} \\
&\leq \left[\sum_{k \geq 1} k^{-\lambda} \left(\sum_{n \geq 1} a_{nk}^2 \right)^{1/2} \right]^2 \leq \left(\sum_{k \geq 1} k^{-\lambda} \frac{1}{\psi_\lambda(k)} \right) \left(\sum_{k \geq 1} k^{-\lambda} \psi_\lambda(k) \sum_{n \geq 1} a_{nk}^2 \right) \\
&\leq C_\varepsilon \sum_{n \geq 1} a_n^2 \sum_{k|n} k^{-\lambda} \psi_\lambda(k). \tag{22}
\end{aligned}$$

We first consider the case $1/2 < \lambda < 1$. Remark that $0 < 2\lambda - 1 < 1$. Using a classical upper-bound for $\sum_{k|n} k^{2\lambda-1}$, cf Krätzel [13], we have for any $\delta > 0$:

$$\begin{aligned}
\sum_{k|n} k^{-\lambda} \psi_\lambda(k) &= \sum_{k|n} k^{1-2\lambda} (\log k)^2 \leq (\log n)^2 \sum_{k|n} k^{1-2\lambda} \leq (\log n)^2 n^{1-2\lambda} \sum_{k|n} k^{2\lambda-1} \\
&\leq (\log n)^2 n^{1-2\lambda} C_\delta n^{2\lambda-1} e^{\frac{(1+\delta)(\log n)^{1-(2\lambda-1)}}{(1-(2\lambda-1)) \log \log n}} \leq C'_\delta n^{\frac{(1+2\delta)(\log n)^{1-2\lambda}}{2(1-\lambda) \log \log n}}.
\end{aligned}$$

In the situation when $\lambda = 1$, we have :

$$\sum_{k|n} k^{-1} \psi_1(k) \leq \psi_1(n) \sum_{k|n} k^{-1} = \psi_1(n) n^{-1} \sum_{k|n} k.$$

We use this time the inequality $\sum_{k|n} k \leq Cn \log \log n$, see again [13]. As a result, for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for any sequence (a_n) :

$$\begin{cases} \left\| \sum a_n g_\lambda(nx) \right\|^2 \leq C_\varepsilon \sum a_n^2 n^{\frac{(1+\varepsilon)(\log n)^{-(2\lambda-1)}}{2(1-\lambda) \log \log n}}, & \text{when } 1/2 < \lambda < 1, \\ \left\| \sum a_n \{nx\} \right\|^2 \leq C_\varepsilon \sum a_n^2 \log n (\log \log n)^{2+\varepsilon}, & \text{when } \lambda = 1. \end{cases} \tag{23}$$

These properties imply the L^2 -convergence of D_λ -series under the assumptions of theorem 1.1.

We turn to the question of the a.e-convergence of D_λ -series. The second item of theorem 1.1 is a consequence of inequalities (23) and of the following proposition. The latter is an adaptation of a method due to Rademacher [16] for the study of series based on a general orthonormal family.

Proposition 4.1

Let $(a_n)_{n \geq 1}$ and $(\varphi(n))_{n \geq 1}$ be such that $\sum a_n^2 \varphi(n) (\log n)^2 < \infty$ and for any $M \leq N$:

$$\left\| \sum_{n=M}^N a_n g_\lambda(nx) \right\|^2 \leq \sum_{n=M}^N a_n^2 \varphi(n).$$

Then $\sum a_n g_\lambda(nx)$ converges a.e.

Proof of the proposition :

We can suppose that \log is the logarithm in base 2. Let $S(n)(x) = \sum_{1 \leq k \leq n} a_k g_\lambda(kx)$. For $m < n$, introduce the notations :

$$S(m, n)(x) = \sum_{m \leq k < n} a_k g_\lambda(kx) \text{ and } \sigma_l(m, n) = \sum_{m \leq k < n} a_k^2 \varphi(k) (\log k)^l, \text{ for } l \in \{0, 1, 2\}.$$

Step 1. We show that $(S(2^n)(x))$ converges for a.e x . Let $0 < N < n$. We have :

$$\begin{aligned} \int_0^1 \sum_{N \leq r < n} S(2^r, 2^n)^2(x) dx &\leq \sum_{N \leq r < n} \sigma_0(2^r, 2^n) = \sum_{N \leq r < n} \sum_{s=r}^{n-1} \sigma_0(2^s, 2^{s+1}) \\ &\leq \sum_{s=N}^{n-1} (s - N + 1) \sigma_0(2^s, 2^{s+1}) \\ &\leq \sum_{s=N}^{n-1} s \sigma_0(2^s, 2^{s+1}) \leq \sigma_1(2^N, 2^n). \end{aligned}$$

By Markov's inequality, $\sum_{N \leq r < n} S(2^r, 2^n)^2(x) \leq \sigma_1(2^N, 2^n)^{2/3}$ for all x in a Borel set $E_{N,n}$ with :

$$\lambda(E_{N,n}) \geq 1 - \sigma_1(2^N, 2^n)^{1/3} \geq 1 - \sigma_1(2^N, \infty)^{1/3}.$$

In particular, for $x \in E_{N,n}$ and all $N \leq r < n$, we have $S(2^r, 2^n)(x) \leq \sigma_1(2^N, 2^n)^{1/3}$. Define a set $E'_{N,n}$ by the condition that for all $N \leq r \leq r' < n$:

$$S(2^r, 2^{r'})(x) \leq 2\sigma_1(2^N, \infty)^{1/3}.$$

Since $E_{N,n} \subset E'_{N,n}$, we have $\lambda(E'_{N,n}) \geq 1 - \sigma_1(2^N, \infty)^{1/3}$. Fixing N , the $E'_{N,n}$ are monotonic in n .

The set D_N defined by the condition that for all $N \leq r \leq r'$, $S(2^r, 2^{r'})(x) \leq 2\sigma_1(2^N, \infty)^{1/3}$ has therefore a measure $\lambda(D_N) \geq 1 - \sigma_1(2^N, \infty)^{1/3}$. Since $\lambda(D_N) \rightarrow 1$, as $N \rightarrow +\infty$, we deduce that $\lambda(\limsup D_N) = 1$. If $x \in \limsup D_N$, the sequence $(S(2^n)(x))$ clearly satisfies the Cauchy criterion, so converges. This concludes *step 1*.

Step 2. To complete the proof, we show that a.e $\sup_{2^r < n < 2^{r+1}} |S(2^r, n)(x)| \rightarrow 0$, as $r \rightarrow +\infty$. Let $2^r < n < 2^{r+1}$ and decompose n in base 2 :

$$n = 2^r + \sum_{l=1}^r \theta_l 2^{r-l}, \text{ with } \theta_l \in \{0, 1\}.$$

Then :

$$S(2^r, n) = \sum_{l=1}^r S \left(2^r + \sum_{m=1}^{l-1} \theta_m 2^{r-m}, 2^r + \sum_{m=1}^l \theta_m 2^{r-m} \right).$$

By convexity :

$$\begin{aligned} S(2^r, n)^2 &\leq r \sum_{l=1}^r S \left(2^r + \sum_{m=1}^{l-1} \theta_m 2^{r-m}, 2^r + \sum_{m=1}^l \theta_m 2^{r-m} \right)^2 \\ &\leq r \sum_{l=1}^r \sum_{h=0}^{2^{r-l}-1} S(2^r + h2^l, 2^r + h2^l + 2^{l-1})^2 =: T(r). \end{aligned}$$

The quantity $T(r)$ is independent on $2^r < n < 2^{r+1}$. Next :

$$\begin{aligned}
\int_0^1 T(r)(x) dx &= r \sum_{l=1}^r \sum_{h=0}^{2^{r-l}-1} \int_0^1 S(2^r + h2^l, 2^r + h2^l + 2^{l-1})^2(x) dx \\
&\leq r \sum_{l=1}^r \sum_{h=0}^{2^{r-l}-1} \sigma_0(2^r + h2^l, 2^r + h2^l + 2^{l-1}) \\
&\leq r \sum_{l=1}^r \sigma_0(2^r, 2^{r+1}) = r^2 \sigma_0(2^r, 2^{r+1}) \leq \sigma_2(2^r, 2^{r+1}).
\end{aligned}$$

Fix N and let $r \geq N$. By Markov's inequality, for x in a Borel set $F_r(N)$ of Lebesgue measure $\lambda(F_r(N)) \geq 1 - \sigma_2(2^r, 2^{r+1})/\sigma_2(2^N, \infty)^{2/3}$, we have :

$$\sup_{2^r < n < 2^{r+1}} S(2^r, n)^2(x) \leq T(r) \leq \sigma_2(2^N, \infty)^{2/3}.$$

Let $G_N = \cap_{r \geq N} F_r(N)$. Then $\lambda(G_N) \geq 1 - \sum_{r \geq N} \sigma_2(2^r, 2^{r+1})/\sigma_2(2^N, \infty)^{2/3} = 1 - \sigma_2(2^N, \infty)^{1/3}$. For $x \in G_N$:

$$\forall r \geq N, \sup_{2^r < n < 2^{r+1}} |S(2^r, n)(x)| \leq \sigma_2(2^N, \infty)^{1/3}.$$

As $\lambda(G_N) \rightarrow 1$, we get $\lambda(\limsup G_N) = 1$. If $x \in \limsup G_N$, then $\sup_{2^r < n < 2^{r+1}} |S(2^r, n)(x)|$ tends to 0, as $r \rightarrow \infty$. This concludes *step 2* and the proof of the proposition. \square

5 Particular classes where L^2 -convergence is true

We consider the proof of theorem 1.2.

5.1 Proof of *i*)

Let (n_k) be lacunary in the sense that $n_{k+1}/n_k \geq c > 1$ and $(a_k) \in l^2$. We first consider the upper bound. We can assume the sequence (a_k) finite. By lemma 2.1, the L^2 -norm of $(2/\zeta(2\lambda))^{1/2} \sum a_k g_\lambda(n_k x)$ is given by :

$$\sum_{k, l \geq 1} a_k a_l \left(\frac{n_k \wedge n_l}{n_k \vee n_l} \right)^\lambda = \sum a_k^2 + 2 \sum_{k < l} a_k a_l \frac{(n_k \wedge n_l)^{2\lambda}}{(n_k n_l)^\lambda}.$$

Using Cauchy-Schwarz's inequality, the second term is bounded by :

$$\begin{aligned}
\sum_{k < l} |a_k a_l| \frac{(n_k \wedge n_l)^{2\lambda}}{(n_k n_l)^\lambda} &\leq \sum_{k < l} |a_k a_l| \frac{n_k^{2\lambda}}{(n_k n_l)^\lambda} \leq \sum_{k < l} |a_k a_l| c^{-\lambda(l-k)} \leq \sum_{k \geq 1} |a_k| \sum_{l \geq 1} c^{-\lambda l} |a_{k+l}| \\
&\leq \left(\sum_{n \geq 1} a_n^2 \right)^{1/2} \left(\sum_{k \geq 1} \left(\sum_{l \geq 1} c^{-\lambda l} |a_{k+l}| \right)^2 \right)^{1/2}.
\end{aligned}$$

Next, still via Cauchy-Schwarz's inequality :

$$\begin{aligned}
\sum_{k \geq 1} \left(\sum_{l \geq 1} c^{-\lambda l} |a_{k+l}| \right)^2 &= \sum_{l, l' \geq 1} c^{-\lambda(l+l')} \sum_{k \geq 1} |a_{k+l} a_{k+l'}| \\
&\leq \sum_{l, l' \geq 1} c^{-\lambda(l+l')} \left(\sum_{k \geq 1} a_{k+l}^2 \right)^{1/2} \left(\sum_{k \geq 1} a_{k+l'}^2 \right)^{1/2} \\
&\leq \left(\sum_{l \geq 1} c^{-\lambda l} \left(\sum_{k \geq 1} a_{k+l}^2 \right)^{1/2} \right)^2 \leq \frac{1}{(c^\lambda - 1)^2} \sum_{k \geq 1} a_k^2.
\end{aligned}$$

As a consequence :

$$\sum_{k < l} |a_k a_l| \frac{(n_k \wedge n_l)^{2\lambda}}{(n_k n_l)^\lambda} \leq \frac{1}{(c^\lambda - 1)} \sum_{k \geq 1} a_k^2.$$

Since $1 + 2/(c^\lambda - 1) = (c^\lambda + 1)/(c^\lambda - 1)$, this completes the proof of the upper-bound.

For the lower bound, one can also suppose that (a_k) is finite. We have $\sum a_k g_\lambda(n_k x) = \sum b_k f_{k, \lambda}$, where (b_k) is also finite. By proposition 2.2 :

$$\| \sum a_k g_\lambda(n_k x) \|^2 = \| \sum b_k f_{k, \lambda} \|^2 \geq \frac{1}{2} \sum b_k^2.$$

Fixing $0 < \varepsilon < 1 - (2\lambda)^{-1}$, giving $2\lambda(1 - \varepsilon) > 1$:

$$\begin{aligned}
\sum a_k^2 &= \sum_{k \geq 1} \left(\sum_{l \geq 1} l^{-\lambda} \mu(l) b_{ln_k} \right)^2 \leq \sum_{k \geq 1} \left(\sum_{l \geq 1} l^{-2\lambda(1-\varepsilon)} \mu(l)^2 \right) \left(\sum_{l \geq 1} l^{-2\lambda\varepsilon} b_{ln_k}^2 \right) \\
&\leq \prod_{i \geq 1} \left(1 + p_i^{-2\lambda(1-\varepsilon)} \right) \left(\sum_{l \geq 1} b_l^2 \sum_{k, n_k | l} \left(\frac{n_k}{l} \right)^{2\lambda\varepsilon} \right) \\
&\leq \prod_{i \geq 1} \left(\frac{1 - p_i^{-4\lambda(1-\varepsilon)}}{1 - p_i^{-2\lambda(1-\varepsilon)}} \right) \left(\sum_{m \geq 0} c^{-2\lambda\varepsilon m} \right) \sum_{l \geq 1} b_l^2 \\
&\leq \frac{\zeta(2\lambda(1-\varepsilon))}{\zeta(4\lambda(1-\varepsilon))} (1 - c^{-2\lambda\varepsilon})^{-1} \sum_{l \geq 1} b_l^2.
\end{aligned}$$

To complete the proof, we first use that $\zeta(4\lambda(1-\varepsilon)) \geq \zeta(4\lambda)$ and $\zeta(2\lambda(1-\varepsilon)) \leq 1 + 1/(2\lambda(1-\varepsilon) - 1)$. Set $\varepsilon = \rho(1 - 1/(2\lambda))$, with $0 < \rho < 1$. We have :

$$\left(1 + \frac{1}{2\lambda(1-\varepsilon) - 1} \right) \frac{1}{1 - c^{-2\lambda\varepsilon}} \leq \frac{2\lambda}{2\lambda - 1} \frac{1}{(1 - \rho)(1 - c^{-(2\lambda-1)\rho})}.$$

Minimizing in ρ , we take $\rho = 1/(1 + \ln c^{2\lambda-1})$. We finally use the inequality $1 - e^{-x} \geq (1 - 1/e)x$, for $0 \leq x \leq 1$, giving $(1 - c^{-(2\lambda-1)\rho}) \geq (1 - 1/e)(1 - \rho)$.

Concerning a.e-convergence, we can now apply proposition 4.1 with $\varphi = 1$.

□

5.2 Proof of *ii*)

We start from (22). For n with $|\text{supp}(n)| \leq N$ and any $0 < \delta < 2\lambda - 1$, we have :

$$\sum_{k|n} k^{-\lambda} \psi_\lambda(k) \leq C_\delta \sum_{k|n} k^{1-2\lambda+\delta} \leq C_\delta \prod_{p|n, p \in \mathcal{P}} (1 - p^{1-2\lambda+\delta})^{-1} \leq C_\delta \prod_{i=1}^N (1 - p_i^{1-2\lambda+\delta})^{-1}.$$

For the lower bound, we use $\|\sum b_n f_{n,\lambda}\|^2 \geq (1/2) \sum b_n^2$, by proposition 2.2. Next :

$$\sum a_k^2 = \sum_{k \geq 1} \left(\sum_{l \geq 1} l^{-\lambda} \mu(l) b_{lk} \right)^2 \leq \sum_{k \geq 1} \left(\sum_{l \geq 1} l^{-\lambda} |b_{lk}| \right)^2 \leq C_\varepsilon \sum_{n \geq 1} b_n^2 \sum_{k|n} k^{-\lambda} \psi_\lambda(k),$$

when proceeding in the same way as for (22). We then conclude as above, using the fact that $b_n = 0$ when $|\text{supp}(n)| > N$. For a.e-convergence, we apply proposition 4.1 with $\varphi = 1$. \square

5.3 Proof of *iii*)

Using the remark after lemma 2.1 we only need to focus on (b_n) . Set $b_{i,n} = b_{p_i^n}$. Multiplicativity implies that the $(b_{i,n})_{i,n \geq 1}$ entirely determine the sequence (b_n) . Via corollary (2.3), we show that :

$$\sum_{n \geq 1} \left(\sum_{k \geq 1} k^{-\lambda} b_{kn} \right)^2 < +\infty. \quad (24)$$

Each term in this series is finite, by Cauchy-Schwarz's inequality. We first claim the equivalence $(b_n) \in l^2 \Leftrightarrow \sum_{i \geq 1} \sum_{n \geq 1} b_{i,n}^2 < +\infty$. Indeed, using that $b_1 = 1$, we have :

$$\sum_{n \geq 1} b_n^2 = 1 + \sum_{k \geq 1} \sum_{1 \leq i_1 < \dots < i_k} \sum_{u_1 \geq 1, \dots, u_k \geq 1} b_{i_1, u_1}^2 \dots b_{i_k, u_k}^2 = \prod_{i \geq 1} \left(1 + \sum_{n \geq 1} b_{i,n}^2 \right). \quad (25)$$

This proves the claim.

For technical reasons, up to considering $\tilde{b}_{i,n} = b_{i,n} + 1/(in)$, $(i, n) \geq 1$, and the corresponding multiplicative sequence (\tilde{b}_n) , which satisfies $(\tilde{b}_n) \in l^2 \Leftrightarrow (b_n) \in l^2$, we assume that $b_{i,n} > 0$ for all indices $(i, n) \geq 1$. Decomposing in prime factors $n = p_{i_1}^{u_1} \dots p_{i_k}^{u_k}$, with $k = 0$ if $n = 1$, and using multiplicativity :

$$\begin{aligned} \sum_{n \geq 1} \left(\sum_{l \geq 1} l^{-\lambda} b_{ln} \right)^2 &= \sum_{k \geq 0} \sum_{1 \leq i_1 < \dots < i_k} \sum_{u_1 \geq 1, \dots, u_k \geq 1} \\ &\quad \prod_{j \notin \{i_1, \dots, i_k\}} \left(1 + \sum_{m \geq 1} p_j^{-\lambda m} b_{j,m} \right) \times \prod_{l=1, \dots, k} \left(\sum_{v_l \geq 0} p_{i_l}^{-\lambda v_l} b_{i_l, u_l + v_l} \right). \end{aligned}$$

The first product term is uniformly bounded from above since for a constant C :

$$\sum_{j, m \geq 1} p_j^{-\lambda m} b_{j,m} \leq \left(\sum_{j, m \geq 1} p_j^{-2\lambda m} \right)^{1/2} \left(\sum_{j, m \geq 1} b_{j,m}^2 \right)^{1/2} \leq C \left(\sum_{j \geq 1} p_j^{-2\lambda} \right)^{1/2} \left(\sum_{j, m \geq 1} b_{j,m}^2 \right)^{1/2} < +\infty.$$

To prove (24), it remains to check the finiteness of :

$$\sum_{k \geq 0} \sum_{1 \leq i_1 < \dots < i_k} \prod_{l=1}^k \left[\sum_{u_l \geq 1} \left(\sum_{v_l \geq 0} p_{i_l}^{-\lambda v_l} b_{i_l, u_l + v_l} \right)^2 \right] = \prod_{i \geq 1} \left[1 + \sum_{u \geq 1} \left(\sum_{v \geq 0} p_i^{-\lambda v} b_{i, u+v} \right)^2 \right].$$

It is equivalent to showing :

$$\sum_{i \geq 1, u \geq 1} \left(\sum_{v \geq 0} p_i^{-\lambda v} b_{i, u+v} \right)^2 < +\infty.$$

Set $c_{i,n} = p_i^{\lambda n} \sum_{v \geq n} p_i^{-\lambda v} b_{i,v}$, which is finite by Cauchy-Schwarz's inequality. We thus verify that $\sum_{i \geq 1, n \geq 1} c_{i,n}^2 < +\infty$. Fixing $0 < \varepsilon < 1 - 2^{-\lambda}$, we prove below that for all $i \geq 1$:

$$\begin{aligned} \sum_{n \geq 1} c_{i,n}^2 &\leq \frac{1}{\varepsilon^2 (1 - (1 - \varepsilon)^{-2} p_i^{-2\lambda})} \sum_{n \geq 1} c_{i,n}^2 \left(1 - \frac{c_{i,n+1}}{p_i^\lambda c_{i,n}} \right)^2 \\ &\leq \frac{1}{\varepsilon^2 (1 - (1 - \varepsilon)^{-2} 2^{-2\lambda})} \sum_{n \geq 1} b_{i,n}^2. \end{aligned} \quad (26)$$

Since $\sum_{i,n \geq 1} b_{i,n}^2 < \infty$, this brings the conclusion.

Fix $i \geq 1$ and introduce $\mathcal{C} = \{n \geq 1, |1 - c_{i,n+1}/(p_i^\lambda c_{i,n})| < \varepsilon\}$. We claim that if \mathcal{C} is infinite, then it does not contain all large integers. Indeed, if $n \in \mathcal{C}$, then $c_{i,n+1} \geq (1 - \varepsilon) p_i^\lambda c_{i,n}$, since $c_{i,n+1} < p_i^\lambda c_{i,n}$. If \mathcal{C} contains some interval $[n_0, +\infty)$, then for $n \geq n_0$:

$$\sum_{v \geq 0} b_{i,v+n} p_i^{-\lambda v} = p_i^{\lambda n} \sum_{v \geq n} b_{i,v} p_i^{-\lambda v} \geq p_i^{\lambda n} (1 - \varepsilon)^{n-n_0} \sum_{v \geq n_0} b_{i,v} p_i^{-\lambda v}.$$

However $\sum_{v \geq n_0} b_{i,v} p_i^{-\lambda v}$ is fixed and > 0 , since $b_{i,v} > 0$. As $p_i^\lambda (1 - \varepsilon) > 1$, a contradiction is given by Cauchy-Schwarz's inequality, because the left-hand side is bounded from above by :

$$\left(\sum_{v \geq 0} p_i^{-2\lambda v} \right)^{1/2} \left(\sum_{v \geq 0} b_{i,n+v}^2 \right)^{1/2} \leq \left(\sum_{v \geq 0} p_i^{-2\lambda v} \right)^{1/2} \left(\sum_{v \geq 0} b_{i,v}^2 \right)^{1/2} < +\infty.$$

Decompose now into disjoint intervals $\mathcal{C} = \cup_{k \geq 1} [a_k, b_k]$ and write in a disjoint union :

$$\{n \geq 1\} = [a_1, b_1] \cup [a'_1, b'_1] \cup \dots \cup [a_k, b_k] \cup [a'_k, b'_k] \cup \dots.$$

Notice that the first interval $[a_1, b_1]$ may be empty, whereas the other ones are not, and that the collection of $([a_k, b_k], [a'_k, b'_k])_k$ may be finite. We have :

$$\sum_{n \notin \mathcal{C}, n \notin \{a'_k, k \geq 1\}} c_{i,n}^2 \left(1 - \frac{c_{i,n+1}}{p_i^\lambda c_{i,n}} \right)^2 \geq \varepsilon^2 \sum_{n \notin \mathcal{C}, n \notin \{a'_k, k \geq 1\}} c_{i,n}^2. \quad (27)$$

Also :

$$\sum_{k \geq 1} \sum_{l=a_k}^{a'_k} c_{i,l}^2 \left(1 - \frac{c_{i,l+1}}{p_i^\lambda c_{i,l}} \right)^2 \geq \varepsilon^2 \sum_{k \geq 1} c_{i,a'_k}^2. \quad (28)$$

Observe finally that :

$$\sum_{l=a_k}^{a'_k} c_{i,l}^2 \leq c_{i,a'_k}^2 \sum_{m \geq 0} (1-\varepsilon)^{-2m} p_i^{-2\lambda m} \leq (1 - (1-\varepsilon)^{-2} p_i^{-2\lambda})^{-1} c_{i,a'_k}^2. \quad (29)$$

Combining (27), (28) and (29), we get (26). This completes the proof of this item. \square

Acknowledgements. We thank Zoltan Buczolic about the first point of theorem 1.1.

References

- [1] BALAZARD, M. AND SAIAS, E., *Notes on the Riemann zeta-function. IV*, Adv. Math. 188 (2004), no. 1, 69–86.
- [2] BOURGIN, D. AND MENDEL, C., *Orthonormal sets of periodic functions of the type $\{f(nx)\}$* , Trans. Amer. Math. Soc. 57, 1945, 332-363.
- [3] DE LA BRETÈCHE, R. AND TENENBAUM, G., *Séries trigonométriques à coefficients arithmétiques*, J. Anal. Math. 92 (2004), 1-79.
- [4] CARLESON, L., *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), 135-157.
- [5] DAVENPORT, H., *On some infinite series involving arithmetical functions*, Quart. J. Math., vol 8, pp. 8-13 (1937).
- [6] DAVENPORT, H., *On some infinite series involving arithmetical functions II*, Quart. J. Math., vol 8, pp. 313-320 (1937).
- [7] FEFFERMAN, C., *Pointwise convergence of Fourier series*, Ann. of Math. 98 (1973), 551-571.
- [8] HARDY, G. AND LITTLEWOOD, J., *Some problems of Diophantine approximation : the analytic properties of certain Dirichlet's series associated with the distribution of numbers of modulus unity*, Trans. Cambridge Philos. Soc., vol. 22 (1923), 519-533.
- [9] HARTMAN, P., *On a class of arithmetical Fourier series*, Amer. J. Math. vol. 60 (1938), 66-74.
- [10] HAUKKANEN, P., WANG, J. AND SILLANP, J., *On Smith's determinant*, Linear Algebra Appl. 258 (1997), 251-269.
- [11] HEDENMALM, H., LINDQVIST, P. AND SEIP., K., *A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$* , Duke Math. J., vol. 86, pp. 1-37 (1997).
- [12] JAFFARD, S., *On Davenport expansions*, Fractal geometry and applications : a jubilee of Benoît Mandelbrot. Part 1, 273–303, Proc. Sympos. Pure Math., 72, Part 1, Amer. Math. Soc., Providence, RI, 2004.
- [13] KRÄTZEL, E., *Zahlentheorie*, Berlin (1981).
- [14] LANG, S., *Introduction to Diophantine approximation*, Springer-Verlag (1995).
- [15] LINDQVIST, P. AND SEIP., K., *Note on some greatest common divisor matrices*, Acta Arith. 84 (1998), no. 2, 149-154.
- [16] RADEMACHER, H., *Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen*, Math. Ann. 87 (1922), 112-138.
- [17] WINTNER, A., *Diophantine approximation and Hilbert's space*, Amer. J. Math., vol. 66 (1944), 564-578.

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