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# DAVENPORT SERIES AND ALMOST-SURE CONVERGENCE

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## Abstract

We consider Davenport-like series with coefficients in  $l^2$  and discuss  $L^2$ -convergence as well as almost-everywhere convergence. We give an example where both fail to hold. We next improve former sufficient conditions under which these convergences are true.

## 1 Introduction

Let  $\mathbb{R}\backslash\mathbb{Z}$  be the Circle and  $L^2$  be the restriction of the space  $L^2(\mathbb{R}\backslash\mathbb{Z} \rightarrow \mathbb{R})$  to odd functions. For a real parameter  $\lambda > 1/2$ , we introduce the map :

$$g_\lambda(x) = \sum_{m \geq 1} \frac{\sin(2\pi m x)}{m^\lambda}.$$

This function is defined everywhere on  $\mathbb{R}\backslash\mathbb{Z}$ . It is continuous, except at 0 when  $1/2 < \lambda \leq 1$ , and belongs to  $L^2$ . For real sequences  $(a_n) \in l^2$ , we consider expansions based on the dilated functions system  $\{g_\lambda(nx)\}_{n \geq 1}$  of the following form :

$$\sum a_n g_\lambda(nx), \tag{1}$$

where we write  $\sum$  for the summation  $\sum_{n=1}^{+\infty}$ . We are interested in the questions of  $L^2$ -convergence and Lebesgue almost-everywhere (a.e) convergence of such series.

This is a natural problem which can be formulated with  $g_\lambda$  replaced by a general  $g \in L^2(\mathbb{R}\backslash\mathbb{Z})$ . A reason for focusing on odd functions is that sin series in general better converge than cos series. When  $g(x) = \sin(2\pi x)$ , the  $L^2$ -convergence of  $\sum a_n g(nx)$  follows from the fact that the  $\{g(nx)\}_{n \geq 1}$  are orthonormal. A.e-convergence in this case is the difficult theorem of Carleson [4]. For a different  $g$  such that the  $\{g(nx)\}_{n \geq 1}$  are complete in  $L^2$ , the  $\{g(nx)\}_{n \geq 1}$  are not orthogonal, see Bourgin and Mendel [2], and the question of  $L^2$ -convergence is already not clear. The case of  $g = g_\lambda$  was introduced by Wintner in [17]. A special motivation comes Arithmetics and the case  $\lambda = 1$ , corresponding to the first Bernoulli polynomial or “sawtooth function”  $\{x\} := x - [x] - 1/2$ , where  $[x]$  is the integer part of  $x \in \mathbb{R}$ . Indeed :

$$\{x\} = -\frac{1}{\pi} \sum_{m \geq 1} \frac{\sin(2\pi m x)}{m}.$$

Series of the form  $\sum a_n \{nx\}$  appear since long ago in the litterature, at the interface of Number Theory and Analysis. We recommend the very detailed presentation of such series by Jaffard in [12], where they are called Davenport series, due to Davenport’s initial systematic study [5, 6]. In

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this article and for simplicity we call  $D_\lambda$ -series a series of the form (1), the case of Davenport series corresponding to  $\lambda = 1$ .

Beginning with a discussion in  $L^2$ , Wintner [17] established that the family  $\{g_\lambda(nx)\}_{n \geq 1}$  is complete in  $L^2$  for any  $\lambda > 1/2$ . We now consider the  $L^2$ -convergence of  $D_\lambda$ -series with  $(a_n) \in l^2$ . According to work by Wintner [17] and next Hedenmalm, Lindqvist and Seip [11], the  $\{g_\lambda(nx)\}_{n \geq 1}$  form a Riesz basis when  $\lambda > 1$ . By a ‘‘Riesz basis’’ we mean a complete sequence  $(\xi_n)$  in  $L^2$  such that for some constant  $C > 0$  :

$$C^{-1} \sum a_n^2 \leq \left\| \sum a_n \xi_n \right\|^2 \leq C \sum a_n^2, \forall (a_n) \in l^2.$$

Here and in the whole article we denote by  $\| \cdot \|$  the usual  $L^2$ -norm, with scalar product  $\langle \cdot, \cdot \rangle$ . Lindqvist and Seip [15] provide the inequalities :

$$\frac{\zeta(2\lambda)}{\zeta(\lambda)^2} \sum a_n^2 \leq \left\| \sum a_n g_\lambda(nx) \right\|^2 \leq \frac{\zeta(\lambda)^2}{\zeta(2\lambda)} \sum a_n^2,$$

where  $\zeta(s) = \sum_{n \geq 1} n^{-s}$ , for  $s > 1$ , is the Riemann Zeta function. Constants are optimal. This settles the question of  $L^2$ -convergence when  $\lambda > 1$ . In this case, the a.e-convergence of  $D_\lambda$ -series for  $(a_n) \in l^2$  follows from Carleson’s theorem [4] on the a.e-convergence of Fourier series. Indeed :

$$\sum_{n=1}^N a_n g_\lambda(nx) = \sum_{m \geq 1} m^{-\lambda} \sum_{n=1}^N a_n \sin(2\pi mnx). \quad (2)$$

For each  $m \geq 1$ ,  $\sum_{n=1}^N a_n \sin(2\pi mnx)$  converges a.e, as  $N \rightarrow +\infty$ , by Carleson’s theorem. Next :

$$\left| \sum_{n=1}^N a_n \sin(2\pi mnx) \right| \leq \sup_{K \geq 1} \left| \sum_{n=1}^K a_n \sin(2\pi mnx) \right| =: M(mx).$$

By classical work on the maximal operator,  $\|M\|_{L^1(\mathbb{R}/\mathbb{Z})} \leq C \sum a_n^2$  (cf for instance Fefferman [7]). Thus  $\sum_{m \geq 1} m^{-\lambda} M(mx)$  is integrable and in particular a.e finite. One can now a.e apply the Lebesgue dominated convergence theorem in (2) to conclude. Of course this argument does not work when  $1/2 < \lambda \leq 1$ . Mention also that Carleson’s theorem uses properties of the Fourier basis. There exists orthonormal bases of  $L^2$  for which  $L^2$ -convergence does not imply a.e-convergence, see Rademacher [16].

For the rest of the article we suppose that  $1/2 < \lambda \leq 1$ . As a consequence of an analysis by Wintner [17] of some Dirichlet series associated to  $D_\lambda$ -series, for any  $1/2 < \lambda \leq 1$  there exists  $(a_n) \in l^2$  such that  $\sum a_n g_\lambda(nx)$  is  $L^2$ -divergent. In particular for  $1/2 < \lambda \leq 1$ , the  $\{g_\lambda(nx)\}_{n \geq 1}$  do not form a Riesz basis of  $L^2$ . We now detail known sufficient conditions for  $L^2$  and a.e-convergence. We are essentially aware of results concerning Davenport series. Wintner [17] proved that  $\sum a_n \{nx\}$  converges in  $L^2$  whenever  $a_n = O(n^{-\kappa})$ , with  $\kappa > 1/2$ . Extending this result, Jaffard [12] showed that for  $(a_n) \in l^2$  the sum  $\sum a_n \{nx\}$  converges in a Sobolev space very close to  $L^2$ . About a.e-convergence, Davenport in his fundamental papers [5, 6] gave non-trivial arithmetical examples where a.e-convergence is true, such as :

$$\sum_{n=1}^{+\infty} \frac{\lambda(n)}{n} \{nx\}, \sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n} \{nx\}, \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^2} \{n^2 x\}, \quad (3)$$

where  $\lambda(n)$ ,  $\Lambda(n)$  and  $\mu(n)$  are respectively Liouville’s function, Von Mangolt’s function and Mobius’ function. When the  $a_n$  are slowly varying, the a.e-convergence of  $\sum a_n \{nx\}$  follows, via an Abel transform, from estimates on  $\sum_{n < N} \{nx\}$ , cf Lang [14]. Jaffard [12] deduced the a.e-convergence of  $\sum a_n \{nx\}$ , whenever  $a_n = O(\log n)^{-\alpha}$  and  $a_{n+1} - a_n = O(n^{-1}(\log n)^{-(1+\alpha)})$  for some  $\alpha > 2$ . In particular, Hecke series :

$$\mathcal{H}_s(x) = \sum_{n=1}^{+\infty} \frac{\{nx\}}{n^s} \quad (4)$$

converge a.e for  $\text{Re}(s) > 0$ , a result already shown by Hardy and Littlewood [8]. For general sequences, Hartman proved in [9] the a.e-convergence of  $\sum a_n \{nx\}$  when  $a_n = O(n^{-\kappa})$ , with  $\kappa > 2/3$ . Mention finally some results going further than a.e-convergence. Using P-summation techniques, de la Bretèche and Tenenbaum [3] proved that (4) is convergent when  $s = 1$  outside a set of Hausdorff dimension zero that they describe. Also the second series in (3) is everywhere convergent.

We now detail the content of the article. We discuss the  $L^2$  and a.e-convergence of  $D_\lambda$ -series of the form (1) for general  $(a_n) \in l^2$ . We first complete the  $L^2$ -divergence result of Wintner [17] by an a.e-divergence result. We next improve former sufficient conditions for  $L^2$  and a.e-convergence.

### Theorem 1.1

Assume that  $1/2 < \lambda \leq 1$ .

i) There exists  $(a_n) \in l^2$  such that  $\sum a_n g_\lambda(nx)$  is simultaneously  $L^2$ -divergent and a.e-divergent.

ii) Suppose that for some  $\varepsilon > 0$  :

$$\begin{cases} \sum a_n^2 n^{\frac{(1+\varepsilon)(\log n) - (2\lambda-1)}{2(1-\lambda) \log \log n}} < \infty, \text{ when } 1/2 < \lambda < 1, \\ \sum a_n^2 (\log n)^3 (\log \log n)^{2+\varepsilon} < \infty, \text{ when } \lambda = 1. \end{cases}$$

Then  $\sum a_n g_\lambda(nx)$  converges in  $L^2$  and a.e.

In particular, the latter conditions are verified if  $\sum a_n^2 n^\varepsilon < \infty$ , for some  $\varepsilon > 0$ . For example, the following series converge in  $L^2$  and a.e when  $s > 1/2$  :

$$\sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s} \{nx\}, \quad \sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n^s} \{nx\} \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{2s}} \{n^2x\}.$$

We note that whereas Wintner's approach is abstract we build here an explicit example. The fact that  $(a_n) \in l^2$  does not imply the a.e-convergence of  $D_\lambda$ -series is not that surprising, since this condition is not the correct one for  $L^2$ -convergence. One can make the second moment explode and develop a probabilistic argument based on the Central Limit Theorem. The true question, more difficult, is whether  $L^2$ -convergence implies a.e-convergence. A weak formulation is as follows :

**Question.** If  $1/2 < \lambda \leq 1$  and  $\sum_{k \geq 1} \left( \sum_{n \geq 1} n^{-\lambda} |a_{kn}| \right)^2 < +\infty$ , does  $\sum a_n g_\lambda(nx)$  converge a.e ?

The above condition is strictly stronger than  $(a_n) \in l^2$ , when  $1/2 < \lambda \leq 1$ . As detailed below, it ensures the  $L^2$ -convergence of  $\sum a_n g_\lambda(nx)$  and is necessary when the  $a_n$  have constant sign.

We next consider three classical situations, for instance Hadamard lacunarity, where we can prove  $L^2$ -convergence, but a.e-convergence only under stronger conditions. We define the support  $\text{supp}(n)$  of an integer  $n$  as its set of prime divisors and write  $|\text{supp}(n)|$  for the cardinal of this set.

### Theorem 1.2

Suppose that  $1/2 < \lambda \leq 1$ .

i) Let  $(n_k)$  check  $n_{k+1}/n_k \geq c$ , where  $c > 1$ . Then  $\sum a_k g_\lambda(n_k x)$  converges in  $L^2$  whenever  $(a_k) \in l^2$  and more precisely :

$$C_1 \sum a_k^2 \leq \left\| \sum a_k g_\lambda(n_k x) \right\|^2 \leq C_2 \sum a_k^2,$$

where :

$$C_1 = (1 - 1/e) \frac{\zeta(4\lambda)}{2} \left( \frac{2\lambda - 1}{2\lambda} \right) \left( \frac{\ln(c^{2\lambda-1})}{1 + \ln(c^{2\lambda-1})} \right)^2 \quad \text{and} \quad C_2 = \frac{\zeta(2\lambda)}{2} \left( \frac{c^\lambda + 1}{c^\lambda - 1} \right).$$

If the stronger condition  $\sum a_k^2 (\log k)^2 < \infty$  is verified, then  $\sum a_k g_\lambda(n_k x)$  is also a.e-convergent.

ii) If  $(a_n) \in l^2$  and  $\{| \text{supp}(n) |, a_n \neq 0\}$  is finite, then  $\sum a_n g_\lambda(n x)$  converges in  $L^2$ . In fact, for  $N \geq 1$  there exists  $C(\lambda, N) > 0$  such that for  $(a_n) \in l^2$  with  $a_n = 0$  if  $| \text{supp}(n) | > N$ , then :

$$C^{-1}(\lambda, N) \sum a_n^2 \leq \| \sum a_n g_\lambda(n x) \|^2 \leq C(\lambda, N) \sum a_n^2.$$

If moreover  $\sum a_n^2 (\log n)^2 < \infty$ , then  $\sum a_n g_\lambda(n x)$  is also a.e-convergent.

iii) Let  $a_n = O(b_n)$ , where  $(b_n)_{n \geq 1} \in l^2 \cap (\mathbb{R}_+)^{\mathbb{N}}$  satisfies  $b_{nm} = b_n b_m$  whenever  $n$  and  $m$  are relatively prime. Then  $\sum a_n g_\lambda(n x)$  converges in  $L^2$ .

A word on the method. The study of the convergence of Davenport series often starts with trying to write  $\sum a_n \{n x\}$  as a Fourier series  $\sum c_m \sin 2\pi m x$ . It was indeed remarked by Davenport [5] that formally the  $(c_m)$  are explicitly given in terms of the  $(a_n)$  and vice-versa. An alternative approach, developed here, when considering  $L^2$ -convergence is to orthonormalize the  $\{g_\lambda(n x)\}_{n \geq 1}$ . The Gram-Schmidt orthonormalisation procedure is explicit and a consequence of Carlitz's lemma on the reduction of quadratic forms. We provide details for simplicity. This furnishes a rather simple characterization of  $L^2$ -convergence. Theorem 1.2 follows via more or less standard computations. Concerning a.e-convergence, the orthonormalisation approach allows to adapt a technique of Rademacher [16] initially developed for the pointwise convergence of series built with general orthonormal systems.

A few notations. We write  $i \wedge j$  and  $i \vee j$  respectively for the greatest common divisor and the smallest common multiple of integers  $i$  and  $j$ . The set of primes is  $\mathcal{P} = \{p_n, n \geq 1\}$ .

## 2 Orthonormalization

Recall that  $1/2 < \lambda \leq 1$ . We first study correlations. The following computation is already contained in Lindqvist and Seip [15].

### Lemma 2.1

Let  $i, j \geq 1$ . Then  $\langle g_\lambda(i \cdot), g_\lambda(j \cdot) \rangle = \frac{\zeta(2\lambda)}{2} \left( \frac{i \wedge j}{i \vee j} \right)^\lambda$ .

*Proof of the lemma :*

Let  $i' = i/i \wedge j$ ,  $j' = j/i \wedge j$ . Since Lebesgue measure on  $\mathbb{R} \setminus \mathbb{Z}$  is invariant by  $x \mapsto px$  for any integer  $p$ , we have  $\langle g_\lambda(i \cdot), g_\lambda(j \cdot) \rangle = \langle g_\lambda(i' \cdot), g_\lambda(j' \cdot) \rangle$ . Using the Fourier expansion of  $g_\lambda$  :

$$\langle g_\lambda(i' \cdot), g_\lambda(j' \cdot) \rangle = \sum_{k, l \geq 1} \int_0^1 \frac{\sin(2\pi k i' x)}{k^\lambda} \frac{\sin(2\pi l j' x)}{l^\lambda} dx = \frac{1}{2} \sum_{m \geq 1} \frac{1}{(m^2 i' j')^\lambda} = \frac{\zeta(2\lambda)}{2} (i' j')^{-\lambda},$$

since a relation  $k i' = l j'$  reduces to  $k = j' m$  and  $l = i' m$  for some integer  $m$ . This concludes the proof of the lemma.  $\square$

*Remark.* — The correlations being positive, if  $\sum b_n g_\lambda(n x)$  is  $L^2$ -convergent with  $(b_n) \in (\mathbb{R}_+)^{\mathbb{N}}$  and if  $a_n = O(b_n)$ , then  $\sum a_n g_\lambda(n x)$  also converges in  $L^2$ .

We turn to the orthonormalization of the  $\{g_\lambda(n x)\}_{n \geq 1}$ . The following proposition is an application of Carlitz's lemma, cf for instance Haukkanen, Wang and Sillanp [10].

Recall that the Möbius function  $\mu$  on the integers is defined by  $\mu(1) = 1$ ,  $\mu(p_{i_1} \cdots p_{i_k}) = (-1)^k$  and  $\mu(n) = 0$  if  $n$  is not square-free. If  $f$  and  $g$  are real maps defined on the integers related by  $f(n) = \sum_{k|n} g(k)$ , then (Möbius inversion formula) we have  $g(n) = \sum_{k|n} \mu(n/k) f(k)$ .

**Proposition 2.2**

i) Let  $f_{n,\lambda}(x) = n^{-\lambda} \sum_{k|n} k^\lambda \mu(n/k) g_\lambda(kx)$ ,  $n \geq 1$ . Then  $\{f_{n,\lambda}\}_{n \geq 1}$  is an orthogonal family with :

$$\|f_{n,\lambda}\|^2 = \frac{\zeta(2\lambda)}{2} \prod_{p|n, p \in \mathcal{P}} (1 - p^{-2\lambda}) \in \left(\frac{1}{2}, \frac{\zeta(2\lambda)}{2}\right).$$

The  $\{f_{n,\lambda}\}_{n \geq 1}$  form an orthogonal Riesz basis of  $L^2$ , with :

$$2\zeta(2\lambda)^{-1} \sum_{n \geq 1} \langle f_{n,\lambda}, h \rangle^2 \leq \|h\|^2 \leq 2 \sum_{n \geq 1} \langle f_{n,\lambda}, h \rangle^2, \forall h \in L^2.$$

ii) An equality  $\sum_{i=1}^n a_i g_\lambda(i) = \sum_{i=1}^n b_i f_{i,\lambda}$  holds if and only if  $b_i = \sum_{k=1}^{[n/i]} k^{-\lambda} a_{ki}$ ,  $1 \leq i \leq n$ . These equalities are reversed into  $a_i = \sum_{k=1}^{[n/i]} k^{-\lambda} \mu(k) b_{ki}$ ,  $1 \leq i \leq n$ .

*Proof of the proposition :*

Let  $n \geq 1$ . We introduce  $n$ -square matrices  $D$  and  $T$ , where  $D$  is diagonal and  $T$  is upper-triangular. Set  $T = (t_{ij})$  with  $t_{ij} = j^{-\lambda} 1_{i|j}$  and write  $D = \text{diag}(d_i)$ , where the  $(d_i)$  are defined below. First :

$$({}^t T D T)_{ij} = \sum_{1 \leq k \leq n} t_{ki} d_k t_{kj} = (ij)^{-\lambda} \sum_{k|i \wedge j} d_k.$$

We choose  $D$  so that  $\sum_{k|m} d_k = (\zeta(2\lambda)/2) m^{2\lambda}$ , which by the Möbius inversion formula corresponds to setting  $d_k = (\zeta(2\lambda)/2) \sum_{l|k} \mu(k/l) l^{2\lambda}$ . Lemma 2.1 gives  $({}^t T D T)_{ij} = \langle g_\lambda(i), g_\lambda(j) \rangle$ .

Next, the inverse of  $T$  is given by  $T_{ij}^{-1} = 1_{i|j} i^\lambda \mu(j/i)$ , since for any  $i \leq j$  :

$$\sum_{k=1}^n 1_{i|k} i^\lambda \mu(k/i) j^{-\lambda} 1_{k|j} = 1_{i|j} i^\lambda j^{-\lambda} \sum_{l|j/i} \mu(l) = 1_{i=j},$$

using that  $\sum_{k|m} \mu(k) = 0$ , if  $m \geq 2$ . Observe that  $f_{i,\lambda} = i^{-\lambda} \sum_{1 \leq k \leq n} ({}^t T^{-1})_{ik} g_\lambda(k)$ , for  $1 \leq i \leq n$ . The  $\{f_{i,\lambda}\}_{i \geq 1}$  are therefore orthogonal in  $L^2$ , with  $\|f_{i,\lambda}\|^2 = d_i i^{-2\lambda}$ . They also form a complete family, since it is the case for the  $\{g_\lambda(nx)\}$ , cf Wintner [17]. Decomposing  $i = p_{l_1}^{\alpha_1} \cdots p_{l_k}^{\alpha_k}$  in prime factors, we have :

$$\|f_{i,\lambda}\|^2 = \frac{\zeta(2\lambda)}{2} \sum_{d|i} d^{-2\lambda} \mu(d) = \frac{\zeta(2\lambda)}{2} \prod_{j=1}^k \sum_{d|p_{l_j}^{\alpha_j}} d^{-2\lambda} \mu(d) = \frac{\zeta(2\lambda)}{2} \prod_{p|i, p \in \mathcal{P}} (1 - p^{-2\lambda}).$$

Finally :

$$\sum_{i=1}^n a_i g_\lambda(i) = \sum_{i=1}^n a_i \sum_{k=1}^n ({}^t T)_{ik} k^\lambda f_{k,\lambda} = \sum_{i=1}^n a_i \sum_{k=1}^n i^{-\lambda} 1_{k|i} k^\lambda f_{k,\lambda} = \sum_{k=1}^n f_{k,\lambda} \sum_{i=1}^{[n/k]} i^{-\lambda} a_{ki}.$$

The reversed formula is proved in a similar way. □

We deduce the following characterization of  $L^2$ -convergence.

**Corollary 2.3**

i) The series  $\sum a_n g_\lambda(nx)$  converges in  $L^2$  if and only if the numerical series  $\sum_{k \geq 1} k^{-\lambda} a_{ki}$  converge for all  $i \geq 1$ , together with the uniformity condition :

$$\sum_{i \geq 1} \left( \sum_{k > [n/i]} k^{-\lambda} a_{ki} \right)^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

ii) A sufficient condition for  $\sum a_n g_\lambda(nx)$  to be  $L^2$ -convergent is :

$$\sum_{i \geq 1} \left( \sum_{k \geq 1} k^{-\lambda} |a_{ki}| \right)^2 < +\infty.$$

This condition is necessary when  $(a_n) \in (\mathbb{R}_+)^{\mathbb{N}}$ .

*Proof of the corollary :*

If  $\sum a_n g_\lambda(nx)$  converges in  $L^2$ , by proposition 2.2 the component  $\sum_{k=1}^{[n/i]} k^{-\lambda} a_{ki}$  with respect to each  $f_{i,\lambda}$  converges as  $n \rightarrow +\infty$ . The  $L^2$ -limit then has to be  $\sum_{i \geq 1} f_{i,\lambda}(\sum_{k \geq 1} k^{-\lambda} a_{ki})$ . The uniformity condition is a consequence from the fact that the norm of  $f_{i,\lambda}$  belongs to  $(1/2, \zeta(2\lambda)/2)$ . The first assertion of the second item is an application of the Lebesgue Dominated Convergence Theorem, whereas the second one follows from the first item.  $\square$

*Remark.* — Corollary 2.3 can also be obtained when considering directly the Fourier expansion of  $\sum a_n g_\lambda(nx)$  given by  $g_\lambda$ . In the sequel, the orthonormalization point of view has the practical advantage to keep finite all partial sums.

### 3 A $l^2$ -example of a $L^2$ -divergent and a.e-divergent series

We prove theorem 1.1 i), using that for  $1/2 < \lambda \leq 1$  the series  $\sum_{p \in \mathcal{P}} p^{-\lambda}$  is divergent. For each integer  $K \geq 1$ , we choose a finite set  $\mathcal{P}_K = \{p_{j,K}\}_{1 \leq j \leq l_K}$  of consecutive primes satisfying :

$$\sum_{j=1}^{l_K} (p_{j,K})^{-\lambda} \geq K. \quad (5)$$

We fix  $m_K \geq 2$  so that  $\left(\frac{m_K - 1}{m_K}\right)^{l_K} \geq 1/2$ . Introduce sets :

$$\begin{cases} F_{1,K} = \left\{ p_{1,K}^{u_1} \cdots p_{l_K,K}^{u_{l_K}}, 1 \leq u_1, \dots, u_{l_K} \leq m_K \right\} \\ F'_{1,K} = \left\{ p_{1,K}^{u_1} \cdots p_{l_K,K}^{u_{l_K}}, 1 \leq u_1, \dots, u_{l_K} \leq m_K - 1 \right\}. \end{cases}$$

We have  $|F_{1,K}| = (m_K)^{l_K}$  and  $|F'_{1,K}| = (m_K - 1)^{l_K}$ . Let  $q_{1,K} = 1$  and take next infinitely many primes  $q_{2,K} < \dots < q_{n,K} < \dots$ , larger than  $p_{l_K,K}$  and subject to the condition :

$$(p_{1,K} \cdots p_{l_K,K})^{m_K} \left( 1 + \frac{(m_K)^{l_K/2}}{K} \right) \sum_{r=2}^{+\infty} \frac{(q_{r-1,K})^{r-1}}{q_{r,K}} \leq \frac{1}{K}. \quad (6)$$

Define the random variable :

$$X_{1,K} = \frac{1}{K |F_{1,K}|^{1/2}} \sum_{n \in F_{1,K}} g_\lambda(nx).$$

It has zero mean and belongs to  $L^2$ . We write  $\sigma_K^2 = \int_0^1 (X_{1,K})^2(x) dx$  for its variance and choose an integer  $T_K \geq K$  so that :

$$\int_0^1 (X_{1,K})^2(x) 1_{\{|X_{1,K}| > (T_K)^{1/12} \sigma_K\}} dx \leq \frac{1}{K}. \quad (7)$$

We define another collection of sets :

$$\left\{ \begin{array}{l} F_{2,K} = q_{2,K} F_{1,K} \\ F'_{2,K} = q_{2,K} F'_{1,K} \end{array} \right. \cdots \left\{ \begin{array}{l} F_{T_K,K} = q_{T_K,K} F_{1,K} \\ F'_{T_K,K} = q_{T_K,K} F'_{1,K} \end{array} \right.$$

Grouping sets, we define :

$$E_K = \bigcup_{r=1}^{T_K} F_{r,K} \text{ and } E'_K = \bigcup_{r=1}^{T_K} F'_{r,K}.$$

We have  $|E_K| = T_K |F_{1,K}|$  and  $|E'_K| = T_K |F'_{1,K}|$ . In particular :

$$\frac{|E'_K|}{|E_K|} = \frac{|F'_{1,K}|}{|F_{1,K}|} = \left( \frac{m_K - 1}{m_K} \right)^{l_K} \geq \frac{1}{2}. \quad (8)$$

When considering the next integer (ie  $K+1$ ) we start with  $p_{1,K+1} > q_{T_K,K} (p_{1,K} \cdots p_{l_K,K})^{m_K}$ . Observe that all the  $(F_{r,K})_{K \geq 1, 1 \leq r \leq T_K}$ , are pairwise disjoint and in particular the  $(E_K)_{K \geq 1}$ , which furthermore are consecutive. We finally set :

$$a_n = \begin{cases} \frac{1}{K|E_K|^{1/2}} & , \text{ when } n \in E_K, \text{ for some } K \geq 1, \\ 0 & , \text{ otherwise.} \end{cases}$$

This completes the definition of the sequence  $(a_n)$ . Formally  $\sum a_n g_\lambda(nx) = \sum_{K \geq 1} Z_K$ , with :

$$Z_K = \frac{1}{\sqrt{T_K}} \sum_{r=1}^{T_K} X_{r,K}(x) \text{ and } X_{r,K}(x) = \frac{1}{K|F_{1,K}|^{1/2}} \sum_{n \in F_{r,K}} g_\lambda(nx). \quad (9)$$

From the previous construction, observe that a partial sum  $\sum_{K=1}^N Z_K(x)$  corresponds to a partial sum of  $\sum a_n g_\lambda(nx)$ . We now proceed to verifications.

i) The sequence  $(a_n)$  belongs to  $l^2$ . Indeed :

$$\sum_{n \geq 1} a_n^2 = \sum_{K \geq 1} \sum_{n \in E_K} \frac{1}{K^2 |E_K|} = \sum_{K \geq 1} \frac{1}{K^2} < \infty.$$

ii) The series  $\sum a_n g_\lambda(nx)$  is  $L^2$ -divergent. Indeed, using (5) and (8) :

$$\begin{aligned} \sum_{n \geq 1} \left( \sum_{k \geq 1} k^{-\lambda} a_{kn} \right)^2 &\geq \sum_{K \geq 1} \sum_{n \in E'_K} \left( \sum_{k \geq 1} k^{-\lambda} a_{kn} \right)^2 \geq \sum_{K \geq 1} \sum_{n \in E'_K} \left( \sum_{k \in \mathcal{P}_K} k^{-\lambda} a_{kn} \right)^2 \\ &\geq \sum_{K \geq 1} \sum_{n \in E'_K} \frac{1}{K^2 |E_K|} \left( \sum_{k \in \mathcal{P}_K} k^{-\lambda} \right)^2 \geq \sum_{K \geq 1} |E'_K| \frac{K^2}{K^2 |E_K|} \geq \sum_{K \geq 1} \frac{1}{2} = +\infty. \end{aligned}$$

Since the  $a_n$  are positive, the conclusion comes from corollary 2.3.



iii) The series  $\sum a_n g_\lambda(nx)$  is a.e.-divergent. This requires longer computations. For a fixed  $K \geq 1$ , all  $(X_{r,K})_{1 \leq r \leq T_K}$  have the same law, due to the invariance of Lebesgue measure on  $\mathbb{R} \setminus \mathbb{Z}$  under multiplication by an integer. They do not form a stationary process, but are nearly independent. Under our hypotheses, it is routine to check that the law of  $(\sigma_K^2 T_K)^{-1/2} \sum_{r=1}^{T_K} X_{r,K}$  is asymptotically normal.

In a first step, we compute the variance  $\sigma_K^2$  and verify that it grows rapidly to infinity, as suggested by ii). Via lemma 2.1, we have :

$$\begin{aligned}
\sigma_K^2 = \text{Var}(X_{1,K}) &= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \sum_{n,m \in F_{1,K}} \left( \frac{n \wedge m}{n \vee m} \right)^\lambda \\
&= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \sum_{1 \leq a_j, b_j \leq m_K, 1 \leq j \leq l_K} \prod_{j=1}^{l_K} (p_{j,K})^{-\lambda|a_j - b_j|} \\
&= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \prod_{j=1}^{l_K} \left( \sum_{a,b=1}^{m_K} (p_{j,K})^{-\lambda|a-b|} \right) \\
&= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \prod_{j=1}^{l_K} \left( m_K + 2(p_{j,K})^{-\lambda(m_K-1)} \sum_{k=1}^{m_K-1} k(p_{j,K})^{\lambda(k-1)} \right).
\end{aligned}$$

For  $x > 1$ , we have  $\sum_{k=1}^{n-1} kx^{k-1} = ((n-1)x^n - nx^{n-1} + 1)/(x-1)^2 = nx^{n-2}(1 + o(1))$ , when  $x$  and  $n$  are large. Inserting this in the previous calculations, we obtain, with a uniform  $o(1)$  :

$$\begin{aligned}
\sigma_K^2 &= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2 (m_K)^{l_K}} \prod_{j=1}^{l_K} (m_K + 2m_K(p_{j,K})^{-\lambda}(1 + o(1))) \\
&= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2} e^{\sum_{j=1}^{l_K} \log(1 + 2(p_{j,K})^{-\lambda}(1 + o(1)))} \\
&= \frac{\zeta(2\lambda)}{2} \frac{1}{K^2} e^{2(\sum_{j=1}^{l_K} (p_{j,K})^{-\lambda})(1 + o(1))} \geq e^K,
\end{aligned} \tag{10}$$

for large  $K$ , using (5).

We now establish the convergence :

$$(\sigma_K^2 T_K)^{-1/2} \sum_{r=1}^{T_K} X_{r,K} \rightarrow \mathcal{N}(0, 1), \text{ in law.} \tag{11}$$

We write  $E$  for the expectation under Lebesgue measure on  $\mathbb{R} \setminus \mathbb{Z}$ . Set  $Y_{r,K} = (\sigma_K^2 T_K)^{-1/2} X_{r,K}$  and  $S_N = \sum_{r=1}^N Y_{r,K}$ . For  $1 \leq r \leq T_K$ , introduce the finite partitions :

$$\mathcal{F}_r = \{[k/q_{r,K}, (k+1)/q_{r,K}), 0 \leq k < q_{r,K}\}.$$

Each  $Y_{r,K}$  being  $(1/q_{r,K})$ -periodic, for a bounded measurable  $f$  :

$$E(f(Y_{r,K})) = E(f(Y_{r,K})|\mathcal{F}_r). \tag{12}$$

For  $t \in \mathbb{R}$  and  $2 \leq N \leq T_K$ , we have :

$$\begin{aligned}
E(e^{itS_N}) &= E(e^{itS_{N-1}} e^{itY_{N,K}}) \\
&= E(E(e^{itS_{N-1}}|\mathcal{F}_N) e^{itY_{N,K}}) + E((e^{itS_{N-1}} - E(e^{itS_{N-1}}|\mathcal{F}_N)) e^{itY_{N,K}}) \\
&= A + B.
\end{aligned}$$

First of all, taking conditional expectation and using (12) :

$$\begin{aligned} A &= E(E(e^{itS_{N-1}}|\mathcal{F}_N)E(e^{itY_{N,K}}|\mathcal{F}_N)) = E(E(e^{itS_{N-1}}|\mathcal{F}_N)E(e^{itY_{N,K}})) \\ &= E(e^{itS_{N-1}})E(e^{itY_{N,K}}). \end{aligned} \quad (13)$$

Next :

$$|B| \leq E(|e^{itS_{N-1}} - E(e^{itS_{N-1}}|\mathcal{F}_N)|). \quad (14)$$

The map  $x \mapsto e^{itx}$  is  $|t|$ -Lipschitz. On each piece of  $\mathcal{F}_N$  which contains no discontinuity of  $S_{N-1}$ , when counting the oscillation we have :

$$\begin{aligned} |e^{itS_{N-1}} - E(e^{itS_{N-1}}|\mathcal{F}_N)| &\leq \frac{|t|}{q_{N,K}} \frac{(\sigma_K^2 T_K)^{-1/2}}{(K(m_K)^{l_K})^{1/2}} \sum_{r=1}^{N-1} (m_K)^{l_K} (p_{1,K} \cdots p_{l_K,K})^{m_K} q_{r,K} \\ &\leq |t| \left( \frac{(m_K)^{l_K/2}}{K} (p_{1,K} \cdots p_{l_K,K})^{m_K} \right) \frac{(N-1)q_{N-1,K}}{q_{N,K}}, \end{aligned} \quad (15)$$

since  $T_K \geq K$  and  $\sigma_K \geq 1$  for large  $K$ , by (10). Next,  $S_{N-1}$  is continuous on the interior of each segment of the partition whose step<sup>-1</sup> is  $q_{1,K} \cdots q_{N-1,K} (p_{1,K} \cdots p_{l_K,K})^{m_K}$ . The total measure of the pieces of  $\mathcal{F}_N$  which may contain a discontinuity of  $S_{N-1}$  is bounded from above by :

$$q_{1,K} \cdots q_{N-1,K} (p_{1,K} \cdots p_{l_K,K})^{m_K} \frac{1}{q_{N,K}} \leq (p_{1,K} \cdots p_{l_K,K})^{m_K} \frac{(q_{N-1,K})^{N-1}}{q_{N,K}}. \quad (16)$$

From (14), (15) and (16), we deduce that :

$$|B| \leq 2(1+|t|)(p_{1,K} \cdots p_{l_K,K})^{m_K} \left( 1 + \frac{(m_K)^{l_K/2}}{K} \right) \frac{(q_{N-1,K})^{N-1}}{q_{N,K}}. \quad (17)$$

Using that for all  $1 \leq N \leq T_K$ , we have  $|E(e^{itY_{N,K}})| \leq 1$ , when iterating the procedure with (13) and (14), we obtain via (6) :

$$\begin{aligned} |E(e^{itS_{T_K}}) - \prod_{r=1}^{T_K} E(e^{itY_{r,K}})| &\leq 2(1+|t|)(p_{1,K} \cdots p_{l_K,K})^{m_K} \left( 1 + \frac{(m_K)^{l_K/2}}{K} \right) \sum_{r=2}^{T_K} \frac{(q_{r-1,K})^{r-1}}{q_{r,K}} \\ &\leq 2(1+|t|) \frac{1}{K}. \end{aligned} \quad (18)$$

As a consequence of (18), in order to show (11) we only need to focus on :

$$\prod_{r=1}^{T_K} E(e^{itY_{r,K}}) = E(e^{itY_{1,K}})^{T_K}. \quad (19)$$

We now use the fact that for all  $t \in \mathbb{R}$  :

$$|e^{it} - (1 + it - t^2/2)| \leq \min\{|t|^3/6, |t|^2\}, \quad (20)$$

which comes from  $e^{it} - (1 + it - t^2/2) = i^3/2 \int_0^t (t-s)^2 e^{is} ds = i^2 \int_0^t (t-s)(e^{is} - 1) ds$ . Via (20) and the property that  $X_{1,K}$  has zero mean, we now deduce the following inequalities :

$$\begin{aligned}
\left| E(e^{itY_{1,K}}) - \left(1 - \frac{t^2}{2T_K}\right) \right| &\leq \left| E\left(e^{itY_{1,K}} - \left(1 + itY_{1,K} - \frac{t^2}{2}Y_{1,K}^2\right)\right) \right| \\
&\leq E\left| \left(e^{itY_{1,K}} - \left(1 + itY_{1,K} - \frac{t^2}{2}Y_{1,K}^2\right)\right) \right| \\
&\leq E(\min\{|tY_{1,K}|^3/6, |tY_{1,K}|^2\}).
\end{aligned}$$

With  $\varepsilon = (T_K)^{-5/12}$  and using (7), as well as  $T_K \geq K$  and  $\sigma_K \geq 1$  for large  $K$  :

$$\begin{aligned}
\left| E(e^{itY_{1,K}}) - \left(1 - \frac{t^2}{2T_K}\right) \right| &\leq \frac{|t|^3}{6} E(|Y_{1,K}|^3 1_{|Y_{1,K}| \leq \varepsilon}) + |t|^2 E(|Y_{1,K}|^2 1_{|Y_{1,K}| > \varepsilon}) \\
&\leq \frac{|t|^3}{6(T_K)^{5/4}} + \frac{|t|^2}{\sigma_K^2 T_K} E(|X_{1,K}|^2 1_{|X_{1,K}| > (T_K)^{1/12} \sigma_K}) \\
&\leq \frac{1}{T_K} \left( \frac{|t|^3}{6(T_K)^{1/4}} + \frac{|t|^2}{K \sigma_K^2} \right) \leq \frac{1}{T_K} \left( \frac{|t|^3}{6K^{1/4}} + \frac{|t|^2}{K} \right). \quad (21)
\end{aligned}$$

Since  $T_K \rightarrow +\infty$ , as  $K \rightarrow +\infty$ , we deduce from (18), (19) and (21) that  $E(e^{itS_{T_K}}) \rightarrow e^{-t^2/2}$ , as  $K \rightarrow +\infty$ , for all  $t \in \mathbb{R}$ . This proves (11).

To conclude, for all  $L \geq 1$  we choose  $K_L \geq L$  so that :

$$P(|S_{T_{K_L}}| \leq 1/L^2) \leq 2 \int_{|t| \leq 1/L^2} d\mathcal{N}(0,1)(t) =: \delta_L.$$

Clearly  $\sum_{L \geq 1} \delta_L < \infty$ , so by Borel-Cantelli's lemma, for a.e  $x$  when  $L$  is large enough we have  $|S_{T_{K_L}}| \geq 1/L^2$ . For such a  $x$ , using (10) and when  $L$  is large enough :

$$|Z_{K_L}| = \left| \frac{1}{\sqrt{T_{K_L}}} \sum_{r=1}^{T_{K_L}} X_{r,K_L}(x) \right| = \sigma_{K_L} |S_{T_{K_L}}| \geq \frac{e^{K_L/2}}{L^2} \geq \frac{e^{L/2}}{L^2}.$$

Since partial sums  $\sum_{K=1}^N Z_K(x)$  are partial sums of  $\sum a_n g_\lambda(nx)$ , this prevents  $\sum a_n g_\lambda(nx)$  from converging at  $x$ . This completes the proof of item *i*) of theorem 1.1.

## 4 Sufficient conditions for $L^2$ and a.e-convergence

We take a finite sequence  $(a_n)$  and write  $\sum a_n g_\lambda(nx) = \sum b_n f_{n,\lambda}(x)$ , where  $(b_n)$  is also finite. By proposition 2.2 :

$$\left\| \sum a_n g_\lambda(nx) \right\|^2 = \left\| \sum b_n f_{n,\lambda} \right\|^2 \leq \frac{\zeta(2\lambda)}{2} \sum b_n^2 \leq \frac{\zeta(2\lambda)}{2} \sum_{n \geq 1} \left( \sum_{k \geq 1} k^{-\lambda} |a_{kn}| \right)^2.$$

Set  $\psi_\lambda(k) = k^{1-\lambda}(\log k)^2$  if  $1/2 < \lambda < 1$  and  $\psi_1(k) = \log k(\log \log k)^{1+\varepsilon}$ , for some  $\varepsilon > 0$ . For simplicity we write  $\log(x)$  for  $\max\{1, \log(x)\}$ . Using Cauchy-Schwarz's inequality :

$$\begin{aligned}
\sum_{n \geq 1} \left( \sum_{k \geq 1} k^{-\lambda} |a_{kn}| \right)^2 &= \sum_{k, k' \geq 1} (kk')^{-\lambda} \sum_{n \geq 1} |a_{nk} a_{nk'}| \leq \sum_{k, k' \geq 1} (kk')^{-\lambda} \left( \sum_{n \geq 1} a_{nk}^2 \right)^{1/2} \left( \sum_{n \geq 1} a_{nk'}^2 \right)^{1/2} \\
&\leq \left[ \sum_{k \geq 1} k^{-\lambda} \left( \sum_{n \geq 1} a_{nk}^2 \right)^{1/2} \right]^2 \leq \left( \sum_{k \geq 1} k^{-\lambda} \frac{1}{\psi_\lambda(k)} \right) \left( \sum_{k \geq 1} k^{-\lambda} \psi_\lambda(k) \sum_{n \geq 1} a_{nk}^2 \right) \\
&\leq C_\varepsilon \sum_{n \geq 1} a_n^2 \sum_{k|n} k^{-\lambda} \psi_\lambda(k). \tag{22}
\end{aligned}$$

We first consider the case  $1/2 < \lambda < 1$ . Remark that  $0 < 2\lambda - 1 < 1$ . Using a classical upper-bound for  $\sum_{k|n} k^{2\lambda-1}$ , cf Krätzel [13], we have for any  $\delta > 0$  :

$$\begin{aligned}
\sum_{k|n} k^{-\lambda} \psi_\lambda(k) &= \sum_{k|n} k^{1-2\lambda} (\log k)^2 \leq (\log n)^2 \sum_{k|n} k^{1-2\lambda} \leq (\log n)^2 n^{1-2\lambda} \sum_{k|n} k^{2\lambda-1} \\
&\leq (\log n)^2 n^{1-2\lambda} C_\delta n^{2\lambda-1} e^{\frac{(1+\delta)(\log n)^{1-(2\lambda-1)}}{(1-(2\lambda-1)) \log \log n}} \leq C'_\delta n^{\frac{(1+2\delta)(\log n)^{1-2\lambda}}{2(1-\lambda) \log \log n}}.
\end{aligned}$$

In the situation when  $\lambda = 1$ , we have :

$$\sum_{k|n} k^{-1} \psi_1(k) \leq \psi_1(n) \sum_{k|n} k^{-1} = \psi_1(n) n^{-1} \sum_{k|n} k.$$

We use this time the inequality  $\sum_{k|n} k \leq Cn \log \log n$ , see again [13]. As a result, for any  $\varepsilon > 0$  there is a constant  $C_\varepsilon > 0$  such that for any sequence  $(a_n)$  :

$$\begin{cases} \|\sum a_n g_\lambda(nx)\|^2 \leq C_\varepsilon \sum a_n^2 n^{\frac{(1+\varepsilon)(\log n)^{-(2\lambda-1)}}{2(1-\lambda) \log \log n}}, & \text{when } 1/2 < \lambda < 1, \\ \|\sum a_n \{nx\}\|^2 \leq C_\varepsilon \sum a_n^2 \log n (\log \log n)^{2+\varepsilon}, & \text{when } \lambda = 1. \end{cases} \tag{23}$$

These properties imply the  $L^2$ -convergence of  $D_\lambda$ -series under the assumptions of theorem 1.1.

We turn to the question of the a.e-convergence of  $D_\lambda$ -series. The second item of theorem 1.1 is a consequence of inequalities (23) and of the following proposition. The latter is an adaptation of a method due to Rademacher [16] for the study of series based on a general orthonormal family.

**Proposition 4.1**

Let  $(a_n)_{n \geq 1}$  and  $(\varphi(n))_{n \geq 1}$  be such that  $\sum a_n^2 \varphi(n) (\log n)^2 < \infty$  and for any  $M \leq N$  :

$$\left\| \sum_{n=M}^N a_n g_\lambda(nx) \right\|^2 \leq \sum_{n=M}^N a_n^2 \varphi(n).$$

Then  $\sum a_n g_\lambda(nx)$  converges a.e.

*Proof of the proposition :*

We can suppose that  $\log$  is the logarithm in base 2. Let  $S(n)(x) = \sum_{1 \leq k \leq n} a_k g_\lambda(kx)$ . For  $m < n$ , introduce the notations :

$$S(m, n)(x) = \sum_{m \leq k < n} a_k g_\lambda(kx) \text{ and } \sigma_l(m, n) = \sum_{m \leq k < n} a_k^2 \varphi(k) (\log k)^l, \text{ for } l \in \{0, 1, 2\}.$$

*Step 1.* We show that  $(S(2^n)(x))$  converges for a.e  $x$ . Let  $0 < N < n$ . We have :

$$\begin{aligned} \int_0^1 \sum_{N \leq r < n} S(2^r, 2^n)^2(x) dx &\leq \sum_{N \leq r < n} \sigma_0(2^r, 2^n) = \sum_{N \leq r < n} \sum_{s=r}^{n-1} \sigma_0(2^s, 2^{s+1}) \\ &\leq \sum_{s=N}^{n-1} (s - N + 1) \sigma_0(2^s, 2^{s+1}) \\ &\leq \sum_{s=N}^{n-1} s \sigma_0(2^s, 2^{s+1}) \leq \sigma_1(2^N, 2^n). \end{aligned}$$

By Markov's inequality,  $\sum_{N \leq r < n} S(2^r, 2^n)^2(x) \leq \sigma_1(2^N, 2^n)^{2/3}$  for all  $x$  in a Borel set  $E_{N,n}$  with :

$$\lambda(E_{N,n}) \geq 1 - \sigma_1(2^N, 2^n)^{1/3} \geq 1 - \sigma_1(2^N, \infty)^{1/3}.$$

In particular, for  $x \in E_{N,n}$  and all  $N \leq r < n$ , we have  $S(2^r, 2^n)(x) \leq \sigma_1(2^N, 2^n)^{1/3}$ . Define a set  $E'_{N,n}$  by the condition that for all  $N \leq r \leq r' < n$  :

$$S(2^r, 2^{r'})(x) \leq 2\sigma_1(2^N, \infty)^{1/3}.$$

Since  $E_{N,n} \subset E'_{N,n}$ , we have  $\lambda(E'_{N,n}) \geq 1 - \sigma_1(2^N, \infty)^{1/3}$ . Fixing  $N$ , the  $E'_{N,n}$  are monotonic in  $n$ .

The set  $D_N$  defined by the condition that for all  $N \leq r \leq r'$ ,  $S(2^r, 2^{r'})(x) \leq 2\sigma_1(2^N, \infty)^{1/3}$  has therefore a measure  $\lambda(D_N) \geq 1 - \sigma_1(2^N, \infty)^{1/3}$ . Since  $\lambda(D_N) \rightarrow 1$ , as  $N \rightarrow +\infty$ , we deduce that  $\lambda(\limsup D_N) = 1$ . If  $x \in \limsup D_N$ , the sequence  $(S(2^n)(x))$  clearly satisfies the Cauchy criterion, so converges. This concludes *step 1*.

*Step 2.* To complete the proof, we show that a.e  $\sup_{2^r < n < 2^{r+1}} |S(2^r, n)(x)| \rightarrow 0$ , as  $r \rightarrow +\infty$ . Let  $2^r < n < 2^{r+1}$  and decompose  $n$  in base 2 :

$$n = 2^r + \sum_{l=1}^r \theta_l 2^{r-l}, \text{ with } \theta_l \in \{0, 1\}.$$

Then :

$$S(2^r, n) = \sum_{l=1}^r S \left( 2^r + \sum_{m=1}^{l-1} \theta_m 2^{r-m}, 2^r + \sum_{m=1}^l \theta_m 2^{r-m} \right).$$

By convexity :

$$\begin{aligned} S(2^r, n)^2 &\leq r \sum_{l=1}^r S \left( 2^r + \sum_{m=1}^{l-1} \theta_m 2^{r-m}, 2^r + \sum_{m=1}^l \theta_m 2^{r-m} \right)^2 \\ &\leq r \sum_{l=1}^r \sum_{h=0}^{2^{r-l}-1} S(2^r + h2^l, 2^r + h2^l + 2^{l-1})^2 =: T(r). \end{aligned}$$

The quantity  $T(r)$  is independent on  $2^r < n < 2^{r+1}$ . Next :

$$\begin{aligned}
\int_0^1 T(r)(x) \, dx &= r \sum_{l=1}^r \sum_{h=0}^{2^{r-l}-1} \int_0^1 S(2^r + h2^l, 2^r + h2^l + 2^{l-1})^2(x) \, dx \\
&\leq r \sum_{l=1}^r \sum_{h=0}^{2^{r-l}-1} \sigma_0(2^r + h2^l, 2^r + h2^l + 2^{l-1}) \\
&\leq r \sum_{l=1}^r \sigma_0(2^r, 2^{r+1}) = r^2 \sigma_0(2^r, 2^{r+1}) \leq \sigma_2(2^r, 2^{r+1}).
\end{aligned}$$

Fix  $N$  and let  $r \geq N$ . By Markov's inequality, for  $x$  in a Borel set  $F_r(N)$  of Lebesgue measure  $\lambda(F_r(N)) \geq 1 - \sigma_2(2^r, 2^{r+1})/\sigma_2(2^N, \infty)^{2/3}$ , we have :

$$\sup_{2^r < n < 2^{r+1}} S(2^r, n)^2(x) \leq T(r) \leq \sigma_2(2^N, \infty)^{2/3}.$$

Let  $G_N = \cap_{r \geq N} F_r(N)$ . Then  $\lambda(G_N) \geq 1 - \sum_{r \geq N} \sigma_2(2^r, 2^{r+1})/\sigma_2(2^N, \infty)^{2/3} = 1 - \sigma_2(2^N, \infty)^{1/3}$ . For  $x \in G_N$  :

$$\forall r \geq N, \sup_{2^r < n < 2^{r+1}} |S(2^r, n)(x)| \leq \sigma_2(2^N, \infty)^{1/3}.$$

As  $\lambda(G_N) \rightarrow 1$ , we get  $\lambda(\limsup G_N) = 1$ . If  $x \in \limsup G_N$ , then  $\sup_{2^r < n < 2^{r+1}} |S(2^r, n)(x)|$  tends to 0, as  $r \rightarrow \infty$ . This concludes *step 2* and the proof of the proposition.  $\square$

## 5 Particular classes where $L^2$ -convergence is true

We consider the proof of theorem 1.2.

### 5.1 Proof of i)

Let  $(n_k)$  be lacunary in the sense that  $n_{k+1}/n_k \geq c > 1$  and  $(a_k) \in l^2$ . We first consider the upper bound. We can assume the sequence  $(a_k)$  finite. By lemma 2.1, the  $L^2$ -norm of  $(2/\zeta(2\lambda))^{1/2} \sum a_k g_\lambda(n_k x)$  is given by :

$$\sum_{k, l \geq 1} a_k a_l \left( \frac{n_k \wedge n_l}{n_k \vee n_l} \right)^\lambda = \sum a_k^2 + 2 \sum_{k < l} a_k a_l \frac{(n_k \wedge n_l)^{2\lambda}}{(n_k n_l)^\lambda}.$$

Using Cauchy-Schwarz's inequality, the second term is bounded by :

$$\begin{aligned}
\sum_{k < l} |a_k a_l| \frac{(n_k \wedge n_l)^{2\lambda}}{(n_k n_l)^\lambda} &\leq \sum_{k < l} |a_k a_l| \frac{n_k^{2\lambda}}{(n_k n_l)^\lambda} \leq \sum_{k < l} |a_k a_l| c^{-\lambda(l-k)} \leq \sum_{k \geq 1} |a_k| \sum_{l \geq 1} c^{-\lambda l} |a_{k+l}| \\
&\leq \left( \sum_{n \geq 1} a_n^2 \right)^{1/2} \left( \sum_{k \geq 1} \left( \sum_{l \geq 1} c^{-\lambda l} |a_{k+l}| \right)^2 \right)^{1/2}.
\end{aligned}$$

Next, still via Cauchy-Schwarz's inequality :

$$\begin{aligned}
\sum_{k \geq 1} \left( \sum_{l \geq 1} c^{-\lambda l} |a_{k+l}| \right)^2 &= \sum_{l, l' \geq 1} c^{-\lambda(l+l')} \sum_{k \geq 1} |a_{k+l} a_{k+l'}| \\
&\leq \sum_{l, l' \geq 1} c^{-\lambda(l+l')} \left( \sum_{k \geq 1} a_{k+l}^2 \right)^{1/2} \left( \sum_{k \geq 1} a_{k+l'}^2 \right)^{1/2} \\
&\leq \left( \sum_{l \geq 1} c^{-\lambda l} \left( \sum_{k \geq 1} a_{k+l}^2 \right)^{1/2} \right)^2 \leq \frac{1}{(c^\lambda - 1)^2} \sum_{k \geq 1} a_k^2.
\end{aligned}$$

As a consequence :

$$\sum_{k < l} |a_k a_l| \frac{(n_k \wedge n_l)^{2\lambda}}{(n_k n_l)^\lambda} \leq \frac{1}{(c^\lambda - 1)} \sum_{k \geq 1} a_k^2.$$

Since  $1 + 2/(c^\lambda - 1) = (c^\lambda + 1)/(c^\lambda - 1)$ , this completes the proof of the upper-bound.

For the lower bound, one can also suppose that  $(a_k)$  is finite. We have  $\sum a_k g_\lambda(n_k x) = \sum b_k f_{k,\lambda}$ , where  $(b_k)$  is also finite. By proposition 2.2 :

$$\left\| \sum a_k g_\lambda(n_k x) \right\|^2 = \left\| \sum b_k f_{k,\lambda} \right\|^2 \geq \frac{1}{2} \sum b_k^2.$$

Fixing  $0 < \varepsilon < 1 - (2\lambda)^{-1}$ , giving  $2\lambda(1 - \varepsilon) > 1$  :

$$\begin{aligned}
\sum a_k^2 &= \sum_{k \geq 1} \left( \sum_{l \geq 1} l^{-\lambda} \mu(l) b_{ln_k} \right)^2 \leq \sum_{k \geq 1} \left( \sum_{l \geq 1} l^{-2\lambda(1-\varepsilon)} \mu(l)^2 \right) \left( \sum_{l \geq 1} l^{-2\lambda\varepsilon} b_{ln_k}^2 \right) \\
&\leq \prod_{i \geq 1} \left( 1 + p_i^{-2\lambda(1-\varepsilon)} \right) \left( \sum_{l \geq 1} b_l^2 \sum_{k, n_k | l} \left( \frac{n_k}{l} \right)^{2\lambda\varepsilon} \right) \\
&\leq \prod_{i \geq 1} \left( \frac{1 - p_i^{-4\lambda(1-\varepsilon)}}{1 - p_i^{-2\lambda(1-\varepsilon)}} \right) \left( \sum_{m \geq 0} c^{-2\lambda\varepsilon m} \right) \sum_{l \geq 1} b_l^2 \\
&\leq \frac{\zeta(2\lambda(1-\varepsilon))}{\zeta(4\lambda(1-\varepsilon))} (1 - c^{-2\lambda\varepsilon})^{-1} \sum_{l \geq 1} b_l^2.
\end{aligned}$$

To complete the proof, we first use that  $\zeta(4\lambda(1-\varepsilon)) \geq \zeta(4\lambda)$  and  $\zeta(2\lambda(1-\varepsilon)) \leq 1 + 1/(2\lambda(1-\varepsilon) - 1)$ . Set  $\varepsilon = \rho(1 - 1/(2\lambda))$ , with  $0 < \rho < 1$ . We have :

$$\left( 1 + \frac{1}{2\lambda(1-\varepsilon) - 1} \right) \frac{1}{1 - c^{-2\lambda\varepsilon}} \leq \frac{2\lambda}{2\lambda - 1} \frac{1}{(1 - \rho)(1 - c^{-(2\lambda-1)\rho})}.$$

Minimizing in  $\rho$ , we take  $\rho = 1/(1 + \ln c^{2\lambda-1})$ . We finally use the inequality  $1 - e^{-x} \geq (1 - 1/e)x$ , for  $0 \leq x \leq 1$ , giving  $(1 - c^{-(2\lambda-1)\rho}) \geq (1 - 1/e)(1 - \rho)$ .

Concerning a.e-convergence, we can now apply proposition 4.1 with  $\varphi = 1$ .

□

## 5.2 Proof of *ii*)

We start from (22). For  $n$  with  $|\text{supp}(n)| \leq N$  and any  $0 < \delta < 2\lambda - 1$ , we have :

$$\sum_{k|n} k^{-\lambda} \psi_\lambda(k) \leq C_\delta \sum_{k|n} k^{1-2\lambda+\delta} \leq C_\delta \prod_{p|n, p \in \mathcal{P}} (1 - p^{1-2\lambda+\delta})^{-1} \leq C_\delta \prod_{i=1}^N (1 - p_i^{1-2\lambda+\delta})^{-1}.$$

For the lower bound, we use  $\|\sum b_n f_{n,\lambda}\|^2 \geq (1/2) \sum b_n^2$ , by proposition 2.2. Next :

$$\sum a_k^2 = \sum_{k \geq 1} \left( \sum_{l \geq 1} l^{-\lambda} \mu(l) b_{lk} \right)^2 \leq \sum_{k \geq 1} \left( \sum_{l \geq 1} l^{-\lambda} |b_{lk}| \right)^2 \leq C_\varepsilon \sum_{n \geq 1} b_n^2 \sum_{k|n} k^{-\lambda} \psi_\lambda(k),$$

when proceeding in the same way as for (22). We then conclude as above, using the fact that  $b_n = 0$  when  $|\text{supp}(n)| > N$ . For a.e-convergence, we apply proposition 4.1 with  $\varphi = 1$ .  $\square$

## 5.3 Proof of *iii*)

Using the remark after lemma 2.1 we only need to focus on  $(b_n)$ . Set  $b_{i,n} = b_{p_i^n}$ . Multiplicativity implies that the  $(b_{i,n})_{i,n \geq 1}$  entirely determine the sequence  $(b_n)$ . Via corollary (2.3), we show that :

$$\sum_{n \geq 1} \left( \sum_{k \geq 1} k^{-\lambda} b_{kn} \right)^2 < +\infty. \quad (24)$$

Each term in this series is finite, by Cauchy-Schwarz's inequality. We first claim the equivalence  $(b_n) \in l^2 \Leftrightarrow \sum_{i \geq 1} \sum_{n \geq 1} b_{i,n}^2 < +\infty$ . Indeed, using that  $b_1 = 1$ , we have :

$$\sum_{n \geq 1} b_n^2 = 1 + \sum_{k \geq 1} \sum_{1 \leq i_1 < \dots < i_k} \sum_{u_1 \geq 1, \dots, u_k \geq 1} b_{i_1, u_1}^2 \dots b_{i_k, u_k}^2 = \prod_{i \geq 1} \left( 1 + \sum_{n \geq 1} b_{i,n}^2 \right). \quad (25)$$

This proves the claim.

For technical reasons, up to considering  $\tilde{b}_{i,n} = b_{i,n} + 1/(in)$ ,  $(i,n) \geq 1$ , and the corresponding multiplicative sequence  $(\tilde{b}_n)$ , which satisfies  $(\tilde{b}_n) \in l^2 \Leftrightarrow (b_n) \in l^2$ , we assume that  $b_{i,n} > 0$  for all indices  $(i,n) \geq 1$ . Decomposing in prime factors  $n = p_{i_1}^{u_1} \dots p_{i_k}^{u_k}$ , with  $k = 0$  if  $n = 1$ , and using multiplicativity :

$$\begin{aligned} \sum_{n \geq 1} \left( \sum_{l \geq 1} l^{-\lambda} b_{ln} \right)^2 &= \sum_{k \geq 0} \sum_{1 \leq i_1 < \dots < i_k} \sum_{u_1 \geq 1, \dots, u_k \geq 1} \\ &\quad \prod_{j \notin \{i_1, \dots, i_k\}} \left( 1 + \sum_{m \geq 1} p_j^{-\lambda m} b_{j,m} \right) \times \prod_{l=1, \dots, k} \left( \sum_{v_l \geq 0} p_{i_l}^{-\lambda v_l} b_{i_l, u_l + v_l} \right). \end{aligned}$$

The first product term is uniformly bounded from above since for a constant  $C$  :

$$\sum_{j,m \geq 1} p_j^{-\lambda m} b_{j,m} \leq \left( \sum_{j,m \geq 1} p_j^{-2\lambda m} \right)^{1/2} \left( \sum_{j,m \geq 1} b_{j,m}^2 \right)^{1/2} \leq C \left( \sum_{j \geq 1} p_j^{-2\lambda} \right)^{1/2} \left( \sum_{j,m \geq 1} b_{j,m}^2 \right)^{1/2} < +\infty.$$

To prove (24), it remains to check the finiteness of :



$$\sum_{k \geq 0} \sum_{1 \leq i_1 < \dots < i_k} \prod_{l=1}^k \left[ \sum_{u_l \geq 1} \left( \sum_{v_l \geq 0} p_{i_l}^{-\lambda v_l} b_{i_l, u_l + v_l} \right)^2 \right] = \prod_{i \geq 1} \left[ 1 + \sum_{u \geq 1} \left( \sum_{v \geq 0} p_i^{-\lambda v} b_{i, u+v} \right)^2 \right].$$

It is equivalent to showing :

$$\sum_{i \geq 1, u \geq 1} \left( \sum_{v \geq 0} p_i^{-\lambda v} b_{i, u+v} \right)^2 < +\infty.$$

Set  $c_{i,n} = p_i^{\lambda n} \sum_{v \geq n} p_i^{-\lambda v} b_{i,v}$ , which is finite by Cauchy-Schwarz's inequality. We thus verify that  $\sum_{i \geq 1, n \geq 1} c_{i,n}^2 < +\infty$ . Fixing  $0 < \varepsilon < 1 - 2^{-\lambda}$ , we prove below that for all  $i \geq 1$  :

$$\begin{aligned} \sum_{n \geq 1} c_{i,n}^2 &\leq \frac{1}{\varepsilon^2 (1 - (1 - \varepsilon)^{-2} p_i^{-2\lambda})} \sum_{n \geq 1} c_{i,n}^2 \left( 1 - \frac{c_{i,n+1}}{p_i^{\lambda} c_{i,n}} \right)^2 \\ &\leq \frac{1}{\varepsilon^2 (1 - (1 - \varepsilon)^{-2} 2^{-2\lambda})} \sum_{n \geq 1} b_{i,n}^2. \end{aligned} \quad (26)$$

Since  $\sum_{i,n \geq 1} b_{i,n}^2 < \infty$ , this brings the conclusion.

Fix  $i \geq 1$  and introduce  $\mathcal{C} = \{n \geq 1, |1 - c_{i,n+1}/(p_i^{\lambda} c_{i,n})| < \varepsilon\}$ . We claim that if  $\mathcal{C}$  is infinite, then it does not contain all large integers. Indeed, if  $n \in \mathcal{C}$ , then  $c_{i,n+1} \geq (1 - \varepsilon) p_i^{\lambda} c_{i,n}$ , since  $c_{i,n+1} < p_i^{\lambda} c_{i,n}$ . If  $\mathcal{C}$  contains some interval  $[n_0, +\infty)$ , then for  $n \geq n_0$  :

$$\sum_{v \geq 0} b_{i,v+n} p_i^{-\lambda v} = p_i^{\lambda n} \sum_{v \geq n} b_{i,v} p_i^{-\lambda v} \geq p_i^{\lambda n} (1 - \varepsilon)^{n-n_0} \sum_{v \geq n_0} b_{i,v} p_i^{-\lambda v}.$$

However  $\sum_{v \geq n_0} b_{i,v} p_i^{-\lambda v}$  is fixed and  $> 0$ , since  $b_{i,v} > 0$ . As  $p_i^{\lambda} (1 - \varepsilon) > 1$ , a contradiction is given by Cauchy-Schwarz's inequality, because the left-hand side is bounded from above by :

$$\left( \sum_{v \geq 0} p_i^{-2\lambda v} \right)^{1/2} \left( \sum_{v \geq 0} b_{i,n+v}^2 \right)^{1/2} \leq \left( \sum_{v \geq 0} p_i^{-2\lambda v} \right)^{1/2} \left( \sum_{v \geq 0} b_{i,v}^2 \right)^{1/2} < +\infty.$$

Decompose now into disjoint intervals  $\mathcal{C} = \cup_{k \geq 1} [a_k, b_k]$  and write in a disjoint union :

$$\{n \geq 1\} = [a_1, b_1] \cup [a'_1, b'_1] \cup \dots \cup [a_k, b_k] \cup [a'_k, b'_k] \cup \dots.$$

Notice that the first interval  $[a_1, b_1]$  may be empty, whereas the other ones are not, and that the collection of  $([a_k, b_k], [a'_k, b'_k])_k$  may be finite. We have :

$$\sum_{n \notin \mathcal{C}, n \notin \{a'_k, k \geq 1\}} c_{i,n}^2 \left( 1 - \frac{c_{i,n+1}}{p_i^{\lambda} c_{i,n}} \right)^2 \geq \varepsilon^2 \sum_{n \notin \mathcal{C}, n \notin \{a'_k, k \geq 1\}} c_{i,n}^2. \quad (27)$$

Also :

$$\sum_{k \geq 1} \sum_{l=a_k}^{a'_k} c_{i,l}^2 \left( 1 - \frac{c_{i,l+1}}{p_i^{\lambda} c_{i,l}} \right)^2 \geq \varepsilon^2 \sum_{k \geq 1} c_{i,a'_k}^2. \quad (28)$$

Observe finally that :

$$\sum_{l=a_k}^{a'_k} c_{i,l}^2 \leq c_{i,a'_k}^2 \sum_{m \geq 0} (1-\varepsilon)^{-2m} p_i^{-2\lambda m} \leq (1 - (1-\varepsilon)^{-2} p_i^{-2\lambda})^{-1} c_{i,a'_k}^2. \quad (29)$$

Combining (27), (28) and (29), we get (26). This completes the proof of this item.  $\square$

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## References

- [1] BALAZARD, M. AND SAIAS, E., *Notes on the Riemann zeta-function. IV*, Adv. Math. 188 (2004), no. 1, 69–86.
- [2] BOURGIN, D. AND MENDEL, C., *Orthonormal sets of periodic functions of the type  $\{f(nx)\}$* , Trans. Amer. Math. Soc. 57, 1945, 332–363.
- [3] DE LA BRETÈCHE, R. AND TENENBAUM, G., *Séries trigonométriques à coefficients arithmétiques*, J. Anal. Math. 92 (2004), 1–79.
- [4] CARLESON, L., *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), 135–157.
- [5] DAVENPORT, H., *On some infinite series involving arithmetical functions*, Quart. J. Math., vol 8, pp. 8–13 (1937).
- [6] DAVENPORT, H., *On some infinite series involving arithmetical functions II*, Quart. J. Math., vol 8, pp. 313–320 (1937).
- [7] FEFERMAN, C., *Pointwise convergence of Fourier series*, Ann. of Math. 98 (1973), 551–571.
- [8] HARDY, G. AND LITTLEWOOD, J., *Some problems of Diophantine approximation : the analytic properties of certain Dirichlet's series associated with the distribution of numbers of modulus unity*, Trans. Cambridge Philos. Soc., vol. 22 (1923), 519–533.
- [9] HARTMAN, P., *On a class of arithmetical Fourier series*, Amer. J. Math. vol. 60 (1938), 66–74.
- [10] HAUKKANEN, P., WANG, J. AND SILLANP, J., *On Smith's determinant*, Linear Algebra Appl. 258 (1997), 251–269.
- [11] HEDENMALM, H., LINDQVIST, P. AND SEIP, K., *A Hilbert space of Dirichlet series and systems of dilated functions in  $L^2(0,1)$* , Duke Math. J., vol. 86, pp. 1–37 (1997).
- [12] JAFFARD, S., *On Davenport expansions*, Fractal geometry and applications : a jubilee of Benoît Mandelbrot. Part 1, 273–303, Proc. Sympos. Pure Math., 72, Part 1, Amer. Math. Soc., Providence, RI, 2004.
- [13] KRÄTZEL, E., *Zahlentheorie*, Berlin (1981).
- [14] LANG, S., *Introduction to Diophantine approximation*, Springer-Verlag (1995).
- [15] LINDQVIST, P. AND SEIP, K., *Note on some greatest common divisor matrices*, Acta Arith. 84 (1998), no. 2, 149–154.
- [16] RADEMACHER, H., *Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen*, Math. Ann. 87 (1922), 112–138.
- [17] WINTNER, A., *Diophantine approximation and Hilbert's space*, Amer. J. Math., vol. 66 (1944), 564–578.

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