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The well-posedness issue in endpoint spaces for an inviscid low-Mach number limit system

Francesco FANELLI and Xian LIAO

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Abstract

The present paper is devoted to the well-posedness issue for a low-Mach number limit system with heat conduction but no viscosity. We will work in the framework of general Besov spaces $B_{p,r}^s(\mathbb{R}^d)$, $d \geq 2$, which can be embedded into the class of Lipschitz functions.

Firstly, we consider the case of $p \in [2, 4]$, with no further restrictions on the initial data. Then we tackle the case of any $p \in]1, \infty]$, but requiring also a finite energy assumption. The extreme value $p = \infty$ can be treated due to a new a priori estimate for parabolic equations. At last we also briefly consider the case of any $p \in]1, \infty[$ but with smallness condition on initial inhomogeneity.

A continuation criterion and a lower bound for the lifespan of the solution are proved as well. In particular in dimension 2, the lower bound goes to infinity as the initial density tends to a constant.

1 Introduction

The free evolution of a compressible, effectively heat-conducting but inviscid fluid obeys the following equations:

$$(1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho v) &= 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p &= 0, \\ \partial_t(\rho e) + \operatorname{div}(\rho v e) - \operatorname{div}(k \nabla \vartheta) + p \operatorname{div} v &= 0, \end{cases}$$

where $\rho = \rho(t, x) \in \mathbb{R}^+$ stands for the mass density, $v = v(t, x) \in \mathbb{R}^d$ for the velocity field and $e = e(t, x) \in \mathbb{R}^+$ for the internal energy per unit mass. The time variable t belongs to \mathbb{R}^+ or to $[0, T]$ and the space variable x is in \mathbb{R}^d with $d \geq 2$. The scalar functions $p = p(t, x)$ and $\vartheta = \vartheta(t, x)$ denote the pressure and temperature respectively. The heat-conducting coefficient $k = k(\rho, \vartheta)$ is supposed to be smooth in both its variables.

We supplement System (1) with the following two equations of states:

$$(2) \quad p = R\rho\vartheta, \quad e = C_v\vartheta,$$

where R, C_v denote the ideal gas constant and the specific heat capacity at constant volume, respectively. That is, we restrict ourselves to perfect gases.

In this paper, we will consider highly subsonic ideal gases strictly away from vacuum, and correspondingly, we will work with the system (see (4) or (7) below) which derives from System (1) by letting the Mach number go to zero. More precisely, just as in [15], suppose (ρ, v, p) to be a solution of System (1) and define the dimensionless Mach number ε to be the ratio of the velocity v by the reference sound speed. Then the rescaled triplet

$$\left(\rho_\varepsilon(t, x) = \rho\left(\frac{t}{\varepsilon}, x\right), \quad v_\varepsilon(t, x) = \frac{1}{\varepsilon}v\left(\frac{t}{\varepsilon}, x\right), \quad p_\varepsilon(t, x) = p\left(\frac{t}{\varepsilon}, x\right) \right)$$

satisfies

$$(3) \quad \begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon v_\varepsilon) &= 0, \\ \partial_t(\rho_\varepsilon v_\varepsilon) + \operatorname{div}(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) + \frac{\nabla p_\varepsilon}{\varepsilon^2} &= 0, \\ \frac{1}{\gamma-1}(\partial_t p_\varepsilon + \operatorname{div}(p_\varepsilon v_\varepsilon)) - \operatorname{div}(k_\varepsilon \nabla \vartheta_\varepsilon) + p_\varepsilon \operatorname{div} v_\varepsilon &= 0. \end{cases}$$

Here $\gamma := C_p/C_v = 1 + R/C_v$ represents the adiabatic index and the constant C_p , denotes the specific heat capacity at constant pressure. The rescaled temperature and heat-conducting coefficient are given by

$$\vartheta_\varepsilon(t, x) = \vartheta\left(\frac{t}{\varepsilon}, x\right), \quad k_\varepsilon(t, x) = \frac{1}{\varepsilon} k\left(\frac{t}{\varepsilon}, x\right).$$

Now let ε go to 0, that is, the pressure p_ε equals to a positive constant P_0 by Equations (3)₂ and (3)₃, and thus System (3) becomes formally the following zero Mach number system immediately (see [2], [25] and [34] for detailed computations):

$$(4) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho v) &= 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla \Pi &= 0, \\ \operatorname{div} v - \frac{\gamma-1}{\gamma P_0} \operatorname{div}(k \nabla \vartheta) &= 0, \end{cases}$$

where $\Pi = \Pi(t, x)$ is a new unknown function. We remark that the justification of this limit process was showed in [2] in the framework of classical solutions.

Now, again similarly as in [15], in order to make use of the almost “divergence-free” condition (4)₃, we reduce System (4) by introducing some new unknowns below. Before going into details, let us suppose here that the density ρ always has positive lower bound and converges to some constant (say “1”) at infinity, in the sense detailed below (see (16)). We also point out that in the following, the fact that $\rho \vartheta \equiv P_0/R$ is a positive constant will be used thoroughly.

Firstly for notational simplicity, we set α to be a positive constant defined by

$$\alpha = \frac{\gamma-1}{\gamma P_0} = \frac{R}{C_p P_0} = \frac{1}{C_p \rho \vartheta}.$$

Then we define the following two coefficients, always viewed as regular functions of ρ :

$$\kappa = \kappa(\rho) = \alpha k \vartheta \quad \text{and} \quad \lambda = \lambda(\rho) = \rho^{-1}.$$

Furthermore for convenience, functions $a = a(\rho)$, $b = b(\rho)$ are introduced to be the antiderivatives of the smooth functions $\kappa(\rho)$ and $-\kappa(\rho)\rho^{-1}$ respectively, such that $a(1) = b(1) = 0$. It is easy to see that

$$(5) \quad \nabla a = \kappa \nabla \rho = \alpha k \vartheta \nabla \rho = -\alpha k \rho \nabla \vartheta = -\rho \nabla b.$$

We moreover define the new “velocity” u and the new “pressure” π respectively as

$$(6) \quad u = v - \alpha k \nabla \vartheta = v - \nabla b = v + \kappa \rho^{-1} \nabla \rho, \quad \pi = \Pi + \alpha k \rho \partial_t \vartheta = \Pi - \partial_t a.$$

Then System (4) finally becomes

$$(7) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho - \operatorname{div}(\kappa \nabla \rho) = 0, \\ \partial_t u + (u + \nabla b) \cdot \nabla u + \lambda \nabla \pi = h, \\ \operatorname{div} u = 0, \end{cases}$$

where

$$(8) \quad h(\rho, u) = \rho^{-1} \operatorname{div}(v \otimes \nabla a) = -u \cdot \nabla^2 b - (u \cdot \nabla \lambda) \nabla a - (\nabla b \cdot \nabla \lambda) \nabla a - \operatorname{div}(\nabla b \otimes \nabla b).$$

Let us just show how (7)₂ comes from (4)₂. Observe that (4)₂ gives

$$(9) \quad \partial_t(\rho u) + \operatorname{div}(\rho v \otimes u) + \partial_t(\rho \nabla b) + \operatorname{div}(v \otimes \rho \nabla b) + \nabla \Pi = 0.$$

It is easy to find that

$$\partial_t(\rho \nabla b) + \nabla \Pi = -\partial_t \nabla a + \nabla \Pi = \nabla \pi \quad \text{and} \quad \operatorname{div}(v \otimes \rho \nabla b) = -\operatorname{div}(v \otimes \nabla a).$$

Thus by view of Equation (4)₁, Equation (9) can be written as

$$(10) \quad \rho \partial_t u + \rho v \cdot \nabla u + \nabla \pi = \operatorname{div}(v \otimes \nabla a).$$

We thus multiply (10) by $\lambda = \rho^{-1}$ to get Equation (7)₂.

Let us note that if $\kappa \equiv 0$, then $a \equiv b \equiv 0$ and hence System (7) becomes the so-called density-dependent Euler equations

$$(11) \quad \begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0, \\ \partial_t v + v \cdot \nabla v + \lambda \nabla \pi = 0, \\ \operatorname{div} v = 0. \end{cases}$$

We can see that, due to the null heat conduction, the density ρ satisfies a transport equation, the “source” term h vanishes and the velocity v itself is divergence-free. In System (7) instead, one encounters a quasilinear parabolic equation for ρ and moreover, the transport velocity u is no longer solenoidal and the “source” term h is a little complicated, as it involves two derivatives of ρ . As early as in 1980, H. Beirão da Veiga and A. Valli in [10, 11, 12] have investigated the local well-posedness issue in some smooth bounded domain of the nonhomogeneous Euler system (11). We also cite here the book [3] as a good survey of the boundary-value problems in mechanics for nonhomogeneous fluids. By use of an energy identity, in [13] R. Danchin studied System (11) in the framework of nonhomogeneous Besov space $B_{p,r}^s(\mathbb{R}^d)$ which can be embedded in $C^{0,1}$. Recently in [14], R. Danchin and the first author treated the end point case where the Lebesgue exponent p in the Besov space $B_{p,r}^s$ is chosen to be ∞ . We notice that if moreover $\rho \equiv 1$ is a constant density state, then System (11) reduces to the classical Euler system, which has been deeply studied.

However, to our knowledge, there are not so many theoretical works devoted to the heat-conducting inviscid zero Mach number System (4), or equivalently System (7). It is interesting to view System (4) as the model for an inviscid fluid consisting of two components, both incompressible, with a diffusion effect obeying Fick’s law. In fact, we can write Equation (4)₃ as the following Fick’s law:

$$\operatorname{div}(v + \kappa \nabla \ln \rho) = 0,$$

with κ denoting the diffusion coefficient. Now ρ , u and v are considered to be the mean density, the mean-volume velocity and the mean-mass velocity of the mixture respectively. As usual, $\nabla \Pi$ denotes some unknown pressure. For more physical backgrounds of this model, see [18]. One can also see [6] for a local existence and uniqueness result of the initial boundary value problem for this model, in the framework of classical solutions. In this paper, we will show some similar well-posedness results as in [13, 14] for the Cauchy problem for System (4), in the scale of Besov spaces which can be embedded into the class of Lipschitzian functions.

Let us mention that the justification of the incompressible limit of Euler equations (i.e. incompressible Euler equations can be viewed as compressible Euler equations when the Mach number goes to zero) has attracted much attention. There are many early works such as [16, 17] by D.G. Ebin, [22, 23] by H. Isozaki, [24] by S. Klainerman and A. Majda, [27] by S. Schochet, [31] by S. Ukai, etc. for the isentropic Euler system case. At the beginning of this century, T. Alazard in [1] and G. Métivier and S. Schochet in [26] treated problems with boundary and without boundary respectively, for the non-isentropic case. But there, the heat conduction did not come into play, and hence the entropy there satisfies a free transport equation. Later T. Alazard in [2] generalized the study to various models which include the case of heat

conductive ideal gases, i.e. he justified rigorously also the low Mach number limit from System (1) to System (4).

Let us also mention that if the fluid is viscous, that is to say there is an additional tensor term

$$-\operatorname{div} \sigma = -\operatorname{div} (\mu(\nabla v + Dv)) - \nabla(\nu \operatorname{div} v)$$

in the evolution equation (4)₂ for the velocity v (here μ, ν denote viscosity coefficients which may depend on ρ), then System (4) becomes the low Mach number limit system of the full Navier-Stokes system. See paper [2] by T. Alazard, [15] by R. Danchin and the second author, and references therein for some relevant results. One notices that such a viscous system can also describe viscous mixture models such as pollutant spreading, snow avalanche, etc.: see [3, 8, 9, 19, 20, 28, 29, 30] for some further discussion.

Our paper is organized in the following way.

In the next section we will present our main results on System (7) (the corresponding results for System (4) will be pointed out, in Remark 2.2 for example), i.e. the local-in-time well-posedness in Besov spaces $B_{p,r}^s \hookrightarrow C^{0,1}$. Firstly, we focus on the case when $p \in [2, 4]$ (see Theorem 2.1); then, we consider also the general instance $p \in]1, +\infty]$, under the additional assumptions of data in L^2 (see Theorem 2.9) or when the initial density is a small perturbation of a constant state (Theorem 2.12). We will also state a continuation criterion for solutions to our system, and a lower bound for the lifespan of the solution in terms of the norms of the initial data only: this will be done in Theorems 2.3 and 2.7. In particular in dimension 2, a better estimation of the lifespan under finite energy hypothesis is given in Theorem 2.11.

Section 3 is devoted to the tools, from Fourier analysis, which we will use in our study. In particular, we will briefly recall the main ideas of Littlewood-Paley decomposition and paradifferential calculus, and some basic properties of Besov spaces.

In Section 4 we will tackle the proof of Theorem 2.1: it will be carried out in a standard way. First of all, we will show a priori estimates for the linearized equations. Then, we will construct inductively a sequence of smooth approximated solutions. Finally, we will show its convergence to a “real solution” to the original equations.

Section 5, instead, is devoted to the proof of the continuation criterion (Theorem 2.3) and of the lower bound for the lifespan of solutions to system (7) (Theorem 2.7).

A new estimate for parabolic equations in Besov spaces $B_{\infty,r}^s$ (see Proposition 6.1) will open Section 6. This will be the key to tackle the case of finite energy initial data in this endpoint space, and so to prove Theorem 2.9. Moreover, in this same instance (finite energy and $p = +\infty$, $s = r = 1$) we will improve the lower bound for the lifespan of the solutions in the case of dimension 2 (see Theorem 2.11). In particular, we will prove that the solutions tend to be globally defined in time whenever the initial density is close (in a suitable sense) to a constant state. In the same section we will also sketch the proof of Theorem 2.12, when the initial density is near a constant state.

Finally, in the Appendix we will give a complete proof of the commutator estimates stated in Lemma 4.2.

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2 Main results

Let us focus on System (7) to introduce our main results. In view of Equation (7)₁, of parabolic type, by maximum principle we can assume that the density ρ (if it exists on the time interval $[0, T]$) has the same positive upper and lower bounds as the initial density ρ_0 :

$$(12) \quad 0 < \rho_* \leq \rho(t, x) \leq \rho^*, \quad \forall t \in [0, T], x \in \mathbb{R}^d.$$

Correspondingly, the coefficients κ and λ can always be assumed to have positive upper and lower bounds too, which ensures that the pressure π satisfies an elliptic equation. As a matter of fact, applying operator “div” to Equation (7)₂ gives the following elliptic equation in divergence form:

$$(13) \quad \operatorname{div}(\lambda \nabla \pi) = \operatorname{div}(h - v \cdot \nabla u).$$

Since by Equation (7)₂ there is no gain of regularity for the velocity u as time goes by, we suppose the initial divergence-free “velocity” field u_0 to belong to some space $B_{p,r}^s$ which can be continuously embedded in $C^{0,1}$, i.e. the triplet $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ only has to satisfy the following condition:

$$(14) \quad s > 1 + \frac{d}{p}, \quad \text{or} \quad s = 1 + \frac{d}{p}, \quad r = 1.$$

This assumption will be enough to have the velocity field u Lipschitz continuous, and so to preserve the initial regularity as a transport velocity. This at least requires the “source” term $h - \lambda \nabla \pi$ to belong to $L^1([0, T]; B_{p,r}^s)$ which, by view of definition (8) of h and bound (12) for ρ , asks at least

$$\nabla^2 \rho \in L^1([0, T]; B_{p,r}^s), \quad \nabla \rho \in L^\infty([0, T]; L^\infty), \quad \text{and} \quad \nabla \pi \in L^1([0, T]; B_{p,r}^s).$$

Keeping in mind that ρ satisfies the parabolic type equation (7)₁, we expect to maintain the initial regularity and to gain also two orders of regularity with respect to the space variable when integrating in time (see Subsection 4.1.1), at least in a small time interval. We thus have to assume the difference of the initial density ρ_0 and a constant (say 1) to be in the same space $B_{p,r}^s$ as above. However in general, we only get $\nabla^2 \rho$ in the time-dependent Besov space $\tilde{L}_T^1(B_{p,r}^s)$, which is a little bit larger than $L_T^1(B_{p,r}^s)$, see Definition 3.2. Therefore in the whole context we will deal with the spaces $\tilde{L}_T^\infty(B_{p,r}^s)$ and $\tilde{L}_T^1(B_{p,r}^s)$.

On the other hand, if $h - v \cdot \nabla u$ is already in space $\tilde{L}_T^1([0, T]; B_{p,r}^s)$, then applying operator “ Δ_j ” (see Section 3 for the definition) to Equation (13) gives

$$\operatorname{div}(\lambda \Delta_j \nabla \pi) = \operatorname{div} \Delta_j(h - v \cdot \nabla u) + \operatorname{div}([\lambda, \Delta_j] \nabla \pi).$$

So, it remains to control the commutator $\operatorname{div}([\lambda, \Delta_j] \nabla \pi)$, hence the low frequencies of $\nabla \pi$, by the classical commutator estimation (see Lemma 4.1 for example). Due to the fact that, for Equation (13) above, we have the a priori estimate

$$\|\nabla \pi\|_{L^p} \leq C \|h - v \cdot \nabla u\|_{L^p}$$

independently of λ only when $p = 2$ (see Lemma 2 of [13]), we have to make sure that $h - v \cdot \nabla u \in L^2$. Hence the fact that h is composed of quadratic forms entails that p has to verify

$$(15) \quad p \in [2, 4].$$

To conclude, we have the following theorem, whose proof is shown in Section 4.

Theorem 2.1. *Let the triplet $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ satisfy conditions (14) and (15).*

Let us take an initial density state ρ_0 and an initial divergence-free velocity field u_0 such that

$$(16) \quad 0 < \rho_* \leq \rho_0 \leq \rho^*, \quad \|\rho_0 - 1\|_{B_{p,r}^s} + \|u_0\|_{B_{p,r}^s} \leq M,$$

for some positive constants ρ_ , ρ^* and M .*

Then there exist a positive time T (depending only on ρ_ , ρ^* , M , d , s , p , r) and a unique solution $(\rho, u, \nabla \pi)$ to System (7) such that $(\varrho, u, \nabla \pi) := (\rho - 1, u, \nabla \pi)$ belongs to the space $E_{p,r}^s(T)$, defined as the set of triplet $(\varrho, u, \nabla \pi)$ such that*

$$(17) \quad \begin{cases} \varrho & \in \tilde{C}([0, T]; B_{p,r}^s) \cap \tilde{L}^1([0, T]; B_{p,r}^{s+2}), \\ u & \in \tilde{C}([0, T]; B_{p,r}^s), \\ \nabla \pi & \in \tilde{L}^1([0, T]; B_{p,r}^s) \cap L^1([0, T]; L^2), \end{cases}$$

with $\tilde{C}_w([0, T]; B_{p,r}^s)$ if $r = +\infty$ (see also Definition 3.2).

Remark 2.2. Let us state briefly here the corresponding wellposedness result for the original system (4). By view of the change of variables (6), we have $u = \mathcal{P}v$, $\nabla b = \mathcal{Q}v$, where \mathcal{P} denotes the Leray projector over divergence-free vector fields and $\mathcal{Q} = \text{Id} - \mathcal{P}$ (see Remark 3.5). Assume Conditions (14) and (15), and the initial datum (ρ_0, v_0) such that

$$0 < \rho_* \leq \rho_0 \leq \rho^*, \quad \nabla b(\rho_0) = \mathcal{Q}v_0, \quad \|\rho_0 - 1\|_{B_{p,r}^s} + \|\mathcal{P}v_0\|_{B_{p,r}^s} \leq M.$$

Then, there exist a positive time T (depending only on $\rho_*, \rho^*, M, d, s, p, r$) and a unique solution $(\rho, v, \nabla \Pi)$ to System (4) such that

$$\begin{cases} \rho - 1 \in \tilde{C}([0, T]; B_{p,r}^s) \cap \tilde{L}^1([0, T]; B_{p,r}^{s+2}), \\ v \in \tilde{C}([0, T]; B_{p,r}^{s-1}) \cap \tilde{L}^2([0, T]; B_{p,r}^s), \quad \mathcal{P}v \in \tilde{C}([0, T]; B_{p,r}^s), \\ \nabla \Pi \in \tilde{L}^1([0, T]; B_{p,r}^s), \end{cases}$$

with $\tilde{C}_w([0, T]; B_{p,r}^s)$ if $r = +\infty$.

One notices from above that the initial velocity v_0 needs not to be in $B_{p,r}^s$ but the velocity $v(t)$ will be in it for almost every $t \in [0, T]$. That is, the initial non-Lipschitzian velocity may evolve into a unique Lipschitzian velocity. On the other side, Theorem 2.1 doesn't say that if the initial datum $(\rho_0 - 1, v_0) \in B_{p,r}^s(\mathbb{R}^d)^{d+1}$, then there exists a unique local-in-time solution $(\rho - 1, v, \nabla \Pi)$ to System (4). In fact, we can just say that if initially, $\rho_0 - 1 \in B_{p,r}^{s+1}$, $v_0 \in B_{p,r}^s$, $\rho_0 \in [\rho_*, \rho^*]$ and $\nabla b(\rho_0) = \mathcal{Q}v_0$.

Let us also point out that since $\partial_t \rho \notin L^1([0, T]; L^2)$ in general, we do not know whether $\nabla \Pi \in L^1([0, T]; L^2)$ (recall also definition (6)). Hence it seems not convenient to deal with System (4) directly since the low frequencies of $\nabla \Pi$ can not be controlled a priori.

Next, one can furthermore get a Beale-Kato-Majda type continuation criterion (see [5] for the original version) for solutions to System (7), similar as the ones in [13] or [14] for the density-dependent Euler System (11). As it is a coupling of a parabolic equation (for the density) and an Euler-type equation (for the velocity field and the pressure), one may expect, a priori, to impose conditions similar, or even weaker, to those found in the quoted papers, because the regularity of the density improves in time evolution. Actually, this is not the case: the criterion holds true under additional conditions, which are motivated by the structure of the nonlinearities in h and $v \cdot \nabla u$. In fact, for instance, we consider $\tilde{L}_T^1(B_{p,r}^s)$ -norm of h and $\nabla b \cdot \nabla u$:

- the term $u \cdot \nabla^2 b$ requires at least $\|\nabla^2 b\|_{L_T^1(L^\infty)}$ and $\|u\|_{L_T^\infty(L^\infty)}$;
- the term $\Delta b \nabla b$ requires $\|\nabla b\|_{L_T^\infty(L^\infty)}$ and $\|\Delta b\|_{L_T^2(L^\infty)}$;
- the transport term $\nabla b \cdot \nabla u$ needs control on $\|\nabla u\|_{L_T^2(L^\infty)}$, since we only have $\nabla b \in \tilde{L}_T^2(B_{p,r}^s)$.

Finally, we have the following statement.

Theorem 2.3. [Continuation Criterion] *Let the triplet $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ satisfy conditions (14) and (15). Let $(\rho, u, \nabla \pi)$ be a solution of (7) on $[0, T[\times \mathbb{R}^d$ such that:*

(i) $\rho - 1 \in \tilde{C}([0, T[; B_{p,r}^s) \cap \tilde{L}_{loc}^1([0, T[; B_{p,r}^{s+2})$ and satisfies

$$(18) \quad \sup_{t \in [0, T[} \|\nabla \rho(t)\|_{L^\infty} + \int_0^T \|\nabla^2 \rho(t)\|_{L^\infty}^2 dt < +\infty;$$

(ii) $u \in \tilde{C}([0, T[; B_{p,r}^s)$ and satisfies

$$(19) \quad \sup_{t \in [0, T[} \|u(t)\|_{L^\infty} + \int_0^T \|\nabla u(t)\|_{L^\infty}^2 dt < +\infty;$$

(iii) $\nabla\pi \in \tilde{L}_{loc}^1([0, T[; B_{p,r}^s)$ verifies, for some $\sigma > 0$,

$$(20) \quad \int_0^T \|\nabla\pi(t)\|_{B_{p,\infty}^{-\sigma} \cap L^\infty} dt < +\infty.$$

Then $(\rho, u, \nabla\pi)$ could be continued beyond T (if T is finite) into a solution of (7) with the same regularity.

The proof of Theorem 2.3 issues from the following fundamental lemma whose proof can be found in Section 5.1:

Lemma 2.4. *Let $s > 0$, $p \in (1, +\infty)$ and $r \in [1, +\infty]$. Let $(\rho, u, \nabla\pi)$ be a solution of (7) over $[0, T[\times \mathbb{R}^d$ such that Hypothesis (i), (ii), (iii) in Theorem 2.3 hold true. If T is finite, then one gets*

$$(21) \quad \|\rho - 1\|_{\tilde{L}_T^\infty(B_{p,r}^s) \cap \tilde{L}_T^1(B_{p,r}^{s+2})} + \|u\|_{\tilde{L}_T^\infty(B_{p,r}^s)} + \|\nabla\pi(t)\|_{\tilde{L}_T^1(B_{p,r}^s)} < +\infty.$$

In fact, from Theorem 2.1, one knows that once Conditions (14), (15) and (16) hold, the initial data will evolve into a unique solution on a positive time interval, which depends only on the bounds on the density, the $B_{p,r}^s$ -norm of the initial data and the indices. Therefore, once we have (21) and besides Conditions (14), (15) hold true, then there exists a time t_0 such that, for any $\tilde{T} < T$, System (7) with initial data $(\rho(\tilde{T}), u(\tilde{T}))$ has a unique solution until the time $\tilde{T} + t_0$. Thus, if we take, for instance, $\tilde{T} = T - (t_0/2)$, then we get a solution until the time $T + (t_0/2)$, which is, by uniqueness, the continuation of $(\rho, u, \nabla\pi)$. Theorem 2.3 follows.

Let us analyse the scaling property of our system in order to show an explicit relationship between the size of the initial data and the lifespan. Recalling (8), it's easy to see that, if $(\rho, u, \nabla\pi)$ is a solution of (7) with initial data (ρ_0, u_0) , then for any $\varepsilon > 0$

$$(22) \quad (\rho^\varepsilon, u^\varepsilon, \nabla\pi^\varepsilon)(t, x) := (\rho, \varepsilon u, \varepsilon^3 \nabla\pi)(\varepsilon^2 t, \varepsilon x)$$

is still a solution of (7), with initial data $(\rho_0^\varepsilon, u_0^\varepsilon)(x) := (\rho_0, \varepsilon u_0)(\varepsilon x)$. So, the following result immediately follows.

Proposition 2.5. *Let $(\rho, u, \nabla\pi)$ be a solution of System (7) with initial data (ρ_0, u_0) on a time interval $[0, T[$. Then the triplet $(\rho^\varepsilon, u^\varepsilon, \nabla\pi^\varepsilon)$, given by (22), is still a solution of (7), defined at least on the time interval $[0, T_\varepsilon[$, with*

$$T_\varepsilon \geq T \varepsilon^{-2}.$$

Remark 2.6. In the homogeneous case, we know that if the initial velocity is of order ε then the lifespan is at least of order ε^{-1} . But here for our system (7) (and hence for System (4)), we have from the above proposition that if the initial data $\rho_0^\varepsilon - 1, u_0^\varepsilon$ are of order $\varepsilon^{s-d/p}$ and $\varepsilon^{s+1-d/p}$ respectively, then the lifespan is at least of order ε^{-2} , due to the fact that

$$\|\rho_0^\varepsilon - 1\|_{B_{p,r}^s} \sim \varepsilon^{s-d/p} \|\rho_0 - 1\|_{B_{p,r}^s}, \quad \|u_0^\varepsilon\|_{B_{p,r}^s} \sim \varepsilon^{s-d/p+1} \|u_0\|_{B_{p,r}^s}.$$

In particular if $s = 1 + d/p$, then the lifespan is of order $O(\varepsilon^{-2})$ for the initial data $\rho_0^\varepsilon - 1 = O(\varepsilon)$, $u_0^\varepsilon = O(\varepsilon^2)$.

Even in the two-dimensional case, it's hard to expect global-in-time well-posedness for this system: the parabolic equation (7)₁ allows to improve regularity for the density term, but such a gain is (roughly speaking) deleted by the nonlinear term in the momentum equation (7)₂. Hence, the obstacles to global existence are the same as those one has to face in considering the density-dependent Euler system (11), which has already been dealt in [14]. However, we manage to establish an explicit lower bound for the lifespan of the solution, depending only on the norms of the initial data, in any dimension $d \geq 2$. The proof will be the matter of Section 5.2.

Theorem 2.7. *Under the hypotheses of Theorem 2.1, there exists a positive constant L , depending only on d, p, r, ρ_* and ρ^* , such that, the lifespan T of the solution to System (7) given by Theorem 2.1 is bounded from below by the quantity*

$$(23) \quad \frac{L}{\|u_0\|_{B_{p,r}^s}} \log \left(\frac{L}{1 + \left(\|\rho_0 - 1\|_{B_{p,r}^s} / \|u_0\|_{B_{p,r}^s}^{1/2} \right)^{\mathfrak{X}}} \right),$$

where $\mathfrak{X} > 2$ is a constant big enough, depending only on s, d, p .

Remark 2.8. Thanks to Theorem 2.3, the lifespan is independent of the regularity. Hence, if we want to get the lower bound of the lifespan, we only deal with the endpoint case $s = 1 + d/p$ ($p \in [2, 4]$) and $r = 1$, whose lifespan is the largest. Therefore, the $B_{p,r}^s$ -norm in (23) can be replaced by $B_{4,1}^{1+d/4}$ -norm.

In Theorem 2.11 we will improve the previous result for 2-D fluids. Under the additional requirements of finite energy initial data (see also below) and taking $p = +\infty$, we will show that the lifespan tends to $+\infty$, i.e. the solution tends to be global in time, if ρ_0 is "close" to (say) 1.

As we have already pointed out, Hypothesis (15) over the index p is required to get $h \in L^2$, and so to solve the elliptic equation for the pressure term. In fact, we can remove it and consider any $p \in]1, +\infty]$, provided that the following energy estimates hold: this immediately gives $\nabla \pi \in L_T^1(L^2)$.

First of all, if the initial datum $\rho_0 - 1 \in L^2(\mathbb{R}^d)$, then by taking the $L^2(\mathbb{R}^d)$ -inner product between $\rho - 1$ and Equation (4)₁, i.e. Equation (7)₁, and integrating in time variable, we arrive at the following:

$$(24) \quad \frac{1}{2} \int_{\mathbb{R}^d} |\rho(t) - 1|^2 + \int_0^t \int_{\mathbb{R}^d} \kappa |\nabla \rho|^2 = \frac{1}{2} \|\rho_0 - 1\|_{L^2}^2.$$

Such an energy identity requires the following equality to hold true:

$$\int_0^t \int_{\mathbb{R}^d} (u \cdot \nabla \rho) (\rho - 1) = 0.$$

In fact for instance, if $u \in L_T^\infty(L^\infty)$, which is always the case under our hypothesis, then for any finite t it holds, by view of $\operatorname{div} u = 0$ and the previous Identity (24).

In the same way, formally, if we take L^2 -inner product between Equation (10) and u , then thanks to Equation (4)₁ and integration by parts, the first two terms entail

$$(25) \quad \int_{\mathbb{R}^d} (\rho \partial_t u + \rho v \cdot \nabla u) \cdot u = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho |u|^2.$$

Thanks to the divergence-free condition of u , the inner product $\langle \nabla \pi, u \rangle_{L^2(\mathbb{R}^d)}$ vanishes. It rests to dealing with $\langle \operatorname{div}(v \otimes \nabla a), u \rangle_{L^2(\mathbb{R}^d)}$. In fact, from Relations (5) and (6) we have

$$\operatorname{div}(v \otimes \nabla a) = \Delta b \nabla a + u \cdot \nabla^2 a + \nabla b \cdot \nabla^2 a.$$

The bound (12) implies that (with C depending on ρ_*, ρ^*)

$$\|\Delta b\|_{L^\infty}, \|\nabla^2 a\|_{L^\infty} \leq C (\|\nabla \rho\|_{L^\infty}^2 + \|\nabla^2 \rho\|_{L^\infty}) \quad \text{and} \quad \|\nabla b\|_{L^2}, \|\nabla a\|_{L^2} \leq C \|\nabla \rho\|_{L^2}.$$

Thus

$$(26) \quad \begin{aligned} |\langle \operatorname{div}(v \otimes \nabla a), u \rangle_{L^2(\mathbb{R}^d)}| &\leq C (\|\nabla \rho\|_{L^\infty}^2 + \|\nabla^2 \rho\|_{L^\infty}) (\|\nabla \rho\|_{L^2} + \|u\|_{L^2}) \|u\|_{L^2} \\ &\leq C (\Theta'(t) \|u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2), \end{aligned}$$

where we have defined

$$\Theta(t) = \int_0^t (\|\nabla \rho\|_{L^\infty}^2 + \|\nabla \rho\|_{L^\infty}^4 + \|\nabla^2 \rho\|_{L^\infty} + \|\nabla^2 \rho\|_{L^\infty}^2) d\tau.$$

Hence Identity (25), Hölder's Inequality, Gronwall's Inequality and Energy Equality (24) give

$$(27) \quad \|u(t)\|_{L^2}^2 \leq Ce^{C\Theta(t)} \left(\|u_0\|_{L^2}^2 + \int_0^t \|\nabla \rho\|_{L^2}^2 \right) \leq Ce^{C\Theta(t)} (\|u_0\|_{L^2}^2 + \|\rho_0 - 1\|_{L^2}^2).$$

On the other hand, if there exists $(\rho - 1, u)$ such that $(17)_1$ and $(17)_2$ hold for some finite time $T > 0$, with (s, p, r) verifying (14), then we have also (18) and (19). Thus, if we assume the initial data (ρ_0, u_0) to satisfy

$$(28) \quad \rho_0 - 1, u_0 \in L^2,$$

then by Energy Estimates (24), (27), we gather $u, \rho - 1 \in L_T^\infty(L^2)$ and $\nabla \rho \in L_T^2(L^2)$. Hence for any finite positive time T ,

$$h - v \cdot \nabla u = \rho^{-1} \operatorname{div} (v \otimes \nabla a) - v \cdot \nabla u = \rho^{-1} \Delta b \nabla a + \rho^{-1} v \cdot \nabla^2 a - v \cdot \nabla u \in L_T^1(L^2),$$

which ensures $\nabla \pi \in L_T^1(L^2)$, by view of Equation (13) and Lemma 3.13 (see below).

To conclude, by the arguments before Theorem 2.1 and above, and a new a priori estimate for linear parabolic equations in Besov spaces $B_{\infty, r}^s$ (see Proposition 6.1), we have the following statement.

Theorem 2.9. *Let the triplet $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ satisfy Condition (14) and $1 < p \leq \infty$.*

Let the initial density ρ_0 and the divergence-free initial velocity u_0 fulfill (28) as well as (16), for some positive constants ρ_ , ρ^* and M .*

Then there exist a positive time T (depending only on ρ_ , ρ^* , M , d , s , p , r) and a unique solution $(\rho, u, \nabla \pi)$ to System (7) with $(\rho - 1, u, \nabla \pi) \in E_{p, r}^s(T)$ such that $\rho - 1, u \in C([0, T]; L^2)$ and $\nabla \rho \in L_T^2(L^2)$.*

Remark 2.10. Notice that we have the embedding $B_{p, r}^s(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ if $s > d/p - d/2$, $p \in [1, 2]$ (see Corollary 3.7 below). Thus if the triplet $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ satisfy Condition (14) and $1 < p \leq 2$, then the initial data ρ_0, u_0 satisfying (16) already verifies finite-energy condition (28).

In fact, if we take $\rho \equiv \bar{\rho}$ constant in system (7), or equivalently in (4), we get the classical homogeneous Euler system. For this system, the global-in-time existence issue in dimension $d = 2$ has been well-known since 1933, due to the pioneering work [33] by Wolibner. For non-homogeneous perfect fluids, see system (11), it's still open if its solutions exist globally in time. However, in [14] it's proved that, for initial densities close to a constant state, the lifespan of the corresponding solutions tends to $+\infty$ (independently of the initial velocity field).

Actually, under the same hypothesis of Theorem 2.9 in dimension 2, we are able to prove a similar result also for our system.

Theorem 2.11. *Let $d = 2$.*

Let us suppose the initial data ρ_0 and u_0 to be such that $\varrho_0 := \rho_0 - 1 \in L^2 \cap B_{\infty, 1}^1$, with $0 < \rho_ \leq \rho_0 \leq \rho^*$, and $u_0 \in L^2 \cap B_{\infty, 1}^1$.*

Then the lifespan of the solution to system (7), given by Theorem 2.9, is bounded from below by the quantity

$$(29) \quad \frac{\tilde{L}}{1 + \|\varrho_0\|_{L^2}^2 + \|u_0\|_{L^2 \cap B_{\infty, 1}^1}} \log \left(1 + \log \left(\frac{\tilde{L}}{\left(1 + \|\varrho_0\|_{B_{\infty, 1}^1}^{\mathfrak{M}} \right) \|\varrho_0\|_{B_{\infty, 1}^1}} \right) \right),$$

for a suitable exponent $\mathfrak{M} > 5$ and a constant \tilde{L} which depends only on ρ_ and ρ^* .*

In particular, the lifespan tends to $+\infty$, i.e. the solution tends to be global, for initial densities close (in the $B_{\infty, 1}^1$ norm) to a constant state.

There is another way to get rid of Condition (15) imposed on p , that is, we assume an additional smallness condition over the initial density, which ensures that the elliptic equation for the pressure is almost the Laplace equation, up to a perturbation term.

Theorem 2.12. *A small constant $c > 0$ exists such that the following statement holds true.*

Let the triplet $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ satisfy Condition (14) and $p \in (1, +\infty)$. Let us take an initial density state ρ_0 and an initial divergence-free velocity field u_0 such that the bounds in (16) are true, for some positive constants ρ_ , ρ^* and M . Assume moreover $\|\rho_0 - 1\|_{B_{p,r}^s} \leq c$.*

Then there exist a positive time T (depending on ρ_ , ρ^* , c , M , d , s , p , r) and a unique solution $(\varrho, u, \nabla \pi)$ to System (7) belonging to the space $I_{p,r}^s(T)$, defined as $E_{p,r}^s(T)$ (recall (17)), but without the condition $\nabla \pi \in L^1([0, T]; L^2)$.*

Remark 2.13. It goes without saying that under hypotheses of Theorem 2.9 or Theorem 2.12, a continuation criterion and a lifespan lower bound analogous to those of Theorem 2.3 and Theorem 2.7 respectively, can be proved. The corresponding results for the original system (4) also hold true, similar as in Remark 2.2, which are omitted here.

We change the word here that in the sequel, C always denotes some “harmless” constant which may only depend on $d, s, p, r, \rho_*, \rho^*$, unless otherwise defined. Notation $A \lesssim B$ means $A \leq CB$ and $A \sim B$ says A equals to B , up to a constant factor. For notational convenience, the notation ϱ always represent $\rho - 1$, unless otherwise specified.

3 An overview on Fourier analysis techniques

Our results mostly rely on Fourier analysis methods which are based on a nonhomogeneous dyadic partition of unity with respect to Fourier variable, the so-called Littlewood-Paley decomposition. Unless otherwise specified, all the results which are presented in this section are proved in [4].

In order to define a Littlewood-Paley decomposition, fix a smooth radial function χ supported in (say) the ball $B(0, \frac{4}{3})$, equals to 1 in a neighborhood of $B(0, \frac{3}{4})$ and such that $r \mapsto \chi(re_r)$ is nonincreasing over \mathbb{R}_+ , and set $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$.

The *dyadic blocks* $(\Delta_j)_{j \in \mathbb{Z}}$ are defined by¹

$$\Delta_j := 0 \text{ if } j \leq -2, \quad \Delta_{-1} := \chi(D) \quad \text{and} \quad \Delta_j := \varphi(2^{-j}D) \text{ if } j \geq 0.$$

We also introduce the following low frequency cut-off:

$$S_j u := \chi(2^{-j}D) = \sum_{j' \leq j-1} \Delta_{j'} \quad \text{for } j \geq 0, \quad S_j u \equiv 0 \quad \text{for } j \leq 0.$$

The following classical properties will be used freely throughout the paper:

- for any $u \in \mathcal{S}'$, the equality $u = \sum_j \Delta_j u$ holds true in \mathcal{S}' ;
- for all u and v in \mathcal{S}' , the sequence $(S_{j-1} u \Delta_j v)_{j \in \mathbb{N}}$ is spectrally supported in dyadic annuli.

One can now define what a Besov space $B_{p,r}^s$ is:

Definition 3.1. Let u be a tempered distribution, s a real number, and $1 \leq p, r \leq \infty$. We set

$$\|u\|_{B_{p,r}^s} := \left(\sum_j 2^{rjs} \|\Delta_j u\|_{L^p}^r \right)^{\frac{1}{r}} \quad \text{if } r < \infty \quad \text{and} \quad \|u\|_{B_{p,\infty}^s} := \sup_j (2^{js} \|\Delta_j u\|_{L^p}).$$

We then define the space $B_{p,r}^s$ as the subset of distributions $u \in \mathcal{S}'$ such that $\|u\|_{B_{p,r}^s}$ is finite.

When solving evolutionary PDEs, it is natural to use spaces of type $L_T^\rho(X) = L^\rho(0, T; X)$ with X denoting some Banach space. In our case, X will be a Besov space so that we will have to localize the equations by Littlewood-Paley decomposition. This will provide us with estimates of the Lebesgue norm of each dyadic block *before* performing integration in time. This leads to the following definition:

¹Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u \mapsto \mathcal{F}^{-1}(f(\xi)\mathcal{F}u(\xi))$.

Definition 3.2. For $s \in \mathbb{R}$, $(q, p, r) \in [1, +\infty]^3$ and $T \in [0, +\infty]$, we set

$$\|u\|_{\tilde{L}_T^q(B_{p,r}^s)} = \left(\sum_{j \geq -1} 2^{rjs} \left(\int_0^T \|\Delta_j u(t)\|_{L^p}^q dt \right)^{\frac{r}{q}} \right)^{\frac{1}{r}},$$

with the usual change if $r = +\infty$ or $q = +\infty$.

We also set $\tilde{C}_T(B_{p,r}^s) = \tilde{L}_T^\infty(B_{p,r}^s) \cap \mathcal{C}([0, T]; B_{p,r}^s)$.

Remark 3.3. From the above definition, it is easy to show that for all $s \in \mathbb{R}$, the Besov space $B_{2,2}^s$ coincides with the nonhomogeneous Sobolev space H^s . Let us also point out that for any $k \in \mathbb{N}$ and $p \in [1, \infty]$, we have the following chain of continuous embedding:

$$B_{p,1}^k \hookrightarrow W^{k,p} \hookrightarrow B_{p,\infty}^k,$$

where $W^{k,p}$ denotes the set of L^p functions with derivatives up to order k in L^p . The following embedding is also true:

$$\tilde{L}_T^q(B_{p,r}^s) \hookrightarrow L_T^q(B_{p,r}^s) \text{ if } q \geq r, \quad \tilde{L}_T^q(B_{p,r}^s) \hookleftarrow L_T^q(B_{p,r}^s) \text{ if } q \leq r.$$

The Besov spaces have many nice properties which will be recalled throughout the paper whenever they are needed. For the time being, let us just recall that if Condition (14) holds true then $B_{p,r}^s$ is an algebra continuously embedded in the set $C^{0,1}$ of bounded Lipschitz functions (see e.g. [4], Chap. 2), and that the gradient operator maps $B_{p,r}^s$ in $B_{p,r}^{s-1}$. The following result will be also needed:

Proposition 3.4. Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth homogeneous function of degree m away from a neighborhood of the origin. Then for all $(p, r) \in [1, \infty]^2$ and $s \in \mathbb{R}$, the operator $F(D)$ maps $B_{p,r}^s$ in $B_{p,r}^{s-m}$.

Remark 3.5. Let \mathcal{P} be the Leray projector over divergence free vector fields and $\mathcal{Q} := \text{Id} - \mathcal{P}$. Recall that in Fourier variables, we have for all vector field u

$$\widehat{\mathcal{P}u}^j(\xi) = \sum_{k=1}^d (\delta_{jk} + 1) \frac{\xi_j \xi_k}{|\xi|^2} \widehat{u}^k(\xi) \quad \text{and} \quad \widehat{\mathcal{Q}u}(\xi) = -\frac{\xi}{|\xi|^2} \xi \cdot \widehat{u}(\xi).$$

Therefore, both $(\text{Id} - \Delta_{-1})\mathcal{P}$ and $(\text{Id} - \Delta_{-1})\mathcal{Q}$ satisfy the assumptions of the above proposition with $m = 0$ hence are self-maps on $B_{p,r}^s$ for any $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$.

The following lemma (referred in what follows as *Bernstein's inequalities*) describes the way derivatives act on spectrally localized functions.

Lemma 3.6. Let $0 < r < R$. A constant C exists so that, for any nonnegative integer k , any couple (p, q) in $[1, \infty]^2$ with $p \leq q$ and any function u of L^p , we have for all $\lambda > 0$,

$$\begin{aligned} \text{Supp } \widehat{u} \subset B(0, \lambda R) &\implies \|\nabla^k u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}; \\ \text{Supp } \widehat{u} \subset \{\xi \in \mathbb{R}^N / r\lambda \leq |\xi| \leq R\lambda\} &\implies C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|\nabla^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}. \end{aligned}$$

The first Bernstein inequality entails the following embedding result, which is a generalization of Remark 3.3:

Corollary 3.7. The space $B_{p_1,r_1}^{s_1}$ is continuously embedded in the space $B_{p_2,r_2}^{s_2}$ whenever $1 \leq p_1 \leq p_2 \leq \infty$ and

$$s_2 < s_1 - d/p_1 + d/p_2 \quad \text{or} \quad s_2 = s_1 - d/p_1 + d/p_2 \quad \text{and} \quad 1 \leq r_1 \leq r_2 \leq \infty.$$

Let us now recall the so-called Bony's decomposition introduced in [7] for the products. Formally, any product of two tempered distributions u and v , may be decomposed into

$$(30) \quad uv = T_u v + T_v u + R(u, v)$$

with

$$T_u v := \sum_j S_{j-1} u \Delta_j v \quad \text{and} \quad R(u, v) := \sum_j \sum_{|j'-j| \leq 1} \Delta_j u \Delta_{j'} v.$$

The above operator T is called “paraproduct” operator whereas R is called “remainder” operator.

We have the following classical estimates in Besov spaces for the paraproduct and remainder operators.

Proposition 3.8. *Let $s, s_1, s_2 \in \mathbb{R}$, $1 \leq r, r_1, r_2, p \leq \infty$ with $\frac{1}{r} \leq \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$.*

- *For the paraproduct we have the following two separate estimates:*

$$\begin{aligned} \|T_u v\|_{B_{p,r}^{s_1+s_2-\frac{d}{p}}} &\lesssim \|u\|_{B_{p,r_1}^{s_1}} \|v\|_{B_{p,r_2}^{s_2}} \quad \text{if } s_1 < \frac{d}{p}, \\ \|T_u v\|_{B_{p,r}^s} &\lesssim \|u\|_{L^\infty} \|v\|_{B_{p,r}^s}. \end{aligned}$$

- *For the remainder, if $s_1 + s_2 + d \min\{0, 1 - \frac{2}{p}\} > 0$, then*

$$\|R(u, v)\|_{B_{p,r}^{s_1+s_2-\frac{d}{p}}} \lesssim \|u\|_{B_{p,r_1}^{s_1}} \|v\|_{B_{p,r_2}^{s_2}}.$$

By using (30), but keeping the operation pertaining to the time variable before taking the ℓ^r -norm, we get the following estimate concerning time-dependent Besov spaces, which will be of constant use in this paper.

Proposition 3.9. *For any $(s, p, r, q, q_1, q_2) \in \mathbb{R} \times [1, \infty]^4$ with*

$$-\min\{\frac{d}{p}, \frac{d}{p'}\} < s \leq \frac{d}{p}, \quad \text{with } r = 1 \text{ if } s = \frac{d}{p}, \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

the following holds:

$$(31) \quad \|uv\|_{\tilde{L}_T^q(B_{p,r}^s)} \leq C \|u\|_{\tilde{L}_T^{q_1}(B_{p,1}^{d/p})} \|v\|_{\tilde{L}_T^{q_2}(B_{p,r}^s)}.$$

Moreover, if $s > 0$, for any $q, q_i \in [1, \infty]$ with $i = 1, \dots, 4$ such that $1/q := 1/q_1 + 1/q_2 = 1/q_3 + 1/q_4$, the following also holds:

$$\|uv\|_{\tilde{L}_T^q(B_{p,r}^s)} \leq C \left(\|u\|_{L_T^{q_1}(L^\infty)} \|v\|_{\tilde{L}_T^{q_2}(B_{p,r}^s)} + \|u\|_{\tilde{L}_T^{q_3}(B_{p,r}^s)} \|v\|_{L_T^{q_4}(L^\infty)} \right).$$

The following results pertain to the composition of functions in Besov spaces: they will be needed for estimating functions depending on the density and coming into play in our equations. The proof of Proposition 3.10 can be found in [4], while for Proposition 3.11 we refer to [13].

Proposition 3.10. *Let f be a smooth function such that $f(0) = 0$, and take $s > 0$ and $(q, p, r) \in [1, +\infty]^3$. For any $u \in C_0^\infty$, the following inequalities hold true:*

$$\|f \circ u\|_{B_{p,r}^s} \leq C \|u\|_{B_{p,r}^s}, \quad \|f \circ u\|_{\tilde{L}_T^q(B_{p,r}^s)} \leq C \|u\|_{\tilde{L}_T^q(B_{p,r}^s)},$$

where the constant C depends on s, p, r, f' and $\|u\|_{L_T^\infty(L^\infty)}$.

Proposition 3.11. *Let I be an open interval of \mathbb{R} and $F : I \rightarrow \mathbb{R}$ a smooth function. Then for all compact subset $J \subset I$, $s > 0$ and $(q, p, r) \in [1, +\infty]^3$ there exists a constant C such that for all function a valued in J and with gradient in $\tilde{L}_T^q(B_{p,r}^{s-1})$, we have $\nabla(F(a)) \in \tilde{L}_T^q(B_{p,r}^{s-1})$ and*

$$\|\nabla(F(a))\|_{\tilde{L}_T^q(B_{p,r}^{s-1})} \leq C \|\nabla a\|_{\tilde{L}_T^q(B_{p,r}^{s-1})}.$$

In the analysis of (7)₂, we will need a priori estimates in Besov spaces for the transport equation

$$(T) \quad \begin{cases} \partial_t a + v \cdot \nabla a = f, \\ a|_{t=0} = a_0. \end{cases}$$

These estimates are provided by the following proposition, whose proof may be found in e.g. [4].

Proposition 3.12. *Let $(p, r) \in [1, +\infty]^2$ and*

$$(32) \quad \sigma \geq -\min \left\{ \frac{d}{p}, \frac{d}{p'} \right\} \quad \text{or} \quad \sigma \geq -1 - \min \left\{ \frac{d}{p}, \frac{d}{p'} \right\} \quad \text{if } \operatorname{div} v = 0,$$

with strict inequality if $r < +\infty$.

Let $a_0 \in B_{p,r}^\sigma$, $f \in L^1([0, T]; B_{p,r}^\sigma)$ and v be a time dependent vector field in $L^q([0, T]; B_{\infty,\infty}^{-M})$ for some $q > 1$ and $M > 0$ such that

$$\begin{aligned} \nabla v &\in L^1([0, T]; B_{p,\infty}^{d/p} \cap L^\infty) & \text{if } \sigma < 1 + d/p, \\ \nabla v &\in L^1([0, T]; B_{p,r}^{\sigma-1}) & \text{if } \sigma > 1 + d/p, \quad \text{or } \sigma = 1 + d/p \text{ and } r = 1. \end{aligned}$$

Then Equation (T) has a unique solution a in

- *the space $\mathcal{C}([0, T]; B_{p,r}^\sigma)$ if $r < \infty$,*
- *the space $\left(\bigcap_{\sigma' < \sigma} \mathcal{C}([0, T]; B_{p,\infty}^{\sigma'}) \right) \cap \mathcal{C}_w([0, T]; B_{p,\infty}^\sigma)$ if $r = \infty$.*

Moreover, for all $t \in [0, T]$, we have

$$(33) \quad e^{-CV(t)} \|a(t)\|_{B_{p,r}^\sigma} \leq \|a_0\|_{B_{p,r}^\sigma} + \int_0^t e^{-CV(t')} \|f(t')\|_{B_{p,r}^\sigma} dt'$$

$$\text{with } V'(t) := \begin{cases} \|\nabla v(t)\|_{B_{p,\infty}^{d/p} \cap L^\infty} & \text{if } \sigma < 1 + d/p, \\ \|\nabla v(t)\|_{B_{p,r}^{\sigma-1}} & \text{if } \sigma > 1 + d/p, \quad \text{or } \sigma = 1 + d/p \text{ and } r = 1, \end{cases}$$

and, if equality holds in (32) and $r = \infty$, $V'(t) = \|\nabla v\|_{B_{p,1}^{d/p}}$.

If $a = v$ then, for all $\sigma > 0$ ($\sigma > -1$ if $\operatorname{div} v = 0$), Estimate (33) holds with $V'(t) := \|\nabla a(t)\|_{L^\infty}$.

Finally, we shall make an extensive use of energy estimates for the following elliptic equation:

$$(34) \quad -\operatorname{div}(a \nabla \Pi) = \operatorname{div} F \quad \text{in } \mathbb{R}^d$$

where $a = a(x)$ is a given suitably smooth bounded function satisfying

$$(35) \quad a_* := \inf_{x \in \mathbb{R}^d} a(x) > 0.$$

We shall use the following result (see the proof in e.g. [13]):

Lemma 3.13. *For all vector field F with coefficients in L^2 , there exists a tempered distribution Π , unique up to constant functions, such that $\nabla \Pi \in L^2$ and Equation (34) is satisfied. In addition, we have*

$$(36) \quad a_* \|\nabla \Pi\|_{L^2} \leq \|F\|_{L^2}.$$

4 Proof of Theorem 2.1

The aim of this section is proving our first well-posedness result, i.e. Theorem 2.1, for System (7). The a priori estimates we establish here will be the basis throughout the following context.

The notation f_j will always denote $\Delta_j f$, unless otherwise specified. We also notice here that by embedding results stated in Remark 3.3 and Corollary 3.7, for any $\epsilon > 0$, the following chain of embeddings holds true:

$$(37) \quad L_t^1(B_{p,1}^s) \hookleftarrow \tilde{L}_t^1(B_{p,1}^{s+\epsilon/2}) \hookleftarrow \tilde{L}_t^1(B_{p,\infty}^{s+\epsilon}),$$

which will be frequently used in our computations.

4.1 Linearized equations

In this subsection we want to establish a priori estimates for the linearized equations associated to original System (7). Subsection 4.1.1 deals with the linearized equation for the density. We first need two lemmas (see Lemma 4.1 and Lemma 4.2 below), corresponding to two types of commutator estimates respectively. Subsection 4.1.2 is dedicated to the velocity equation, where we shall pay attention to the pressure term, whose low frequencies can be controlled by a direct use of Hölder's Inequality.

Let us point out here that in order to prove the uniqueness, we will consider the difference of two solutions to System (7). Noticing that Equation (7)₂ for the velocity u will cause one derivative loss, we therefore also have to look for a priori estimates for the unknown in $B_{p,r}^s$, under a weaker condition on the indices (see (47) below), rather than (14). One notices also in this case, the “given” transport velocity field w in the linear equation for u (see (56) below) is of higher regularity than u , and hence we will use throughly the following equality (which holds due to $\operatorname{div} u \equiv 0$):

$$(38) \quad \operatorname{div} (w \cdot \nabla u) \equiv \operatorname{div} (u \cdot \nabla w - u \operatorname{div} w).$$

In addition, in proving a continuation criterion, we only need L^∞ -norm instead of $B_{p,r}^{s-1}$ -norm (with (s, r) satisfying (14)). Therefore, in the present paragraph we will give also a priori estimates involving L^∞ norm, i.e. we will not only make a rough use of the embedding $B_{p,r}^{s-1} \hookrightarrow L^\infty$.

4.1.1 A priori estimates for the density

First of all, we consider the linearized equation for the density term, which is actually the same for both ρ and ϱ . Let us suppose the right hand side of the first equation of (7) to be some scalar function f , a more general case which turns out to be useful in the sequel:

$$(39) \quad \partial_t \varrho + u \cdot \nabla \varrho - \operatorname{div} (\kappa \nabla \varrho) = f.$$

This is a parabolic equation and the treatment of the transport term $u \cdot \nabla \varrho$ is very classical. In fact, after applying the operator Δ_j , it rests to handle the commutator $[u, \Delta_j] \cdot \nabla \varrho$, for which we have the following lemma (see [4], Chapter 2).

Lemma 4.1. *Let us take the triple (s, p, r) verifying*

$$(40) \quad s > -d \min\left\{\frac{1}{p}, \frac{1}{p'}\right\} \quad \text{and} \quad (p, r) \in [1, +\infty]^2, \quad \text{with} \quad r = 1 \quad \text{if} \quad s = 1 + \frac{d}{p}.$$

Then we have (with some positive constant C depending only on d, s, p, r)

$$(41) \quad \int_0^t \|2^{js} \|[\varphi, \Delta_j] \nabla \psi\|_{L^p} \|_{\ell^r} d\tau \leq C \int_0^t \Phi^s(\tau) \|\nabla \psi\|_{B_{p,r}^{s-1}} d\tau,$$

where $\{\Phi^s\}_s$ is defined by

$$(42) \quad \Phi^s(t) = \begin{cases} \|\nabla\varphi(t)\|_{B_{p,1}^{\frac{d}{p}}} & \text{if } s \in (-d \min\{\frac{1}{p}, \frac{1}{p'}\}, 1 + \frac{d}{p}), \\ \|\nabla\varphi(t)\|_{B_{p,r}^{s-1}} & \text{otherwise.} \end{cases}$$

Moreover, if $s > 0$ we can also infer

$$(43) \quad \int_0^t \|2^{js} \|[\varphi, \Delta_j] \nabla\psi\|_{L^p} \|_{\ell^r} d\tau \leq C \int_0^t \left(\|\nabla\varphi\|_{L^\infty} \|\psi\|_{B_{p,r}^s} + \|\nabla\varphi\|_{B_{p,r}^{s-1}} \|\nabla\psi\|_{L^\infty} \right) d\tau.$$

On the other side, the term $\text{div}(\kappa \nabla \varrho)$ gives us the commutator $\text{div}([\kappa, \Delta_j] \nabla \varrho)$. We can do similarly as in [15]. But since we work in the Besov space $B_{p,r}^s$ with r not necessarily being 1, we have to resort to a new commutator estimate to get the norm $\|\cdot\|_{\tilde{L}_t^1(\cdot)}$ (instead of $\|\cdot\|_{L_t^1(\cdot)}$) of ϱ , which will be absorbed by the left hand side. Compared with Lemma 4.1, this is nothing but taking the interpolation before the integral with respect to the time variable. More precisely, we have the following lemma, whose proof can be found in Appendix.

Lemma 4.2. *Let the triple $(\sigma + 1, p, r)$ satisfy (40) with $\varepsilon > 0$ and $\sigma_1 < \sigma + 1 < \sigma_2$. Define $\theta \in]0, 1]$ such that*

$$\sigma + 1 = \theta \sigma_1 + (1 - \theta) \sigma_2.$$

Then, for any space derivative ∂_k , with $1 \leq k \leq d$, we have (with some positive constant C depending only on d, σ, p, r)

$$(44) \quad \left\| 2^{j\sigma} \int_0^t \|\partial_k([\varphi, \Delta_j] \nabla\psi)\|_{L^p} d\tau \right\|_{\ell^r} \leq \frac{C\theta}{\varepsilon^{(1-\theta)/\theta}} (\tilde{\Phi}^{\sigma+1}(t))^{\frac{1}{\theta}} \|\nabla\psi\|_{\tilde{L}_t^\infty(B_{p,r}^{\sigma_1-1})} + (1-\theta)\varepsilon \|\nabla\psi\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2-1})},$$

with $\tilde{\Phi}^{\sigma+1}$ defined by

$$(45) \quad \tilde{\Phi}^{\sigma+1}(t) = \begin{cases} \|\nabla\varphi(t)\|_{L_t^{\frac{1}{\theta}}(B_{p,\infty}^{\frac{d}{p}} \cap L^\infty)} & \text{if } \sigma \in (-1 - d \min\{\frac{1}{p}, \frac{1}{p'}\}, \frac{d}{p}), \\ \|\nabla\varphi\|_{\tilde{L}_t^{\frac{1}{\theta}}(B_{p,r}^{\sigma})} & \text{if } \sigma = \frac{d}{p} \text{ with } r = 1 \text{ or } \sigma > \frac{d}{p}, \end{cases}$$

if in addition $\sigma_i, i = 1, 2$, satisfy

$$\sigma_i < 1 + \frac{d}{p} \quad \text{if} \quad \sigma + 1 < 1 + \frac{d}{p} \quad \text{or} \quad \sigma_i > 1 + \frac{d}{p} \quad \text{if} \quad \sigma + 1 > 1 + \frac{d}{p}.$$

If moreover $\sigma > 0$ and, for some ζ_1, ζ_2 and $\eta \in]0, 1]$, we have also $\sigma + 1 = \eta \zeta_1 + (1 - \eta) \zeta_2$, then

$$(46) \quad \left\| 2^{j\sigma} \int_0^t \|\partial_k([\varphi, \Delta_j] \nabla\psi)\|_{L^p} d\tau \right\|_{\ell^r} \leq \frac{C\theta}{\varepsilon^{(1-\theta)/\theta}} \int_0^t \|\nabla\varphi\|_{L^\infty}^{1/\theta} \|\psi\|_{B_{p,r}^{\sigma_1}} d\tau + \frac{C\eta}{\varepsilon^{(1-\eta)/\eta}} \int_0^t \|\nabla\psi\|_{L^\infty}^{1/\eta} \|\nabla\varphi\|_{B_{p,r}^{\zeta_1-1}} d\tau + (1-\theta)\varepsilon \|\psi\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2})} + (1-\eta)\varepsilon \|\nabla\varphi\|_{\tilde{L}_t^1(B_{p,r}^{\zeta_2-1})}.$$

Next lemma is in the same spirit of the previous one, by view of Proposition 3.9 which gives estimates for the product of two functions (see again the Appendix for the proof).

Lemma 4.3. *Let $\sigma > 0$ and $(p, r) \in [1, +\infty]^2$. Fix $\varepsilon > 0$ and $\sigma_1 < \sigma < \sigma_2$. Define $\theta \in]0, 1]$ such that $\sigma = \theta \sigma_1 + (1 - \theta) \sigma_2$.*

There exists a constant C , depending only on d, σ, p and r , such that, for all functions $f, g \in \mathcal{S}$, one has

$$\begin{aligned} \|f g\|_{\tilde{L}_t^1(B_{p,r}^\sigma)} &\leq \frac{C \theta}{\varepsilon^{(1-\theta)/\theta}} \int_0^t \left(\|f\|_{L^\infty}^{1/\theta} \|g\|_{B_{p,r}^{\sigma_1}} + \|g\|_{L^\infty}^{1/\theta} \|f\|_{B_{p,r}^{\sigma_1}} \right) d\tau \\ &\quad + (1-\theta) \varepsilon \left(\|f\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2})} + \|g\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2})} \right). \end{aligned}$$

Remark 4.4. Let us point out that the previous results can be also generalized in the following sense. For instance, as in commutator's estimate (46), take this time θ and $\eta \in]0, 1]$ such that

$$\sigma = \theta \sigma_1 + (1-\theta) \sigma_2 = \eta \zeta_1 + (1-\eta) \zeta_2,$$

with again $\sigma_1 < \sigma_2$ and $\zeta_1 < \zeta_2$. Then we have, for any ε and $\delta \in]0, 1[$,

$$\begin{aligned} \|f g\|_{\tilde{L}_t^1(B_{p,r}^\sigma)} &\leq \frac{C \theta}{\varepsilon^{(1-\theta)/\theta}} \int_0^t \|f\|_{L^\infty}^{1/\theta} \|g\|_{B_{p,r}^{\sigma_1}} d\tau + \frac{C \eta}{\delta^{(1-\eta)/\eta}} \int_0^t \|g\|_{L^\infty}^{1/\eta} \|f\|_{B_{p,r}^{\zeta_1}} d\tau + \\ &\quad + (1-\theta) \varepsilon \|f\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2})} + (1-\eta) \delta \|g\|_{\tilde{L}_t^1(B_{p,r}^{\zeta_2})}, \end{aligned}$$

for a constant C which depends only on the dimension d and on the indices σ, p and r .

Let us come back to the linear equation (39). Thanks to inequality (41) and (46) in Lemmas 4.1 and 4.2, the following a priori estimates hold true, similar to Proposition 4.1 in [15].

Proposition 4.5. *Let the triple (s, p, r) verify*

$$(47) \quad s \geq \frac{d}{p}, \quad p \in]1, +\infty[, \quad r \in [1, +\infty], \quad \text{with} \quad r = 1 \quad \text{if} \quad s = 1 + \frac{d}{p} \quad \text{or} \quad \frac{d}{p}.$$

Let u be a smooth divergence-free vector field such that $\nabla u \in B_{p,r}^{s-1}$, and κ a smooth real function such that $\kappa \geq \kappa_* > 0$, $\nabla \kappa \in B_{p,r}^s$. Moreover, let us assume the external force $f \in \tilde{L}_{T_0}^1(B_{p,r}^s)$.

Then there exists a positive constant C_1 depending only on κ_*, d, s, p, r such that, for every smooth solution ϱ of (39) with initial condition $\varrho|_{t=0} = \varrho_0$, the following estimate holds true for every $t \in [0, T_0]$:

$$(48) \quad \|\varrho\|_{\tilde{L}_t^\infty(B_{p,r}^s)} + \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})} \leq C_1 e^{C_1 K(t)} \left(\|\varrho_0\|_{B_{p,r}^s} + \|f\|_{\tilde{L}_t^1(B_{p,r}^s)} \right),$$

where we have defined $K(0) = 0$ and

$$(49) \quad K'(t) := 1 + \|\nabla u\|_{B_{p,1}^{\frac{d}{p}} \cap B_{p,r}^{s-1}} + \|\nabla \kappa\|_{B_{p,r}^s} + \|\nabla \kappa\|_{L^\infty}^2.$$

Proof. We follow the proof of Proposition 4.1 in [15]: we apply the operator Δ_j to the equation, we integrate first in space and then in time; then we use the commutator estimates and Gronwall's Inequality to get the result.

Applying the localization operator Δ_j to Equation (39) yields

$$(50) \quad \partial_t \varrho_j + u \cdot \nabla \varrho_j - \operatorname{div}(\kappa \nabla \varrho_j) = f_j + \mathcal{R}_j^1 - \mathcal{R}_j^2,$$

where we have set $\varrho_j := \Delta_j \varrho$, $f_j := \Delta_j f$, $\mathcal{R}_j^1 := [u, \Delta_j] \cdot \nabla \varrho$ and $\mathcal{R}_j^2 := \operatorname{div}[\kappa, \Delta_j] \nabla \varrho$.

According to the following Bernstein type inequality given in Appendix B of [13] (which holds true for any $p \in (1, \infty)$ and $j \geq 0$)

$$(51) \quad \begin{aligned} - \int_{\mathbb{R}^d} \operatorname{div}(\kappa \nabla \varrho_j) |\varrho_j|^{p-2} \varrho_j &= (p-1) \int_{\mathbb{R}^d} \kappa |\nabla \varrho_j|^2 |\varrho_j|^{p-2} \\ &\geq C(d, p, \kappa_*) 2^{2j} \int_{\mathbb{R}^d} |\varrho_j|^p, \end{aligned}$$

and taking the L^2 scalar product between (50) and $|\varrho_j|^{p-2}\varrho_j$ gives us, for all $j \geq 0$,

$$\frac{d}{dt} \|\varrho_j\|_{L^p}^p + C 2^{2j} \|\varrho_j\|_{L^p}^p \leq C \|\varrho_j\|_{L^p}^{p-1} \left(\|f_j\|_{L^p} + \|\mathcal{R}_j^1\|_{L^p} + \|\mathcal{R}_j^2\|_{L^p} \right).$$

Now, we divide both members of the previous inequality by $\|\varrho_j\|_{L^p}^{p-1}$ and we integrate in time; then we multiply by 2^{js} and take the ℓ^r norm with respect to j . Hence, for all $t \in [0, T_0]$ we get:

$$(52) \quad \begin{aligned} \|\varrho\|_{\tilde{L}_t^\infty(B_{p,r}^s)} + \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})} &\leq C \left(\|\varrho_0\|_{B_{p,r}^s} + 2^{-(s+2)} \|\Delta_{-1}\varrho\|_{L_t^1(L^p)} + \|f\|_{\tilde{L}_t^1(B_{p,r}^s)} + \right. \\ &\quad \left. + \left\| 2^{js} \int_0^t \|\mathcal{R}_j^1(\tau)\|_{L^p} d\tau \right\|_{\ell^r} + \left\| 2^{js} \int_0^t \|\mathcal{R}_j^2(\tau)\|_{L^p} d\tau \right\|_{\ell^r} \right). \end{aligned}$$

The low-frequency term $\Delta_{-1}\varrho$ can be easily bounded in $[0, T_0]$. As a matter of fact, by definition we immediately have, for all t ,

$$(53) \quad 2^{-(s+2)} \|\Delta_{-1}\varrho\|_{L_t^1(L^p)} \leq C \int_0^t \|\varrho\|_{B_{p,r}^s} d\tau.$$

For the first commutator term, by Lemma 4.1, we get

$$(54) \quad \int_0^t \left\| 2^{js} \|\mathcal{R}_j^1\|_{L^p} \right\|_{\ell^r} d\tau \lesssim \begin{cases} \int_0^t \|\nabla u\|_{B_{p,1}^{\frac{d}{p}}} \|\varrho\|_{B_{p,r}^s} & \text{if } s \in (-d \min\{\frac{d}{p}, \frac{d}{p'}\}, 1 + \frac{d}{p}), \\ \int_0^t \|\nabla u\|_{B_{p,r}^{s-1}} \|\varrho\|_{B_{p,r}^s} & \text{if } s \geq 1 + \frac{d}{p}, \text{ and } r = 1 \text{ if } s = 1 + \frac{d}{p}. \end{cases}$$

For the second commutator term, instead, we apply (46) with $\sigma = \sigma_1 = s$, $\sigma_2 = s + 2$ and $\eta = 1$, $\zeta_1 = s + 1$, and we get, for any small $\varepsilon > 0$ and some corresponding constant C_ε ,

$$(55) \quad \left\| 2^{js} \int_0^t \|\mathcal{R}_j^2\|_{L^p} \right\|_{\ell^r} d\tau \lesssim \varepsilon \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})} + C_\varepsilon \int_0^t \left(\|\nabla \kappa\|_{B_{p,r}^s} + \|\nabla \kappa\|_{L^\infty}^2 \right) \|\varrho\|_{B_{p,r}^s} d\tau$$

for all $s \geq d/p$, with $r = 1$ if $s = d/p$, such that embedding $B_{p,r}^{d/p} \hookrightarrow L^\infty$ holds true.

We put (53), (54) and (55) into (52), choose sufficiently small ε and apply Gronwall's lemma to arrive at (48). \square

4.1.2 The linearized equation for velocity field and pressure term

The linearized equation for the velocity reads

$$(56) \quad \begin{cases} \partial_t u + w \cdot \nabla u + \lambda \nabla \pi = h, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where the initial datum u_0 , the transport vector field w , the coefficient λ and the source term h are all smooth and decrease rapidly at infinity. We have the following result.

Proposition 4.6. *Let*

$$(57) \quad s > \frac{d}{p} - \frac{d}{4}, \quad p \in [2, 4], \quad r \in [1, +\infty], \quad \text{with} \quad r = 1 \quad \text{if} \quad s = \frac{d}{p} \quad \text{or} \quad 1 + \frac{d}{p}.$$

Suppose $0 < \lambda_* \leq \lambda(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$.

Then there exists a positive constant C_2 such that for any smooth solution u of (56), the following estimates hold true:

$$(58) \quad \|u\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq C_2 e^{C_2 W(t)} \left(\|u_0\|_{B_{p,r}^s} + \|h\|_{\tilde{L}_t^1(B_{p,r}^s) \cap L_t^1(L^2)} \right),$$

$$(59) \quad \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s) \cap L_t^1(L^2)} \leq C_2 \left(\|h\|_{\tilde{L}_t^1(B_{p,r}^s) \cap L_t^1(L^2)} + W(t) \|u\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \right),$$

where $W(0) = 0$ and

$$(60) \quad W'(t) = \begin{cases} \|\nabla w(t)\|_{B_{p,1}^{\frac{d}{p}}} & \text{if } s \in (\frac{d}{p} - \frac{d}{4}, 1 + \frac{d}{p}], \\ \|\nabla w(t)\|_{B_{p,r}^{s-1}} & \text{if } s > 1 + \frac{d}{p}, \end{cases}$$

and the constant C_2 depends on $d, p, s, r, \lambda_*, \lambda^*$ with

$$(61) \quad \lambda^*(t) := \|\lambda\|_{L_t^\infty(L^\infty)} + \begin{cases} \|\nabla \lambda\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{d}{p}})} & \text{if } s \in (\frac{d}{p} - \frac{d}{4}, 1 + \frac{d}{p}], \\ \|\nabla \lambda\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} & \text{if } s > 1 + \frac{d}{p}. \end{cases}$$

Proof. From Proposition 3.12 we easily get the following estimate for u with $W(t)$ defined by (60):

$$(62) \quad \|u(t)\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq e^{C(d,p,s)W(t)} \left(\|u_0\|_{B_{p,r}^s} + \|h - \lambda \nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} \right).$$

By decomposing $\lambda \nabla \pi$ into

$$(\Delta_{-1} \lambda) \nabla \pi + ((Id - \Delta_{-1}) \lambda) \nabla \pi,$$

and by product estimates given in Proposition 3.9, we have

$$(63) \quad \|\lambda \nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} \leq C \lambda^* \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)}, \text{ if } s > -\min\{\frac{d}{p}, \frac{d}{p'}\}, \text{ with } r = 1 \text{ if } s = \frac{d}{p},$$

with λ^* defined by (61). Hence it is sufficient to estimate $\|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)}$, in order to obtain Estimate (58).

By view of Equality (38), applying the operator “div” to the first equation of (56) yields the elliptic equation for π :

$$(64) \quad \operatorname{div}(\lambda \nabla \pi) = \operatorname{div}(h - w \cdot \nabla u) = \operatorname{div}(h - u \cdot \nabla w + u \operatorname{div} w).$$

Similar as to get (52), we apply the localization operator Δ_j to (64), we multiply $|\pi_j|^{p-2} \pi_j$, then we intergrate with respect to space variable and we use the Bernstein type Inequality (51) and Hölder’s Inequality, and we finally find

$$\lambda_* 2^{2j} \|\pi_j\|_{L^p} \lesssim \|\Delta_j \operatorname{div}(h - w \cdot \nabla u)\|_{L^p} + \|\operatorname{div}[\lambda, \Delta_j] \nabla \pi\|_{L^p}, \quad \forall j \geq 0.$$

On the other hand, for $p \geq 2$, Bernstein’s Inequalities given in Lemma 3.6 entails

$$\|\Delta_{-1} \nabla \pi\|_{L^p} \leq C \|\nabla \pi\|_{L^2}, \quad 2^j \|\pi_j\| \sim \|\nabla \pi_j\|, \quad \forall j \geq 0,$$

which ensure that

$$(65) \quad \begin{aligned} \lambda_* \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} &\lesssim \lambda_* \|\nabla \pi\|_{L_t^1(L^2)} + \|\operatorname{div}(h - w \cdot \nabla u)\|_{\tilde{L}_t^1(B_{p,r}^{s-1})} \\ &\quad + \|2^{j(s-1)} \|\operatorname{div}[\lambda, \Delta_j] \nabla \pi\|_{L_t^1(L^p)}\|_{\ell^r}. \end{aligned}$$

Let us focus on the above commutator term for a while. Commutator Estimate (44) with $\theta = 1$ in Lemma 4.2 implies that it can be controlled by

$$\|2^{j(s-1)} \operatorname{div} [\lambda, \Delta_j] \nabla \pi\|_{L_t^1(L^p)} \lesssim \lambda^* \|\nabla \pi\|_{L_t^1(B_{p,r}^{s-1})}, \text{ if } s > -\min\{\frac{d}{p}, \frac{d}{p'}\}.$$

Motivated by the embeddings (37) and

$$L^2 \hookrightarrow B_{p,\infty}^{\frac{d}{p}-\frac{d}{2}}, \quad \forall p \geq 2,$$

we apply interpolation inequalities between Besov spaces, obtaining

$$\|\nabla \pi\|_{L_t^1(B_{p,r}^{s-1})} \lesssim \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^{s+\epsilon-1})} \lesssim \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,\infty}^{\frac{d}{p}-\frac{d}{2}})}^{1-\eta} \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)}^\eta \lesssim \|\nabla \pi\|_{L_t^1(L^2)}^{1-\eta} \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)}^\eta,$$

for some $\epsilon \in (0, 1)$, such that

$$\eta = \frac{s + \epsilon - 1 - (d/p - d/2)}{s - (d/p - d/2)} \in (0, 1), \quad \text{i.e.} \quad s > \frac{d}{p} - \frac{d}{2} + 1 - \epsilon.$$

Therefore, one can use directly Young's Inequality on the above bound for commutator term

$$\|2^{j(s-1)} \operatorname{div} [\lambda, \Delta_j] \nabla \pi\|_{L_t^1(L^p)} \leq C(d, s, p, r, \epsilon)(\lambda^*)^{1/(1-\eta)} \|\nabla \pi\|_{L_t^1(L^2)} + \varepsilon \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)},$$

such that for some small enough $\varepsilon > 0$, Estimate (65) becomes

$$(66) \quad \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} \leq C(d, s, p, r, \epsilon, \lambda_*)(\lambda^*)^{1/(1-\eta)} \|\nabla \pi\|_{L_t^1(L^2)} + C \|\operatorname{div} (h - w \cdot \nabla u)\|_{\tilde{L}_t^1(B_{p,r}^{s-1})}.$$

Thanks to Lemma 3.13, we have already got, by Equation (64),

$$\begin{aligned} \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} &\leq C(d, s, p, r, \lambda_*, \lambda^*) \|h - u \cdot \nabla w + u \operatorname{div} w\|_{L_t^1(L^2)} \\ &\quad + C \|\operatorname{div} (h - w \cdot \nabla u)\|_{\tilde{L}_t^1(B_{p,r}^{s-1})}. \end{aligned}$$

It rests to dealing with

$$\|u \cdot \nabla w\|_{L_t^1(L^2)}, \quad \|u \operatorname{div} w\|_{L_t^1(L^2)} \quad \text{and} \quad \|\operatorname{div} (w \cdot \nabla u)\|_{\tilde{L}_t^1(B_{p,r}^{s-1})}.$$

We can easily find, by Hölder's Inequality and embedding results, that

$$\|u \cdot \nabla w\|_{L_t^1(L^2)} \leq \int_0^t \|u\|_{L^4} \|\nabla w\|_{L^4} \lesssim \int_0^t \|u\|_{B_{4,\infty}^{s_1}} \|\nabla w\|_{B_{4,\infty}^{s_2}} d\tau, \quad \forall s_1, s_2 > 0.$$

Hence for $p \leq 4$, $s > \frac{d}{p} - \frac{d}{4}$, we have

$$(67) \quad \|u \cdot \nabla w\|_{L_t^1(L^2)} \lesssim \int_0^t \|u\|_{B_{p,r}^s} \|\nabla w\|_{B_{p,\infty}^{\frac{d}{p}}} d\tau.$$

The term $u \operatorname{div} w$ is actually analogous to the previous one.

On the other hand, recalling the divergence-free condition over u , it is easy to decompose $\|\operatorname{div} (w \cdot \nabla u)\|_{B_{p,r}^{s-1}}$ into

$$(68) \quad \|T_{\partial_i w^j} \partial_j u^i + T_{\partial_j u^i} \partial_i w^j + \operatorname{div} (R(w^j, \partial_j u))\|_{B_{p,r}^{s-1}},$$

which can be controlled, according to Proposition 3.8, by

$$W'(t) \|\nabla u\|_{B_{p,r}^{s-1}}, \quad \text{if } s > -\min\{\frac{d}{p}, \frac{d}{p'}\},$$

with $W'(t)$ defined by (60).

To conclude, in the case $p \in [2, 4]$, $s > \frac{d}{p} - \frac{d}{4}$, estimate (59) holds and so does estimation (58), by view of (62) and Gronwall's Inequality. \square

4.2 Proof of the existence

In this section we will follow the standard procedure to prove the local existence of the solution to System (7): we construct a sequence of approximate solutions, we show uniform bounds and we prove the convergence to a unique solution. We note here some related key points:

- Since we admit also large initial density ρ_0 , we will introduce the large linear part ρ_L of the solution ρ as the solution to the free heat equation with the same initial datum, which is explicit and has positive lower bound. The remainder part $\tilde{\rho} := \rho - \rho_L$ is small and hence easier to handle.
- In the convergence part, since there are quantities like $\nabla u^n \in B_{p,r}^{s-1}$ appearing in the source term of the equation for the “difference” sequence δu^n , we first show that the built sequence converges to a solution in a space with lower regularity (i.e. in space $E_{p,r}^{d/p}(T)$, see (17)) and hence in the desired Besov space because of the uniform bounds for the sequence.
- Whenever the indices s, r satisfy $s > \frac{d}{p}$ or $s \geq \frac{d}{p}$, $r = 1$, the inequalities

$$(69) \quad \|uv\|_{B_{p,r}^s} \leq C(d, s, p, r) \|u\|_{B_{p,r}^s} \|v\|_{B_{p,r}^s}, \quad \|f(\rho)\|_{B_{p,r}^s} \leq C(f', \|\rho\|_{L^\infty}) \|\varrho\|_{B_{p,r}^s}, \quad \text{with } f(1) = 0,$$

and their time-dependent version

$$\begin{aligned} \|uv\|_{\tilde{L}_t^q(B_{p,r}^s)} &\leq C(d, s, p, r) \|u\|_{\tilde{L}_t^{q_1}(B_{p,r}^s)} \|v\|_{\tilde{L}_t^{q_2}(B_{p,r}^s)}, \quad \text{with } \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \\ \|f(\rho)\|_{\tilde{L}_t^q(B_{p,r}^s)} &\leq C(f', \|\rho\|_{L_t^\infty(L^\infty)}) \|\varrho\|_{\tilde{L}_t^q(B_{p,r}^s)}, \quad \text{with } f(1) = 0, \end{aligned}$$

always hold true, thanks to Propositions 3.9 and 3.10.

In the coming proof, C always denotes some “harmless” constant which may depend only on $\rho_*, \rho^*, d, s, p, r$ and for simplicity we assume $\kappa(1) = \lambda(1) = 1$.

4.2.1 Step 1 – Construction of a sequence of approximate solutions

In this step, we take (s, p, r) such that Conditions (14) and (15) hold true. Let us introduce the approximate solution sequence $\{(\varrho^n, u^n, \nabla \pi^n)\}_{n \geq 0}$ by induction.

Without loss of generality we can assume

$$(70) \quad \frac{\rho_*}{2} \leq S_n \rho_0, \quad \forall n \in \mathbb{N};$$

then, first of all we set $(\varrho^0, u^0, \nabla \pi^0) := (S_0 \varrho_0, S_0 u_0, 0)$. Let us note that these functions are smooth and fast decaying at infinity.

Now, we assume by induction that the triplet $(\varrho^{n-1}, u^{n-1}, \nabla \pi^{n-1})$ of smooth and fast decaying functions has been constructed. Let us suppose also that there exists a sufficiently small parameter τ (to be determined later), a positive time T^* (which may depend on τ) and a positive constant C_M (which may depend on M) such that

$$(71) \quad \frac{\rho_*}{2} \leq \rho^{n-1} := 1 + \varrho^{n-1}, \quad \|\varrho^{n-1}\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} \leq C_M, \quad \|\varrho^{n-1}\|_{\tilde{L}_{T^*}^2(B_{p,r}^{s+1}) \cap \tilde{L}_{T^*}^1(B_{p,r}^{s+2})} \leq \tau,$$

$$(72) \quad \|u^{n-1}\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} \leq C_M, \quad \|u^{n-1}\|_{L_{T^*}^2(B_{p,r}^s) \cap L_{T^*}^1(B_{p,r}^s)} \leq \tau, \quad \|\nabla \pi^{n-1}\|_{\tilde{L}_{T^*}^1(B_{p,r}^s) \cap L_{T^*}^1(L^2)} \leq \tau^{1/2}.$$

Let us immediately remark that the above estimates (71) and (72) obviously hold true for $(\varrho^0, u^0, \nabla \pi^0)$, if T^* is assumed small enough.

Now we define $(\varrho^n, u^n, \nabla \pi^n)$ as the unique smooth global solution of the linear system

$$(73) \quad \begin{cases} \partial_t \varrho^n + u^{n-1} \cdot \nabla \varrho^n - \operatorname{div}(\kappa^{n-1} \nabla \varrho^n) = 0, \\ \partial_t u^n + (u^{n-1} + \nabla b^{n-1}) \cdot \nabla u^n + \lambda^{n-1} \nabla \pi^n = h^{n-1}, \\ \operatorname{div} u^n = 0, \\ (\varrho^n, u^n)|_{t=0} = (S_n \varrho_0, S_n u_0), \end{cases}$$

where we have set

$$b^{n-1} = b(\rho^{n-1}), \quad \kappa^{n-1} = \kappa(\rho^{n-1}), \quad \lambda^{n-1} = \lambda(\rho^{n-1}), \quad h^{n-1} = h(\rho^{n-1}, u^{n-1}).$$

We want to show that also the triplet $(\varrho^n, u^n, \nabla \pi^n)$ verifies (71) and (72).

First of all, keeping in mind (70), we apply the maximum principle to the linear parabolic equation for ϱ^n , yielding $\rho^n := 1 + \varrho^n \geq \rho_*/2$.

Now, we introduce ϱ_L as the solution of the heat equation with the initial datum $\varrho_0 \in B_{p,r}^s$:

$$\begin{cases} \partial_t \varrho_L - \Delta \varrho_L = 0 \\ (\varrho_L)|_{t=0} = \varrho_0. \end{cases}$$

Then, it's easy to see (e.g. applying Proposition 4.5) that the global solution $\varrho_L = e^{t\Delta} \varrho_0$ satisfies, for any positive time $T < +\infty$ and some constant $C_T > 0$ depending on T ,

$$(74) \quad \|\varrho_L\|_{\tilde{L}_T^\infty(B_{p,r}^s)} + \|\varrho_L\|_{\tilde{L}_T^1(B_{p,r}^{s+2})} \leq C_T \|\varrho_0\|_{B_{p,r}^s}.$$

We claim that, given $\tau > 0$, we can choose $T^* < +\infty$ such that one has

$$(75) \quad \|\varrho_L\|_{\tilde{L}_{T^*}^2(B_{p,r}^{s+1}) \cap \tilde{L}_{T^*}^1(B_{p,r}^{s+2})} \leq \tau^2.$$

Indeed, we can write

$$\|\varrho_L\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s+2})} = \left\| \left(2^{j(s+2)} \int_0^{T^*} \|e^{t\Delta} \Delta_j \varrho_0\|_{L^p} dt \right)_j \right\|_{\ell^r}.$$

Now, we use the fact that (see e.g. Chapter 2 of [4] for the proof) for any $j \geq -1$, the operator $e^{t\Delta} \Delta_j$ belongs to $\mathcal{L}(L^p) := \{A : L^p \rightarrow L^p \text{ linear and bounded}\}$. Furthermore, for $j \geq 0$, its norm can be bounded in the following way:

$$\|e^{t\Delta} \Delta_j\|_{\mathcal{L}(L^p)} \leq C e^{-Ct 2^{2j}}.$$

On the other side, we can decompose ϱ_0 into low-frequency, large part $\varrho_{0,l}$ and high-frequency, small part $\varrho_{0,h}$, such that, for some N large enough,

$$\widehat{\varrho}_{0,l} = \widehat{\varrho}_0 \text{ on } 2^N \mathcal{B}, \quad \widehat{\varrho}_{0,l} = 0 \text{ outside } 2^N \mathcal{B}, \quad \varrho_{0,l} + \varrho_{0,h} = \varrho_0, \quad \|\varrho_{0,h}\|_{B_{p,r}^s} \leq \tau^3,$$

where \mathcal{B} is the unitary ball centered at the origin. Therefore, once one notices that

$$\left\| \left(\int_0^{T^*} 2^{2j} e^{-Ct 2^{2j}} dt \right)_{j \leq N} \right\|_{\ell^\infty} \leq C(1 - e^{-C2^{2N} T^*}), \quad \left\| \left(\int_0^{T^*} 2^{2j} e^{-Ct 2^{2j}} dt \right)_{j > N} \right\|_{\ell^\infty} \leq C,$$

we get

$$\begin{aligned} \|\varrho_L\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s+2})} &\leq C(1 - e^{-C2^{2N} T^*}) \left\| (2^{js} \|\Delta_j \varrho_0\|_{L^p})_{j \leq N} \right\|_{\ell^r} + C \left\| (2^{js} \|\Delta_j \varrho_0\|_{L^p})_{j > N} \right\|_{\ell^r} \\ &\leq C(1 - e^{-C2^{2N} T^*}) \|\varrho_0\|_{B_{p,r}^s} + C\tau^3. \end{aligned}$$

So one can choose sufficiently small T^* such that $\|\varrho_L\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s+2})} \leq \tau^2$. The term $\|\varrho_L\|_{\tilde{L}_{T^*}^2(B_{p,r}^{s+1})}$ can be handled in the same way or by interpolation inequality. Hence, our claim (75) is proved.

Now we define the sequence $\varrho_L^n = S_n \varrho_L$: it too solves the free heat equation, but with initial data $S_n \varrho_0$. Hence, it too satisfies (74) and (75), for some $T^* > 0$ independent of n .

We next consider the small remainder $\bar{\varrho}^n := \varrho^n - \varrho_L^n$. We claim that it fulfills, for all $n \in \mathbb{N}$,

$$(76) \quad \|\bar{\varrho}^n\|_{\tilde{L}_{T^*}^2(B_{p,r}^{s+1})} \leq \|\bar{\varrho}^n\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} + \|\bar{\varrho}^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s+2})} \leq \tau^{3/2}.$$

In fact, $\bar{\varrho}^n = \varrho^n - \varrho_L^n$ solves

$$(77) \quad \begin{cases} \partial_t \bar{\varrho}^n + u^{n-1} \cdot \nabla \bar{\varrho}^n - \operatorname{div}(\kappa^{n-1} \nabla \bar{\varrho}^n) = -u^{n-1} \cdot \nabla \varrho_L^n + \operatorname{div}((\kappa^{n-1} - 1) \nabla \varrho_L^n), \\ \bar{\varrho}^n|_{t=0} = 0. \end{cases}$$

So, if we define

$$K^{n-1}(t) := t + \|\nabla u^{n-1}\|_{L_t^1(B_{p,r}^{s-1})} + \|\nabla \kappa^{n-1}\|_{L_t^2(L^\infty)}^2 + \|\nabla \kappa^{n-1}\|_{L_t^1(B_{p,r}^s)},$$

by (48) we infer that

$$\|\bar{\varrho}^n\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s) \cap \tilde{L}_{T^*}^1(B_{p,r}^{s+2})} \leq C e^{CK^{n-1}(T^*)} \left\| -u^{n-1} \cdot \nabla \varrho_L^n + \operatorname{div}((\kappa^{n-1} - 1) \nabla \varrho_L^n) \right\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)}.$$

Now, inductive assumptions (71) and (72) tell us that $K^{n-1}(T^*) \leq C\tau$ if $T^* \leq \tau$ and τ is small enough. One also refers to Proposition 3.9 and Proposition 3.10 to bound

$$\left\| -u^{n-1} \cdot \nabla \varrho_L^n + \operatorname{div}((\kappa^{n-1} - 1) \nabla \varrho_L^n) \right\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)}$$

by the quantity (up to a constant factor)

$$\begin{aligned} & \|u^{n-1}\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} \|\nabla \varrho_L^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)} \\ & + \|\varrho^{n-1}\|_{\tilde{L}_{T^*}^2(B_{p,r}^{s+1})} \|\nabla \varrho_L^n\|_{L_{T^*}^2(L^\infty)} + \|\varrho^{n-1}\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} \|\nabla \varrho_L^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s+1})}, \end{aligned}$$

and hence by $C_M \tau^2$. Therefore, (76) is proved, and hence (71) holds for $\varrho^n = \bar{\varrho}^n + \varrho_L^n$, for sufficiently small τ .

We now want to get (72). Our starting point is Proposition 4.6, and in particular Inequalities (58) and (59). The estimates for products and for functions of the density, together with the embedding result (37), give us

$$W^{n-1}(T^*) := \int_0^{T^*} \|\nabla u^{n-1} + \nabla^2 b^{n-1}\|_{B_{p,r}^{s-1}} \leq \|u^{n-1}\|_{L_{T^*}^1(B_{p,r}^s)} + C \|\varrho^{n-1}\|_{L_{T^*}^1(B_{p,r}^{s+1})} \leq C\tau,$$

$$\|h^{n-1}\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)} \leq C(\|u^{n-1}\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} + 1)(\|\varrho^{n-1}\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s+2})} + \|\varrho^{n-1}\|_{\tilde{L}_{T^*}^2(B_{p,r}^{s+1})}^2) \leq C\tau,$$

and also

$$\|h^{n-1}\|_{L_{T^*}^1(L^2)} \leq C(\|u^{n-1}\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} + \|\varrho^{n-1}\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)}) \|\varrho^{n-1}\|_{L_{T^*}^1(B_{p,r}^{s+1})} \leq C\tau,$$

by use of these two inequalities:

$$\|\cdot\|_{L^4} \lesssim \|\cdot\|_{B_{p,\infty}^{s-1}} \quad \text{and} \quad \|\nabla \varrho^{n-1}\|_{L^\infty} \leq \|\varrho^{n-1}\|_{B_{p,r}^s}.$$

Hence applying (58) and (59) to the system (73) we deduce

$$\begin{aligned} \|u^n\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} & \leq C(\|S_n u_0\|_{B_{p,r}^s} + C\tau) \leq C_M, \\ \|\nabla \pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^s) \cap L_{T^*}^1(L^2)} & \leq C\tau + C_M C\tau. \end{aligned}$$

Hence also (72) holds true for small τ and T^* .

4.2.2 Step 2 – Convergence of the sequence

In this step we will consider the “difference” sequence

$$(\delta \varrho^n, \delta u^n, \nabla \delta \pi^n) := (\varrho^n - \varrho^{n-1}, u^n - u^{n-1}, \nabla \pi^n - \nabla \pi^{n-1}), \quad \forall n \geq 1.$$

Since taking the difference of the transport term $(u + \nabla b) \cdot \nabla u$ will cause one derivative loss because of $\nabla u^n \in B_{p,r}^{s-1}$, we will consider the above difference sequence in the Banach space $E_{p,1}^{\frac{d}{p}}(T^*)$ (recall (17) for its definition).

First of all, by System (73), $(\delta \varrho^n, \delta u^n, \nabla \delta \pi^n)$ solves

$$(78) \quad \begin{cases} \partial_t \delta \varrho^n + u^{n-1} \cdot \nabla \delta \varrho^n - \operatorname{div}(\kappa^{n-1} \nabla \delta \varrho^n) = F^{n-1}, \\ \partial_t \delta u^n + (u^{n-1} + \nabla b^{n-1}) \cdot \nabla \delta u^n + \lambda^{n-1} \nabla \delta \pi^n = H^{n-1}, \\ \operatorname{div} \delta u^n = 0, \\ (\delta \varrho^n, \delta u^n)|_{t=0} = (\Delta_n \varrho_0, \Delta_n u_0), \end{cases}$$

where

$$\begin{aligned} F^{n-1} &= -\delta u^{n-1} \cdot \nabla \varrho^{n-1} + \operatorname{div}((\kappa^{n-1} - \kappa^{n-2}) \nabla \varrho^{n-1}), \\ H^{n-1} &= h^{n-1} - h^{n-2} - (\delta u^{n-1} + \nabla \delta b^{n-1}) \cdot \nabla u^{n-1} - (\lambda^{n-1} - \lambda^{n-2}) \nabla \pi^{n-1}. \end{aligned}$$

Next we apply a priori estimates (48), (58) and (59) with $s = d/p$, $p \in [2, 4]$ and $r = 1$, to $\delta \varrho^n$ and $(\delta u^n, \nabla \delta \pi^n)$ respectively. The use of uniform bounds (71) and (72) for the approximated solutions sequence gives us

$$(79) \quad \|\delta \varrho^n\|_{L_{T^*}^\infty(B_{p,1}^{d/p}) \cap L_{T^*}^1(B_{p,1}^{d/p+2})} \leq C \left(\|\Delta_n \varrho_0\|_{B_{p,1}^{d/p}} + \|F^{n-1}\|_{L_{T^*}^1(B_{p,1}^{d/p})} \right),$$

$$(80) \quad \|\delta u^n\|_{\tilde{L}_{T^*}^\infty(B_{p,1}^{d/p})} + \|\nabla \delta \pi^n\|_{L_{T^*}^1(B_{p,1}^{d/p} \cap L^2)} \leq C \left(\|\Delta_n u_0\|_{B_{p,1}^{d/p}} + \|H^{n-1}\|_{L_{T^*}^1(B_{p,1}^{d/p} \cap L^2)} \right).$$

Now we use the following fact, coming from Proposition 3.10: for any smooth function $f = f(\rho)$ and a given sequence $(\rho^m)_m$, if we denote $f^m := f(\rho^m)$ and $\delta f^m := f^m - f^{m-1}$, then

$$\|\delta f^m\|_{B_{p,1}^s} \leq C(\|\varrho^m\|_{B_{p,1}^s}, \|\varrho^{m-1}\|_{B_{p,1}^s}) \|\delta \varrho^m\|_{B_{p,1}^s}, \quad \text{if } s \geq \frac{d}{p}.$$

Therefore, one easily gets

$$\begin{aligned} \|F^{n-1}\|_{L_{T^*}^1(B_{p,1}^{d/p})} &\leq C(\|\delta \varrho^{n-1}\|_{L_{T^*}^2(B_{p,1}^{d/p+1})}, \|\varrho^{n-1}\|_{L_{T^*}^2(B_{p,1}^{d/p+1})}) \\ &\quad + \int_0^{T^*} \|\delta u^{n-1}\|_{B_{p,1}^{d/p}} \|\nabla \varrho^{n-1}\|_{B_{p,1}^{d/p}} + \|\delta \varrho^{n-1}\|_{B_{p,1}^{d/p}} \|\varrho^{n-1}\|_{B_{p,1}^{d/p+2}}, \\ \|H^{n-1}\|_{L_{T^*}^1(B_{p,1}^{d/p} \cap L^2)} &\leq C(\|\delta \varrho^{n-1}\|_{L_{T^*}^1(B_{p,1}^{d/p+2})}) \\ &\quad + \|\delta \varrho^{n-1}\|_{L_{T^*}^2(B_{p,1}^{d/p+1})} \left(\|(\varrho^{n-1}, \varrho^{n-2})\|_{L_{T^*}^2(B_{p,1}^{d/p+2})} + \|(u^{n-1}, u^{n-2})\|_{L_{T^*}^2(B_{p,1}^{d/p+1})} \right) \\ &\quad + \int_0^{T^*} \|\delta \varrho^{n-1}\|_{B_{p,1}^{d/p}} \|\nabla \pi^{n-1}\|_{B_{p,1}^{d/p}} + \|\delta u^{n-1}\|_{B_{p,1}^{d/p}} (\|\varrho^{n-1}\|_{B_{p,1}^{d/p+2}} + \|u^{n-1}\|_{B_{p,1}^{d/p+1}}). \end{aligned}$$

Hence putting the uniform estimate (71) into (79) entails

$$\|\delta \varrho^n\|_{L_{T^*}^\infty(B_{p,1}^{d/p}) \cap L_{T^*}^1(B_{p,1}^{d/p+2})} \leq C \left(\|\Delta_n \varrho_0\|_{B_{p,1}^{d/p}} + \tau \|\delta \varrho^{n-1}\|_{L_{T^*}^2(B_{p,1}^{d/p+1})} + \int_0^{T^*} \|\delta u^{n-1}\|_{B_{p,1}^{d/p}} \|\nabla \varrho^{n-1}\|_{B_{p,1}^{d/p}} \right).$$

Apply the above estimate to the term $\|\delta \varrho^{n-1}\|_{L_{T^*}^1(B_{p,1}^{d/p+2})}$ appearing in the bound for H^{n-1} . This makes (80) become

$$\begin{aligned} \|\delta u^n\|_{\tilde{L}_{T^*}^\infty(B_{p,1}^{d/p})} + \|\nabla \delta \pi^n\|_{\tilde{L}_{T^*}^1(B_{p,1}^{d/p})} &\leq C(\|(\Delta_n u_0, \Delta_{n-1} \varrho_0)\|_{B_{p,1}^{d/p}} + \tau \|(\delta \varrho^{n-1}, \delta \varrho^{n-2})\|_{L_{T^*}^2(B_{p,1}^{d/p+1})}) \\ &\quad + \int_0^{T^*} \|(\delta \varrho^{n-1}, \delta u^{n-1}, \delta u^{n-2})\|_{B_{p,1}^{d/p}} \left(\|\nabla \pi^{n-1}\|_{B_{p,1}^{d/p}} + \|(\varrho^{n-1}, \varrho^{n-2})\|_{B_{p,1}^{d/p+2}} + \|u^{n-1}\|_{B_{p,1}^{d/p+1}} \right). \end{aligned}$$

Let us now define

$$B^n(t) := \|\delta \varrho^n\|_{L_t^\infty(B_{p,1}^{d/p})} + \|\delta \varrho^n\|_{L_t^1(B_{p,1}^{d/p+2})} + \|\delta u^n\|_{L_t^\infty(B_{p,1}^{d/p})} + \|\nabla \delta \pi^n\|_{L_t^1(B_{p,1}^{d/p} \cap L^2)};$$

then, from previous inequalities we gather

$$\begin{aligned} B^n(t) &\leq C \|(\Delta_{n-1} \varrho_0, \Delta_n \varrho_0, \Delta_n u_0)\|_{B_{p,1}^{d/p}} + \\ &\quad + \tau^{\frac{1}{2}} (B^{n-1}(t) + B^{n-2}(t)) + C \int_0^t (B^{n-1} + B^{n-2}) D(\sigma) d\sigma, \end{aligned}$$

with $\|D(t)\|_{L^1([0, T^*])} \leq C$. Let us note that, by spectral localization, there exists a constant $C > 0$ for which, for all $n \geq 0$, we have

$$\|(\Delta_n \varrho_0, \Delta_n u_0)\|_{B_{p,1}^{d/p}} \leq C 2^{n(d/p)} \|(\Delta_n \varrho_0, \Delta_n u_0)\|_{L^p}.$$

Keeping in mind this fact, we claim that the previous estimate implies $\sum_n B^n(t) < +\infty$ uniformly in $[0, T^*]$. As a matter of fact, for all $N \geq 3$ we have

$$\sum_{n=1}^N B^n(t) \leq C \|(\varrho_0, u_0)\|_{B_{p,1}^{d/p}} + 2\tau^{1/2} \sum_{n=1}^N B^n(t) + \int_0^t \sum_{n=1}^N B^n(\sigma) D(\sigma) d\sigma + \mathcal{B},$$

where we have denoted by \mathcal{B} a constant which depends only on B^1 and B^2 . We can suppose τ to have been chosen small enough such that, moreover, we can absorb the second term of the right hand side into the left hand side. Therefore, Gronwall's inequality entails

$$\sum_{n=1}^N B^n(t) \leq C_{T^*} \|(\varrho_0, u_0)\|_{B_{p,1}^{d/p}},$$

and passing to the limit for $N \rightarrow +\infty$ we get our claim.

Hence, we gather that the sequence $(\varrho^n, u^n, \nabla \pi^n)$ is a Cauchy sequence in the functional space $E_{p,1}^{d/p}(T^*)$. Then, it converges to some $(\varrho, u, \nabla \pi)$, which actually belongs to the space $E_{p,r}^s(T^*)$ by Fatou property. Hence, by interpolation, we discover that convergence holds true in every intermediate space between $E_{p,r}^s(T^*)$ and $E_{p,1}^{d/p}(T^*)$, and this is enough to pass to the limit in our equations. So, $(\varrho, u, \nabla \pi)$ is actually a solution of System (7).

4.3 Uniqueness

Let us now prove the uniqueness part in Theorem 2.1.

We take two solutions $(\varrho_1, u_1, \nabla \pi_1)$ and $(\varrho_2, u_2, \nabla \pi_2)$ in $E_{p,r}^s(T^*)$ with the initial data $(\varrho_{i,0}, u_{i,0})$ for $i = 1, 2$. Then the difference $(\delta \varrho, \delta u, \nabla \delta \pi) = (\varrho_1 - \varrho_2, u_1 - u_2, \nabla \pi_1 - \nabla \pi_2)$ solves

$$(81) \quad \begin{cases} \partial_t \delta \varrho + u_1 \cdot \nabla \delta \varrho - \operatorname{div}(\kappa_1 \nabla \delta \varrho) = -\delta u \cdot \nabla \varrho_2 + \operatorname{div}((\kappa_1 - \kappa_2) \nabla \varrho_2), \\ \partial_t \delta u + (u_1 + \nabla b_1) \cdot \nabla \delta u + \lambda_1 \nabla \delta \pi = h_1 - h_2 - (\delta u + \nabla(b_1 - b_2)) \cdot \nabla u_2 - (\lambda_1 - \lambda_2) \nabla \pi_2, \\ \operatorname{div} \delta u = 0, \\ (\delta \varrho, \delta u)|_{t=0} = (\delta \varrho_0, \delta u_0) = (\varrho_{1,0} - \varrho_{2,0}, u_{1,0} - u_{2,0}), \end{cases}$$

with the notation $\kappa_i = \kappa(\varrho_i)$ and analogous for b_i , λ_i and h_i .

As $(\varrho_1, u_1, \nabla \pi_1), (\varrho_2, u_2, \nabla \pi_2) \in E_{p,r}^s(T^*)$, then for any $\varepsilon > 0$ to be determined later, there exists T_ε such that

$$\|\varrho_i\|_{L_{T_\varepsilon}^2(B_{p,1}^{d/2+2}) \cap L_{T_\varepsilon}^1(B_{p,1}^{d/p+3})}, \|u_i\|_{L_{T_\varepsilon}^1(B_{p,1}^{d/p+1})}, \|\nabla \pi_i\|_{L_{T_\varepsilon}^1(B_{p,1}^{d/p+1})} \leq \varepsilon, \quad i = 1, 2.$$

Let us define (as done before)

$$B(t) := \|\delta \varrho\|_{L_t^\infty(B_{p,1}^{d/p})} + \|\delta \varrho\|_{L_t^1(B_{p,1}^{d/p+2})} + \|\delta u\|_{L_t^\infty(B_{p,1}^{d/p})} + \|\nabla \delta \pi\|_{L_t^1(B_{p,1}^{d/p} \cap L^2)}.$$

Then, the same proof as in paragraph 4.2.2 shows that

$$B(t) \leq C \left(\|\delta \varrho_0\|_{B_{p,1}^{d/p}} + \|\delta u_0\|_{B_{p,1}^{d/p}} + \varepsilon B(t) + \int_0^t B(s) D(s) ds \right),$$

which implies, if ε is taken small enough,

$$B(t) \leq e^C \left(\|\delta \varrho_0\|_{B_{p,1}^{d/p}} + \|\delta u_0\|_{B_{p,1}^{d/p}} \right).$$

Hence, uniqueness holds for small time. Now, standard continuity arguments show uniqueness on the whole time interval $[0, T^*]$.

5 Proof of Theorems 2.3 and 2.7

In this section we try to get a continuation criterion and a lower bound of the lifespan for the local-in-time solutions given by Theorem 2.1. It is only a matter of repeating a priori estimates established previously, but in an “accurate” way (we use L^∞ -norm instead of $B_{p,r}^{s-1}$ -norm) for obtaining the continuation criterion, whereas in a “rough” way (we use (85), (86) below) for bounding the lifespan from below.

5.1 Proof of the continuation criterion

From the arguments after Theorem 2.3, it rests us to prove Lemma 2.4. As argued before Theorem 2.3, in order to ensure $u \in \tilde{L}^\infty(B_{p,r}^s)$, the source terms such as $u \cdot \nabla^2 b$, $\operatorname{div}(\nabla b \otimes \nabla b)$ require at least $u, \nabla b \in L^\infty(L^\infty)$ since $\nabla^2 b \in \tilde{L}^1(B_{p,r}^s)$ only, and also $\nabla^2 b \in L^2(L^\infty)$ since $\nabla b \in \tilde{L}^2(B_{p,r}^s)$. Similarly, the transport term $\nabla b \cdot \nabla u$ requires $\nabla u \in L^2(L^\infty)$. Therefore in the following, we do not have to optimize all the inequalities for the commutators or products and sometimes it simplifies the argument.

Proof of Lemma 2.4. It’s only a matter of repeating a priori estimates previously established, but in a more accurate way. Roughly speaking, in the estimates we use L^∞ -norm instead of Besov norm, in order to require lower regularity.

Let us consider the density term. Our starting point is (50), with this time $f = 0$, and we argue as in proving Proposition 4.5, but control commutators \mathcal{R}_j^1 and \mathcal{R}_j^2 (see (50) for definition) by use of Commutator Estimates (43) and (46) instead. Inequality (43) gives us for $s > 0$,

$$\int_0^t \left\| 2^{js} \|\mathcal{R}_j^1\|_{L^p} \right\|_{\ell^r} d\tau \leq \int_0^t \left(\|\nabla u\|_{L^\infty} \|\varrho\|_{B_{p,r}^s} + \|\nabla \varrho\|_{L^\infty} \|u\|_{B_{p,r}^s} \right) d\tau.$$

We apply instead (46) to control the second commutator term: with $\sigma = s$, $\theta = 1/2$ and $\varepsilon > 0$ to be fixed later, it entails for $s > 0$,

$$\left\| 2^{js} \int_0^t \|\mathcal{R}_j^2\|_{L^p} \right\|_{\ell^r} d\tau \leq \frac{C}{\varepsilon} \int_0^t \|\nabla \varrho\|_{L^\infty}^2 \|\varrho\|_{B_{p,r}^s} d\tau + \varepsilon \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})}.$$

So, if ε is small enough, $p \in (1, \infty)$, putting these inequalities in (52) and keeping in mind (53), for all $t \in [0, T[$ we get the following estimate:

$$(82) \quad \|\varrho\|_{\tilde{L}_t^\infty(B_{p,r}^s) \cap \tilde{L}_t^1(B_{p,r}^{s+2})} \leq C \left(\|\varrho_0\|_{B_{p,r}^s} + \int_0^t \|\varrho\|_{B_{p,r}^s} d\tau + \int_0^t \|\nabla \varrho\|_{L^\infty} \|u\|_{B_{p,r}^s} d\tau + \int_0^t (\|\nabla u\|_{L^\infty} + \|\nabla \varrho\|_{L^\infty}^2) \|\varrho\|_{B_{p,r}^s} d\tau \right).$$

Let us now consider velocity field and pressure term: we have

$$(83) \quad \|u\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq C \left(\|u_0\|_{B_{p,r}^s} + \|\lambda \nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} + \|h\|_{\tilde{L}_t^1(B_{p,r}^s)} + \left\| 2^{js} \int_0^t \|\mathcal{R}_j(\tau)\|_{L^p} d\tau \right\|_{\ell^r} \right),$$

where we have set, $\mathcal{R}_j := [u + \nabla b(\rho), \Delta_j] \cdot \nabla u$. To control the commutator term, a direct application of Lemma 4.1 yields, for $s > 0$,

$$\left\| 2^{js} \int_0^t \|[u, \Delta_j] \cdot \nabla u\|_{L^p} d\tau \right\|_{\ell^r} \leq C \int_0^t \|\nabla u\|_{L^\infty} \|u\|_{B_{p,r}^s} d\tau.$$

For the second part $[\nabla b(\rho), \Delta_j] \cdot \nabla u$, as in the density case, one can resort to Lemma 4.2 to get, for $s > 0$,

$$\begin{aligned} \left\| 2^{js} \int_0^t \|[\nabla b(\rho), \Delta_j] \cdot \nabla u\|_{L^p} d\tau \right\|_{\ell^r} &\leq C \int_0^t \|\nabla^2 \varrho\|_{L^\infty} \|u\|_{B_{p,r}^s} d\tau + \\ &+ C \int_0^t \|\nabla u\|_{L^\infty}^2 \|\varrho\|_{B_{p,r}^s} d\tau + \varepsilon \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})}. \end{aligned}$$

Let us immediately consider the pressure term:

$$\begin{aligned} \|\lambda(\rho) \nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} &\leq C \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} + \|(\lambda(\rho) - \lambda(1)) \nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} \\ &\leq C \left(\|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} + \int_0^t \|\nabla \pi\|_{L^\infty} \|\varrho\|_{B_{p,r}^s} d\tau \right). \end{aligned}$$

To bound $\|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)}$, we argue as in Subsection 4.1.2: we take the divergence of second equation of System (7), we localize in frequencies by operator Δ_j and we perform a weighed summation. Hence we discover, instead of (65), that

$$\begin{aligned} \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} &\lesssim \int_0^t \|\Delta_{-1} \nabla \pi\|_{L^p} + \|h\|_{\tilde{L}_t^1(B_{p,r}^s)} + \|\operatorname{div}((u + \nabla b) \cdot \nabla u)\|_{\tilde{L}_t^1(B_{p,r}^{s-1})} + \\ &+ \left\| 2^{j(s-1)} \int_0^t \|\operatorname{div}([\lambda, \Delta_j] \nabla \pi)\|_{L^p} d\tau \right\|_{\ell^r}. \end{aligned}$$

Inequality (43) in Lemma 4.1, entails the control for the commutator term:

$$\left\| 2^{j(s-1)} \int_0^t \|\operatorname{div}([\lambda, \Delta_j] \nabla \pi)\|_{L^p} d\tau \right\|_{\ell^r} \lesssim \int_0^t (\|\nabla \varrho\|_{L^\infty} \|\nabla \pi\|_{B_{p,r}^{s-1}} + \|\nabla \varrho\|_{B_{p,r}^{s-1}} \|\nabla \pi\|_{L^\infty}) d\tau.$$

By view of (37), interpolation inequality helps us to control the above first term on the right hand side as follows:

$$\begin{aligned} \int_0^t \|\nabla \varrho\|_{L^\infty} \|\nabla \pi\|_{B_{p,r}^{s-1}} d\tau &\lesssim \|\nabla \varrho\|_{L_t^\infty(L^\infty)} \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^{s-1+\epsilon})}, \quad \text{for some fixed } \epsilon \in (0, 1), \\ &\leq C_\varepsilon \|\nabla \varrho\|_{L_t^\infty(L^\infty)}^{(s+\sigma)/(1-\epsilon)} \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^{s-\sigma})} + \varepsilon \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} \end{aligned}$$

with any positive (large) σ . We point out here that for any $\sigma \in \mathbb{R}$, there exists C depending on σ such that the following holds:

$$\|\Delta_{-1} \nabla \pi\|_{L_t^1(L^p)} \leq C \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,\infty}^{-\sigma})}.$$

Next, by the divergence-free condition over u and Lemma 4.3, we infer ($\operatorname{div} u = 0$ implies that we only need $s > 0$)

$$\begin{aligned} \|\operatorname{div}((u + \nabla b) \cdot \nabla u)\|_{\tilde{L}_t^1(B_{p,r}^{s-1})} &\leq \|\nabla u : \nabla u\|_{\tilde{L}_t^1(B_{p,r}^{s-1})} + \|\nabla^2 b : \nabla u\|_{\tilde{L}_t^1(B_{p,r}^{s-1})} \\ &\leq C \left(\int_0^t (\|\nabla u\|_{L^\infty} + \|\nabla^2 \varrho\|_{L^\infty}) \|u\|_{B_{p,r}^s} d\tau + \right. \\ &\quad \left. + \int_0^t \|\nabla u\|_{L^\infty}^2 \|\varrho\|_{B_{p,r}^s} d\tau + \varepsilon \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})} \right). \end{aligned}$$

The control of the non-linear term h is quite similar as above. We come back to definition (8) and consider its terms one by one. As usual, estimation for products in Lemma 4.3 ensures that

$$\begin{aligned} \|u \cdot \nabla^2 b\|_{\tilde{L}_t^1(B_{p,r}^s)} &\lesssim \int_0^t \|\nabla^2 \varrho\|_{L^\infty} \|u\|_{B_{p,r}^s} d\tau + \|u\|_{L_t^\infty(L^\infty)} \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})}, \\ \|(u \cdot \nabla \lambda) \nabla a\|_{\tilde{L}_t^1(B_{p,r}^s)} &\lesssim \int_0^t \|u\|_{L^\infty}^2 \|\nabla \varrho\|_{L^\infty}^2 \|\varrho\|_{B_{p,r}^s} d\tau + \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})} + \int_0^t \|\nabla \varrho\|_{L^\infty}^2 \|u\|_{B_{p,r}^s} d\tau, \end{aligned}$$

and

$$\|(\nabla b \cdot \nabla \lambda) \nabla a\|_{\tilde{L}_t^1(B_{p,r}^s)} \lesssim \int_0^t \|\nabla \varrho\|_{L^\infty}^4 \|\varrho\|_{B_{p,r}^s} d\tau + \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})}.$$

Moreover, the last element of h can be treated in the following way:

$$\begin{aligned} \|\operatorname{div}(\nabla b \otimes \nabla b)\|_{\tilde{L}_t^1(B_{p,r}^s)} &= \|\Delta b \nabla b + \nabla b \cdot \nabla^2 b\|_{\tilde{L}_t^1(B_{p,r}^s)} \\ &\lesssim \int_0^t \|\nabla^2 \varrho\|_{L^\infty}^2 \|\varrho\|_{B_{p,r}^s} d\tau + (1 + \|\nabla \varrho\|_{L_t^\infty(L^\infty)}) \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})}. \end{aligned}$$

Let us collect all these informations: up to multiplication by a constant, we gather

$$\begin{aligned} (84) \quad \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} &\lesssim C(\|\nabla \varrho\|_{L_t^\infty(L^\infty)}, s, \sigma) \|\nabla \pi\|_{L_t^1(B_{p,\infty}^{-\sigma})} \\ &\quad + \int_0^t (\|\nabla u\|_{L^\infty} + \|\nabla^2 \varrho\|_{L^\infty} + \|\nabla \varrho\|_{L^\infty}^2 + \|\nabla \varrho\|_{L^\infty}^4) \|u\|_{B_{p,r}^s} d\tau \\ &\quad + \int_0^t (\|\nabla u\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \|\nabla \varrho\|_{L^\infty}^2 + \|\nabla^2 \varrho\|_{L^\infty}^2 + \|\nabla \pi\|_{L^\infty}) \|\varrho\|_{B_{p,r}^s} d\tau \\ &\quad + (1 + \|\nabla \varrho\|_{L_t^\infty(L^\infty)} + \|u\|_{L_t^\infty(L^\infty)}) \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})}, \end{aligned}$$

and the same control actually holds true also for $\|h\|_{\tilde{L}_t^1(B_{p,r}^s)}$.

In the end, we discover from (83) that $\|u\|_{\tilde{L}_t^\infty(B_{p,r}^s)}$ satisfies also Inequality (84), just with an additional term $\|u_0\|_{B_{p,r}^s}$ on the right-hand side. Recalling Estimate (82) for the density, we replace $\|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})}$ in Inequality (84) by the right-hand side of it.

Thus, we can sum up (82) and the (modified) estimate (84) for the velocity u , yielding the thesis by Gronwall's Lemma. \square

5.2 Lower bounds for the lifespan of the solution

The aim of the present subsection is analyzing the lifespan of the solutions to system (7). We want to show, as carefully as possible, the dependence of the lifespan T on the initial data. This can be done by

repeating the a priori estimates previously established, but in a “rough” way. For example, we will use thoroughly the following inequalities:

$$(85) \quad \|ab\|_{B_{p,r}^{s-i}} \lesssim \|a\|_{B_{p,r}^{s-i}} \|b\|_{B_{p,r}^{s-i}}, \quad \|a^2\|_{B_{p,r}^s} \lesssim \|a\|_{L^\infty} \|a\|_{B_{p,r}^s}, \quad \|a\|_{L^\infty}, \|\nabla a\|_{L^\infty} \lesssim \|a\|_{B_{p,r}^s},$$

with $i = 0, 1$, and thanks to Conditions (14) and (15),

$$(86) \quad \|ab\|_{L^2} \leq \|a\|_{L^4} \|b\|_{L^4} \lesssim \|a\|_{B_{p,r}^{s-1}} \|b\|_{B_{p,r}^{s-1}}.$$

Let us point out that, thanks to Theorem 2.3 and embedding results, without any loss of generality from now on throughout this subsection we assume

$$s = 1 + \frac{d}{p}, \quad p = 4 \quad \text{and} \quad r = 1.$$

In particular, for any $\sigma \in \mathbb{R}$ we have $\|\cdot\|_{\tilde{L}_t^1(B_{p,1}^\sigma)} \sim \|\cdot\|_{L_t^1(B_{p,1}^\sigma)}$, and this really simplifies our computations. We also assume that C always denotes some large enough constant.

For notational convenience, we define $R_0 := \|\varrho_0\|_{B_{p,1}^s}$ and $U_0 := \|u_0\|_{B_{p,1}^s}$,

$$R(t) = \|\varrho\|_{L_t^\infty(B_{p,1}^s)}, \quad S(t) := \|\varrho\|_{L_t^1(B_{p,1}^{s+2})} \quad \text{and} \quad U(t) := \|u\|_{L_t^\infty(B_{p,1}^s)}.$$

From (82), we infer that

$$R(t) + S(t) \leq C \left(R_0 + \int_0^t R(\tau)(U(\tau) + 1) d\tau + \int_0^t R^3(\tau) d\tau \right).$$

Now, if we define

$$(87) \quad T_R := \sup \left\{ t > 0 \mid \int_0^t R^3(\tau) d\tau \leq 2R_0 \right\},$$

by Gronwall’s Lemma, we get

$$(88) \quad R(t) + S(t) \leq C R_0 \exp \left(C \left(t + \int_0^t U(\tau) d\tau \right) \right).$$

So, from now on we work with $t \in [0, T_R]$.

Let us now focus on the velocity field and the pressure term. We immediately point out that the nonlinear term h will make density terms with critical regularity appear. Therefore, unlike what has been done in previous sections, we decided not to use systematically interpolation inequalities to isolate the term $S(t)$: we consider instead $S'(t)$, which controls high regularity of the density.

Our starting point is estimate (83):

$$\|u\|_{L_t^\infty(B_{p,1}^s)} \leq C \left(\|u_0\|_{B_{p,1}^s} + \|h - \lambda \nabla \pi\|_{L_t^1(B_{p,1}^s)} + \left\| 2^{js} \int_0^t \|\mathcal{R}_j(\tau)\|_{L^p} d\tau \right\|_{\ell^1} \right),$$

where, as before, $\mathcal{R}_j := [u + \nabla b(\rho), \Delta_j] \cdot \nabla u$. Now we apply the classical results, e.g. lemma 2.100 of [4], to control the commutator term, and we get

$$(89) \quad U(t) \leq C \left(U_0 + \int_0^t \|h\|_{B_{p,1}^s} d\tau + \int_0^t \|\lambda \nabla \pi\|_{B_{p,1}^s} d\tau + \int_0^t U^2(\tau) d\tau + \int_0^t U(\tau) S'(\tau) d\tau \right).$$

Let us now establish some stationary estimates, which will be useful to complete the bound for $U(t)$. We will often use (85) to deal with products.

(i) Let us start with $\|h\|_{B_{p,r}^s}$. We consider its terms one by one, and we get

$$\begin{aligned} \|u \cdot \nabla^2 b\|_{B_{p,1}^s} &\leq C U S' \\ \|(u \cdot \nabla \lambda) \nabla a\|_{B_{p,1}^s} &\leq C U R S' \\ \|(\nabla b \cdot \nabla \lambda) \nabla a\|_{B_{p,1}^s} &\leq C R^2 S' \\ \|\operatorname{div} (\nabla b \otimes \nabla b)\|_{B_{p,1}^s} &\leq 2 \|\nabla^2 b \otimes \nabla b\|_{B_{p,1}^s} \leq C R S'. \end{aligned}$$

Therefore, one infers that

$$\|h\|_{B_{p,1}^s} \leq C (U S' + U R S' + R^2 S' + R S').$$

(ii) Now we focus on the Besov norm of the pressure term. First of all, we have

$$\|\lambda \nabla \pi\|_{B_{p,1}^s} \leq C (1 + R) \|\nabla \pi\|_{B_{p,r}^s}.$$

Now, we can handle $\|\nabla \pi\|_{B_{p,r}^s}$. Since $r = 1$ here, we do not have to worry about the order of taking the integration with respect to time L_t^1 and taking the ℓ^r norm of the dyadic sequence.

Hence, we can first “forget about” the time variable and discuss $\|\nabla \pi\|_{B_{p,1}^s}$. In order to control it, we separate low and high frequencies and use Bernstein’s inequalities: we get

$$\|\nabla \pi\|_{B_{p,1}^s} \leq \|\Delta_{-1} \nabla \pi\|_{B_{p,1}^s} + \|(\operatorname{Id} - \Delta_{-1}) \nabla \pi\|_{B_{p,1}^s} \leq C \left(\|\nabla \pi\|_{L^2} + \|\Delta \pi\|_{B_{p,1}^{s-1}} \right).$$

The equation for the Laplacian of the pressure comes from (64):

$$-\Delta \pi = \nabla(\log \lambda) \cdot \nabla \pi + \lambda^{-1} \operatorname{div} (v \cdot \nabla u - h),$$

from which it immediately follows

$$\begin{aligned} \|\Delta \pi\|_{B_{p,1}^{s-1}} &\leq \|\nabla(\log \lambda) \cdot \nabla \pi\|_{B_{p,1}^{s-1}} + (\|\varrho\|_{B_{p,1}^{s-1}} + 1) \left(\|\operatorname{div} (v \cdot \nabla u)\|_{B_{p,1}^{s-1}} + \|h\|_{B_{p,1}^s} \right) \\ &\leq \|\nabla \varrho\|_{B_{p,1}^{s-1}} \|\nabla \pi\|_{B_{p,1}^{s-1}} + (\|\varrho\|_{B_{p,1}^{s-1}} + 1) \left(\|\nabla v : \nabla u\|_{B_{p,1}^{s-1}} + \|h\|_{B_{p,1}^s} \right) \end{aligned}$$

(where we used also that $\operatorname{div} u = 0$). So, applying the interpolation inequality for the embeddings $B_{p,r}^s \hookrightarrow B_{p,1}^{s-1} \hookrightarrow B_{p,1}^{-d/2}$, then the inclusion $L^2 \hookrightarrow B_{p,1}^{-d/2}$ and finally the Young inequality, we gather

$$\|\nabla \pi\|_{B_{p,1}^s} \leq C \left((1 + R^\delta) \|\nabla \pi\|_{L^2} + (1 + R) \|\nabla v : \nabla u\|_{B_{p,1}^{s-1}} + (1 + R) \|h\|_{B_{p,1}^s} \right),$$

for some $\delta > 1$, thanks to $\operatorname{div} u \equiv 0$. Then, we have also

$$(90) \quad \|\lambda \nabla \pi\|_{B_{p,1}^s} \leq C \left((1 + R) (1 + R^\delta) \|\nabla \pi\|_{L^2} + (1 + R)^2 U(U + S') + (1 + R)^2 \|h\|_{B_{p,1}^s} \right).$$

(iii) Now we focus on the L^2 norm of the pressure term. Applying Lemma 3.13 to equation (64) immediately gives

$$\|\nabla \pi\|_{L^2} \leq C (\|h\|_{L^2} + \|v \cdot \nabla u\|_{L^2}).$$

Thanks to (86), we find

$$\|v \cdot \nabla u\|_{L^2} \leq C \|u + \nabla b\|_{B_{p,1}^{s-1}} \|\nabla u\|_{B_{p,1}^{s-1}} \leq C (U^2 + U R).$$

Using also (85) and (86), it’s easy to control the L^2 norm of h :

$$\begin{aligned} \|u \cdot \nabla^2 b\|_{L^2} &\leq C \|u\|_{B_{p,1}^{s-1}} \|\nabla^2 b\|_{B_{p,1}^{s-1}} \leq C U S' \\ \|(u \cdot \nabla \lambda) \nabla a\|_{L^2} &\leq C \|u\|_{B_{p,1}^{s-1}} \|\nabla \varrho\|_{B_{p,1}^{s-1}}^2 \leq C U R^2 \\ \|(\nabla b \cdot \nabla \lambda) \nabla a\|_{L^2} &\leq C \|\nabla \varrho\|_{B_{p,1}^{s-1}}^3 \leq C R^3 \\ \|\operatorname{div} (\nabla b \otimes \nabla b)\|_{L^2} &\leq 2 \|\nabla^2 b \otimes \nabla b\|_{L^2} \leq C R S'. \end{aligned}$$

Before putting all these inequalities together into (89), let us note the following point. Due to the fact that $\delta > 1$, for any $m \geq 0$ we have

$$R^m (1 + R^\delta) \leq C_m (1 + R^{m+\delta}),$$

and we have to deal with only a finite number of powers of R (the biggest m is actually 4, by previous stationary estimates). Hence, if we define (denoting by η some big enough exponent)

$$\mathcal{E}(t) := 1 + R_0^\eta \exp\left(C \left(t + \int_0^t U(\tau) d\tau\right)\right),$$

thanks to (88) we can bound all the terms of the form R^m and $R^m(1 + R^\delta)$ which occurs in our estimate by $C\mathcal{E}$, with $C > 0$ suitably large.

Now, we are ready to complete the bound for $\|\lambda \nabla \pi\|_{B_{p,1}^s}$. From the previous step (iii), we get

$$(1 + R) (1 + R^\delta) \|\nabla \pi\|_{L^2} \leq C (\mathcal{E} U^2 + \mathcal{E} U + \mathcal{E} U S' + \mathcal{E} S'),$$

while from the step (i) we infer

$$(1 + R^2) \|h\|_{B_{p,r}^s} \leq C (\mathcal{E} U S' + \mathcal{E} S').$$

Then from (90) we finally get

$$\|\lambda \nabla \pi\|_{B_{p,r}^s} \leq C (\mathcal{E} U^2 + \mathcal{E} U + \mathcal{E} U S' + \mathcal{E} S').$$

Let us note that, as $\mathcal{E} \geq 1$, we can bound also $\|h\|_{B_{p,r}^s}$ by the previous quantity.

Now we put this last inequality in (89). Using again the fact that $\mathcal{E} \geq 1$, we find

$$(91) \quad U(t) \leq C \left(U_0 + \int_0^t (\mathcal{E} U^2 + \mathcal{E} U + \mathcal{E} U S' + \mathcal{E} S') d\tau \right)$$

for all $t \in [0, T_R]$. Therefore, if we define

$$(92) \quad T_U := \sup \left\{ t > 0 \mid \int_0^t (\mathcal{E} U^2 + \mathcal{E} U + \mathcal{E} U S' + \mathcal{E} S') d\tau \leq 2U_0 \right\},$$

then in $[0, T_R] \cap [0, T_U]$ we get $U(t) \leq C U_0$ and we manage to close the estimates.

So, our next goal is to prove that, if we define T as the quantity in (23), then $T \leq \min\{T_R, T_U\}$, i.e. both conditions in (87) and (92) are fulfilled.

Let's first tackle the case $U_0 \equiv 1$ and then we will see how to deal with the general case by use of Proposition 2.5. First of all, from (88), (91) and the definition of \mathcal{E} , in the interval $[0, T_R] \cap [0, T_U]$ we have (if $U_0 \equiv 1$)

$$(93) \quad R(t) + S(t) \leq C R_0 e^{Ct} \quad \text{and} \quad \mathcal{E}(t) \leq (1 + R_0^\eta) e^{Ct},$$

for some suitable positive constant C . Note that in $[0, T_R] \cap [0, T_U]$ we have

$$\int_0^t (\mathcal{E} U^2 + \mathcal{E} U + \mathcal{E} U S' + \mathcal{E} S') d\tau \leq 2C (1 + R_0^{\eta+1}) e^{2Ct}.$$

Therefore, T_U , defined by (92), is bigger than any time t for which the quantity $2C (1 + R_0^{\eta+1}) e^{2Ct}$ is controlled by 2. Hence, for suitable values of L and with $\mathfrak{X} = \eta + 1$, $U_0 \equiv 1$ in (23), we have that $T \leq T_U$.

Now, let us consider T_R . By (93), it's easy to see that the left-hand side of the condition in (87) can be controlled in the following way:

$$\int_0^t R^3(\tau) d\tau \leq C R_0^3 \int_0^t e^{3C\tau} d\tau \leq C \frac{R_0^3}{3C} e^{3Ct}.$$

Hence, T_R is greater than any time t for which $R_0^2 e^{3\mu t} \leq \tilde{C}$, for some convenient constant \tilde{C} . But this condition is always verified in $[0, T]$, up to change the values of L in (23) (recall that $\mathfrak{X} > 2$). Therefore, in the end we gather that $T \leq \min\{T_R, T_U\}$, and Theorem 2.7 is proved for $U_0 \equiv 1$.

For arbitrary initial data $\rho_0 - 1, u_0$ in $B_{4,1}^{1+d/4}$, we have a unique local-in-time solution $(\rho, u, \nabla\pi)$. If we now set

$$\varepsilon^2 = \|u_0\|_{B_{4,1}^{1+d/4}},$$

then we can see $(\rho, u, \nabla\pi)$ as a rescaling of some solution $(\tilde{\rho}, \tilde{u}, \nabla\tilde{\pi})$ having initial data

$$(\tilde{\rho}_0, \tilde{u}_0)(x) := (\rho_0, \varepsilon^{-1}u_0)(\varepsilon^{-1}x).$$

In particular $\tilde{U}_0 := \|\tilde{u}_0\|_{B_{4,1}^{1+d/4}} \equiv 1$ and hence the lifespan of the solution $(\tilde{\rho}, \tilde{u}, \nabla\tilde{\pi})$ is bigger than

$$L \log\left(\frac{L}{1 + \|\tilde{\rho}_0 - 1\|_{B_{4,1}^{1+d/4}}^{\mathfrak{X}}}\right) = L \log\left(\frac{L}{1 + (\varepsilon^{-1}\|\rho_0 - 1\|_{B_{4,1}^{1+d/4}})^{\mathfrak{X}}}\right).$$

In virtue of Proposition 2.5, the lifespan of the solution $(\rho, u, \nabla\pi)$ is larger than

$$\frac{L}{\varepsilon^2} \log\left(\frac{L}{1 + (\varepsilon^{-1}\|\rho_0 - 1\|_{B_{4,1}^{1+d/4}})^{\mathfrak{X}}}\right).$$

This completes the proof of Theorem 2.7, thanks to Remark 2.8.

6 Other cases

We want now to deal with finite energy initial data and with small initial densities. Before passing to the proof of Theorems 2.9 and 2.12, we need an estimate for parabolic equations in Besov spaces of type $B_{\infty,r}^s$, in order to tackle also the case $p = \infty$.

6.1 Linearized parabolic equations in $B_{\infty,r}^s$

The aim of this paragraph is to show an estimate for parabolic equations in Besov spaces of type $B_{\infty,r}^s$. Such an estimate will be fundamental in the sequel.

In the present subsection we resort to notations and tools from Homogeneous Paradifferential Calculus, which enjoy most of the properties we have seen for the non-homogeneous case (see also Section 3). We refer to Chapter 2 of [4] for a detailed description.

The result we want to prove in the present subsection is the following.

Proposition 6.1. *Let $\rho \in \mathcal{S}$ solve the following linear parabolic-type system*

$$(94) \quad \begin{cases} \partial_t \rho - \operatorname{div}(\kappa \nabla \rho) = f, \\ \rho|_{t=0} = \rho_0, \end{cases}$$

with $\kappa, f, \rho_0 \in \mathcal{S}$ and

$$0 < \rho_* \leq \rho_0 \leq \rho^*, \quad 0 < \kappa_* \leq \kappa(t, x) \leq \kappa^*.$$

If $s > 0$, $r \in [1, +\infty]$, then the following estimate holds true:

$$(95) \quad \|\rho\|_{\tilde{L}_t^\infty(B_{\infty,r}^s) \cap \tilde{L}_t^1(B_{\infty,r}^{s+2})} \leq C e^{CK_\infty(t)} \times \left(\|\rho_0\|_{B_{\infty,r}^s} + \|f\|_{\tilde{L}_t^1(B_{\infty,r}^s)} \right),$$

where C is a constant depending on $d, s, r, \rho_*, \rho^*, \kappa_*, \kappa^*$ and $\|\kappa\|_{L_t^\infty(\dot{C}^\epsilon)}$ for some $\epsilon \in (0, 1)$, and

$$K_\infty(t) := \int_0^t \left(1 + \|\nabla \kappa\|_{L^\infty}^2 + \|\nabla \kappa\|_{B_{\infty,r}^{\max\{2/(1+s), 1\}}}^{\max\{2/(1+s), 1\}} \right) d\tau.$$

Remark 6.2. Let us point out that the above proposition provides us with estimates analogous to those in Proposition 4.5 (for instance, one can compare the expression (49) with the definition of K_∞). Although here we are considering a parabolic equation with no transport term, the result can be easily extended to that case by a direct use of product estimates as well as interpolation and Gronwall inequalities.

We will first focus on the case $s \in (0, 1)$ and $r = \infty$, i.e. we take solutions in time-dependent Hölder's spaces $\tilde{L}_T^\infty(C^\epsilon) \cap \tilde{L}_T^1(C^{2+\epsilon})$, $\epsilon \in (0, 1)$. Note that Maximum Principle applied to parabolic equations has already given us the control on low frequencies of the solution:

$$(96) \quad \|\rho\|_{L_t^\infty(L^\infty)} \leq \|\rho_0\|_{L^\infty} + \int_0^t \|f\|_{L^\infty};$$

besides, the classical a priori estimates for heat equations are simpler (at least formally, see Proposition 6.3) in homogeneous Besov spaces. For these reasons, we only have to focus on the homogeneous Hölder's space:

$$\dot{E}^\epsilon := \tilde{L}_T^\infty(\dot{C}^\epsilon) \cap \tilde{L}_T^1(\dot{C}^{2+\epsilon}), \quad \epsilon \in (0, 1).$$

We can localize the function ρ into countable functions ϱ_n , each of which is supported on some ball $B(x_n, \delta)$, with the small radius $\delta \in (0, 1)$ to be determined in the proof. It is useful to notice that time-dependent Hölder's spaces may be described in terms of finite differences (see Proposition 6.5 below). Roughly speaking, up to a constant, the \dot{E}^ϵ -norm can be determined locally, which reduces to bound $\{\varrho_n\}$ instead of the whole ρ . The systems for $\{\varrho_n\}$ (see System (101) below) derive from (94) after multiplying by some partition of unity and thus the coefficient κ can be viewed as a small perturbation of a function which depends only on time t , because ϱ_n is supported on a ball with sufficiently small radius. Consequently, changing the time variable and making use of estimates for heat equations entail an a priori estimate for ϱ_n in \dot{E}^ϵ , with the “source” terms being either small or of lower regularity, and hence easy to control. At last we will show how the Hölder case will yield the general one.

We agree that in this subsection $\{\varrho_n(t, x)\}$ always denote localized functions of $\rho(t, x)$ in x -space, while ρ_j as usual, denotes $\Delta_j \rho$ (localization in the phase space).

Before going on, we recall a priori estimates in homogeneous Hölder's spaces for heat equations with constant coefficients (see Chapter 3 of [4]):

Proposition 6.3. *For any $s \in \mathbb{R}$, there exists a constant C_0 such that*

$$(97) \quad \|F\|_{L_T^\infty(\dot{C}^s) \cap \tilde{L}_T^1(\dot{C}^{2+s})} \leq C_0 (\|f_0\|_{\dot{C}^s} + \|f\|_{\tilde{L}_T^1(\dot{C}^s)}),$$

where $f_0, f, F \in \mathcal{S}(\mathbb{R}^d)$ and are linked by the relation

$$F(t, x) = e^{t\Delta} f_0 + \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau.$$

It is also convenient to show an a priori estimate for the paraproduct $\tilde{T}_v u$ in the space $\tilde{L}_T^1(\dot{C}^s)$, similar to Lemma 4.3.

Lemma 6.4. *For any $s > 0$, $\varepsilon > 0$, $a, b > 0$, there exists a constant $C_\varepsilon \sim \varepsilon^{-a/b}$ such that*

$$(98) \quad \|\dot{T}_v u\|_{\tilde{L}_T^1(\dot{C}^s)} \leq C_\varepsilon \int_0^T \|u(t)\|_{\dot{C}^s}^{\frac{a+b}{b}} \|v\|_{\dot{C}^{-a}} + \varepsilon \|v\|_{\tilde{L}_T^1(\dot{C}^b)}.$$

Proof. First, let us notice that,

$$\|\dot{T}_v u\|_{\tilde{L}_T^1(\dot{C}^s)} \leq \int_0^T \sum_{j \in \mathbb{Z}} \|u\|_{\dot{C}^s} \|\dot{\Delta}_j v\|_{L^\infty}.$$

For any $\varepsilon > 0$, $a, b > 0$ and $t \in (0, T)$, we fix an integer

$$N_t = \left\lceil \frac{1}{b} \log_2(\varepsilon^{-1} \|u(t)\|_{\dot{C}^s}) \right\rceil + 1,$$

then noticing that $\varepsilon^{-1} \|u(t)\|_{\dot{C}^s} \sim 2^{N_t b}$, we have

$$\begin{aligned} \int_0^T \sum_{j \in \mathbb{Z}} \|u\|_{\dot{C}^s} \|\dot{\Delta}_j v\|_{L^\infty} &\leq \int_0^T \sum_{j \leq N_t} 2^{ja} \|u(t)\|_{\dot{C}^s} 2^{-ja} \|\dot{\Delta}_j v\|_{L^\infty} + \sum_{j \geq N_t+1} 2^{-jb} \|u(t)\|_{\dot{C}^s} 2^{jb} \|\dot{\Delta}_j v\|_{L^\infty} \\ &\lesssim \int_0^T 2^{N_t a} \|u(t)\|_{\dot{C}^s} \|v\|_{\dot{C}^{-a}} + \sum_{j \geq N_t+1} 2^{-(j-N_t)b} \varepsilon 2^{jb} \|\dot{\Delta}_j v\|_{L^\infty} \\ &\lesssim \int_0^T \varepsilon^{-a/b} \|u(t)\|_{\dot{C}^s}^{(a+b)/b} \|v\|_{\dot{C}^{-a}} + \varepsilon \sum_{j \geq 1} 2^{-jb} 2^{(j+N_t)b} \|\dot{\Delta}_{j+N_t} v\|_{L^\infty} \\ &\lesssim \varepsilon^{-a/b} \int_0^T \|u(t)\|_{\dot{C}^s}^{(a+b)/b} \|v\|_{\dot{C}^{-a}} + \varepsilon \sup_j \int_0^T 2^{jb} \|\dot{\Delta}_j v\|_{L^\infty}. \end{aligned}$$

Thus the lemma follows. \square

Recall now the characterization of homogeneous Hölder spaces and time-dependent homogeneous Hölder spaces:

Proposition 6.5. *$\forall \epsilon \in (0, 1)$, there exists a constant C such that for all $u \in \mathcal{S}$,*

$$(99) \quad C^{-1} \|u\|_{\dot{C}^\epsilon} \leq \left\| \frac{\|u(x+y) - u(x)\|_{L_x^\infty}}{|y|^\epsilon} \right\|_{L_y^\infty} \leq C \|u\|_{\dot{C}^\epsilon},$$

and

$$(100) \quad C^{-1} \|u\|_{\tilde{L}_t^1(\dot{C}^\epsilon)} \leq \left\| \int_0^t \frac{\|u(\tau, x+y) - u(\tau, x)\|_{L_x^\infty}}{|y|^\epsilon} d\tau \right\|_{L_y^\infty} \leq C \|u\|_{\tilde{L}_t^1(\dot{C}^\epsilon)}.$$

Proof. The proof of (99) can be found in Chap. 2 of [4].

Let us just show the left-hand inequality of (100). Since

$$\dot{\Delta}_j u(t, x) = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) (u(t, x-y) - u(t, x)) dy,$$

then

$$\begin{aligned} \int_0^t 2^{j\epsilon} \|\dot{\Delta}_j u(\tau, \cdot)\|_{L_x^\infty} &\leq 2^{jd} \int_{\mathbb{R}^d} 2^{j\epsilon} |y|^\epsilon |h(2^j y)| \int_0^t \frac{\|u(\tau, x-y) - u(\tau, x)\|_{L_x^\infty}}{|y|^\epsilon} d\tau dy \\ &\leq C \left\| \int_0^t \frac{\|u(\tau, x+y) - u(\tau, x)\|_{L_x^\infty}}{|y|^\epsilon} \right\|_{L_y^\infty}, \end{aligned}$$

which ensures the left-hand inequality of (100). The inverse inequality follows immediately after similar changes with respect to time in the classical proof. \square

Next we will prove Proposition 6.1 in three steps. In Step 1 we will mainly deal with the localized solutions $\{\varrho_n\}$ or equivalently, after a one-to-one transformation in time variables, with $\{\tilde{\varrho}_n\}$, which solve heat equations. Thus Proposition 6.3 ensures an a priori estimate for $\tilde{\varrho}_n$ and Lemma 6.4 provides the control on “source” terms. Finally, thanks to Proposition 6.5, we can carry the results from $\{\varrho_n\}$ to ρ : this will be done in Step 2. In order to handle general Besov spaces of form $B_{\infty,r}^s$, we again localize the system, but in Fourier variables, in Step 3. Then we apply the result of Step 2 to ρ_j and a careful calculation on commutator terms will yield the thesis.

Step 1 Estimate for ϱ_n in \dot{E}^ϵ

Let us take first a smooth (e.g. C^3 is enough) partition of unity $\{\psi_n\}_{n \in \mathbb{N}}$ subordinated to a locally finite covering of \mathbb{R}^d . We suppose also that the ψ_n ’s satisfy the following conditions:

- (i) $\text{Supp } \psi_n \subset B(x_n, \delta) \triangleq B_n$, $\forall n \in \mathbb{N}$, with $\delta < 1$ to be determined later;
- (ii) $\sum_n \psi_n \equiv 1$;
- (iii) $0 \leq \psi_n \leq 1$, $\forall n \in \mathbb{N}$ with $\psi_n \equiv 1$ on $B(x_n, \delta/2)$;
- (iv) $\|\nabla^\eta \psi_n\|_{L^\infty} \leq C|\delta|^{-|\eta|}$, $\forall n \in \mathbb{N}$, for $|\eta| \leq 3$;
- (v) for each $x \in \mathbb{R}^d$, there are N_d (depending on the dimension d) elements in $\{\psi_n\}_{n \in \mathbb{N}}$ covering the ball $B(x, \delta/2)$.

Now by multiplying ψ_n to Equation (94), we get the equation for compactly supported function $\varrho_n \triangleq \rho \psi_n$ which is supported on B_n :

$$(101) \quad \begin{cases} \partial_t \varrho_n - \bar{\kappa}_n \Delta \varrho_n = (\kappa - \bar{\kappa}_n) \Delta \varrho_n + \nabla \kappa \cdot \nabla \varrho_n + g_n, \\ \varrho_n|_{t=0} = \varrho_{0,n} = \psi_n \rho_0, \end{cases}$$

where

$$\bar{\kappa}_n(t) \triangleq \frac{1}{\text{vol}(B_n)} \int_{B_n} \kappa(t, y) dy$$

is a function depending only on t , and

$$(102) \quad g_n = -2\kappa \nabla \psi_n \cdot \nabla \rho - (\kappa \Delta \psi_n + \nabla \kappa \cdot \nabla \psi_n) \rho + f \psi_n.$$

For convenience we suppose that there exists a positive constant $C_\kappa \sim \|\kappa\|_{L_{t_0}^\infty(\dot{C}^\epsilon)}$ such that

$$(103) \quad |\kappa(t, x) - \kappa(t, y)| \leq C_\kappa |x - y|^\epsilon, \quad \forall x, y \in \mathbb{R}^d, t \in [0, t_0].$$

Noticing that, by (103), we have $\bar{\kappa}_n \geq \kappa_* > 0$, this ensures that, for all $t \in [0, t_0]$,

$$(104) \quad \|\kappa/\bar{\kappa}_n - 1\|_{L^\infty(B_n)} \leq \kappa_*^{-1} \left\| \frac{1}{\text{vol}(B_n)} \int_{B_n} \kappa(t, x) - \kappa(t, y) dy \right\|_{L^\infty(B_n)} \leq C_\kappa \kappa_*^{-1} \delta^\epsilon.$$

In order to get rid of the variable coefficient $\bar{\kappa}_n(t)$, let us make the one-to-one change in time variable

$$(105) \quad \tau \triangleq \tau(t) = \int_0^t \bar{\kappa}_n(t') dt'.$$

Therefore, the new unknown

$$\tilde{\varrho}_n(\tau, x) \triangleq \varrho_n(t, x),$$

satisfies (observe that $\frac{d\tau}{dt} = \bar{\kappa}_n(t)$)

$$(106) \quad \begin{cases} \partial_\tau \tilde{\varrho}_n - \Delta \tilde{\varrho}_n = \left(\frac{\tilde{\kappa}(\tau)}{\tilde{\kappa}_n(\tau)} - 1 \right) \Delta \tilde{\varrho}_n + \frac{\nabla \tilde{\kappa}(\tau)}{\tilde{\kappa}_n(\tau)} \cdot \nabla \tilde{\varrho}_n + \frac{\tilde{g}_n(\tau)}{\tilde{\kappa}_n(\tau)}, \\ \tilde{\varrho}_n|_{\tau=0} = \rho_{0,n}, \end{cases}$$

where $\tilde{\kappa}(\tau, x) = \kappa(t, x)$, $\tilde{\kappa}_n(\tau) = \bar{\kappa}_n(t)$, $\tilde{\rho}(\tau, x) = \rho(t, x)$, $\tilde{g}_n(\tau, x) = g_n(t, x)$.

This is a heat equation and hence by view of Proposition 6.3, it rests to bound the “source” terms. Estimate (98) and the following one,

$$\|\dot{T}_u v + \dot{R}(u, v)\|_{\tilde{L}_T^1(\dot{C}^\epsilon)} \leq C \|u\|_{L_T^\infty(L^\infty)} \|v\|_{\tilde{L}_T^1(\dot{C}^\epsilon)},$$

imply that the first source term of Equation (106) can be controlled by

$$\begin{aligned} \left\| \left(\frac{\tilde{\kappa}(\tau, \cdot)}{\tilde{\kappa}_n(\tau)} - 1 \right) \Delta \tilde{\varrho}_n(\tau, \cdot) \right\|_{\tilde{L}_T^1(\dot{C}^\epsilon)} &\leq C \|\tilde{\kappa}/\tilde{\kappa}_n - 1\|_{L_T^\infty(L^\infty(B_n))} \|\Delta \tilde{\varrho}_n\|_{\tilde{L}_T^1(\dot{C}^\epsilon)} \\ &\quad + C_{\eta_1} \int_0^T \|\tilde{\kappa}/\tilde{\kappa}_n - 1\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}} \|\Delta \tilde{\varrho}_n\|_{\dot{C}^{\epsilon-2}} + \eta_1 \|\Delta \tilde{\varrho}_n\|_{\tilde{L}_T^1(\dot{C}^\epsilon)}, \end{aligned}$$

for any $\eta_1 \in (0, 1)$ with $C_{\eta_1} \sim \eta_1^{\frac{\epsilon-2}{\epsilon}}$. Besides, Inequality (104) ensures that for all $\tau \in [0, \tau_0]$, with $\tau_0 = \tau(t_0)$,

$$\|\tilde{\kappa}/\tilde{\kappa}_n - 1\|_{L_{\tau_0}^\infty(B_n)} \leq C_{\kappa} \kappa_*^{-1} \delta^\epsilon,$$

which implies, for $T \in [0, \tau_0]$,

$$(107) \quad \left\| \left(\frac{\tilde{\kappa}(\tau, \cdot)}{\tilde{\kappa}_n(\tau)} - 1 \right) \Delta \tilde{\varrho}_n(\tau, \cdot) \right\|_{\tilde{L}_T^1(\dot{C}^\epsilon)} \leq C_{\eta_1} \int_0^T \|\tilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}} \|\tilde{\varrho}_n\|_{\dot{C}^\epsilon} + (C C_{\kappa} \kappa_*^{-1} \delta^\epsilon + \eta_1) \|\tilde{\varrho}_n\|_{\tilde{L}_T^1(\dot{C}^{2+\epsilon})}.$$

By Lemma 4.3, for any $\eta > 0$, there exists $C_\eta \sim \eta^{-1}$ such that

$$\|T_u v + R(u, v)\|_{\tilde{L}_T^1(\dot{C}^\epsilon)} \leq C_\eta \int_0^T \|u\|_{L^\infty}^2 \|v\|_{\dot{C}^{\epsilon-1}} + \eta \|v\|_{\tilde{L}_T^1(\dot{C}^{\epsilon+1})}.$$

Thus, also by use of Lemma 6.4 with $a = 1 - \epsilon$ and $b = 1 + \epsilon$, for any $\eta_2 \in (0, 1)$ we have the following (with $C_{\eta_2} \sim \eta_2^{-1}$):

$$(108) \quad \left\| \frac{\nabla \tilde{\kappa}(\tau, \cdot)}{\tilde{\kappa}_n(\tau)} \cdot \nabla \tilde{\varrho}_n(\tau, \cdot) \right\|_{\tilde{L}_T^1(\dot{C}^\epsilon)} \leq C_{\eta_2} \int_0^T \left(\|\nabla \tilde{\kappa}\|_{L^\infty}^2 + \|\nabla \tilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}} \right) \|\tilde{\varrho}_n\|_{\dot{C}^\epsilon} + \eta_2 \|\tilde{\varrho}_n\|_{\tilde{L}_T^1(\dot{C}^{2+\epsilon})}.$$

Now let us choose δ, η_1, η_2 such that

$$(109) \quad C_0 C C_{\kappa} \kappa_*^{-1} \delta^\epsilon, \quad C_0 \eta_1, \quad C_0 \eta_2 \leq 1/6,$$

with the same C_0 in (97). Then, from Proposition 6.3 and estimates (107), (108), for any $t \in [0, T_0]$ we get, for some “harmless” constant still denoted by C ,

$$\|\tilde{\varrho}_n\|_{L_T^\infty(\dot{C}^\epsilon) \cap \tilde{L}_T^1(\dot{C}^{2+\epsilon})} \leq C \left(\|\varrho_{0,n}\|_{\dot{C}^\epsilon} + \int_0^T \left(\|\tilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}} + \|\nabla \tilde{\kappa}\|_{L^\infty}^2 + \|\nabla \tilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}} \right) \|\tilde{\varrho}_n\|_{\dot{C}^\epsilon} + \|\tilde{g}_n\|_{\tilde{L}_T^1(\dot{C}^\epsilon)} \right).$$

Since $\kappa_* \leq \bar{\kappa}_n \leq \kappa^*$, after transformation in time (105) we arrive at

$$(110) \quad \|\varrho_n\|_{L_T^\infty(\dot{C}^\epsilon) \cap \tilde{L}_T^1(\dot{C}^{2+\epsilon})} \leq C \left(\|\varrho_{0,n}\|_{\dot{C}^\epsilon} + \int_0^T K_1 \|\varrho_n\|_{\dot{C}^\epsilon} + \|g_n\|_{\tilde{L}_T^1(\dot{C}^\epsilon)} \right)$$

for all $T \in [0, t_0]$, with

$$K_1 = \|\kappa\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}} + \|\nabla \kappa\|_{L^\infty}^2 + \|\nabla \kappa\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}},$$

provided

$$(111) \quad \delta^{-\epsilon} = C_\delta C_\kappa \leq C_\delta \|\kappa\|_{L_{t_0}^\infty(\dot{C}^\epsilon)} \quad \text{for some constant } C_\delta \text{ depending only on } d, \epsilon.$$

Step 2 Estimate for ρ

Now we come back to consider $\rho = \sum_n \varrho_n$. By assumptions on the partition of unity $\{\psi_n\}$, for any x there exist N_d balls of our covering which cover the small ball $B(x, \delta/4)$. Therefore, from Inequality (99) we have

$$\begin{aligned} \|\rho\|_{\tilde{L}_t^1(\dot{C}^\epsilon)} &\leq C \left\| \int_0^t \frac{\|\rho(\tau, x+y) - \rho(\tau, x)\|_{L_x^\infty}}{|y|^\epsilon} \right\|_{L_y^\infty} \\ &\leq C \sup_{|y| > \delta/4} \int_0^t \frac{\|\rho(x+y) - \rho(x)\|_{L_x^\infty}}{|y|^\epsilon} + C \sup_{|y| \leq \delta/4} \int_0^t \frac{\|\rho(x+y) - \rho(x)\|_{L_x^\infty}}{|y|^\epsilon}, \end{aligned}$$

whose second term can be controlled by

$$N_d C \sup_{|y-z| \leq \delta/4} \int_0^t \frac{\sup_n \|\varrho_n(x+y) - \varrho_n(x+z)\|_{L_x^\infty}}{|y-z|^\epsilon}.$$

Thus we find

$$\|\rho\|_{\tilde{L}_t^1(\dot{C}^\epsilon)} \leq C \delta^{-\epsilon} \int_0^t \|\rho\|_{L^\infty} + N_d C \sup_n \|\varrho_n\|_{\tilde{L}_t^1(\dot{C}^\epsilon)}.$$

Similarly, we have

$$\|\rho\|_{\tilde{L}_t^\infty(\dot{C}^\epsilon)} \leq C \delta^{-\epsilon} \|\rho\|_{L_t^\infty(L^\infty)} + C \sup_n \|\varrho_n\|_{\tilde{L}_t^\infty(\dot{C}^\epsilon)}.$$

Since $\nabla^2 \rho = \sum_n (\nabla^2 \varrho_n)$, from the same arguments as before we infer

$$\|\rho\|_{\tilde{L}_t^1(\dot{C}^{2+\epsilon})} \leq C \|\nabla^2 \rho\|_{\tilde{L}_t^1(\dot{C}^\epsilon)} \leq C \delta^{-\epsilon} \int_0^t \|\nabla^2 \rho\|_{L^\infty} + C \sup_n \|\varrho_n\|_{\tilde{L}_t^1(\dot{C}^{2+\epsilon})}.$$

Therefore, to sum up, for all $t \in [0, t_0]$,

$$\begin{aligned} \|\rho\|_{\tilde{L}_t^\infty(\dot{C}^\epsilon) \cap \tilde{L}_t^1(\dot{C}^{2+\epsilon})} &\leq C \delta^{-\epsilon} \left(\|\rho\|_{L_t^\infty(L^\infty)} + \int_0^t \|\nabla^2 \rho\|_{L^\infty} \right) + C \sup_n \|\varrho_n\|_{\tilde{L}_t^\infty(\dot{C}^\epsilon) \cap \tilde{L}_t^1(\dot{C}^{2+\epsilon})} \\ &\leq C \delta^{-\epsilon} \left(\|\rho\|_{L_t^\infty(L^\infty)} + \int_0^t \|\nabla^2 \rho\|_{L^\infty} \right) + \\ &\quad + C \sup_n \left(\|\varrho_{0,n}\|_{\dot{C}^\epsilon} + \int_0^t K_1 \|\varrho_n\|_{\dot{C}^\epsilon} + \|g_n\|_{\tilde{L}_t^1(\dot{C}^\epsilon)} \right), \end{aligned}$$

with the second inequality deriving from Estimate (110). Thanks to (96) and the fact that

$$\|\varrho_n\|_{C^\epsilon} = \|\rho \psi_n\|_{C^\epsilon} \leq C \|\rho\|_{C^\epsilon} \|\psi_n\|_{C^\epsilon} \leq C \delta^{-\epsilon} \|\rho\|_{C^\epsilon},$$

we thus have the following estimate for ρ in the nonhomogeneous Hölder space

$$\begin{aligned} \|\rho\|_{\tilde{L}_t^\infty(C^\epsilon) \cap \tilde{L}_t^1(C^{2+\epsilon})} &\leq C \|\rho\|_{L_t^\infty(L^\infty)} + C \int_0^t \|\rho\|_{L^\infty} + \|\rho\|_{\tilde{L}_t^\infty(\dot{C}^\epsilon) \cap \tilde{L}_t^1(\dot{C}^{2+\epsilon})} \\ &\leq C \delta^{-\epsilon} \left(\|\rho_0\|_{C^\epsilon} + \int_0^t (\|\nabla^2 \rho\|_{L^\infty} + \|\rho\|_{L^\infty} + \|f\|_{L^\infty}) + \int_0^t K_1 \|\rho\|_{C^\epsilon} \right) \\ &\quad + C \sup_n \|g_n\|_{\tilde{L}_t^1(\dot{C}^\epsilon)}, \end{aligned}$$

It rests us to bound g_n uniformly. In fact, starting from definition (102) of g_n , we follow the same

method to get (107) and (108) to arrive at

$$\begin{aligned} \|g_n\|_{\tilde{L}_t^1(\dot{C}^\epsilon)} &\leq C \int_0^t \left(\eta^{-1} (\|\kappa \nabla \psi_n\|_{L^\infty}^2 + \|\kappa \nabla \psi_n\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}}) + \|\kappa \Delta \psi_n + \nabla \kappa \cdot \nabla \psi_n\|_{L^\infty} \right) \|\rho\|_{\dot{C}^\epsilon} \\ &\quad + \eta \|\rho\|_{\tilde{L}_t^1(\dot{C}^{2+\epsilon})} + C \int_0^t \|\kappa \Delta \psi_n + \nabla \kappa \cdot \nabla \psi_n\|_{\dot{C}^\epsilon} \|\rho\|_{L^\infty} + C \delta^{-\epsilon} \int_0^t \|f\|_{L^\infty} + C \|f\|_{\tilde{L}_t^1(\dot{C}^\epsilon)} \\ &\leq C_\eta \delta^{-2} \int_0^t \left(1 + \|\kappa\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}} + \|\kappa\|_{C^{1+\epsilon}} \right) \|\rho\|_{C^\epsilon} + \eta \|\rho\|_{\tilde{L}_t^1(\dot{C}^{2+\epsilon})} + C \delta^{-\epsilon} \|f\|_{\tilde{L}_t^1(\dot{C}^\epsilon)}, \end{aligned}$$

where we have used $\|f\|_{L_t^1(L^\infty) \cap \tilde{L}_t^1(\dot{C}^\epsilon)} \leq C \|f\|_{\tilde{L}_t^1(\dot{C}^\epsilon)}$.

We finally get a priori estimates for ρ :

$$\|\rho\|_{\tilde{L}_t^\infty(C^\epsilon) \cap \tilde{L}_t^1(C^{2+\epsilon})} \leq C \delta^{-\epsilon} \left(\|\rho_0\|_{C^\epsilon} + \int_0^t \|\rho\|_{C^2} + \|f\|_{\tilde{L}_t^1(C^\epsilon)} \right) + C \delta^{-2} \int_0^t K_2 \|\rho\|_{C^\epsilon},$$

with

$$K_2 = 1 + \|\kappa\|_{C^{1+\epsilon}}^{\frac{2}{1+\epsilon}} \geq C \left(K_1 + 1 + \|\kappa\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}} + \|\kappa\|_{C^{1+\epsilon}} \right).$$

Thus, by a direct interpolation inequality, that is to say

$$\delta^{-\epsilon} \|\rho\|_{L_t^1(C^2)} \leq C_\eta \delta^{-2} \int_0^t \|\rho\|_{C^\epsilon} + \eta \|\rho\|_{\tilde{L}_t^1(C^{2+\epsilon})},$$

Gronwall's Inequality tells us

$$(112) \quad \|\rho\|_{\tilde{L}_t^\infty(C^\epsilon) \cap \tilde{L}_t^1(C^{2+\epsilon})} \leq C \delta^{-\epsilon} \exp \left(C \delta^{-2} \int_0^t K_2 \right) \left(\|\rho_0\|_{C^\epsilon} + \|f\|_{\tilde{L}_t^1(C^\epsilon)} \right),$$

which is just Estimate (95) when $s = \epsilon$ and $r = \infty$.

Step 3 The general case $B_{\infty,r}^s$

Now we want to deal with the general case $B_{\infty,r}^s$. Let us apply $\widetilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$, $j \geq 0$, to System (94), yielding

$$(113) \quad \begin{cases} \partial_t \bar{\rho}_j - \operatorname{div}(\kappa \nabla \bar{\rho}_j) = \bar{f}_j - \bar{R}_j, \\ \bar{\rho}_j|_{t=0} = \bar{\rho}_{0,j}, \end{cases}$$

with

$$\bar{\rho}_j = \widetilde{\Delta}_j \rho, \quad \bar{f}_j = \widetilde{\Delta}_j f, \quad \bar{R}_j = \operatorname{div}([\kappa, \widetilde{\Delta}_j] \nabla \rho), \quad \bar{\rho}_{0,j} = \widetilde{\Delta}_j \rho_0.$$

We apply the a priori estimate (112) to the solution $\bar{\rho}_j$ of System (113), for some positive $\epsilon < \min\{s, 1\}$, entailing

$$\|\bar{\rho}_j\|_{\tilde{L}_t^\infty(C^\epsilon) \cap \tilde{L}_t^1(C^{2+\epsilon})} \leq C \delta^{-\epsilon} \exp \left(C \delta^{-2} \int_0^t K_2 \right) \left(\|\bar{\rho}_{0,j}\|_{C^\epsilon} + \|\bar{f}_j - \bar{R}_j\|_{\tilde{L}_t^1(C^\epsilon)} \right).$$

Let us notice that for $j \geq 0$, denoted by $\rho_j = \Delta_j \rho$ and $\rho_q = \Delta_q \rho$ as usual, then we have

$$\Delta_j \bar{\rho}_j = \rho_j \quad \text{and} \quad \Delta_q \bar{\rho}_j \equiv 0 \text{ if } |q - j| \geq 2.$$

Hence the above inequality gives

$$\begin{aligned} 2^{j\epsilon} \|\rho_j\|_{L_t^\infty(L^\infty)} + 2^{j(2+\epsilon)} \int_0^t \|\rho_j\|_{L^\infty} &\leq \\ &\leq C \delta^{-\epsilon} \exp \left(C \delta^{-2} \int_0^t K_2 \right) \left(2^{j\epsilon} \sum_{|j-q| \leq 1} \left(\|\rho_{0,q}\|_{L^\infty} + \int_0^t \|f_q\|_{L^\infty} \right) + \|\bar{R}_j\|_{\tilde{L}_t^1(C^\epsilon)} \right). \end{aligned}$$

Let us focus on $\|\bar{R}_j\|_{\tilde{L}_t^1(C^\epsilon)}$ for a while. As usual, we decompose \bar{R}_j into four parts:

$$\begin{aligned}\bar{R}_j^1 &= \sum_{|j-q|\leq 3} \operatorname{div}([S_{q-1}\tilde{\kappa}, \widetilde{\Delta_j}] \nabla \Delta_q \rho), & \bar{R}_j^2 &= \sum_{q\geq j-3} \operatorname{div} \widetilde{\Delta_j}(\Delta_q \tilde{\kappa} S_{q+1} \nabla \rho), \\ \bar{R}_j^3 &= \sum_{q\geq j-3} \operatorname{div}(\Delta_q \tilde{\kappa} \widetilde{\Delta_j} S_{q+1} \nabla \rho), & \bar{R}_j^4 &= \operatorname{div}([\Delta_{-1}\kappa, \widetilde{\Delta_j}] \nabla \rho), \quad \text{with } \tilde{\kappa} = \kappa - \Delta_{-1}\kappa.\end{aligned}$$

It is easy to see that the Fourier transform of the terms \bar{R}_j^1 and \bar{R}_j^2 is supported near the ring $2^j\mathcal{C}$; thus, by Lemma 2.100 of [4], we get (for some sequence $(c_j)_j \in \ell^r$)

$$\begin{aligned}\|\bar{R}_j^1\|_{\tilde{L}_t^1(C^\epsilon)} &\leq C2^{j(1+\epsilon)} \sum_{|j-q|\leq 3} \int_0^t \|[S_{q-1}\tilde{\kappa}, \widetilde{\Delta_j}] \nabla \Delta_q \rho\|_{L^\infty} \\ &\leq C2^{j\epsilon} \sum_{|j-q|\leq 3} \int_0^t \|\nabla \kappa\|_{L^\infty} \|\nabla \Delta_q \rho\|_{L^\infty} \\ &\leq C2^{j(\epsilon-s)} c_j \int_0^t \|\nabla \kappa\|_{L^\infty} \|\nabla \rho\|_{B_{\infty,r}^s} \end{aligned}$$

and, for some $s > -1$ and $(c_j)_j \in \ell^r$

$$\begin{aligned}\|\bar{R}_j^2\|_{\tilde{L}_t^1(C^\epsilon)} &\leq C2^{j(1+\epsilon)} \sum_{q\geq j-3} \int_0^t \|\Delta_q \tilde{\kappa}\|_{L^\infty} \|S_{q+1} \nabla \rho\|_{L^\infty} \\ &\leq C2^{j(\epsilon-s)} \sum_{q\geq j-3} 2^{(j-q)(1+s)} c_q \int_0^t \|\nabla \kappa\|_{B_{\infty,r}^s} \|\nabla \rho\|_{L^\infty}, \\ &\leq C2^{j(\epsilon-s)} c_j \int_0^t \|\nabla \kappa\|_{B_{\infty,r}^s} \|\nabla \rho\|_{L^\infty}.\end{aligned}$$

Let us now consider \bar{R}_j^3 . Each $\Delta_q \tilde{\kappa} \widetilde{\Delta_j} S_{q+1} \nabla \rho$ has Fourier transform supported near a ball centered at origin with radius of size 2^q . Therefore, arguing as above, we find, for some $(c_j)_j \in \ell^r$

$$\begin{aligned}\|\bar{R}_j^3\|_{\tilde{L}_t^1(C^\epsilon)} &\leq \sum_{q\geq j-3} \sup_{j'\leq q-2} 2^{j'(1+\epsilon)} \int_0^t \|\Delta_q \tilde{\kappa}\|_{L^\infty} \|\widetilde{\Delta_j} \nabla \rho\|_{L^\infty} \\ &\leq C \sum_{q\geq j-3} 2^{q(\epsilon+1)} \int_0^t 2^{-q} \|\nabla \kappa\|_{L^\infty} c_j 2^{-js} \|\nabla \rho\|_{B_{\infty,r}^s} \\ &\leq C2^{j(\epsilon-s)} c_j \int_0^t \|\nabla \kappa\|_{L^\infty} \|\nabla \rho\|_{B_{\infty,r}^s}.\end{aligned}$$

Since \bar{R}_j^4 Fourier transform is always supported near a ring $2^j\mathcal{C}$, arguing for instance as in the proof of Lemma 4.2 (see the Appendix), we can bound $\|\bar{R}_j^4\|_{\tilde{L}_t^1(C^\epsilon)}$ also by the above quantity.

To conclude, we have got a priori estimate for ρ_j :

$$\begin{aligned}\|\rho_j\|_{L_t^\infty(L^\infty)} + 2^{2j} \int_0^t \|\rho_j\|_{L^\infty} &\leq C\delta^{-\epsilon} \exp\left(C\delta^{-2} \int_0^t K_2\right) \times \\ &\times \left(\sum_{|j-q|\leq 2} \left(\|\rho_{0,q}\|_{L^\infty} + \int_0^t \|f_q\|_{L^\infty}\right) + 2^{-js} c_j \int_0^t \|\nabla \kappa\|_{B_{\infty,r}^s} \|\nabla \rho\|_{L^\infty} + \int_0^t \|\nabla \kappa\|_{L^\infty} \|\nabla \rho\|_{B_{\infty,r}^s} \right).\end{aligned}$$

Therefore, we multiply both sides by 2^{js} (for $s > -1$) and then take ℓ^r norm, to arrive at

$$\begin{aligned} \|\rho\|_{\tilde{L}_t^\infty(B_{\infty,r}^s) \cap \tilde{L}_t^1(B_{\infty,r}^{s+2})} &\leq C\delta^{-\epsilon} \exp\left(C\delta^{-2} \int_0^t K_2\right) \times \\ &\times \left(\|\rho_0\|_{B_{\infty,r}^s} + \|f\|_{\tilde{L}_t^1(B_{\infty,r}^s)} + \int_0^t \|\nabla \kappa\|_{L^\infty} \|\nabla \rho\|_{B_{\infty,r}^s} + \|\nabla \kappa\|_{B_{\infty,r}^s} \|\nabla \rho\|_{L^\infty}\right). \end{aligned}$$

The direct application of interpolation inequalities and embedding results reduces the above estimate into (95).

6.2 The case of finite energy data

The proof of Theorem 2.9 is just as the proof presented in Subsection 4.2, with some changes pertaining to the energy. We will follow the standard process in proving the local existence result also: we construct a sequence of approximate smooth solutions which have uniform bounds and then we show the convergence to a unique solution. In each step we will try to sketch the analysis first and then present the proof into details.

Let us make some simplification. Sometimes a few estimates may depend on the existing time T^* , and hence a priori we suppose that $T^* \leq 1$. We also assume that all the constants appearing in the sequel, such as C, C_M, C_E , are bigger than 1. We always denote $\delta b^n = b(\rho^n) - b(\rho^{n-1})$ and $\delta a^n = a(\rho^n) - a(\rho^{n-1})$.

6.2.1 Construction of a sequence of approximate solutions

As usual, after fixing $(\varrho^0, u^0, \nabla \pi^0) = (\varrho_0, u_0, 0)$, we consider inductively the n -th approximate density ϱ^n to be the unique global solution of the following linear system

$$(114) \quad \begin{cases} \partial_t \varrho^n + u^{n-1} \cdot \nabla \varrho^n - \operatorname{div}(\kappa^{n-1} \nabla \varrho^n) = 0, \\ \varrho^n|_{t=0} = \varrho_0, \end{cases}$$

with $\kappa^{n-1} = \kappa(\rho^{n-1})$ and the n -th approximate velocity and pressure $(u^n, \nabla \pi^n)$ satisfying

$$(115) \quad \begin{cases} \partial_t u^n + (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n) \cdot \nabla u^n + \lambda^n \nabla \pi^n = h^{n-1}, \\ \operatorname{div} u^n = 0, \\ u^n|_{t=0} = u_0, \end{cases}$$

where we have set as before, $\lambda^n = \lambda(\rho^n)$ and the same h^{n-1} defined in (73):

$$(116) \quad h^{n-1} = (\rho^{n-1})^{-1} \left(\Delta b(\rho^{n-1}) \nabla a(\rho^{n-1}) + u^{n-1} \cdot \nabla^2 a(\rho^{n-1}) + \nabla b(\rho^{n-1}) \cdot \nabla^2 a(\rho^{n-1}) \right).$$

We pay attention that, compared with System (73), the coefficients $u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n$ and λ^n of System (115) here are chosen to keep accordance with Equation (114), in order to get the energy identity for u^n . Indeed, noticing $\operatorname{div} u^n = 0$ and Equation (114) for ρ^n , we can take the $L^2(\mathbb{R}^d)$ -inner product between Equation (115)₁ and $\rho^n u^n$, getting at least formally ²

$$(117) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho^n |u^n|^2 = \int_{\mathbb{R}^d} \rho^n h^{n-1} \cdot u^n.$$

Furthermore, the initial data for $\{\varrho^n\}$ are chosen the same. We will see later that this choice ensures us to estimate the difference sequence $\{\delta \varrho^n = \varrho^n - \varrho^{n-1}\}_{n \geq 2}$ in Space $C_{T^*}(H^1)$ (T^* denotes the existence time),

²We can also follow the scheme (73) which, gives the following by taking L^2 -inner product between (73)₂ and $\rho^{n-1} u^n$

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho^{n-1} |u^n|^2 + \frac{1}{2} \int_{\mathbb{R}^d} \operatorname{div} \left(\rho^{n-1} (u^{n-2} - u^{n-1}) - (\kappa^{n-2} - \kappa^{n-1}) \nabla \rho^{n-1} \right) |u^n|^2 = \int_{\mathbb{R}^d} \rho^{n-1} h^{n-1} \cdot u^n.$$

For simplicity we choose iterative linear systems (114) and (115) here.

although the initial density ϱ_0 belongs to L^2 only. This estimate makes it possible to bound the difference of the source term $h^n - h^{n-1}$ (notice it involves terms such as $\Delta \delta b^n \nabla a^n$) in Space $L_{T^*}^1(L^2)$. Therefore the convergence of the velocity sequence in Space $C_{T^*}(L^2)$ follows. We will explain the convergence process in detail in Subsection 6.2.2 below.

In this paragraph, we aim at proving the existence of the solution sequence $(\rho^n, u^n, \nabla \pi^n)$ and uniform estimates for it. We want to show estimates (71) and (72), with a change pertaining to $\nabla \pi^n$:

$$(118) \quad \|\nabla \pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^s) \cap L_{T^*}^1(L^2)} \leq \tau^{\frac{1}{2s_p}}, \quad \text{with} \quad s_p = \max \left\{ s, s - \frac{d}{p} + \frac{d}{2} \right\}.$$

We also prove the following inductive estimate involving energy:

$$(119) \quad \|\varrho^n\|_{L_{T^*}^\infty(L^2)} + \|\nabla \varrho^n\|_{L_{T^*}^2(L^2)} + \|u^n\|_{L_{T^*}^\infty(L^2)} \leq C_E E_0,$$

with $E_0 := \|\varrho_0\|_{L^2} + \|u_0\|_{L^2}$ and some constant C_E depending on d, s, ρ_*, ρ^* .

In fact, the subtlety still comes out when dealing with the pressure term $\nabla \pi^n$. Getting inductive estimate (118) relies on the divergence-free condition of u^n , which helps us to write the equations for π^n in different forms, such as (121), (122) and (123) in the following. Informations on low and high frequencies issue from these equations separately. This yields estimates for $\nabla \pi$ itself finally, thanks to

$$(120) \quad \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} \leq C \|\Delta_{-1} \nabla \pi\|_{L_t^1(L^p)} + \|\Delta \pi\|_{\tilde{L}_t^1(B_{p,r}^{s-1})}.$$

In Subsection 4.2, where $p \in [2, 4]$, by view of the divergence-form elliptic equation for π^n , the quantity $\|\nabla \pi^n\|_{L_t^1(L^2)}$ (involving the low frequency information) can be controlled by a simple use of Hölder's Inequality $\|fg\|_{L^2} \leq \|f\|_{L^4} \|g\|_{L^4}$ to the quadratic "source" terms. Now, we also apply "div" to Equation (115)₁, getting the following equation for π^n :

$$(121) \quad \operatorname{div}(\lambda^n \nabla \pi^n) = \operatorname{div} \left(h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n) \cdot \nabla u^n \right).$$

But here, thanks to Estimate (119), one uses $\|fg\|_{L^2} \leq \|f\|_{L^2} \|g\|_{L^\infty}$ to bound $\|\nabla \pi^n\|_{L_{T^*}^1(L^2)}$ and hence the low frequencies, when $p \geq 2$.

If $p \in (1, 2)$, instead, we multiply Equation (115)₁ by ρ^n and then apply "div" to it. Recalling that $\operatorname{div} \partial_t u^n = 0$, we get a Laplace equation for π^n :

$$(122) \quad \Delta \pi^n = -\operatorname{div} \left((\rho^n - 1) \partial_t u^n + (\rho^n u^{n-1} - \kappa^{n-1} \nabla \rho^n) \cdot \nabla u^n \right) + \operatorname{div}(\rho^n h^{n-1}).$$

One observes that if $p \in (1, 2)$, then

$$B_{p,r}^{s-1-d/q} \hookrightarrow L^q, \quad \forall q \geq 2.$$

Thus we perform $\|fg\|_{L^p} \leq \|f\|_{L^2} \|g\|_{L^{p^*}}$ with $1/p^* + 1/2 = 1/p$ to these right-hand side quadratic terms. Since ϱ^n is already in L^q for all $q \geq 2$ according to Maximum Principle and Energy Estimate, this requires $\|\partial_t u^n\|_{L_t^1(L^2)}$. Luckily it is related to $\|\nabla \pi^n\|_{L_t^1(L^2)}$ by Equation (115)₁.

In order to control the high frequency, it is enough to show $\Delta \pi^n$ in $\tilde{L}_t^1(B_{p,r}^{s-1})$. Let us rewrite Equation (121) as

$$(123) \quad \Delta \pi^n = \nabla \log \rho^n \cdot \nabla \pi^n + \rho^n \operatorname{div} \left(h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n) \cdot \nabla u^n \right).$$

The quantity $\|\nabla \pi^n\|_{\tilde{L}_t^1(B_{p,r}^{s-1})}$ can be interpolated between $\tilde{L}_t^1(B_{p,r}^s)$ and $L_t^1(L^p)$ when $p \in (1, 2)$ (resp. $L_t^1(L^2)$ when $p \in [2, \infty)$ since $L^2 \hookrightarrow B_{p,\infty}^{d/p-d/2}$) and the index s_p will come out from these interpolation.

As in [14], let us also notice that the limit functional space $B_{\infty,r}^{s-1}$ with $s = r = 1$ is no longer an algebra as spaces $B_{p,r}^{s-1}$ with $p < +\infty$. But we still have the following product estimates by considering paraproducts and remainder separately:

$$(124) \quad \|fg\|_{B_{\infty,1}^0} \leq C \|f\|_{B_{\infty,1}^\epsilon} \|g\|_{B_{\infty,1}^0}, \quad \forall \epsilon > 0.$$

Thus we just have to modify the above interpolation a little to make quantity $\|\nabla \pi^n\|_{\tilde{L}_{T^*}^1(B_{\infty,1}^\varepsilon)}$ appear. We refer also to [14] for the inequality (due to $\operatorname{div} u^n = 0$)

$$(125) \quad \|\operatorname{div}(u^{n-1} \cdot \nabla u^n)\|_{B_{\infty,1}^0} \leq C \|u^{n-1}\|_{B_{\infty,1}^1} \|\nabla u^n\|_{B_{\infty,1}^0}.$$

Now let us realize the above analysis.

Since $(\varrho^0, u^0, \nabla \pi^0) = (\varrho_0, u_0, 0)$, then by choosing small T^* , Estimates (71), (72), (118) and (119) all hold for $n = 0$. Next we suppose $(\varrho^{n-1}, u^{n-1}, \nabla \pi^{n-1})$ to belong to the functional space E defined by

$$(126) \quad \left(C(\mathbb{R}^+; B_{p,r}^s \cap L^2) \cap L_{\text{loc}}^2(H^1) \cap \tilde{L}_{\text{loc}}^1(B_{p,r}^{s+2}) \right) \times \left(C(\mathbb{R}^+; B_{p,r}^s \cap L^2) \right) \times \left(\tilde{L}_{\text{loc}}^1(B_{p,r}^s) \cap L_{\text{loc}}^1(L^2) \right),$$

and such that the inductive assumptions hold. We just have to show that the n -th unknown $(\varrho^n, u^n, \nabla \pi^n)$ defined by System (114) and (115) belongs to the same space, satisfying the same conditions.

Instead of working with ϱ^n and u^n together, we first consider ϱ^n independently. By smoothing out the initial datum and coefficients, applying Proposition 4.5 or Proposition 6.1 and then showing the convergence, we can get the unique global solution ϱ^n of the linear system (114). The process is quite standard and we omit it. One observes that Estimates (48) and (95) imply $\varrho^n \in \tilde{L}_t^\infty(B_{p,r}^s) \cap \tilde{L}_t^1(B_{p,r}^{s+2})$ for any finite $t > 0$. Thus in particular $\varrho^n \in C(\mathbb{R}^+; B_{p,r}^s)$.

On the other hand, since $u^{n-1} \in L_{\text{loc}}^\infty(L^\infty)$ and $\operatorname{div} u^{n-1} = 0$ (noticing Equation (115)₂), energy inequality (24) for ϱ^n follows. Thus $\varrho^n \in C(\mathbb{R}^+; L^2) \cap L_{\text{loc}}^2(H^1)$, and we have

$$\|\varrho^n\|_{L_{T^*}^\infty(L^2)} + \|\nabla \varrho^n\|_{L_{T^*}^2(L^2)} \leq \frac{1}{2} C_E \|\varrho_0\|_{L^2}.$$

It is easy to see that, by Maximum Principle,

$$\rho_* \leq 1 + \varrho^n(t, x) = \rho^n(t, x) \leq \rho^*, \quad \forall t \in \mathbb{R}^+, x \in \mathbb{R}^d.$$

As in Subsection 4.2, we introduce ϱ_L to be the solution of the free heat equation with initial datum ϱ_0 , which satisfies (74) and (75) also. Correspondingly, the remainder $\bar{\varrho}^n := \varrho^n - \varrho_L$ solves System (77), with ϱ_L instead of ϱ_L^n . Propositions 4.5 and 6.1 thus imply that for $p \in (1, \infty]$,

$$\|\bar{\varrho}^n\|_{\tilde{L}_t^\infty(B_{p,r}^s) \cap \tilde{L}_t^1(B_{p,r}^{s+2})} \leq \left(C^{n-1}(t) e^{C^{n-1}(t) \mathcal{K}^{n-1}(t)} \right) \|f^n\|_{\tilde{L}_t^1(B_{p,r}^s)},$$

where

$$f^n = -u^{n-1} \cdot \nabla \bar{\varrho}^n - u^{n-1} \cdot \nabla \varrho_L + \operatorname{div}((\kappa^{n-1} - 1) \nabla \varrho_L).$$

Here $C^{n-1}(t)$ depends also on $\|\varrho^{n-1}\|_{L_t^\infty(B_{p,r}^s)}$ when $p = \infty$, and by embeddings, we can take

$$\mathcal{K}^{n-1}(t) := t + \|\nabla \kappa^{n-1}\|_{L_t^2(L^\infty)}^2 + \|\nabla \kappa^{n-1}\|_{L_t^1(B_{p,r}^s)}.$$

Since the inductive assumption (71) holds, we derive on the whole time interval $[0, T^*]$

$$C^{n-1}(t) e^{C^{n-1}(t) \mathcal{K}^{n-1}(t)} \leq C_{\mathcal{K}},$$

for some constant $C_{\mathcal{K}}$ depending only³ on M . Furthermore, product estimates, interpolation inequality and Estimate (75) ensure that

$$\begin{aligned} \|f^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)} &\leq C \|u^{n-1}\|_{\tilde{L}_{T^*}^2(B_{p,r}^s)} \|\nabla \bar{\varrho}^n\|_{\tilde{L}_{T^*}^2(B_{p,r}^s)} + C C_M \tau^2 \\ &\leq C_\varepsilon C_{\mathcal{K}} \|u^{n-1}\|_{\tilde{L}_{T^*}^2(B_{p,r}^s)}^2 \|\bar{\varrho}^n\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} + C_{\mathcal{K}}^{-1} \varepsilon \|\bar{\varrho}^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s+2})} + C C_M \tau^2. \end{aligned}$$

Therefore the smallness statement (76) pertaining to $\bar{\varrho}^n$ is verified and hence inductive assumption (71) holds for ϱ^n .

³In fact, $C_{\mathcal{K}}$ also depends on T^* , which can be “omitted” since we have supposed a priori $T^* \leq 1$.

As to solve System (115), a convenient way is to view it as a transport equation of the velocity u^n . For each finite time t , if there exists a constant C_t (depending on t , $\|\varrho^k, u^{n-1}, \nabla \pi^{n-1}\|_{E_{p,r}^s(t)}$, $\|\varrho^k\|_{L_t^\infty(L^2) \cap L_t^2(H^1)}$, $k = n-1, n$, and $\|u^{n-1}\|_{L_t^\infty(L^2)}$) such that the $\tilde{L}_t^1(B_{p,r}^s)$ -norm of the “source” term $-\lambda^n \nabla \pi^n + h^{n-1}$ is bounded by $C_t(1 + \mathcal{U}^n(t))$ with

$$\mathcal{U}^n(t) = \|u^n\|_{\tilde{L}_t^\infty(B_{p,r}^s)},$$

then a standard proof gives a unique global solution $u^n \in C(\mathbb{R}^+; B_{p,r}^s)$ (see the proof of Theorem 3.19 in [4]). If furthermore $\nabla \pi^n, h^n \in L_{\text{loc}}^1(L^2)$, then $u^n \in C(\mathbb{R}^+; L^2)$, by view of Energy Identity (117). Therefore, it reduces to get a priori estimates of $\nabla \pi^n, h^{n-1}$ in $\tilde{L}_{\text{loc}}^1(B_{p,r}^s) \cap L_{\text{loc}}^1(L^2)$, by use of C_t and \mathcal{U}^n . In fact, it will immediately follow by observing the estimates in the demonstration below of the inductive estimates for u^n and $\nabla \pi^n$.

In the following, we a priori demonstrate inductive estimates for u^n and $\nabla \pi^n$. The idea is that, by terms of τ and

$$\Pi^n \triangleq \|\nabla \pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)},$$

we are going to bound the following quantities in the following order:

$$\begin{aligned} \|u^n\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} &\rightsquigarrow \|\nabla \pi^n\|_{L_{T^*}^1(L^2)} \rightsquigarrow \|\partial_t u^n\|_{L_{T^*}^1(L^2)} \\ &\rightsquigarrow \|\nabla \pi^n\|_{L_{T^*}^1(L^p)} \rightsquigarrow \|\nabla \pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s-1})} \rightsquigarrow \|\Delta \pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s-1})}, \end{aligned}$$

which include informations on both low frequencies $\|\nabla \pi^n\|_{L_{T^*}^1(L^p)}$ and high frequencies $\|\Delta \pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s-1})}$. One also notices that if $p \geq 2$, then $\|\nabla \pi^n\|_{L_{T^*}^1(L^2)}$ readily offers bound on $\|\Delta_{-1} \nabla \pi^n\|_{L_{T^*}^1(L^p)}$.

(i) First of all, by view of Equation (115)₁, Lemma 4.1 ensures that

$$\|u^n\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq C e^{C\mathcal{W}^{n-1}(t)} \left(\|u_0\|_{B_{p,r}^s} + \|h^{n-1} - \lambda^n \nabla \pi^n\|_{\tilde{L}_t^1(B_{p,r}^s)} \right),$$

where

$$\mathcal{W}^{n-1}(t) := \int_0^t \|u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n\|_{B_{p,r}^s}.$$

By virtue of inductive assumptions (71) for ϱ^{n-1} , ϱ^n and (72) for u^{n-1} , we easily derive

$$\mathcal{W}^{n-1}(T^*), \|h^{n-1}\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)} \leq CC_M \tau.$$

Thus (as $\|\lambda^n - 1\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq C C_M$) for sufficiently small parameter τ , we have

$$(127) \quad \|u^n\|_{\tilde{L}_{T^*}^\infty(B_{p,r}^s)} \leq C \left(\|u_0\|_{B_{p,r}^s} + CC_M \tau + CC_M \|\nabla \pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)} \right) \leq CC_M (1 + \Pi^n).$$

(ii) Secondly, inductive assumptions (72) and (119) imply

$$\|h^{n-1}\|_{L_{T^*}^1(L^2)} \leq CC_E E_0 \tau.$$

Hence, by view of $\|\nabla u^n\|_{L_{T^*}^2(L^\infty)} \leq (T^*)^{1/2} \|u^n\|_{L_{T^*}^\infty(B_{p,r}^s)}$ and Estimate (127), if $(T^*)^{1/2} \leq \tau$, then Equation (121) entails

$$(128) \quad \|\nabla \pi^n\|_{L_{T^*}^1(L^2)} \leq CC_E E_0 (\tau + \tau \|u^n\|_{L_{T^*}^\infty(B_{p,r}^s)}) \leq CC_E C_M E_0 \tau (1 + \Pi^n).$$

Correspondingly, $\|\partial_t u^n\|_{L_{T^*}^1(L^2)}$ is bounded also by above, with some change of the constant C .

- (iii) Next we also want to bound $\|\nabla\pi^n\|_{L_{T^*}^1(L^p)}$ for $p \in (1, 2)$, which controls the low frequency. It relies on Equation (122). Firstly, since

$$\Delta b(\rho^{n-1}), \nabla^2 a(\rho^{n-1}) \in \tilde{L}_{T^*}^2(B_{p,r}^{s-1}) \hookrightarrow L_{T^*}^2(L^{p^*}), \quad \text{with } p^* = \frac{2p}{2-p} \geq 2,$$

we have furthermore

$$\|h^{n-1}\|_{L_{T^*}^1(L^p)} \leq CC_E E_0 \tau.$$

Similarly, we have

$$\|(\rho^n - 1)\partial_t u^n\|_{L_{T^*}^1(L^p)}, \|(\rho^n u^{n-1} - \kappa^{n-1} \nabla \rho^n) \cdot \nabla u^n\|_{L_{T^*}^1(L^p)} \leq CC_E C_M^2 E_0 \tau (1 + \Pi^n).$$

Hence Equation (122) implies, for $p \in (1, 2)$,

$$(129) \quad \|\nabla\pi^n\|_{L_{T^*}^1(L^p)} \leq CC_E C_M^2 E_0 \tau (1 + \Pi^n) \leq C_\Pi (1 + \Pi^n) \tau,$$

with notation C_Π denoting some constant depending on $s, d, p, \rho_*, \rho^*, C_E, C_M, E_0$, to be precisely determined later.

- (iv) Now, one observes that Estimate (129) for $\nabla\pi^n$ implies moreover

$$\|\nabla\pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s-1})} \leq C \|\nabla\pi^n\|_{L_{T^*}^1(L^p)}^{1/s} \|\nabla\pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s-1})}^{(s-1)/s} \leq C_\Pi (1 + \Pi^n) \tau^{1/s}, \quad p \in (1, 2),$$

with some change of constant C_Π . On the other hand, if $p \geq 2$, then Embedding $L^2(\mathbb{R}^d) \hookrightarrow B_{p,\infty}^{d/p-d/2}(\mathbb{R}^d)$ and (128) also ensure similar interpolation inequality, for any $\eta \in [0, 1]$:

$$\begin{aligned} \|\nabla\pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s-1+\eta})} &\leq C \|\nabla\pi^n\|_{L_{T^*}^1(B_{p,\infty}^{d/p-d/2})}^{\frac{1-\eta}{s_p}} \|\nabla\pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)}^{\frac{s_p-1+\eta}{s_p}} \\ &\leq C \|\nabla\pi^n\|_{L_{T^*}^1(L^2)}^{\frac{1-\eta}{s_p}} (\Pi^n)^{\frac{s_p-1+\eta}{s_p}} \\ &\leq C_\Pi (1 + \Pi^n) \tau^{\frac{1-\eta}{s_p}}. \end{aligned}$$

- (v) At last, Equation (123) and Estimates (124) and (125) ensure for some $\eta \in (0, 1)$ (still with some appropriated constant C_Π)

$$\|\Delta\pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s-1})} \leq CC_M \|\nabla\pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^{s-1+\eta})} + CC_M \tau + CC_M \tau^2 \leq C_\Pi (1 + \Pi^n) \tau^{\frac{1-\eta}{s_p}},$$

which, together with (120), (128), (129) and the definition of Π^n , implies, for τ and T^* small enough,

$$\|\nabla\pi^n\|_{\tilde{L}_{T^*}^1(B_{p,r}^s)} \leq \tau^{1/2s_p}.$$

Therefore by virtue of Estimates (127) and (128), inductive assumption (72) and (118) for u^n and $\nabla\pi^n$ follow respectively.

- (vi) From above, Energy Identity (117) holds and hence we have

$$\|u^n\|_{L_{T^*}^\infty(L^2)} \leq C(\|u_0\|_{L^2} + \|h^{n-1}\|_{L_{T^*}^1(L^2)}) \leq \frac{1}{2} C_E E_0.$$

Inductive assumption (119) is then verified.

6.2.2 Convergence Part

Let us turn to establish that the above sequence converges to the solution. As in Subsection 4.2.2, we introduce the difference sequence

$$(\delta \varrho^n, \delta u^n, \nabla \delta \pi^n) = (\varrho^n - \varrho^{n-1}, u^n - u^{n-1}, \nabla \pi^n - \nabla \pi^{n-1}), \quad n \geq 1.$$

When $n \geq 2$, it verifies the following system:

$$(130) \quad \begin{cases} \partial_t \delta \varrho^n + u^{n-1} \cdot \nabla \delta \varrho^n - \operatorname{div}(\kappa^{n-1} \nabla \delta \varrho^n) = F^{n-1}, \\ \partial_t \delta u^n + (u^{n-1} - \kappa^{n-1} \nabla \log \rho^n) \cdot \nabla \delta u^n + \lambda^n \nabla \delta \pi^n = H_e^{n-1}, \\ \operatorname{div} \delta u^n = 0, \\ (\delta \varrho^n, \delta u^n)|_{t=0} = (0, 0), \end{cases}$$

where

$$\begin{aligned} F^{n-1} &= -\delta u^{n-1} \cdot \nabla \varrho^{n-1} + \operatorname{div}(\delta \kappa^{n-1} \nabla \varrho^{n-1}), \\ H_e^{n-1} &= \delta h^{n-1} - (\delta u^{n-1} - \delta \kappa^{n-1} \nabla \log \rho^n - \kappa^{n-2} \nabla \delta(\log \rho)^n) \cdot \nabla u^{n-1} - \delta \lambda^n \nabla \pi^{n-1}, \end{aligned}$$

with

$$\delta \kappa^{n-1} = \kappa^{n-1} - \kappa^{n-2}, \quad \delta h^{n-1} = h^{n-1} - h^{n-2}, \quad \delta(\log \rho)^n = \log \rho^n - \log \rho^{n-1}, \quad \delta \lambda^n = \lambda^n - \lambda^{n-1}.$$

Firstly, since in the case $p \in (1, 4]$ we have the embedding $B_{p,r}^s \hookrightarrow B_{4,1}^{1+d/4}$, we just have to establish that $\{(\varrho^n, u^n, \nabla \pi^n)\}_{n \geq 0}$ is a Cauchy sequence in the functional space $E_{4,1}^{d/4}(T^*)$ (see (17) for the definition). In fact, we just have to follow exactly Subsection 4.2.2, with some necessary changes in the coefficients and source terms switching from System (78) to System (130).

If p is big enough and we consider the limit case $(s, p, r) = (1, \infty, 1)$, then it's no more true that $\{(\varrho^n, u^n, \nabla \pi^n)\}_{n \geq 0}$ is a Cauchy sequence in $E_{\infty,1}^0(T^*)$. Indeed, applying “div” to Equation (130)₂ entails

$$(131) \quad \operatorname{div}(\lambda^n \nabla \delta \pi^n) = \operatorname{div} H_e^{n-1} - \operatorname{div}((u^{n-1} - \kappa^{n-1} \nabla \log \rho^n) \cdot \nabla \delta u^n).$$

As presented in the proof of Proposition 4.6 in Subsection 4.1.2 (see (68) for details), the following estimate

$$\|\operatorname{div}(u^{n-1} \cdot \nabla \delta u^n)\|_{B_{p,1}^{d/p-1}} \leq \|u^{n-1}\|_{B_{p,1}^{d/p+1}} \|\nabla \delta u^n\|_{B_{p,1}^{d/p-1}}$$

only holds for $p < \infty$. Thus we cannot have $\nabla \delta \pi^n \in L_{T^*}^1(B_{\infty,1}^0)$ in general.

Therefore in the general case, we have to consider the difference sequence in the energy space. One wants H_e^{n-1} in $L_{T^*}^1(L^2)$. One pays attention to the following terms in H_e^{n-1} :

$$(\rho^{n-1})^{-1} \Delta \delta b^{n-1} \nabla a^{n-1} \quad \text{and} \quad (\rho^{n-1})^{-1} \nabla b^{n-2} \cdot \nabla^2 \delta a^{n-1}.$$

We only have $\nabla a^{n-1}, \nabla b^{n-1}$ in $L_{T^*}^\infty(L^\infty)$, and thus one requires $\Delta \delta b^{n-1}, \nabla^2 \delta a^{n-1}$ in $L_{T^*}^1(L^2)$ and hence $\delta \varrho^{n-1}$ in $L_{T^*}^1(H^2)$.

Firstly, it is easy to see that $F^{n-1} \in L_{\text{loc}}^2(L^2)$ and hence taking L^2 inner product between Equation (130)₁ and $\delta \varrho^n$ gives controls on $\delta \varrho^n$ by $\delta \varrho^{n-1}, \delta u^{n-1}$, with small coefficient if restricted on the small time interval $[0, T^*]$.

Next, thanks to the null initial datum for $\delta \varrho^n$, by taking derivation of Equation (130)₁, we expect to get energy estimates for $\nabla \delta \varrho^n$. In fact, the equation for $\nabla \delta \varrho^n$, $n \geq 2$ reads

$$(132) \quad \partial_t \nabla \delta \varrho^n + u^{n-1} \cdot \nabla^2 \delta \varrho^n - \operatorname{div}(\kappa^{n-1} \nabla^2 \delta \varrho^n) = -\nabla \delta \varrho^n \cdot \nabla u^{n-1} + \operatorname{div}(\nabla \delta \varrho^n \otimes \nabla \kappa^{n-1}) + \nabla F^{n-1}.$$

The first two terms of the right-hand side are of lower order, while the third one is in $L_{\text{loc}}^2(H^{-1})$, thus taking L^2 inner product works. Therefore, $\delta \varrho^n \in L_{\text{loc}}^\infty(H^1) \cap L_{\text{loc}}^2(H^2)$ ensures $\delta \varrho^n \in C(\mathbb{R}^+; L^2)$ and $H_e^{n-1} \in L_{\text{loc}}^1(L^2)$. Thus energy inequality for δu^n also follows and its energy is bounded in terms of $\delta \varrho^{n-1}, \delta \varrho^n, \delta u^{n-1}$.

Thanks to the small time T^* , we thus can demonstrate that $\{(\rho^n, u^n)\}_{n \geq 1}$ is a Cauchy sequence in $C([0, T^*]; L^2)$ and hence converges to some unique limit (ρ, u) . Furthermore $\|\rho - \rho^n\|_{L_{T^*}^\infty(H^1) \cap L_{T^*}^2(H^2)}$ goes to 0 as n goes to ∞ . Then by use of the high regularity of the solution sequence, we can show that (ρ, u) is a solution.

Now we begin to make the above analysis in detail.

Our goal is to demonstrate that $\{\rho^n - \rho_0\}_n, \{u^n\}_n$ are Cauchy sequences in $C([0, T^*]; L^2)$ and the limit really solves System (114) and System (115).

Since $\delta\varrho^n \in E$ (see (126) for definition), we can take $L^2(\mathbb{R}^d)$ inner product between Equation (130)₁ and $\delta\varrho^n$, $n \geq 2$, entailing

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\delta\varrho^n|^2 + \int_{\mathbb{R}^d} \kappa^{n-1} |\nabla \delta\varrho^n|^2 = - \int_{\mathbb{R}^d} \delta u^{n-1} \cdot \nabla \rho^{n-1} \delta\varrho^n - \int_{\mathbb{R}^d} \delta \kappa^{n-1} \nabla \rho^{n-1} \cdot \nabla \delta\varrho^n.$$

Thus integration in time and Young's Inequality give

$$\begin{aligned} \|\delta\varrho^n\|_{L_{T^*}^\infty(L^2)} + \|\nabla \delta\varrho^n\|_{L_{T^*}^2(L^2)} &\leq C(\|\delta u^{n-1}\|_{L_{T^*}^2(L^2)} + \|\delta\varrho^{n-1}\|_{L_{T^*}^2(L^2)}) \|\nabla \varrho^{n-1}\|_{L_{T^*}^2(L^\infty)} \\ (133) \quad &\leq C\tau(\|\delta\varrho^{n-1}\|_{L_{T^*}^\infty(L^2)} + \|\delta u^{n-1}\|_{L_{T^*}^2(L^2)}). \end{aligned}$$

By view of the analysis above, $\delta\varrho^n \in C(\mathbb{R}^+; L^2) \cap L_{\text{loc}}^2(H^2)$, $n \geq 2$, (notice that it is not clear that $\delta\varrho^1 = \rho^1 - \rho_0 \in L_{\text{loc}}^2(H^2)$). So we can still take $L^2(\mathbb{R}^d)$ inner product between Equation (132) and $\nabla \delta\varrho^n$:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla \delta\varrho^n|^2 + \int_{\mathbb{R}^d} \kappa^{n-1} |\nabla^2 \delta\varrho^n|^2 = - \int \nabla \delta\varrho^n \cdot \nabla u^{n-1} \cdot \nabla \delta\varrho^n + \nabla \delta\varrho^n \cdot \nabla^2 \delta\varrho^n \cdot \nabla \kappa^{n-1} + F^{n-1} \Delta \delta\varrho^n.$$

Integrating in time also implies

$$\begin{aligned} \|\nabla \delta\varrho^n\|_{L_{T^*}^\infty(L^2)} + \|\nabla^2 \delta\varrho^n\|_{L_{T^*}^2(L^2)} &\leq \\ C \Big(\|\nabla u^{n-1}\|_{L_{T^*}^2(L^\infty)} \|\nabla \delta\varrho^n\|_{L_{T^*}^\infty(L^2)} &+ \|\nabla \rho^{n-1}\|_{L_{T^*}^2(L^\infty)} \|\nabla \delta\varrho^n\|_{L_{T^*}^\infty(L^2)} + \|F^{n-1}\|_{L_{T^*}^2(L^2)} \Big). \end{aligned}$$

Since

$$\|F^{n-1}\|_{L_{T^*}^2(L^2)} \leq C(\tau \|\delta u^{n-1}\|_{L_{T^*}^2(L^2)} + \tau \|\delta\varrho^{n-1}\|_{L_{T^*}^\infty(L^2)} + C_M \|\nabla \delta\varrho^{n-1}\|_{L_{T^*}^2(L^2)}),$$

for the above small τ and T^* ,

$$\|\nabla \delta\varrho^n\|_{L_{T^*}^\infty(L^2)} + \|\nabla^2 \delta\varrho^n\|_{L_{T^*}^2(L^2)} \leq C\tau(\|\delta\varrho^{n-1}, \delta u^{n-1}\|_{L_{T^*}^\infty(L^2)} + CC_M \|\nabla \delta\varrho^{n-1}\|_{L_{T^*}^2(L^2)}).$$

We can substitute (133) into the term $\|\nabla \delta\varrho^{n-1}\|_{L_{T^*}^2(L^2)}$ above, and then sum up these two inequalities, entailing

$$(134) \quad \|\delta\varrho^n\|_{L_{T^*}^\infty(H^1)} + \|\nabla \delta\varrho^n\|_{L_{T^*}^2(H^1)} \leq CC_M \tau (\|\delta\varrho^{n-1}, \delta\varrho^{n-2}, \delta u^{n-1}, \delta u^{n-2}\|_{L_{T^*}^\infty(L^2)}).$$

Now we turn to δu^n . We rewrite δh^{n-1} as

$$\begin{aligned} \frac{1}{\rho^{n-1}} (\Delta \delta b^{n-1} \nabla a^{n-1} + \Delta b^{n-2} \nabla \delta a^{n-1} + \delta u^{n-1} \cdot \nabla^2 a^{n-1} + u^{n-2} \cdot \nabla^2 \delta a^{n-1} + \nabla \delta b^{n-1} \cdot \nabla^2 a^{n-1} \\ + \nabla b^{n-2} \cdot \nabla^2 \delta a^{n-1}) + \left((\rho^{n-1})^{-1} - (\rho^{n-2})^{-1} \right) (\Delta b^{n-2} \cdot \nabla a^{n-2} + u^{n-2} \cdot \nabla^2 a^{n-2} + \nabla b^{n-2} \cdot \nabla^2 a^{n-2}). \end{aligned}$$

By virtue of $\|\Delta \delta b^{n-1}\|_{L^2} \leq C\|\delta\varrho^{n-1}\|_{H^2}$ and

$$\|\Delta b^{n-1}\|_{L_{T^*}^2(L^\infty)} \leq C \|\nabla \rho^{n-1}\|_{L_{T^*}^2(L^\infty)} + C \|\Delta \rho^{n-1}\|_{L_{T^*}^2(L^\infty)} \leq CC_M \tau,$$

we have also from the above inductive estimates that

$$\|\delta h^{n-1}\|_{L_{T^*}^1(L^2)} \leq CC_M \tau (\|\delta\varrho^{n-1}\|_{L_{T^*}^2(H^2)} + \|\delta u^{n-1}\|_{L_{T^*}^\infty(L^2)}) + CC_E E_0 \tau \|\delta\varrho^{n-1}\|_{L_{T^*}^\infty(L^2)}.$$

Similarly,

$$\|H_e^{n-1}\|_{L_{T^*}^1(L^2)} \leq C(C_M + C_E E_0) \tau (\|\delta\varrho^{n-1}\|_{L_{T^*}^2(H^2)} + \|\delta u^{n-1}\|_{L_{T^*}^\infty(L^2)}) + C\tau^{1/2s_p} \|\delta\varrho^n\|_{L_{T^*}^\infty(L^2)}.$$

By view of Equation (114) and $\operatorname{div} \delta u^n = 0$, taking the $L^2(\mathbb{R}^d)$ inner product between Equation (130)₂ and $\rho^n u^n$ and then integrating on the time interval $[0, T^*]$ yields

$$(135) \quad \|\delta u^n\|_{L_{T^*}^\infty(L^2)} \leq C \|H_e^{n-1}\|_{L_{T^*}^1(L^2)}.$$

Combining Estimate (134) and (135) entails, for sufficiently small τ , depending only on the space dimension d , on the indices (s, p, r) and on the constants C_M, C_E, E_0 ,

$$\|\delta \varrho^n\|_{L_{T^*}^\infty(H^1) \cap L_{T^*}^2(H^2)} + \|\delta u^n\|_{L_{T^*}^\infty(L^2)} \leq \frac{1}{6} \|(\delta \varrho^{n-1}, \delta \varrho^{n-2}, \delta \varrho^{n-3}, \delta u^{n-1}, \delta u^{n-2}, \delta u^{n-3})\|_{L_{T^*}^\infty(L^2)}.$$

Thus $\sum \|(\delta \varrho^n, \delta u^n)\|_{L_{T^*}^\infty(L^2)}$ converges. Since $\delta \varrho^n \in C(\mathbb{R}^+; L^2)$, the Cauchy sequences $\{\varrho^n\}$ and $\{u^n\}$ converge to ϱ and u in $C([0, T^*]; L^2)$ respectively. It is also easy to see that

$$\sum_{n \geq 2} \|\delta \varrho^n\|_{L_{T^*}^\infty(H^1) \cap L_{T^*}^2(H^2)}, \quad \sum_{n \geq 2} \|\delta h^n\|_{L_{T^*}^1(L^2)}, \quad \sum_{n \geq 2} \|H_e^{n-1}\|_{L_{T^*}^1(L^2)} < +\infty.$$

Writing

$$\operatorname{div} ((u^{n-1} - \kappa^{n-1} \nabla \log \rho^n) \cdot \nabla \delta u^n) = \operatorname{div} \left(\delta u^n \cdot \nabla (u^{n-1} - \kappa^{n-1} \nabla \log \rho^n) + \delta u^n \operatorname{div} (\kappa^{n-1} \nabla \log \rho^n) \right),$$

from Equation (131) we get

$$\|\nabla \delta \pi^n\|_{L_{T^*}^1(L^2)} \leq C (\|H_e^{n-1}\|_{L_{T^*}^1(L^2)} + C_M \|\delta u^n\|_{L_{T^*}^\infty(L^2)}).$$

Thus $\sum_2^\infty \|\nabla \delta \pi^n\|_{L_{T^*}^1(L^2)}$ also converges and hence $\nabla \pi^n$ converges to the unique limit $\nabla \pi$ in $L_{T^*}^1(L^2)$.

Now one notices that, by interpolation between L^2 and $B_{p,r}^s$, u^n converges to u in $C([0, T^*]; B_{4,1}^{1/2})$, for instance, and $u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n$ is at least in $L_{\text{loc}}^2(B_{4,1}^{1/2})$. It entails

$$(u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n) \cdot \nabla u^n - (u - \kappa \nabla \rho) \cdot \nabla u \rightarrow 0 \quad \text{in} \quad L_{T^*}^2(B_{4,\infty}^{-d/4}).$$

Thus a direct computation ensures that $(\varrho, u, \nabla \pi)$ solves System (7) and is in $E_{p,r}^s(T^*)$ by Fatou property.

6.2.3 Uniqueness Part

The proof of uniqueness just follows the idea of the convergence part. More precisely, as in Subsection 4.3, take two solutions $(\rho_1, u_1, \nabla \pi_1), (\rho_2, u_2, \nabla \pi_2) \in E_{p,r}^s(T^*)$ of System (7) with the same initial data, such that $\rho_1 - 1, \rho_2 - 1, u_1, u_2 \in L_{T^*}^\infty(L^2)$, $\nabla \rho_1, \nabla \rho_2 \in L_{T^*}^2(L^2)$. Then the difference $(\delta \rho, \delta u, \nabla \delta \pi) = (\rho_1 - \rho_2, u_1 - u_2, \nabla \pi_1 - \nabla \pi_2)$ verifies

$$\begin{cases} \partial_t \delta \varrho + u_1 \cdot \nabla \delta \varrho - \operatorname{div} (\kappa_1 \nabla \delta \varrho) = -\delta u \cdot \nabla \varrho_2 + \operatorname{div} ((\kappa_1 - \kappa_2) \nabla \varrho_2), \\ \partial_t \delta u + (u_1 + \nabla b_1) \cdot \nabla \delta u + \lambda_1 \nabla \delta \pi = h_1 - h_2 - (\delta u + \nabla \delta b) \cdot \nabla u_2 - \delta \lambda \nabla \pi_2, \\ \operatorname{div} \delta u = 0, \\ (\delta \varrho, \delta u)|_{t=0} = (0, 0), \end{cases}$$

with the notation $\kappa_i = \kappa(\varrho_i)$ and analogous for b_i, λ_i and h_i .

Similarly as Convergence Part, we can get

$$\begin{aligned} \|\delta \rho\|_{L_t^\infty(L^2)} + \|\nabla \delta \rho\|_{L_t^2(L^2)} &\leq C (\|\nabla \rho_2\|_{L_t^2(L^\infty)} \|\delta \rho\|_{L_t^\infty(L^2)} + \|\nabla \rho_2\|_{L_t^1(L^\infty)} \|\delta u\|_{L_t^\infty(L^2)}), \\ \|\nabla \delta \rho\|_{L_t^\infty(L^2)} + \|\nabla^2 \delta \rho\|_{L_t^2(L^2)} &\leq C \left((\|\nabla u_1\|_{L_t^1(L^\infty)} + \|\nabla \kappa_1\|_{L_t^2(L^\infty)}) \|\nabla \delta \rho\|_{L_t^\infty(L^2)} \right. \\ &\quad + (\|\Delta \rho_2\|_{L_t^2(L^\infty)} + \|\nabla \rho_1\|_{L_t^2(L^\infty)} \|\nabla \rho_2\|_{L_t^\infty(L^\infty)}) \|\delta \rho\|_{L_t^\infty(L^2)} \\ &\quad \left. + \|\nabla \rho_2\|_{L_t^2(L^\infty)} \|\delta u\|_{L_t^\infty(L^2)} + \|\nabla \rho_1\|_{L_t^\infty(L^\infty)} \|\nabla \delta \rho\|_{L_t^2(L^2)} \right), \end{aligned}$$

and some similar estimate for $\|\delta u\|_{L_t^\infty(L^2)}$, which we omit here. Thus, on sufficiently small interval $[0, t]$, $\delta \rho \equiv \delta u \equiv 0$, the uniqueness holds. Then we recover uniqueness on the whole existence time interval $[0, T^*]$ by use of classical arguments.

6.3 Remark on the lifespan in the 2-D case

In this subsection, we want to give a better estimate for the lower bound for the lifespan of the solution in the case of dimension $d = 2$.

As mentioned before, the global-in-time existence issue in dimension $d = 2$ for the classical homogeneous Euler system (i.e. $\rho \equiv \bar{\rho}$ constant in system (7)) has been well-known (see [33] by Wolibner, or also Chapter 7 of [4] and the references therein for a survey on this topic). The key to the proof is the fact that, if we define the *vorticity* of the fluid as

$$(136) \quad \omega := \partial_1 u^2 - \partial_2 u^1,$$

then this quantity is conserved along the trajectories of the fluid particles, i.e. it fulfills the free transport equation

$$(V) \quad \partial_t \omega + u \cdot \nabla \omega = 0.$$

For non-homogeneous perfect fluids, see system (11), the previous relation (V) is no more true, due to a density term which comes into play combined with the pressure. Hence, it's not clear if solutions to (11) exist globally in time. However, in [14] it's proved that, for initial densities close to a constant state, the lifespan of the corresponding solutions tends to $+\infty$.

Theorem 2.11 gives us the analogous result for our model. The idea is to resort to the vorticity in order to control the high frequencies of the velocity field. The vorticity ω of the fluid is still defined by formula (136), where u solves (7). However, it's easy to see that actually

$$\omega \equiv \partial_1 v^2 - \partial_2 v^1;$$

now, from (4)₂ the equation for ω immediately follows:

$$(137) \quad \partial_t \omega + v \cdot \nabla \omega + \omega \Delta b + \nabla \lambda \wedge \nabla \Pi = 0,$$

where we have set $\nabla \lambda \wedge \nabla \Pi = \partial_1 \lambda \partial_2 \Pi - \partial_2 \lambda \partial_1 \Pi$.

In order to bound the vorticity, it's fundamental to take advantage of a new version of refined estimates for transport equations in borderline Besov spaces of the type $B_{\infty,1}^0$, proved first by Vishik in [32] and then generalized by Hmidi and Keraani in [21] (see also Chapter 3 of [4]). They state that the $B_{\infty,1}^0$ norm of the vorticity grows linearly (and not exponentially, as in the general case) with respect to the Lipschitz norm of the solenoidal velocity field. Even if here we don't have the divergence-free condition for the transport velocity v , the proof in [21] still works, except that one asks for an additional regularity on $\operatorname{div} v$. More precisely, the following lemma holds.

Lemma 6.6. *Let us consider the following linear transport equation:*

$$(138) \quad \begin{cases} \partial_t \omega + v \cdot \nabla \omega = g, \\ \omega|_{t=0} = \omega_0. \end{cases}$$

For any $\beta > 0$, there exists a constant C depending only on d, β such that the following a priori estimate holds true:

$$(139) \quad \|\omega(t)\|_{B_{\infty,1}^0} \leq C \left(\|\omega_0\|_{B_{\infty,1}^0} + \|g\|_{L_t^1(B_{\infty,1}^0)} \right) (1 + \mathcal{V}(t)),$$

with

$$\mathcal{V}(t) := \int_0^t \|\nabla v\|_{L^\infty} + \|\operatorname{div} v\|_{B_{\infty,\infty}^\beta} dt'.$$

Proof. We will follow the proof of [21]. Firstly we can write the solution ω of the transport equation (138) as a sum: $\omega = \sum_{k \geq -1} \omega_k$, with ω_k satisfying

$$(140) \quad \begin{cases} \partial_t \omega_k + v \cdot \nabla \omega_k = \Delta_k g, \\ \omega_k|_{t=0} = \Delta_k \omega_0. \end{cases}$$

We obviously have from above that

$$(141) \quad \|\omega_k(t)\|_{L^\infty} \leq \|\Delta_k \omega_0\|_{L^\infty} + \int_0^t \|\Delta_k g\|_{L^\infty} dt'.$$

By Proposition 3.12, for any $\epsilon \in (0, 1)$, we have the following a priori estimate in Besov space $B_{\infty,1}^\epsilon$:

$$(142) \quad \|\omega_k(t)\|_{B_{\infty,1}^\epsilon} \leq \left(\|\Delta_k \omega_0\|_{B_{\infty,1}^\epsilon} + \|\Delta_k g\|_{L_t^1(B_{\infty,1}^\epsilon)} \right) \exp\left(C\|\nabla v\|_{L_t^1(L^\infty)}\right).$$

In order to get a priori estimate in Besov space $B_{\infty,1}^{-\epsilon}$, after applying the operator Δ_j to Equation (140), we write the commutator $[v, \Delta_j] \cdot \nabla \omega_k$ as follows (recalling Bony's decomposition (30) and denoting $\tilde{v} := v - \Delta_{-1}v$)

$$[T_{\tilde{v}}, \Delta_j] \cdot \nabla \omega_k + T_{\Delta_j \nabla \omega_k} \tilde{v} + R(\Delta_j \nabla \omega_k, \tilde{v}) - \Delta_j(T_{\nabla \omega_k} \tilde{v}) - \Delta_j \operatorname{div}(R(\omega_k, \tilde{v})) + \Delta_j R(\omega_k, \operatorname{div} \tilde{v}) + [\Delta_{-1}v, \Delta_j] \cdot \nabla \omega_k.$$

Then, $\forall \beta > \epsilon$, the L^∞ -norm of all the above terms can be bounded by (for some nonnegative sequence $\|(c_j)\|_{\ell^1} = 1$):

$$C(d, \beta) 2^{-j\epsilon} c_j \mathcal{V}'(t) \|\omega_k\|_{B_{\infty,1}^{-\epsilon}}.$$

Thus, we have the following a priori estimate in the space $B_{\infty,1}^{-\epsilon}$:

$$(143) \quad \|\omega_k(t)\|_{B_{\infty,1}^{-\epsilon}} \leq \left(\|\Delta_k \omega_0\|_{B_{\infty,1}^{-\epsilon}} + \|\Delta_k g\|_{L_t^1(B_{\infty,1}^{-\epsilon})} \right) \exp\left(C\mathcal{V}(t)\right).$$

On the other side, one has the following, for some positive integer N to be determined hereafter:

$$\|\omega\|_{B_{\infty,1}^0} \leq \sum_{j,k \geq -1} \|\Delta_j \omega_k\|_{L^\infty} = \sum_{|j-k| < N} \|\Delta_j \omega_k\|_{L^\infty} + \sum_{|j-k| \geq N} \|\Delta_j \omega_k\|_{L^\infty}.$$

Estimate (141) implies

$$\sum_{|j-k| < N} \|\Delta_j \omega_k\|_{L^\infty} \leq N \sum_k \left(\|\Delta_k \omega_0\|_{L^\infty} + \|\Delta_k g\|_{L_t^1(L^\infty)} \right) \leq N \left(\|\omega_0\|_{B_{\infty,1}^0} + \|g\|_{L_t^1(B_{\infty,1}^0)} \right),$$

while Estimates (142) and (143) entail the following (for some nonnegative sequence $(c_j) \in \ell^1$):

$$\|\Delta_j \omega_k\|_{L^\infty} \leq 2^{-\epsilon|k-j|} c_j \left(\|\Delta_k \omega_0\|_{L^\infty} + \|\Delta_k g\|_{L_t^1(L^\infty)} \right) \exp\left(C\mathcal{V}(t)\right),$$

which issues immediately

$$\sum_{|j-k| \geq N} \|\Delta_j \omega_k\|_{L^\infty} \leq 2^{-N\epsilon} \left(\|\omega_0\|_{B_{\infty,1}^0} + \|g\|_{L_t^1(B_{\infty,1}^0)} \right) \exp\left(C\mathcal{V}(t)\right).$$

Therefore, for any $\beta > 0$, we can choose $\epsilon \in (0, 1)$ and $N \in \mathbb{N}$ such that $\epsilon < \beta$ and $N\epsilon \log 2 \sim 1 + C\mathcal{V}(t)$. Thus the lemma follows from the above estimates. \square

With the above lemma in hand, we can now begin to prove Theorem 2.11. In view of the continuation criterion, without loss of generality, we will always assume in the sequel

$$s = 1, \quad p = \infty, \quad r = 1.$$

In what follows, we will resort to the notation of Subsection 5.2. That is, we set

$$R(t) = \|\varrho\|_{L_t^\infty(B_{\infty,1}^1)}, \quad S(t) = \|\varrho\|_{L_t^1(B_{\infty,1}^3)}, \quad U(t) = \|u\|_{L_t^\infty(B_{\infty,1}^1)},$$

and the time T_R as defined by (87).

First of all, let us consider the density term. Under our hypothesis, we get the energy equality (24). Moreover, arguing as in the previous subsection, we recover estimate (88) in the time interval $[0, T_R]$.

Now, let us consider the velocity field. First of all, let us summarize the following inequalities for the nonlinear terms, which will be frequently used in the sequel:

$$(144) \quad \|\nabla^2 b(\rho)\|_{B_{\infty,1}^1} \lesssim \|b\|_{B_{\infty,1}^3} \lesssim \|\varrho\|_{B_{\infty,1}^3} = S';$$

$$(145) \quad \|\Delta b \nabla a\|_{L^2} \lesssim \|b\|_{B_{\infty,1}^2} \|\nabla a\|_{L^2} \lesssim \|\varrho\|_{B_{\infty,1}^2} \|\nabla \rho\|_{L^2} \lesssim R^{1/2} (S')^{1/2} \|\nabla \rho\|_{L^2} \leq R \|\nabla \rho\|_{L^2}^2 + S';$$

$$(146) \quad \|(u + \nabla b) \cdot \nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} (\|\nabla \rho\|_{L^2} + \|u\|_{L^2}) \lesssim U (\|\nabla \rho\|_{L^2} + \|u\|_{L^2}).$$

Similarly as the above inequality (145), one has

$$(147) \quad \|\nabla b \cdot \nabla^2 a\|_{L^2} \lesssim R \|\nabla \rho\|_{L^2}^2 + S', \quad \|u \cdot \nabla^2 a\|_{L^2} \lesssim R \|u\|_{L^2}^2 + S'.$$

It is time to bound the velocity field u by use of the above inequalities. Firstly, by separating low and high frequencies of u and by use of Bernstein's inequalities and of a Fourier multiplier of order -1 , we get

$$(148) \quad U(t) \leq C \left(\|u\|_{L^2} + \|u\|_{B_{\infty,1}^0} \right).$$

In order to handle the energy of the velocity field, one takes the L^2 scalar product between (10) and u and performing standard computations (recall also (25) and the following arguments), getting

$$\|u(t)\|_{L^2} \leq C \left(\|u_0\|_{L^2} + \int_0^t \|\operatorname{div} (v \otimes \nabla a)\|_{L^2} d\tau \right).$$

Therefore, due to Inequalities (145) and (147), it follows that

$$(149) \quad \|u(t)\|_{L^2} \leq C \left(\|u_0\|_{L^2} + \int_0^t \left(R \|\nabla \rho\|_{L^2}^2 + \|u\|_{L^2}^2 + S' \right) d\tau \right).$$

Now, applying Lemma 6.6 with $\beta = 1$ to Equation (137), we find

$$\|\omega(t)\|_{B_{\infty,1}^0} \leq C \left(\|\omega_0\|_{B_{\infty,1}^0} + \int_0^t \left\| \nabla \lambda \wedge \nabla \Pi + \omega \Delta b \right\|_{B_{\infty,1}^0} d\tau \right) \left(1 + \int_0^t \left(\|\nabla u\|_{L^\infty} + \|\nabla^2 b\|_{B_{\infty,1}^1} \right) d\tau \right).$$

By use of Bony's paraproduct decomposition (see also Section 4.2 of [14] for the first inequality)

$$\begin{aligned} \|\nabla \lambda \wedge \nabla \Pi\|_{B_{\infty,1}^0} &\lesssim \|\nabla \rho\|_{B_{\infty,1}^0} \|\nabla \Pi\|_{B_{\infty,1}^0} \\ \|\omega \Delta b\|_{B_{\infty,1}^0} &\lesssim \|\omega\|_{B_{\infty,1}^0} \|\Delta b\|_{B_{\infty,1}^1}. \end{aligned}$$

Hence, by virtue of $\|\omega\|_{B_{\infty,1}^0} \lesssim U$ and Inequality (144), we immediately gather

$$(150) \quad \|\omega(t)\|_{B_{\infty,1}^0} \leq C \left(U_0 + \int_0^t \left(R \|\nabla \Pi\|_{B_{\infty,1}^0} + U S' \right) d\tau \right) \left(1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau + S \right).$$

It remains to deal with the pressure term. First of all, we have

$$(151) \quad \|\nabla \Pi\|_{B_{\infty,1}^0} \leq \|\nabla \pi\|_{B_{\infty,1}^0} + \|\partial_t \nabla a\|_{B_{\infty,1}^0} \lesssim \|\nabla \pi\|_{B_{\infty,1}^0} + \|\partial_t \varrho\|_{B_{\infty,1}^1}.$$

Thanks to equation (7)₁, and by use of Proposition 3.10, we get

$$\|\partial_t \varrho\|_{B_{\infty,1}^1} \lesssim \|u \cdot \nabla \varrho\|_{B_{\infty,1}^1} + \|\kappa \nabla \varrho\|_{B_{\infty,1}^2} \lesssim U S' + S'.$$

Let us now focus on $\|\nabla \pi\|_{B_{\infty,1}^0}$. Actually, we will bound the $B_{\infty,1}^1$ norm, as it's not clear for us how to get advantage of the weaker norm in (150). The analysis is mostly the same performed in the previous

subsection for the general case: so we cut low and high frequency, and we are reconducted to consider $\|\nabla\pi\|_{L^2}$ and $\|\Delta\pi\|_{B_{\infty,1}^0}$. Finally, as $\delta > 1$, we have

$$\|\nabla\pi\|_{B_{\infty,1}^1} \leq C (1 + R^\delta) \left(\|\nabla\pi\|_{L^2} + \|\operatorname{div}(v \cdot \nabla u)\|_{B_{\infty,1}^0} + \|h\|_{B_{\infty,1}^1} \right),$$

with the controls (by what we established in the case of higher dimension)

$$\|h\|_{B_{\infty,1}^1} \lesssim (1 + R) U S' + (1 + R) R S'$$

and, as $\operatorname{div}(u \cdot \nabla u) = \sum_{i,j} 2T_{\partial_j u^i} \partial_i u^j + \partial_j R(u^i, \partial_i u^j)$ (thanks to the divergence-free condition over u) and analogous for $\operatorname{div}(\nabla b \cdot \nabla u)$,

$$\|\operatorname{div}(v \cdot \nabla u)\|_{B_{\infty,1}^0} \lesssim U^2 + U S'.$$

On the other hand, observing that $\|h\|_{L^2} \lesssim \|\operatorname{div}(v \otimes \nabla a)\|_{L^2}$ and using Inequalities (145), (146) and (147), one has also

$$\|\nabla\pi\|_{L^2} \lesssim \|h\|_{L^2} + \|v \cdot \nabla u\|_{L^2} \lesssim R(\|\nabla\rho\|_{L^2}^2 + \|u\|_{L^2}^2) + S' + U(\|\nabla\rho\|_{L^2} + \|u\|_{L^2}).$$

Let us define

$$X(t) := U(t) + \|u(t)\|_{L^2} = \|u(t)\|_{L^2 \cap B_{\infty,1}^1}.$$

So we get

$$(152) \quad \|\nabla\pi\|_{B_{\infty,1}^1} \leq C (1 + R^{\delta+2}) \left(\|\nabla\rho\|_{L^2}^2 + S' + X^2 + X S' \right)$$

and, by (151), the same holds true also for $\|\nabla\Pi\|_{B_{\infty,1}^0}$.

Therefore, Estimate (150) for the vorticity becomes (denoting $X_0 = X(0)$)

$$\|\omega(t)\|_{B_{\infty,1}^0} \leq C \left(1 + S + \int_0^t X d\tau \right) \left(X_0 + \int_0^t (1 + R^{\delta+3}) \left(R \|\nabla\rho\|_{L^2}^2 + R S' + R X^2 + X S' \right) d\tau \right).$$

It's now time to insert the above estimate and (149) into (148). Keeping in mind that, in $[0, T_R]$,

$$(153) \quad R(t) + S(t) \leq C R_0 \exp \left(C \int_0^t (1 + X(\tau)) d\tau \right),$$

we finally find

$$(154) \quad \begin{aligned} X(t) \leq C \left(1 + S + \int_0^t X d\tau \right) & \left(\underbrace{X_0 + R_0(1 + R_0^{\delta+3}) e^{C \int_0^t (1+X)}}_{\Gamma_1} \int_0^t \|\nabla\rho\|_{L^2}^2 + \right. \\ & \left. + \underbrace{R_0(1 + R_0^{\delta+4}) e^{C \int_0^t (1+X)}}_{\Gamma_2} + \underbrace{R_0(1 + R_0^{\delta+3}) e^{C \int_0^t (1+X)}}_{\Gamma_3} \int_0^t X^2 + \underbrace{(1 + R_0^{\delta+3}) e^{C \int_0^t (1+X)}}_{\Gamma_4} \int_0^t X S' d\tau \right). \end{aligned}$$

We define T_X as the following quantity (with Γ_i , $i = 1, \dots, 4$ defined as above)

$$T_X := \sup \left\{ t \mid \Gamma_1(t) \leq \|\varrho_0\|_{L^2}^2, \quad \Gamma_2(t) \leq 1, \quad \Gamma_3(t), \Gamma_4(t) \leq 1 + \|\varrho\|_{L^2}^2 + X_0 \right\}.$$

Then, noticing $S \leq \Gamma_2$, one easily arrives at the following bound for $X(t)$ with t in $[0, T_R] \cap [0, T_X]$ (with some positive constant still denoted by C)

$$X(t) \leq C(1 + \|\varrho_0\|_{L^2}^2 + X_0) \left(1 + \int_0^t X(\tau) d\tau \right).$$

Hence, set $\Gamma_0 := C(1 + \|\varrho_0\|_{L^2}^2 + X_0)$, then by Gronwall's lemma we get

$$(155) \quad X(t) \leq \Gamma_0 e^{\Gamma_0 t},$$

and the norm of the solution can be controlled by the norm of the initial data only.

Our next task, in order to complete the argument, is then to prove that T , defined by (29) with $\mathfrak{M} = \delta + 4$ and small enough constant \tilde{L} , is smaller than both T_R and T_X . First of all, thanks to (155), for $t \in [0, T_X] \cap [0, T_R]$, we have

$$\int_0^t X \leq e^{\Gamma_0 t}, \quad \int_0^t X^2 \leq \frac{\Gamma_0}{2} e^{2\Gamma_0 t}, \quad \int_0^t (1 + X) \leq 2e^{\Gamma_0 t}, \quad e^{C \int_0^t (1+X)} \leq e^{2Ce^{\Gamma_0 t}}, \quad R + S \leq CR_0 e^{2Ce^{\Gamma_0 t}}.$$

With the above bounds in hand, one just has to show

$$\Gamma_1(T) \leq \|\varrho_0\|_{L^2}^2, \quad \Gamma_2(T) \leq 1, \quad \Gamma_3(T), \Gamma_4(T) \leq 1 + \|\varrho_0\|_{L^2}^2 + X_0 \quad \text{and} \quad \int_0^T R^3 \leq 2R_0.$$

We will check the above bounds one by one.

By use of the energy equality (24) for the density, it's then easy to see that $\Gamma_1(T) \leq \|\varrho_0\|_{L^2}^2$, with T defined by (29). It is also easy to find that $\Gamma_2(T) \leq 1$. Now noticing that $(\sigma \leq e^\sigma)$

$$\Gamma_3(t) \leq R_0(1 + R_0^{\delta+3})e^{2Ce^{\Gamma_0 t}} \frac{\Gamma_0}{2} e^{2\Gamma_0 t} \leq \frac{\Gamma_0}{2} R_0(1 + R_0^{\delta+3})e^{3Ce^{2\Gamma_0 t}},$$

we hence have $\Gamma_3(T) \leq 1 + \|\varrho_0\|_{L^2}^2 + X_0$. Similarly, since one has

$$\Gamma_4(t) \leq (1 + R_0^{\delta+3})e^{2Ce^{\Gamma_0 t}} \Gamma_0 e^{\Gamma_0 t} \int_0^t S' \leq \Gamma_0(1 + R_0^{\delta+3})e^{5Ce^{\Gamma_0 t}} CR_0,$$

thus $\Gamma_4(t) \leq 1 + \|\varrho_0\|_{L^2}^2 + X_0$. Finally, it's not hard to check that

$$\int_0^T R^3 \leq C^3 R_0^3 e^{6Ce^{\Gamma_0 T}} \leq 2R_0.$$

This completes the proof of the theorem.

6.4 The case when the density is near 1

Let us give here a sketch of the proof to Theorem 2.12. In fact, we will establish only a priori estimates, the rest of the proof to existence and uniqueness being similar to that of Theorem 2.1.

So, recall that $p \in (1, +\infty)$ and $\|\varrho_0\|_{B_{p,r}^s} \leq c$, where c is a small positive constant. Moreover we will use the convention that $r = 1$ if $s = 1 + d/p$.

We first focus on the density equation (7)₁. We start from inequality (52), in which we use also the second relation of (54) to bound the first commutator term.

We apply instead (46) to control the second commutator term: with $\sigma = s$, $\theta = \eta = 1/2$ and $\varepsilon = 1/2$, it entails

$$\left\| 2^{js} \int_0^t \|\mathcal{R}_j^2\|_{L^p} \right\|_{\ell^r} d\tau \leq C \int_0^t \|\nabla \varrho\|_{L^\infty}^2 \|\varrho\|_{B_{p,r}^s} d\tau + \frac{1}{2} \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})}.$$

Putting all these relations together, we get

$$\|\varrho\|_{\tilde{L}_t^\infty(B_{p,r}^s)} + \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})} \leq C \|\varrho_0\|_{B_{p,r}^s} \exp \left(Ct + C \int_0^t \|\nabla u\|_{B_{p,r}^{s-1}} + C \|\nabla \varrho\|_{L^\infty}^2 d\tau \right),$$

for constant C depending only on indices d, s, p, r and on ρ_*, ρ^* .

Now, if we take $T \in]0, T_0]$ small enough, such that, for instance,

$$(156) \quad \exp \left(Ct + C \int_0^t \|\nabla u\|_{B_{p,r}^{s-1}} + C \|\nabla \varrho\|_{L^\infty}^2 d\tau \right) \leq \log 2,$$

then the previous inequality becomes

$$(157) \quad \|\varrho\|_{\tilde{L}_t^\infty(B_{p,r}^s)} + \|\varrho\|_{\tilde{L}_t^1(B_{p,r}^{s+2})} \leq 2C c.$$

Let us now consider Equations (7)₂ and (7)₃ for the velocity field and the pressure term. Going along the lines of the proof to Proposition 4.6, we can see that (62) and (63) still hold true, with the transport velocity w being $u + \nabla b$.

Here we handle $\nabla \pi$ as done in the previous section, that is, we make use of (120). Firstly, notice that the equation for π can be rewritten as

$$(158) \quad \Delta \pi = \operatorname{div} \left(h - (u + \nabla b) \cdot \nabla u + \frac{\varrho}{\rho} \nabla \pi \right),$$

which (formally) implies

$$\nabla \pi = \nabla (-\Delta)^{-1} \operatorname{div} \left(-h + (u + \nabla b) \cdot \nabla u - \frac{\varrho}{\rho} \nabla \pi \right).$$

Hence, by Calderon-Zygmund theory and (120) we infer (with $w = u + \nabla b$)

$$(159) \quad \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} \leq C \left(\|h\|_{\tilde{L}_t^1(B_{p,r}^s)} + \|\partial_i w^j \partial_j u^i\|_{\tilde{L}_t^1(B_{p,r}^{s-1})} + \|w \cdot \nabla u\|_{L_t^1(L^p)} + \left\| \frac{\varrho}{\rho} \nabla \pi \right\|_{\tilde{L}_t^1(B_{p,r}^s)} \right).$$

The term $\|\partial_i w^j \partial_j u^i\|_{\tilde{L}_t^1(B_{p,r}^{s-1})}$ can be controlled as done in (68). For $\|w \cdot \nabla u\|_{L_t^1(L^p)}$ we notice that

$$\|w \cdot \nabla u\|_{L_t^1(L^p)} \leq \int_0^t \|w\|_{L^p} \|\nabla u\|_{L^\infty} \leq \int_0^t (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^p}) \|u\|_{B_{p,r}^s}.$$

Finally, we have

$$\left\| \frac{\varrho}{\rho} \nabla \pi \right\|_{\tilde{L}_t^1(B_{p,r}^s)} \leq C \|\varrho\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)}.$$

Therefore, putting all these inequalities together into (159) and using (157), we finally arrive, for c small enough, at the following relation: for all $t \in [0, T]$,

$$(160) \quad \|\nabla \pi\|_{\tilde{L}_t^1(B_{p,r}^s)} \leq C \left(\|h\|_{\tilde{L}_t^1(B_{p,r}^s)} + \int_0^t (\|\nabla u\|_{B_{p,r}^{s-1}} + \|\nabla \rho\|_{B_{p,r}^s}) \|u\|_{B_{p,r}^s} d\tau \right).$$

Now, the analysis of the nonlinear term h can be performed as before, and, up to taking a smaller T than the one defined in (156), this allows us to close the estimates on some suitable time interval $[0, T]$, for sufficiently small c .

A Appendix

First of all, let us recall an easy version of Young inequality: for all $\theta \in]0, 1[$, all $\varepsilon > 0$ and all real numbers $a, b > 0$, we have

$$(1) \quad ab \leq \theta \varepsilon^{-(1-\theta)/\theta} a^{1/\theta} + (1-\theta) \varepsilon b^{1/(1-\theta)}.$$

It is the main ingredient to the proof of Lemmas 4.2 and 4.3.

Proof of Lemma 4.2. We decompose the commutator by use of Bony's paraproduct:

$$(2) \quad [\varphi, \Delta_j] \nabla \psi = R_j^1(\varphi, \psi) + R_j^2(\varphi, \psi) + R_j^3(\varphi, \psi) + R_j^4(\varphi, \psi) + R_j^5(\varphi, \psi),$$

where, setting $\tilde{\varphi} = (\text{Id} - \Delta_{-1})\varphi$, we have defined

$$\begin{aligned} R_j^1(\varphi, \psi) &:= [T_{\tilde{\varphi}}, \Delta_j] \nabla \psi \\ R_j^2(\varphi, \psi) &:= T'_{\Delta_j \nabla \psi} \tilde{\varphi} = \sum_k S_{k+2} \Delta_j \nabla \psi \Delta_k \tilde{\varphi} \\ R_j^3(\varphi, \psi) &:= -\Delta_j T_{\nabla \psi} \tilde{\varphi} \\ R_j^4(\varphi, \psi) &:= -\Delta_j R(\tilde{\varphi}, \nabla \psi) \\ R_j^5(\varphi, \psi) &:= [\Delta_{-1} \varphi, \Delta_j] \nabla \psi. \end{aligned}$$

Let us point out here that, in the following, in general we will derive two types of estimates for each term, corresponding to (44) and (46) respectively. For simplicity, in proving (46) we will consider only the case $\theta = \eta$; the general result follows from easy changes in the proof.

In the following, for the notational simplicity, ∂_x will always refer to one of the partial derivatives ∂_{x_l} , $l = 1, \dots, d$. Let us first consider the term

$$\mathcal{R}^1 := \left\| \left(2^{j\sigma} \int_0^t \|\partial_x R_j^1\|_{L^p} d\tau \right)_j \right\|_{\ell^r},$$

where one finds already

$$R_j^1(\varphi, \psi) = \sum_{\nu \sim j} \int_{\mathbb{R}_z^d} 2^{-j} \left(\int_0^1 h(z) z \cdot \nabla S_{\nu-1} \tilde{\varphi}(x - 2^{-j} \lambda z) \Delta_\nu \nabla \psi(x - 2^{-j} z) d\lambda \right) dz.$$

As it is supported in a ball of radius 2^j , the derivative ∂_x gives (by Bernstein's inequalities) a factor 2^j . Hence we get

$$\mathcal{R}^1 \lesssim \left(\sum_{j \geq -1} \left(2^{j\sigma} \int_0^t \|\nabla S_{j-1} \tilde{\varphi}\|_{L^\infty} \|\Delta_j \nabla \psi\|_{L^p} d\tau \right)^r \right)^{1/r};$$

we can decompose the terms in the integral in the following way:

$$2^{j\sigma} \|\nabla S_{j-1} \tilde{\varphi}\|_{L^\infty} \|\Delta_j \nabla \psi\|_{L^p} \lesssim 2^{j\theta\sigma_1} \|\nabla S_{j-1} \tilde{\varphi}\|_{L^\infty} \|\Delta_j \psi\|_{L^p}^\theta 2^{j(1-\theta)\sigma_2} \|\Delta_j \psi\|_{L^p}^{1-\theta}.$$

Now, we apply Young inequality (1) to separate the two factors; therefore, using Minkowski inequality for the first term, we get, for some constant C :

$$(3) \quad \mathcal{R}^1 \leq \frac{C\theta}{\varepsilon(1-\theta)/\theta} \int_0^t \|\nabla \varphi\|_{L^\infty}^{1/\theta} \|\psi\|_{B_{p,r}^{\sigma_1}} d\tau + (1-\theta)\varepsilon \|\psi\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2})}.$$

Let us now handle

$$\begin{aligned} \mathcal{R}^2 &:= \left\| 2^{j\sigma} \int_0^t \|\partial_x R_j^2\|_{L^p} d\tau \right\|_{\ell^r} \leq \left\| 2^{j\sigma} \int_0^t \sum_{\mu \geq j-2} \|\partial_x S_{\mu+2} \nabla \Delta_j \psi\|_{L^\infty} \|\Delta_\mu \tilde{\varphi}\|_{L^p} d\tau \right\|_{\ell^r} \\ &\quad + \left\| 2^{j\sigma} \int_0^t \sum_{\mu \geq j-2} \|S_{\mu+2} \nabla \Delta_j \psi\|_{L^\infty} \|\partial_x \Delta_\mu \tilde{\varphi}\|_{L^p} d\tau \right\|_{\ell^r}. \end{aligned}$$

One notices that, since $\mu \geq j-2$, we have

$$\|\partial_x S_{\mu+2} \nabla \Delta_j \psi\|_{L^\infty} \|\Delta_\mu \tilde{\varphi}\|_{L^p} \lesssim 2^\mu \|\nabla \Delta_j \psi\|_{L^\infty} \|\Delta_\mu \tilde{\varphi}\|_{L^p} \lesssim \|\nabla \Delta_j \psi\|_{L^\infty} \|\nabla \Delta_\mu \tilde{\varphi}\|_{L^p}.$$

Hence

$$\begin{aligned} \mathcal{R}^2 &\lesssim \left\| 2^{j\sigma} \int_0^t \sum_{\mu \geq j-2} \|\nabla \Delta_j \psi\|_{L^\infty} \|\nabla \Delta_\mu \tilde{\varphi}\|_{L^p} d\tau \right\|_{\ell^r} \\ &\lesssim \left\| \int_0^t \|\nabla \psi\|_{L^\infty} \sum_{\mu \geq j-2} 2^{(j-\mu)\sigma} 2^{\mu\sigma} \|\nabla \Delta_\mu \tilde{\varphi}\|_{L^p} d\tau \right\|_{\ell^r}. \end{aligned}$$

On one side, we just do exactly as above (the way to obtain (3)): if $\sigma > 0$, then we have

$$(4) \quad \mathcal{R}^2 \leq \frac{C\theta}{\varepsilon(1-\theta)/\theta} \int_0^t \|\nabla \psi\|_{L^\infty}^{1/\theta} \|\nabla \tilde{\varphi}\|_{B_{p,r}^{\sigma_1-1}} d\tau + (1-\theta)\varepsilon \|\nabla \tilde{\varphi}\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2-1})}.$$

On the other side, by the relationship $2^{-j\frac{d}{p}} \|\Delta_j f\|_{L^\infty} \lesssim \|\Delta_j f\|_{L^p}$ and Young's inequality, we further derive

$$\begin{aligned} \mathcal{R}^2 &\lesssim \left\| \int_0^t 2^{j(\sigma+1-\frac{d}{p})} \|\Delta_j \psi\|_{L^\infty} \sum_{\mu \geq j-2} 2^{(j-\mu)\frac{d}{p}} \left(2^{\mu\frac{d}{p}} \|\nabla \Delta_\mu \tilde{\varphi}\|_{L^p} \right) d\tau \right\|_{\ell^r} \\ &\lesssim \left\| \sum_{\mu \geq j-2} 2^{(j-\mu)\frac{d}{p}} \left\| 2^{j(\sigma+1)} \|\Delta_j \psi\|_{L^p} \right\|_{L_t^{1/(1-\theta)}} \left\| 2^{\mu\frac{d}{p}} \|\nabla \Delta_\mu \tilde{\varphi}\|_{L^p} \right\|_{L_t^{1/\theta}} \right\|_{\ell^r} \\ &\lesssim \left\| \sum_{\mu \geq j-2} 2^{(j-\mu)\frac{d}{p}} c_\mu \|\nabla \varphi\|_{\tilde{L}_t^{\frac{1}{\theta}}(B_{p,\infty}^{\frac{d}{p}})} 2^{j(\theta\sigma_1+(1-\theta)\sigma_2)} \|\Delta_j \psi\|_{L_t^\infty(L^p)}^\theta \|\Delta_j \psi\|_{L_t^1(L^p)}^{1-\theta} \right\|_{\ell^r}, \quad (c_j)_j \in \ell^\infty \\ (5) \quad &\lesssim \frac{C\theta}{\varepsilon(1-\theta)/\theta} \|\nabla \varphi\|_{\tilde{L}_t^{\frac{1}{\theta}}(B_{p,\infty}^{\frac{d}{p}})}^{\frac{1}{\theta}} \|\psi\|_{\tilde{L}_t^\infty(B_{p,r}^{\sigma_1})} + (1-\theta)\varepsilon \|\psi\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2})}. \end{aligned}$$

Moreover, since

$$\mathcal{R}^3 := \left\| 2^{j\sigma} \int_0^t \|\partial_x R_j^3\|_{L^p} d\tau \right\|_{\ell^r} \leq \left\| 2^{j(\sigma+1)} \int_0^t \sum_{\mu \sim j} \|S_{\mu-1} \nabla \psi\|_{L^\infty} \|\Delta_\mu \tilde{\varphi}\|_{L^p} d\tau \right\|_{\ell^r}$$

we can immediately see that (4) holds also for \mathcal{R}^3 . In order to get the form like (5), we consider the two cases $\sigma < \frac{d}{p}$ and $\sigma > \frac{d}{p}$ separately. Here we notice that if $\sigma = \frac{d}{p}$ and $r = 1$, then taking the integral in time and the ℓ^r norm can commute and hence the classical result for commutators (see Chapter 2 of [4]) gives this lemma. Now if $\sigma_i < 1 + \frac{d}{p}$, $i = 1, 2$, then

$$\mathcal{R}^3 \lesssim \left\| \int_0^t \left(2^{j\theta(\sigma_1-1-\frac{d}{p})} \|S_{j-1} \nabla \psi\|_{L^\infty}^\theta \right) \left(2^{j(1-\theta)(\sigma_2-1-\frac{d}{p})} \|S_{j-1} \nabla \psi\|_{L^\infty}^{1-\theta} \right) 2^{j(\frac{d}{p}+1)} \|\Delta_j \tilde{\varphi}\|_{L^p} \right\|_{\ell^r},$$

which gives (5) for \mathcal{R}^3 . On the other hand, $\sigma_i > 1 + \frac{d}{p}$, $i = 1, 2$, then as above

$$\begin{aligned} \mathcal{R}^3 &\lesssim \left\| \int_0^t \|\nabla \psi\|_{L^\infty} \left(2^{j(\sigma+1)} \|\Delta_j \tilde{\varphi}\|_{L^p} \right) \right\|_{\ell^r} \\ &\leq \frac{C\theta}{\varepsilon(1-\theta)/\theta} \|\nabla \varphi\|_{\tilde{L}_t^{\frac{1}{\theta}}(B_{p,r}^{\sigma_1+1})}^{\frac{1}{\theta}} \|\nabla \psi\|_{L_t^\infty(L^\infty)} + (1-\theta)\varepsilon \|\nabla \psi\|_{L_t^1(L^\infty)} \\ (6) \quad &\leq \frac{C\theta}{\varepsilon(1-\theta)/\theta} \|\nabla \varphi\|_{\tilde{L}_t^{\frac{1}{\theta}}(B_{p,r}^{\sigma_1+1})}^{\frac{1}{\theta}} \|\psi\|_{L_t^\infty(B_{p,r}^{\sigma_1})} + (1-\theta)\varepsilon \|\nabla \psi\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2-1})}, \end{aligned}$$

where the last inequality follows by the embeddings $L_t^1(L^\infty) \hookleftarrow \tilde{L}_t^1(B_{p,1}^{d/p}) \hookleftarrow \tilde{L}_t^1(B_{p,r}^{\sigma_2-1})$.

The same holds true also for

$$\mathcal{R}^4 := \left\| 2^{j\sigma} \int_0^t \|\partial_x R_j^4\|_{L^p} d\tau \right\|_{\ell^r}.$$

Let us first sketch the case $\sigma > \frac{d}{p}$. We can bound \mathcal{R}^4 in the following way:

$$\mathcal{R}^4 \lesssim \left\| \int_0^t \|\nabla \psi\|_{L^\infty} \sum_{\mu \geq j-2} 2^{(j-\mu)(\sigma+1)} \left(2^{\mu(\sigma+1)} \|\Delta_\mu \tilde{\varphi}\|_{L^p} \right) d\tau \right\|_{\ell^r}.$$

Hence (4) and (6) follow immediately for \mathcal{R}^4 . If $\sigma < \frac{d}{p}$, then we consider two cases $p \leq 2$ and $p > 2$ separately, which both give (5). In fact, if $p \leq 2$, then $p' \geq p$ and we have for $\sigma + 1 > -\frac{d}{p'}$,

$$\begin{aligned} \mathcal{R}^4 &\lesssim \left\| \int_0^t 2^{j(\sigma+1+\frac{d}{p'})} \|R_j^4\|_{L^1} d\tau \right\|_{\ell^r} \\ &\lesssim \left\| \int_0^t \sum_{\mu \geq j-3} 2^{(j-\mu)(\sigma+1+\frac{d}{p'})} \left(2^{\mu(\frac{d}{p}+1)} \|\Delta_j \tilde{\varphi}\|_{L^p} \right) \left(2^{\mu(\sigma+\frac{d}{p'}-\frac{d}{p})} \|\Delta_\mu \nabla \psi\|_{L^{p'}} \right) d\tau \right\|_{\ell^r}, \end{aligned}$$

which gives (5). Otherwise if $p > 2$, then we have for $\sigma + 1 > -\frac{d}{p}$,

$$\begin{aligned} \mathcal{R}^4 &\lesssim \left\| \int_0^t 2^{j(\sigma+1+\frac{d}{p})} \|R_j^4\|_{L^{p/2}} d\tau \right\|_{\ell^r} \\ &\lesssim \left\| \int_0^t \sum_{\mu \geq j-3} 2^{(j-\mu)(\sigma+1+\frac{d}{p})} \left(2^{\mu(\frac{d}{p}+1)} \|\Delta_j \tilde{\varphi}\|_{L^p} \right) (2^{\mu\sigma} \|\Delta_\mu \nabla \psi\|_{L^p}) d\tau \right\|_{\ell^r}, \end{aligned}$$

which yields (5) also.

Finally, last term

$$\mathcal{R}^5 := \left\| 2^{j\sigma} \int_0^t \|\partial_x R_j^5\|_{L^p} d\tau \right\|_{\ell^r}$$

can be handled as \mathcal{R}^1 , leading us to the same estimate as (3) and so to the end of the proof. \square

Proof of Lemma 4.3. We have to estimate

$$\|f g\|_{\tilde{L}_t^1(B_{p,r}^\sigma)} = \left(\sum_{j \geq -1} \left(2^{\sigma j} \int_0^t \|\Delta_j(f g)\|_{L^p} d\tau \right)^r \right)^{1/r}.$$

Using Bony's paraproduct decomposition, we can write

$$(7) \quad \Delta_j(f g) = \Delta_j T_f g + \Delta_j T_g f + \Delta_j R(f, g).$$

Let us consider the first term in the right-hand side of the previous relation. Due to spectral localization, we infer that there exists a positive integer ι_0 such that, for all j ,

$$\Delta_j T_f g = \sum_{|\nu-j| \leq \iota_0} \Delta_j (S_{\nu-1} f \Delta_\nu g) \implies \|\Delta_j T_f g\|_{L^p} \leq C \|f\|_{L^\infty} \|\Delta_j g\|_{L^p},$$

for a constant C which depends only on ι_0 . Now we integrate in time, we apply Young's inequality (1) and we pass to the ℓ^r norm: using also Minkowski's inequality we gather

$$\|T_f g\|_{\tilde{L}_t^1(B_{p,r}^\sigma)} \leq \frac{C_1 \theta}{\varepsilon(1-\theta)/\theta} \int_0^t \|f\|_{L^\infty}^{1/\theta} \|g\|_{B_{p,r}^{\sigma_1}} d\tau + C_2 \varepsilon (1-\theta) \|g\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2})}.$$

The term $\Delta_j T_g f$ can be treated in the same way, and this leads us to an analogous estimate, in which however the roles of f and g are reversed.

Let us now consider the remainder term. Thanks again to spectral localization, we have, for some positive integer j_0 independent of j ,

$$\Delta_j R(f, g) = \sum_{\nu \geq j-j_0} \sum_{|\mu-\nu| \leq 1} \Delta_j (\Delta_\mu f \Delta_\nu g).$$

Therefore it follows that

$$\|R(f, g)\|_{\tilde{L}_t^1(B_{p,r}^\sigma)} \leq C \left(\sum_{j \geq -1} \left(\sum_{\nu \geq j-j_0} 2^{(j-\nu)\sigma} 2^{\nu\sigma} \int_0^t \|\Delta_\nu f\|_{L^\infty} \|\Delta_\nu g\|_{L^p} d\tau \right)^r \right)^{1/r}.$$

Now we apply (1) to the product in the integral and Young's inequality for convolutions: we finally arrive to

$$\|R(f, g)\|_{\tilde{L}_t^1(B_{p,r}^\sigma)} \leq \frac{C_1 \theta}{\varepsilon(1-\theta)/\theta} \int_0^t \|f\|_{L^\infty}^{1/\theta} \|g\|_{B_{p,r}^{\sigma_1}} d\tau + C_2 \varepsilon (1-\theta) \|g\|_{\tilde{L}_t^1(B_{p,r}^{\sigma_2})},$$

and this completes the proof of the lemma.

Let us point out also that Remark 4.4 easily follows slightly modifying the previous proof. \square

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