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# AN APPLICATION OF SHADOW SYSTEMS TO MAHLER'S CONJECTURE.

MATTHIEU FRADELIZI, MATHIEU MEYER AND ARTEM ZVAVITCH

ABSTRACT. We elaborate on the use of shadow systems to prove a particular case of the conjectured lower bound of the volume product  $\mathcal{P}(K) = \min_{z \in \text{int}(K)} |K| |K^z|$ , where  $K \subset \mathbb{R}^n$  is a convex body and  $K^z = \{y \in \mathbb{R}^n : (y - z) \cdot (x - z) \leq 1 \text{ for all } x \in K\}$  is the polar body of  $K$  with respect to the center of polarity  $z$ . In particular, we show that if  $K \subset \mathbb{R}^3$  is the convex hull of two 2-dimensional convex bodies, then  $\mathcal{P}(K) \geq \mathcal{P}(\Delta^3)$ , where  $\Delta^3$  is a 3-dimensional simplex, thus confirming the 3-dimensional case of Mahler conjecture, for this class of bodies. A similar result is provided for the symmetric case, where we prove that if  $K \subset \mathbb{R}^3$  is symmetric and the convex hull of two 2-dimensional convex bodies, then  $\mathcal{P}(K) \geq \mathcal{P}(B_\infty^3)$ , where  $B_\infty^3$  is the unit cube.

## 1. INTRODUCTION AND PRELIMINARIES

As usual, we denote by  $x \cdot y$  the inner product of two vectors  $x, y \in \mathbb{R}^n$  and by  $|x|$  the length of vector  $x \in \mathbb{R}^n$ . For two non-empty subsets  $K, L \subset \mathbb{R}^n$  we define their Hausdorff distance  $d_H(K, L)$  by

$$d_H(K, L) = \max\left\{ \sup_{x \in K} \inf_{y \in L} |x - y|, \sup_{y \in L} \inf_{x \in K} |x - y| \right\}.$$

A *convex body* is a compact convex subset of  $\mathbb{R}^n$  with non empty interior and  $\mathcal{K}_n$  is the set of all convex bodies in  $\mathbb{R}^n$  endowed with the Hausdorff metric. We say that a set  $K$  is *symmetric* if it is centrally symmetric with center at the origin, i.e.  $K = -K$ .

We write  $|A|$  for the  $k$ -dimensional Lebesgue measure (volume) of a measurable set  $A \subset \mathbb{R}^n$ , where  $k = 1, \dots, n$  is the dimension of the minimal flat containing  $A$ . We denote by  $\overline{\text{conv}}(A)$  the *closed* convex hull of a set  $A \subset \mathbb{R}^n$ , and  $\overline{\text{conv}}(A, B, C, \dots)$  the closed convex hull of  $A \cup B \cup C, \dots$ . For  $a, b \in \mathbb{R}^n$ , we denote  $[a, b]$  the segment joining  $a$  to

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$b$ :  $[a, b] = \{(1-t)a + tb : t \in [0, 1]\}$ . If  $K \in \mathcal{K}_n$ , we denote by  $\mathcal{E}(K)$  the set of its extreme points. By Caratheodory's theorem, one has  $K = \overline{\text{conv}}(\mathcal{E}(K))$ . We recommend [Gr] and [Sc] as a general references for convex bodies and polytopes and their properties.

By  $\text{int}(K)$  we denote the interior of  $K$ . If  $K$  is a convex body in  $\mathbb{R}^n$  and  $z \in \text{int}(K)$  the *polar body*  $K^z$  of  $K$  with the center of polarity  $z$  is defined by

$$K^z = \{y \in \mathbb{R}^n : (y - z) \cdot (x - z) \leq 1 \text{ for all } x \in K\}.$$

If the center of polarity is taken to be the origin, we denote by  $K^\circ$  the polar body of  $K$ , thus  $K^z = (K - z)^\circ + z$ . The bipolar theorem says that  $(K^z)^z = K$ , for  $z \in \text{int}(K)$  (see [Gr], p. 47).

A well known result of Santaló [S] (see also [Sc], p. 419) states that in every convex body  $K$  in  $\mathbb{R}^n$ , there exists a unique point  $s(K)$ , called the *Santaló point* of  $K$ , such that

$$|K^{s(K)}| = \min_{z \in \text{int}(K)} |K^z|.$$

The *volume product* of  $K$  is defined by

$$\mathcal{P}(K) = \inf\{|K||K^z| : z \in \text{int}(K)\} = |K| |K^{s(K)}|.$$

The volume product is affinely invariant, that is,  $\mathcal{P}(A(K)) = \mathcal{P}(K)$  for every affine isomorphism  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Observe that if we denote  $L = K^{s(K)}$  then

$$\mathcal{P}(K^{s(K)}) = |L||L^{s(L)}| \leq |L||L^{s(K)}| = |K^{s(K)}||K| = \mathcal{P}(K).$$

The set of all convex bodies in  $\mathbb{R}^n$  is compact with respect to the Banach-Mazur distance and  $K \mapsto \mathcal{P}(K)$  is continuous (see Lemma 3, below), so that it is natural to ask for a maximal and minimal values of  $\mathcal{P}(K)$ . The Blaschke-Santaló inequality states that

$$\mathcal{P}(K) \leq \mathcal{P}(B_2^n),$$

where  $B_2^n$  is the Euclidean unit ball. The equality in the above inequality is possible only for ellipsoids ([S], [P], see [MP] or also [MR2] for a simple proof of both the inequality and the case of equality).

The main focus of this paper is the conjecture about minimality of  $\mathcal{P}(K)$ , often called Mahler's conjecture [Ma1, Ma2], which states that, for every convex body  $K$  in  $\mathbb{R}^n$ ,

$$(1) \quad \mathcal{P}(K) \geq \mathcal{P}(\Delta^n) = \frac{(n+1)^{n+1}}{(n!)^2},$$

where  $\Delta^n$  is an  $n$ -dimensional simplex. It is also conjectured that equality in (1) is attained only if  $K$  is a simplex.

We shall also pay a special attention to the symmetric case of Mahler conjecture which states that for every convex *symmetric* body  $K \subset \mathbb{R}^n$ :

$$(2) \quad \mathcal{P}(K) \geq \mathcal{P}(B_1^n) = \mathcal{P}(B_\infty^n) = \frac{4^n}{n!},$$

where  $B_1^n$  and  $B_\infty^n$  are cross-polytope and its dual - cube, respectively.

The inequalities (1) and (2) for  $n = 2$  were proved by Mahler [Ma1] with the case of equality proved by Meyer [M2] in the general case and by Reisner [R1] in the symmetric case. Other cases, like e.g. the bodies of revolution, were treated in [MR1]. Several special cases of the (mostly) symmetric case of the conjecture can be found in [SR, R1, GMR, M1, R2, NPRZ, KR, FGMR, RSW, GM]. Not many special cases in which (1) is true seem to be known, one such is the case of convex bodies having hyperplane symmetries which fix only one common point [BF]; another one is the  $n$  dimensional polytopes with at most  $n + 3$  vertices (or facets) proved in [MR2]. The proof of this last result is based on the method of shadow systems. We shall here elaborate on this method, applying it alternatively to a body and its polar.

Observe that an isomorphic version of reverse Santaló inequality was proved by Bourgain and Milman [BM], see also [Pi], p.100:

$$\mathcal{P}(K) \geq c^n \mathcal{P}(B_2^n),$$

where  $c$  is a positive constant; Kuperberg [Ku] gave a new proof of this result with a better constant (see also [N], [GPV] for different proofs of the inequality).

It is conjectured that in the centrally symmetric case, the convex bodies which are minimal for the volume product are the unit balls of Lima spaces, *i.e.* the finite dimensional normed spaces with the 3 – 2 intersection property: every three translates of the unit ball which intersect 2 by 2, actually intersect. As normed spaces, Lima spaces are characterized by the fact that they can be decomposed into  $\ell_\infty$  or  $\ell_1$  sums of normed spaces of smaller dimension satisfying the same decomposition property [HL]. In the special cases of convex bodies symmetric with respect to  $n$  independent hyperplanes, it was proved in [M1, R2], that the unit ball of Lima spaces are the minimizers of  $\mathcal{P}(K)$ . The unit ball of Lima spaces have the special property (which does not characterize them): any facet of their unit ball contains half of the extreme points so that they are the convex hull of any of two of their opposite facets (see [HL]). The purpose of this paper was originally to try to prove that unit balls of finite dimensional spaces which have this last property satisfy Mahler conjecture. We proved it in dimension 3.

Then we realized that our method of proof can be used for a much more general result: if a convex body in  $\mathbb{R}^3$  is the convex hull of two of its hyperplane sections, then it satisfies Mahler's conjecture (Theorem 1).

Thus the main goal of this paper is to prove the following special cases of inequalities (1) and (2):

**Theorem 1.** *Consider two different planes  $H_1$  and  $H_2$  in  $\mathbb{R}^3$  and a convex body  $K$  in  $\mathbb{R}^3$  such that*

$$K = \overline{\text{conv}}(K \cap H_1, K \cap H_2).$$

*Then*

$$\mathcal{P}(K) \geq \mathcal{P}(\Delta_3) = \frac{64}{9}.$$

*If, moreover,  $K$  is centrally symmetric then*

$$\mathcal{P}(K) \geq \mathcal{P}(B_\infty^3) = \frac{32}{3}.$$

**Remark 1.** *Notice that if  $K = \overline{\text{conv}}(K \cap H_1, K \cap H_2)$  is centrally symmetric then*

*- either  $H_1$  is parallel to  $H_2$ , the central symmetry maps  $H_1$  onto  $H_2$  and  $K \cap H_1$  onto  $K \cap H_2$ .*

*- or  $H_1$  and  $H_2$  intersect and  $K \cap H_1$  and  $K \cap H_2$  are centrally symmetric.*

From Theorem 1 and the first case of Remark 1 we deduce the following corollary.

**Corollary 1.** *Let  $K$  be a centrally symmetric convex body in  $\mathbb{R}^3$  whose extreme points lie into two parallel planes. Then*

$$\mathcal{P}(L) \geq \mathcal{P}(B_\infty^3) = \frac{32}{3}.$$

We would like to note that most of the tools presented in Section 2 are stated and proved in dimension  $n \geq 2$ . Still, unfortunately, the proof of Theorem 1, in Section 3, requires to use the special geometrical structure of 3-dimensional polytopes as well as the assumption that the Mahler conjecture is true in  $\mathbb{R}^{n-1}$ , thus restricting to the case  $n = 3$ .

## 2. THE TOOLS

As the main tool in the proof of Theorem 1, we will use shadow systems of convex sets: a *shadow system of convex sets* along a direction  $\theta \in \mathbb{S}^{n-1}$  is a family of convex sets  $L_t \in \mathbb{R}^n$  which are defined by

$$L_t = \overline{\text{conv}}\{x + \alpha(x)t\theta : x \in B\},$$

where  $B \subset \mathbb{R}^n$  is a bounded set, called the basis of the shadow system,  $\alpha : B \rightarrow \mathbb{R}$  is a bounded function, called the speed of the shadow system and  $t$  belongs to an open interval in  $\mathbb{R}$ . We say that a shadow system is *non-degenerate*, if all the convex sets  $L_t$  have non-empty interior. The shadow systems were introduced by Rogers and Shephard [RS]. Campi and Gronchi [CG] proved that if  $L_t$  is a symmetric shadow system then  $t \mapsto |L_t^\circ|^{-1}$  is a convex function of  $t$ . In [MR2], Reisner and the second author generalized this result to the non-symmetric case and studied the equality case. The following proposition is the key tool in the proof of Theorem 1:

**Proposition 1** ([MR2]). *Let  $L_t$ ,  $t \in [-a, a]$ , be a non-degenerate shadow system in  $\mathbb{R}^n$ , with direction  $\theta \in \mathbb{S}^{n-1}$ , then  $t \mapsto |L_t^{s(L_t)}|^{-1}$  is a convex function on  $[-a, a]$ .*

*If, moreover,  $t \mapsto |L_t|$  is affine on  $[-a, a]$  and  $t \mapsto \mathcal{P}(L_t)$  is constant on  $[-a, a]$ , then there exists  $w \in \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$ , such that for every  $t \in [-a, a]$ , one has  $L_t = A_t(L_0)$ , where  $A_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the affine map defined by*

$$A_t(x) = x + t(w \cdot x + \alpha)\theta.$$

The following corollary is a variation of the above proposition that we shall need.

**Corollary 2.** *Let  $L_t$ ,  $t \in [-a, a]$ , be a non-degenerate shadow system in  $\mathbb{R}^n$  such that  $t \mapsto |L_t|$  is affine on  $[-a, a]$ . Then*

- 1)  $t \mapsto \mathcal{P}(L_t)$  is quasi-concave on  $[-a, a]$ , i.e. for every interval  $[c, d] \subset [-a, a]$  one has  $\min_{[c, d]} \mathcal{P}(L_t) = \min\{\mathcal{P}(L_c), \mathcal{P}(L_d)\}$ ;
- 2) if for some  $t_0 \in (-c, c) \subset [-a, a]$ , one has  $\mathcal{P}(L_{t_0}) = \min_{t \in [-c, c]} \mathcal{P}(L_t)$ , then either  $\mathcal{P}(L_t) = \mathcal{P}(L_{t_0})$  for every  $t \in [t_0, c]$ , or  $\mathcal{P}(L_t) = \mathcal{P}(L_{t_0})$  for every  $t \in [-c, t_0]$ .

*Proof of Corollary 2.*

- 1) The volume product of  $L_t$  is the quotient of the affine positive function  $f(t) := |L_t|$  by the convex (from Proposition 1) positive function  $g(t) := |L_t^{s(L_t)}|^{-1}$ . Such a quotient is quasi-concave because for every interval  $[c, d] \subset [a, b]$  and any  $\lambda \in [0, 1]$  one has

$$\frac{f((1-\lambda)c + \lambda d)}{g((1-\lambda)c + \lambda d)} \geq \frac{(1-\lambda)f(c) + \lambda f(d)}{(1-\lambda)g(c) + \lambda g(d)} \geq \min\left(\frac{f(c)}{g(c)}, \frac{f(d)}{g(d)}\right).$$

- 2) Notice first that if  $h$  is a quasi-concave function on some interval  $[c, d]$  and  $\min_{[c, d]} h = h(t_0)$ , for some  $t_0 \in (c, d)$ , then  $h$  is constant on either  $[c, t_0]$  or on  $[t_0, d]$ . Indeed if  $h$  was not constant on any of these intervals, there would exist  $u \in [c, t_0]$  and  $v \in (t_0, d]$  such that

$h(u) > h(t_0)$  and  $h(v) > h(t_0)$ . But since  $t_0 \in (u, v)$  this contradicts the fact that  $\min_{[u,v]} h = \min \{h(u), h(v)\}$ . It follows that  $h$  is constant on at least one of these intervals.  $\square$

We need now to define a particular form of shadow system:

**Definition.** Consider  $K \in \mathcal{K}_n$ ,  $x \in \mathcal{E}(K)$  and  $\theta \in \mathbb{S}^{n-1}$ . Let  $R_x(K) = \overline{\text{conv}}(\mathcal{E}(K) \setminus \{x\})$ . Define  $x_t = x + t\theta$  and  $K_t = \overline{\text{conv}}(R_x(K), x_t)$ . Then  $t \mapsto K_t$  is called *the shadow movement of  $K$  based on  $x$  with direction  $\theta$* . Such a shadow movement is actually a shadow system, with  $B = \mathcal{E}(K)$  and speed  $\alpha : \mathcal{E}(K) \rightarrow \mathbb{R}$  defined by  $\alpha(y) = 0$  if  $y \neq x$  and  $\alpha(x) = 1$ . Observe that one has of course  $K_0 = K$ . We say that a shadow movement  $(K_t)$  is *volume affine on  $[-c, c]$*  if  $t \mapsto |K_t|$  is affine on  $[-c, c]$ .

We shall need the following easy lemma (see, for example, Lemma 11 in [MR2]), where we use the fact that inequalities (1) and (2) hold in dimension 2, by the original Mahler's result.

**Lemma 1.** *Let  $H$  be a hyperplane in  $\mathbb{R}^n$ ,  $B$  be a convex body in  $H$ ,  $x \notin H$ .*

1) *Let  $C$  be the cone with apex  $x$  and basis  $B$ , i.e.  $C = \overline{\text{conv}}(B, \{x\})$ . Then one has  $\mathcal{P}(C) \geq \frac{(n+1)^{n+1}}{n^{n+2}} \mathcal{P}(B)$ . In particular, if  $n = 3$ , then  $\mathcal{P}(C) \geq \mathcal{P}(\Delta^3)$ .*

2) *If  $B$  is symmetric, let  $D = \overline{\text{conv}}(B, \{x\}, \{-x\})$  be a double-cone with basis  $B$ . Then one has  $\mathcal{P}(D) \geq \frac{4}{n} \mathcal{P}(B)$  and in particular, if  $n = 3$ , then  $\mathcal{P}(D) \geq \mathcal{P}(B_1^3) = \mathcal{P}(B_\infty^3)$ .*

**Lemma 2.** *Let  $t \mapsto K_t$ ,  $t \in [-c, c]$  be a volume affine shadow movement of a convex body  $K \subset \mathbb{R}^n$  based on  $x \in \mathcal{E}(K)$  with direction  $\theta \in \mathbb{S}^{n-1}$ . Suppose, moreover, that*

$$\mathcal{P}(K) = \min_{t \in [-c, c]} \mathcal{P}(K_t).$$

*Then  $\mathcal{E}(K) \setminus \{x\}$  is contained in a hyperplane,  $K$  is a cone with apex  $x$  and basis  $\overline{\text{conv}}(\mathcal{E}(K) \setminus \{x\})$  and*

$$\mathcal{P}(K) \geq \frac{(n+1)^{n+1}}{n^{n+2}} \min_{M \in \mathcal{K}_{n-1}} \mathcal{P}(M).$$

*Proof.* Let  $R_x(K) = \overline{\text{conv}}(\mathcal{E}(K) \setminus \{x\})$ . Observe that  $x \notin R_x(K)$ . Otherwise, one would have  $K \subset K_t$ , hence  $|K| \leq |K_t|$  for every  $t$ , since  $t \mapsto |K_t|$  is affine on interval  $[-c, c]$ , it may have minimum at  $t = 0$  only if it is a constant on  $[-c, c]$ . Thus  $K = K_t$ , which implies that  $x_t \in K$  for every  $t \in [-c, c]$ , which contradicts the extremality of  $x \in K$ .

It follows that  $d(x, R_x(K)) > 0$ , hence there exist  $\gamma > 0$  and  $c' > 0$  such that  $d(x + t\theta, R_x(K)) > \gamma$  for all  $t \in [-c', c']$ . Thus, for all  $t \in [-c', c']$ ,  $x + t\theta$  is an extreme point of  $K_t$ . Decreasing  $c$ , we may suppose that  $c' = c$ .

By the second part of Corollary 2, we may suppose also that  $\mathcal{P}(K_t) = \mathcal{P}(K)$  for every  $t \in [0, c]$ . Let  $d := \frac{c}{2}$  and  $L = K_d$ . We define the shadow movement of  $L$  based on  $x + d\theta$  with direction  $\theta$ . We notice that for  $s \in [-d, d]$ , one has  $L_s = K_{d+s}$ . Of course the shadow movement  $(L_s)$  is also volume affine on  $[-d, d]$  and satisfy  $s \mapsto \mathcal{P}(L_s)$  is constant on  $[-d, d]$ .

By Proposition 1, there exist  $\alpha \in \mathbb{R}$  and  $w \in \mathbb{R}^n$  such that for all  $s \in [-d, d]$ , one has  $L_s = A_s(L)$ , where  $A_s : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the affine isomorphism defined by  $A_s(z) = z + s(z \cdot w + \alpha)\theta$ .

Now, since  $A_s$  is affine, it maps the extreme points of  $L$  onto the extreme points of  $L_s$ ,  $|s| \leq d$ . Thus  $A_s(x + d\theta)$  is an extreme point of  $L_s$ , that is  $A_s(x + d\theta) \in R_x(K) \cup \{x + (d + s)\theta\}$ . But for  $|s|$  small enough  $A_s(x + d\theta)$  is close to  $x + d\theta$ , and thus far from  $R_x(K)$ . It follows that  $A_s(x + d\theta) = x + (d + s)\theta$  for all  $s$  small enough and thus also  $A_s(R_x(K)) = R_x(K)$ . Then  $A_s$  is an isometry (with respect to the Euclidean scalar product), when  $R_x(K)$  has non-empty interior. From the special form of  $A_s$  this can only happen if  $A_s$  is the identity. But  $A_s(x + d\theta) = x + (d + s)\theta \neq x + d\theta$ , for  $s \neq 0$ , thus we get a contradiction and  $R_x(K)$  must have an empty interior. Being convex  $R_x(K)$ , must be contained in a hyperplane. Thus  $K = \overline{\text{conv}}(R_x(K), x)$  is a cone with apex  $x$  and basis  $R_x(K)$ . The result follows from Lemma 1.  $\square$

For sake of completeness, we prove the following classical result, which is needed to restrict the proof of Theorem 1 to the case of polytopes:

**Lemma 3.**  $K \mapsto \mathcal{P}(K)$  is a continuous function on  $\mathcal{K}_n$ .

*Proof.* Let  $K_m, m \geq 0$  and  $K$  be compact convex bodies such that  $K_m \rightarrow K$ . By John's Theorem (see, for example, [Pi], p. 33) there is an affine isomorphism  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $B_2^n \subset L := A(K) \subset nB_2^n$ . If  $L_m := A(K_m)$ , one has still  $L_m \rightarrow L$ . There exist constants  $c, d > 0$  and  $M \geq 0$  such that for any  $m \geq M$ , one has  $cB_2^n \subset L_m \subset dB_2^n$  and  $cB_2^n \subset L \subset dB_2^n$ . Since  $L_m \rightarrow L$ , for every  $\varepsilon > 0$  there exists  $M'$  such that  $L_m \subset L + \varepsilon B_2^n$  and  $L \subset L_m + \varepsilon B_2^n$ , for every  $m \geq M'$ . It follows that  $L_m \subset (1 + \eta)L$  and  $L \subset (1 + \eta)L_m$ , for  $m \geq \max\{M, M'\}$  and  $\eta = \frac{\varepsilon}{c}$ . Thus  $L^{s(L_m)} \subset (1 + \eta)L_m^{s(L_m)}$  and

$$\mathcal{P}(L) = |L| |L^{s(L)}| \leq |L| |L^{s(L_m)}| \leq (1 + \eta)^{2n} |L_m| |L_m^{s(L_m)}| = (1 + \eta)^{2n} \mathcal{P}(L_m),$$



for  $m \geq M$ . Similarly,  $L_m^{s(L)} \subset (1 + \eta)L^{s(L)}$  and

$$\mathcal{P}(L_m) = |L_m| |L_m^{s(L_m)}| \leq |L_m| |L_m^{s(L)}| \leq (1 + \eta)^{2n} |L| |L^{s(L)}| = (1 + \eta)^{2n} \mathcal{P}(L).$$

Since  $\mathcal{P}(L_m) = \mathcal{P}(A(K_m)) = \mathcal{P}(K_m)$  and  $\mathcal{P}(L) = \mathcal{P}(A(K)) = \mathcal{P}(K)$ , we can conclude.  $\square$

**Proposition 2.** *Suppose that  $\mathcal{C}$  is a closed subset of  $\mathcal{K}_n$  such that for any  $K \in \mathcal{C}$ , any  $z \in \text{int}(K)$ , any  $y \in \text{int}(K^z)$  and any affine isomorphism  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , one has  $(K^z)^y \in \mathcal{C}$  and  $A(K) \in \mathcal{C}$ . Then*

1) *there exists  $Q \in \mathcal{C}$  such that  $\mathcal{P}(Q) = \min_{P \in \mathcal{C}} \mathcal{P}(P)$ ;*

2) *Moreover, if one of the following hypothesis holds*

(i) *there exists a volume affine shadow movement  $(Q_t)_{t \in [-c, c]}$  of  $Q$  with basis  $x \in \mathcal{E}(Q)$  such that  $Q_t \in \mathcal{C}$  for  $t \in [-c, c]$ ,*

(ii) *there exists a volume affine shadow movement  $(L_t)_{t \in [-c, c]}$  of  $L = Q^{s(Q)}$  with basis  $y \in \mathcal{E}(L)$ ,  $t \in [-c, c]$  such that  $(L_t)^{s(L)} \in \mathcal{C}$  for  $t \in [-c, c]$ ,*

*then for every  $P \in \mathcal{C}$ , one has  $\mathcal{P}(P) \geq \frac{(n+1)^{n+1}}{n^{n+2}} \min_{M \in \mathcal{K}_{n-1}} \mathcal{P}(M)$ .*

*Proof.* Let  $\gamma = \inf_{K \in \mathcal{C}} \mathcal{P}(K)$ . Then there exists a sequence  $(P_m) \in \mathcal{C}$  such that  $\mathcal{P}(P_m) \rightarrow \gamma$ . Applying John's Theorem, one can find a sequence  $\{A_m\}$  of affine isomorphisms such that

$$B_2^n \subset Q_m := A_m(P_m) \subset nB_2^n, \text{ for all } m \in \mathbb{N}.$$

But  $Q_m \in \mathcal{C}$ , from the assumption of Proposition 2 and  $\mathcal{P}(Q_m) = \mathcal{P}(P_m) \rightarrow \gamma$ . The set  $\{K \in \mathcal{K}_n : B_2^n \subset K \subset nB_2^n\}$  is compact with respect to the Hausdorff distance. Thus we may assume that  $\{Q_m\}_m$  converges to some  $Q \in \mathcal{K}_n$ . Since  $\mathcal{C}$  is closed,  $Q \in \mathcal{C}$  and since, by Lemma 3,  $K \mapsto \mathcal{P}(K)$  is continuous on  $\mathcal{K}_n$ , we get  $\mathcal{P}(Q) = \gamma$ . Under hypothesis (i), the result follows immediately from Lemma 2. Now, under hypothesis (ii), with the above notation, we get

$$M_t = (L_t)^{s(L)} \in \mathcal{C}, L_t = (M_t)^{s(L)} \text{ and } L_t^{s(L_t)} = ((M_t)^{s(L)})^{s(L_t)},$$

for small enough  $|t|$ . It follows that  $L_t^{s(L_t)} \in \mathcal{C}$ . Thus

$$\mathcal{P}(Q) \leq \mathcal{P}(L_t^{s(L_t)}) \leq \mathcal{P}(L_t).$$

Applying this to  $t = 0$  and since  $L_0 = L = Q^{s(Q)}$ , one has also

$$\mathcal{P}(Q) \leq \mathcal{P}(L_0) = \mathcal{P}(L) = \mathcal{P}(Q^{s(Q)}) \leq \mathcal{P}(Q).$$

Thus  $L_t$  is a volume affine shadow movement whose volume product is minimized at 0. From Lemma 2, we conclude that  $L = Q^{s(Q)}$  is a cone

and for every  $P \in \mathcal{C}$  one has

$$\mathcal{P}(P) \geq \mathcal{P}(Q) \geq \mathcal{P}(Q^{s(Q)}) \geq \frac{(n+1)^{n+1}}{n^{n+2}} \min_{M \in \mathcal{K}_{n-1}} \mathcal{P}(M).$$

□

**Definition.** For  $N \geq n+1$ , we define  $\mathcal{P}_N$  to be the set of all convex polytopes in  $\mathbb{R}^n$  with non empty interior, having at most  $N$  vertices,  $\mathcal{C}_N$  to be the set of all polytopes in  $\mathcal{P}_N$  which are the convex hull of two of their hyperplane sections and  $\mathcal{D}_N$  to be the set of all polytopes in  $\mathcal{P}_N$  which are the convex hull of two of their parallel facets.

**Lemma 4.** *Let  $N \geq n+1$ . Then  $\mathcal{P}_N$ ,  $\mathcal{C}_N$  and  $\mathcal{D}_N$  are closed subsets of  $\mathcal{K}_n$ . Moreover, if  $P$  is a polytope in  $\mathbb{R}^n$ ,  $z \in \text{int}(P)$ ,  $y \in \text{int}(P^z)$  and  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an affine isomorphism and  $P \in \mathcal{P}_N$  (resp.  $\mathcal{C}_N, \mathcal{D}_N$ ), then so do  $(P^z)^y$  and  $A(P)$ .*

*Proof.* We use standard arguments of compactness for the fact that these classes are closed. It also follows from the definition that the classes are stable under affine isomorphisms. Next we observe that if  $K \in \mathcal{K}_n$ ,  $z \in \text{int}(K)$  and  $y \in \text{int}(K^z)$ , then

$$(K^z)^y = \left\{ \frac{x-z}{1-(x-z) \cdot (y-z)} + y : x \in K \right\}.$$

The above formula follows immediately by applying polarity with respect to  $y$  to both sets. To show that  $(P^z)^y$  is in the same class as  $P$ , we define  $F : K \rightarrow (K^z)^y$  by

$$F(x) = \frac{x-z}{1-(x-z) \cdot (y-z)} + y.$$

We note that  $F$  is a bijection preserving segments. Indeed, let  $x_1, x_2 \in K$  and  $\lambda \in [0, 1]$ . We set

$$\mu = \frac{\lambda(1-(x_2-z) \cdot (y-z))}{(1-\lambda)(1-(x_1-z) \cdot (y-z)) + \lambda(1-(x_2-z) \cdot (y-z))}.$$

Then  $F((1-\lambda)x_1 + \lambda x_2) = (1-\mu)F(x_1) + \mu F(x_2)$ . Hence,  $F([x_1, x_2]) = [F(x_1), F(x_2)]$  and  $F$  maps extreme points into extreme points and hyperplanes to hyperplanes, hence preserves  $\mathcal{P}_N$ ,  $\mathcal{C}_N$  and  $\mathcal{D}_N$ . □

**Lemma 5.** *Let  $P$  be a convex polytope in  $\mathbb{R}^n$  and  $x$  be a vertex of  $P$ . Suppose that for some  $0 \leq k \leq n-1$ , all but  $k$  facets  $F_1, \dots, F_k$  of  $P$  containing  $x$  are  $(n-1)$ -dimensional cones with apex  $x$ . Let  $H_1, \dots, H_k$  be their supporting hyperplanes,  $E = \bigcap_{i=1}^k H_i$  and  $Q = \overline{\text{conv}}(\mathcal{E}(P) \setminus \{x\})$ . For  $y \in \mathbb{R}^n$ , let  $P(y) = \overline{\text{conv}}(Q, y)$ . Then*

1)  $P(x) = P$ ,  $\dim(E) \geq n - k$  and  $y \mapsto |P(y)|$  is affine on a neighborhood of  $x$  in  $E$ .

2) For any direction  $\theta \in \mathbb{S}^{n-1}$  parallel to  $E$ , there exist  $c > 0$  and a shadow movement  $(P_t)$  of  $P$ , with basis  $x$  and direction  $\theta$ , which is volume affine on  $[-c, c]$ .

*Proof.* Without loss of generality we may assume that  $0 \in \text{int}(Q)$ . Let  $(G_m)$ ,  $1 \leq m \leq M$ , be the facets of  $Q$ , with supporting hyperplanes  $\{z : z \cdot v_m = h_m\}$ ,  $v_m \in \mathbb{S}^{n-1}$ ,  $h_m > 0$  so that

$$Q = \bigcap_{m=1}^M \{z : z \cdot v_m \leq h_m\} \text{ and } |Q| = \frac{1}{n} \sum_{m=1}^M h_m |G_m|.$$

Then

$$|P(y)| = |Q| + \frac{1}{n} \sum_{m=1}^M (y \cdot v_m - h_m)_+ |G_m| = \frac{1}{n} \sum_{m=1}^M \max\{y \cdot v_m, h_m\} |G_m|.$$

Observe that if  $1 \leq m \leq k$ ,  $F_m \cap Q$  is a facet of  $Q$ . With a reordering, we may suppose that  $G_m = F_m \cap Q$ , for  $1 \leq m \leq k$ . Moreover, for  $k+1 \leq m \leq M$ , the supporting hyperplane  $H_m$  of  $G_m$  does not contain  $x$  (indeed if  $x \cdot v_m = h_m$ , then  $H_m \cap P$  is a facet of  $P$  which is not a cone with apex  $x$ ). Thus,

$$|P(y)| = \frac{1}{n} \sum_{m=k+1}^M \max\{y \cdot v_m, h_m\} |G_m| + \frac{1}{n} \sum_{m=1}^k h_m |G_m|, \text{ for } y \in E,$$

and  $|P(y)|$  is affine function in the neighborhood  $V(x) \subset E$  defined by

$$V(x) = \{y \in E : (y \cdot v_m - h_m) \cdot (x \cdot v_m - h_m) > 0 : k+1 \leq m \leq M\}.$$

□

### 3. PROOF OF THEOREM 1

It is enough to prove Theorem 1 for the case of polytopes. The result for general bodies follows from Lemma 3 and a standard approximation procedure (see, for example, [Gru], [SW]).

We first prove the general case and we shall treat the symmetric case afterwards. Let  $N \geq 4$ , and  $\mathcal{C}_N$  be the set of convex polytopes  $P$  in  $\mathbb{R}^3$ , with not more than  $N$  vertices, such that for some different hyperplanes  $H_1$  and  $H_2$  one has  $P = \overline{\text{conv}}(P \cap H_1, P \cap H_2)$ . We notice that this class is closed in the Hausdorff metric, thus by Proposition 2 and Lemma 4, there exists  $Q \in \mathcal{C}_N$  such that

$$\mathcal{P}(Q) = \min_{P \in \mathcal{C}_N} \mathcal{P}(P).$$

We shall prove that  $\mathcal{P}(Q) \geq \mathcal{P}(\Delta^3)$ . To do this we shall distinguish three cases according to the dimension of  $H_1 \cap H_2 \cap Q$ :

**Case 1:**  $H_1 \cap H_2 \cap Q$  is a segment  $[p, m]$ . We shall consider 3 subcases.

**Subcase 1.i: the point  $m$  is in the interior of a facet of  $Q$ .**

Then  $m = [a_1, b_1] \cap [a_2, b_2]$ , where  $[a_i, b_i]$  are edges of  $H_i \cap Q$ , with  $m \neq a_i, b_i$ ,  $i = 1, 2$ . Thus  $F := \overline{\text{conv}}(a_1, b_1, a_2, b_2)$  is a face of  $Q$  containing  $m$ . Next we shall study the edges of  $Q$ , which are not edges of  $F$ , but have a common vertex with  $F$ . More precisely, consider  $a'_i, b'_i \in H_i \cap Q$ , such that  $a'_i \notin \{a_i, b_i\}$ ,  $b'_i \notin \{a_i, b_i\}$  satisfy that  $[a'_i, a_i]$  and  $[b'_i, b_i]$  are edges of  $Q_i := Q \cap H_i$ . If  $a'_i = b'_i = p$  for  $i = 1$  or  $i = 2$ , then  $Q$  is a cone, and we can conclude applying Lemma 1. For  $i = 1, 2$ , let  $m_i$  be the intersection of the lines supporting  $[a_i, a'_i]$  and  $[p, m]$  and  $n_i$  be the intersection of the lines supporting  $[b_i, b'_i]$  and  $[p, m]$ . We denote  $\alpha_i = \overline{mm_i}$  and  $\beta_i = \overline{mn_i}$ , the oriented lengths of the segments  $[m, m_i]$  and  $[n, n_i]$ . Observe that  $\alpha_i, \beta_i \neq 0$ . We set  $\alpha_i = +\infty$  or  $\beta_i = +\infty$  if  $[a_i, a'_i]$  is parallel to  $[p, m]$  or  $[b_i, b'_i]$  is parallel to  $[p, m]$ . We set  $\frac{1}{+\infty} = 0$ .

We shall use the following easy facts:

**Fact 1.**  $(a_1, a'_1, a_2, a'_2)$  (resp.  $(a_1, a'_1, b_2, b'_2)$ ,  $(b_1, b'_1, a_2, a'_2)$ ,  $(b_1, b'_1, b_2, b'_2)$ ) are the vertices of a face of  $Q$  if and only if  $\alpha_1 = \alpha_2$  (resp.  $\alpha_1 = \beta_2$ ,  $\beta_1 = \alpha_2$ ,  $\beta_1 = \beta_2$ ).

**Fact 2.** If  $\frac{1}{\alpha_2} < \frac{1}{\alpha_1}$ , then  $(a_1, a'_1, a_2)$  are the vertices of a triangular face of  $Q$  and  $(a_2, a'_2, a'_1)$  are vertices of another face of  $Q$ . If  $\frac{1}{\alpha_1} < \frac{1}{\alpha_2}$ , then  $(a_2, a'_2, a_1)$  are the vertices of a triangular face of  $Q$  and  $(a_1, a'_1, a'_2)$  are vertices of another face of  $Q$ . A similar statement holds with  $\beta_i$  instead of  $\alpha_i$ ,  $b_i$  instead of  $a_i$  and  $b'_i$  instead of  $a'_i$ , for  $i = 1$  or  $2$ .

We say that  $a_i$ ,  $i = 1, 2$  is a *simple point* if  $\overline{\text{conv}}(a_i, a'_i, a_j)$  and  $\overline{\text{conv}}(a_i, a'_i, b_j)$ ,  $j \neq i$ , are side faces of  $Q$  and similarly that  $b_i$ ,  $i = 1, 2$  is a *simple point* if  $\overline{\text{conv}}(b_i, b'_i, b_j)$  and  $\overline{\text{conv}}(b_i, b'_i, a_j)$ ,  $j \neq i$ , are side faces of  $Q$ . Observe that if  $a_i$  or  $b_i$ ,  $i = 1, 2$ , is simple, then these are the only two faces of  $Q$  together with  $F$  containing this point.

**Fact 3.** For  $i = 1, 2$ ,  $a_i$  is a simple point if and only if  $\frac{1}{\alpha_i} > \max(\frac{1}{\beta_j}, \frac{1}{\alpha_j})$ ,  $j \neq i$ , and  $b_i$  is a simple point if an only if  $\frac{1}{\beta_i} > \max(\frac{1}{\alpha_j}, \frac{1}{\beta_j})$ ,  $j \neq i$ .

**Fact 4.** Assume that at least one of the points (say  $a_1$ ) among  $a_i, b_i$ ,  $i = 1, 2$ , is simple. Thus there are exactly 3 faces, two of which are triangles, connected to  $a_1$ . We define a shadow movement  $Q_t$  with direction  $[a_1, b_1]$ , based on  $a_1$ . From Lemma 5, this shadow movement

of  $Q$  is volume affine on some interval  $[-c, c]$ ,  $c > 0$ . We conclude using the minimality of  $Q$  and Lemmas 2 and 4.

**Conclusion.** It follows from **Fact 3** and **Fact 4** that there is a simple point among  $a_i, b_i$ ,  $i = 1, 2$  if and only if

$$\max\left(\frac{1}{\beta_1}, \frac{1}{\alpha_1}\right) \neq \max\left(\frac{1}{\beta_2}, \frac{1}{\alpha_2}\right).$$

Suppose now that none of the points  $a_i, b_i$ ,  $i = 1, 2$ , is simple. Then one has

$$\max\left(\frac{1}{\beta_1}, \frac{1}{\alpha_1}\right) = \max\left(\frac{1}{\beta_2}, \frac{1}{\alpha_2}\right).$$

Without loss of generality, we may then suppose that

$$\frac{1}{\beta_1}, \frac{1}{\beta_2} \leq \frac{1}{\alpha_1} = \frac{1}{\alpha_2}.$$

From **Fact 1** and **Fact 2**, one deduces that  $\overline{\text{conv}}(a_1, a'_1, a_2, a'_2)$  is a face of  $Q$  and  $\overline{\text{conv}}(b_2, a_1, a'_1)$  and  $\overline{\text{conv}}(b_1, a_2, a'_2)$  are part of faces of  $Q$ . So  $a_1$  and  $a_2$  are the intersection of exactly 3 faces of  $Q$ . Let  $L = Q^{s(Q)}$ . Then  $L$  is a polytope. Let  $f$  be the vertex of  $L$  corresponding to the face  $\overline{\text{conv}}(a_1, a_2, b_1, b_2)$  of  $Q$ . Using the correspondence between the face lattices of a polytope and its dual (see, for example, [Mat], Chapter 5.1 or [Gr], Chapter 3.4), we see that there are exactly 4 faces, say,  $F_1, F_2, G_1, G_2$  of  $L$  containing  $f$ , corresponding to  $a_1, a_2, b_1, b_2$ . Moreover,  $F_1$  and  $F_2$  are triangles (because,  $a_1$  and  $a_2$  are the intersection of exactly 3 faces of  $Q$ ). Let  $\ell$  be the intersection of the supporting hyperplanes of  $G_1$  and  $G_2$ , and let  $\theta \in \mathbb{S}^{n-1}$  be the direction of  $\ell$ . We define the shadow movement of  $L$  with basis  $f$  and direction  $\theta$ . By Lemma 5 for some  $c > 0$ , this shadow movement is volume affine on  $[-c, c]$ . We conclude using Lemmas 2 and 4.  $\square$

**Subcase 1.ii: The point  $m$  is in the interior of an edge of  $Q$ .**

We may suppose that  $m$  is an interior point of an edge  $[a_1, b_1]$  of  $H_1 \cap Q$  and is an extreme point of  $H_2 \cap Q$ . Let  $a_2 \neq b_2$  such that  $[a_2, m]$  and  $[m, b_2]$  are edges of  $H_2 \cap Q$ . Then the two faces of  $Q$  containing the edge  $[a_1, b_1]$  of  $Q$  are the triangles  $(a_1, b_1, a_2)$  and  $(a_1, b_1, b_2)$ . Observe that by  $a_1$  cannot pass more than two non triangular faces, and that these non-triangular faces must have  $[a'_1, a_1]$  as an edge, where as before, we denote by  $a'_1$  the vertex of  $H_1 \cap Q$  different from  $b_1$  such that  $[a_1, a'_1]$  is an edge of  $H_1 \cap Q$ . Define a shadow movement  $Q_t$  with basis  $a_1$  and direction  $\theta$  parallel to the line supporting  $[a_1, a'_1]$ . Then, from Lemma 5, for some  $c > 0$ ,  $Q_t$  is volume affine and  $Q_t \in \mathcal{C}_N$  on  $[-c, c]$ . We conclude again using Lemmas 2 and 4.  $\square$

**Subcase 1.iii: The point  $m$  is a vertex of  $Q$ .** If  $m$  is a vertex of  $Q$ , then all the faces containing  $m$  are triangles. We move  $m$  on the segment  $[m, p]$  and, from Lemma 5, we get a shadow movement  $Q_t$  such that  $t \mapsto |Q_t|$  is affine on  $[-c, c]$  for some  $c > 0$ , and  $Q_t \in \mathcal{C}_N$ . We conclude as above.

**Case 2:  $H_1 \cap H_2 \cap Q$  is one point  $m$ .** It is easy to see that if  $Q$  is not a double-cone nor a cone then  $m$  is a vertex of  $H_1 \cap Q$  and  $H_2 \cap Q$ . Let  $a_i, b_i \in Q \cap H_i$  be vertices of  $Q \cap H_i$  such that  $[a_i, m]$  and  $[b_i, m]$  are edges of  $Q \cap H_i$  (and thus of  $Q$ ),  $i = 1, 2$ . Then the faces of  $Q$  containing  $m$  are the two triangles  $(a_1, m, a_2)$ ,  $(b_1, m, b_2)$  and the two other faces  $Q \cap H_i$ ,  $i = 1, 2$ . We define the affine shadow movement with basis  $m$  in the direction of the line  $H_1 \cap H_2$  and we conclude as above.

**Case 3:  $H_1 \cap H_2 \cap Q = \emptyset$ .** In this case  $K_i = H_i \cap Q$  are faces of  $Q$ . We call the other faces of  $Q$  *side faces*. It is clear that the side faces of  $Q$  can be only triangles or quadrilaterals: any side face of  $Q$  is either the convex hull of an edge of  $K_1$  and an edge of  $K_2$  or the convex hull of an edge of  $K_1$  and a vertex of  $K_2$ , or the convex hull of an edge of  $K_2$  and a vertex of  $K_1$ . We distinguish two cases:

**Subcase 3.i:  $Q$  has a side face  $F$  which is a triangle.** Without loss of generality, we may suppose that  $F = \overline{\text{conv}}(u_1, v_1, v_2)$ , where  $[u_1, v_1]$  is an edge of  $K_1$  and  $v_2$  is a vertex of  $K_2$ . If  $K_2 = \{v_2\}$  we conclude applying Lemma 1. Otherwise, among the faces of  $Q$  containing  $v_1$ , those which have  $[u_1, v_1]$  as an edge are  $F$  and  $K_1$ . If there is another side face  $G$  of  $Q$  containing  $v_1$  which is a quadrilateral, it must be of the form  $G = \overline{\text{conv}}(v_1, a_2, b_2, w_1)$ , where  $[v_1, w_1]$  is the other edge of  $K$  containing  $v_1$ , and  $[a_2, b_2]$  is an edge of  $K_2$  (with possibly  $a_2 = v_2$ ). So there are at most 2 faces of  $Q$  passing through  $v_1$  which are not triangles. We are thus in position to apply Lemma 5. With the notation of this lemma, we define the volume affine shadow movement  $K_t$  with basis  $v_1$  and in the direction of  $[v_1, w_1]$ . By Lemma 5 there exists  $c > 0$  such that  $(K_t)$  is volume affine on  $[-c, c]$ . We conclude using Lemmas 4 and 2.

**Subcase 3.ii: There is no triangle among the side faces.** Thus all the side faces of  $Q$  are quadrilateral. Because of the special configuration of  $Q$ , we see that each vertex of  $Q$  belongs to exactly three faces, two among them are quadrilateral side faces. Thus  $Q$  is a simple polytope, and its polar body  $L := Q^{s(Q)}$  is a simplicial polytope (all its faces are triangles). The polytope  $L$  has two special vertices  $k_1$  and  $k_2$  (dual to faces  $H_1 \cap Q$  and  $H_2 \cap Q$  of  $Q$ ) such that all the faces of

$L$  contain either  $k_1$  or  $k_2$ . Through all the other vertices of  $L$  pass exactly four triangular faces of  $L$ , two of them containing  $k_1$  and the two others  $k_2$ . Let  $r \neq k_1, k_2$  be a vertex of  $L$ . Fix any  $\theta \in \mathbb{S}^2$  and define a shadow movement  $(L_t)$  based on  $r$  with direction  $\theta$ . We also note that  $t \mapsto |L_t|$  is affine and  $L_t^{s(L)} \in \mathcal{C}_N$ , for small enough  $|t|$ . We use Lemmas 2 and 4 to conclude this case.  $\square$

**Proof of the centrally symmetric case:**

The proof is exactly the same as the proof of the general case, except that the movements will now keep the symmetry. Moreover, as mentioned in Remark 1 after Theorem 1, there are only two cases: either  $H_1 \cap H_2 \cap Q$  is a segment or it is empty, in which case the planes are parallel. Case 2 of the preceding discussion does not occur. Let us show how the arguments may be adapted to the symmetric case in the case 3, when  $H_1$  and  $H_2$  are parallel. By an adaptation of Lemma 4 there is a polyhedron  $Q$  with minimal volume product in the set  $\{K \in \mathcal{D}_{2N} : K = -K\}$ . We reduce to the case when  $Q$  is formed as follows:  $Q = \overline{\text{conv}}(M + e_3, -M - e_3)$  where  $M$  is a polygon in the hyperplane  $\{x_3 = 0\}$  with  $N$  vertices. We shall show that

$$\mathcal{P}(Q) = \mathcal{P}(Q_1) = \mathcal{P}(Q_2) = \frac{32}{3},$$

where  $Q_i = \overline{\text{conv}}(Q_i + e_3, -Q_i - e_3)$  with  $Q_1$  a triangle and  $Q_2$  a parallelogram. Actually  $Q_1$  is affinely equivalent to  $B_1^3$  and  $Q_2$  to  $B_\infty^3$ , which are the two, non affinely equivalent, conjectured minima for the volume product of centrally symmetric bodies in  $\mathbb{R}^3$ . We begin as in the proof of Theorem 1, but we apply now shadow systems which keep the central symmetry of  $Q$ .

**Subcase i: There is a triangle  $F$  among the side faces of  $Q$ .** Since  $Q$  is centrally symmetric, the face  $-F$  symmetric of  $F$  is also a triangle. One may suppose that  $F = \overline{\text{conv}}(u + e_3, -v - e_3, -w - e_3)$  where  $u, v, w$  are different extreme points of  $Q$  and  $[v, w]$  is an edge of  $Q$ . Consider now the shadow system constructed as follows. Define

$$Q_t = \overline{\text{conv}}(\mathcal{E}(Q) \setminus \{v + e_3, -v - e_3\}, v + e_3 + t\theta, -v - e_3 - t\theta),$$

for  $t \in \mathbb{R}$  and  $\theta \in \mathbb{S}^2$  is the direction of the edge  $[v, w]$ . We notice that at most two faces of  $Q$  which are not triangles can contain  $v + e_3$  (the same is true for  $-v - e_3$ ), and we may apply Lemma 5. Thus, for some  $c > 0$ ,  $t \mapsto |Q_t|$  is affine for  $t \in [-c, c]$ . Applying the minimality of  $\mathcal{P}(Q)$  we get  $\mathcal{P}(Q) = \mathcal{P}(Q_0) = \mathcal{P}(Q_t)$ ; it follows from Proposition 1 that  $Q_t$  is an affine image of  $Q$ , say  $Q_t = A_t(Q)$ , with  $A_t$  of the type described in

Proposition 1. Thus  $Q$  is a double cone with apices  $v + e_3, -v - e_3$  and we conclude the proof applying Lemma 2.

**Subcase ii: There is no triangle among the side faces of  $Q$ .** Then they are all quadrangles. We pass to the polar body  $L = Q^\circ$ , which is of course also centrally symmetric, and has all the properties mentioned in Subcase 3 ii, in the proof of the non-symmetric case. We modify it with a "symmetric shadow movement"  $L_t$ , moving two of its vertices  $v$  and  $-v$ , along one edge. As before, by minimality this shadow movement must satisfy that  $L_t = A_t(L)$  for  $t \in [-c, c]$ , for some  $c > 0$ , where  $A_t$  is an affine isomorphism of  $\mathbb{R}^3$ . It is easy to see that this can only happen when  $L$  is a double cone with apices  $-v$  and  $v$ . So that, by the minimality of the volume product, its basis must be a quadrilateral. So that  $L$  is affinely isomorphic to  $B_1^3$  and  $P = L^\circ$  is affinely isomorphic to  $B_\infty^3$ .  $\square$

## REFERENCES

- [BF] F. BARTHE, M. FRADELIZI, *The volume product of convex bodies with many hyperplane symmetries*, to appear in Amer. J. Math..
- [BM] J. BOURGAIN, V. D. MILMAN, *New volume ratio properties for convex symmetric bodies in  $\mathbb{R}^n$* , Invent. Math. 88 (1987), 319–340.
- [CG] S. CAMPI, P. GRONCHI, *On volume product inequalities for convex sets*, Proc. Amer. Math. Soc. 134 (2006), no. 8, 2393-2402.
- [FGMR] M. FRADELIZI, Y. GORDON, M. MEYER AND S. REISNER, *The case of equality for an inverse Santaló functional inequality*, Advances in Geometry 10 (2010), 621-630.
- [GPV] A. GIANOPOULOS, G. PAOURIS AND B-H. VRITSIOU, *The isotropic position and the reverse Santaló inequality*, Israel Journal of Mathematics (to appear).
- [GM] Y. GORDON, M. MEYER, *On the minima of the functional Mahler product*, Houston J. Math., to appear.
- [GMR] Y. GORDON, M. MEYER AND S. REISNER, *Zonoids with minimal volume-product - a new proof*, Proceedings of the American Math. Soc. 104 (1988), 273-276.
- [Gru] P.M. GRUBER, *Approximation of convex bodies*, Convexity and its Applications. P.M. Gruber, J.M. Wills (eds.), Birkhäuser, 131-162, (1983).
- [Gr] B. GRÜNBAUM, *Convex Polytopes*, Graduate Texts in mathematics, 221, Springer, 2003.
- [HL] A. HANSEN AND A. LIMA, *The structure of finite-dimensional Banach spaces with the 3.2. intersection property*, Acta Math. 146 (1981), no. 1-2, 1-23.
- [KR] J. KIM AND S. REISNER, *Local minimality of the volume-product at the simplex*, Mathematika, 57 (2011), 121–134.
- [Ku] G. KUPERBERG, *From the Mahler Conjecture to Gauss Linking Integrals*, Geometric And Functional Analysis 18 (2008), 870-892.
- [Ma1] K. MAHLER, *Ein Minimalproblem für konvexe Polygone*, Mathematica (Zutphen) B7 (1939), 118-127.



- [Ma2] K. MAHLER, *Ein Übertragungsprinzip für konvexe Körper*, Časopis Pěst. Mat. Fys. 68 (1939), 93-102.
- [Mat] J. MATOUSEK, *Lectures on discrete geometry*, Graduate Texts in Mathematics, Vol. 212, Springer (2002).
- [M1] M. MEYER, *Une caractérisation volumique de certains espaces normés*, Israel J. Math. 55 (1986), 317-326.
- [M2] M. MEYER, *Convex bodies with minimal volume product in  $\mathbb{R}^2$* , Monatsh. Math. 112 (1991), 297-301.
- [MP] M. MEYER AND A. PAJOR, *On Santaló inequality*, Geometric aspects of functional analysis (1987-88), Lecture Notes in Math. 1376, Springer Ver. (1989), 261-263.
- [MR1] M. MEYER AND S. REISNER, *Inequalities involving integrals of polar-conjugate concave functions*, Monatsh. Math. 125 (1998), 219-227.
- [MR2] M. MEYER AND S. REISNER, *Shadow systems and volumes of polar convex bodies*, Mathematika 53 (2006), no. 1, 129-148 (2007).
- [N] F. NAZAROV, *The Hörmander proof of the Bourgain-Milman theorem*, preprint, 2009.
- [NPRZ] F. NAZAROV, F. PETROV, D. RYABOGIN, A. ZVAVITCH, *A remark on the Mahler conjecture: local minimality of the unit cube*, Duke Math. J., 154 (2010), no 3, 419-430.
- [P] C. M. PETTY, *Affine isoperimetric problems*, Ann. New York Acad. Sci. 440 (1985), 113-127.
- [Pi] G. PISIER, *The volume of convex bodies and Banach space geometry*, Cambridge Tracts in Mathematics, 94. Cambridge University Press, Cambridge, 1989.
- [R1] S. REISNER, *Zonoids with minimal volume-product*, Math. Zeitschrift 192 (1986), 339-346.
- [R2] S. REISNER, *Minimal volume product in Banach spaces with a 1-unconditional basis*, J. London Math. Soc. 36 (1987), 126-136.
- [RSW] S. REISNER, C. SCHÜTT AND E. M. WERNER, *Mahler's conjecture and curvature*, International Math. Research Notices, to appear
- [RS] C.A. ROGERS AND G.C. SHEPHARD, *Some external problems for convex bodies*, Mathematika 5 (1958), 93-102.
- [S] L. A. SANTALÓ, *Un invariante afin para los cuerpos convexos del espacio de  $n$  dimensiones*, Portugal. Math. 8 (1949), 155-161.
- [SR] J. SAINT RAYMOND, *Sur le volume des corps convexes symétriques*. Séminaire d'Initiation à l'Analyse, 1980-81, Université Paris VI, 1981.
- [Sc] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.
- [SW] R. SCHNEIDER AND J. A. WIEACKER, *Approximation of convex bodies by polytopes*, Bull. London Math. Soc. 13 (1981), 149-156.

UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE, LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES (UMR 8050) CITÉ DESCARTES - 5, BD DESCARTES, CHAMPS-SUR-MARNE 77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE

*E-mail address*: Matthieu.Fradelizi@univ-mlv.fr, Mathieu.Meyer@univ-mlv.fr

AN APPLICATION OF SHADOW SYSTEMS TO MAHLER'S CONJECTURE. 17

DEPARTMENT OF MATHEMATICAL SCIENCES, KENT STATE UNIVERSITY, KENT,  
OH 44242, USA

*E-mail address:* `zvavitch@math.kent.edu`