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# A predictive maintenance strategy based on mean residual life for systems subject to competing failures due to degradation and shocks

I.T. Castro<sup>a\*</sup>, K.T. Huynh<sup>b</sup>, A. Barros<sup>b</sup>, C. Bérenguer<sup>c</sup>

<sup>a</sup>Departamento de Matemáticas, University of Extremadura, Cáceres, Spain

<sup>b</sup>Université de technologie de Troyes, ICD - STMR UMR CNRS 6279, Troyes, France

<sup>c</sup>Gipsa-lab, Control Systems Department, Grenoble Institute of Technology, Grenoble, France

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**Abstract:** In condition-based maintenance area, mean residual life function is recognized as a promising reliability indicator for maintenance decision-making. However, very few maintenance strategies based on the mean residual life have been developed. This paper therefore analyzes a predictive maintenance strategy based on the mean residual life given the degradation level collected periodically in different inspection times. This strategy is developed for systems which are subject to competing and dependent failures due to degradation and traumatic shocks. Numerical examples are given to illustrate the analytical results of the strategy.

**Keywords:** Mean residual life, Condition-based maintenance, Degradation-threshold-shock model, Stochastic processes.

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## 1. INTRODUCTION

Condition based maintenance is a maintenance program that utilizes condition monitoring technologies in order to construct a reliability indicator representing the health state of a system and, through that knowledge, set up proper maintenance decisions. Synthesizing, the literature shows that a robust reliability indicator has to characterize precisely not only the current, but also the future state of systems. The system degradation level is a traditional indicator, based on which numerous maintenance models have been analyzed and their performance has been proved through practical applications (see e.g., [1] and [2]). However, since the degradation level can only reflect the current system state, it does not suffice to be a robust indicator. From the degradation level, one can estimate the mean residual life (MRL) of system. Since the MRL can provide an idea about how long a system at any particular age and deterioration is expected to survive, it can also reflect the system future state. The MRL is therefore considered as a very promising indicator for maintenance decision-making. This is why we seek in the present paper a maintenance strategy based on this reliability indicator.

It is well known that most of the industrial systems degrade physically over time, and a stochastic process is often used to describe this phenomenon [3]. But, considering a degradation process only seems to be unsatisfactory for modeling dynamic systems, because they can fail due to another causes rather than the degradation [4]. When the system is subject to the competing risks of degradation and shocks, one can show that Degradation-Threshold-Shock (DTS) models are relevant for the modeling purpose [5]. For these models, the system is regarded as failed when its degradation reaches a critical threshold or when a catastrophic shock occurs although the degradation process has not reached the threshold. The “classical” DTS models assume that the degradation process and the shock process are independent [6, 7]. However, in many practical situations, the dependence between them is of importance and should not be neglected in the modeling [8]. Recently, Huynh *et al.* provide in [9] and [10] an extension of DTS models by introducing the dependence of the shock process on the degradation process. The authors show that such a model can be seen as a combination and a more versatile and hopefully realistic extension of many classical failure models based either only on degradation or only on parametric lifetime distributions. Hence, we will develop our MRL-based maintenance strategy under the assumption of this kind of model.

The performance of the considered MRL-based maintenance strategy is assessed by a cost criterion

which is the long-run expected maintenance cost rate  $C_\infty$ . We assume that a maintenance action, either preventive or corrective, restores the system to an as-good-as-new condition. The behavior of the maintained system is then assimilated to a regenerative process. As such, by applying the well-known renewal-reward theorem [11], the long-run expected maintenance cost rate can be calculated as the expected cost in a renewal cycle divided by the expected length of the cycle

$$C_\infty = \lim_{t \rightarrow \infty} \frac{E[C(t)]}{t} = \frac{E[C(S)]}{E[S]}, \quad (1)$$

where  $C(t)$  is the cumulated maintenance cost at time  $t$  and  $S$  is the length of the first renewal cycle.

The remainder of this paper is organized as follows. The detailed description of the maintained system according to the MRL-based maintenance strategy is introduced in section 2. Section 3 is devoted to computing the mean residual life function and to analyzing its monotony. In section 4, we formulate the cost model of the MRL-based maintenance strategy for the considered system. Section 5 shows the maintenance strategy optimization and some numerical examples. Finally, section 6 concludes the paper.

## 2. SYSTEM DEFINITION AND MAINTENANCE STRATEGY

We consider a system subject to the competing and dependent failure modes due to shocks and degradation. Such a system, under the proposed predictive maintenance strategy based on MRL, is described as follows.

1. The system is subject to an accumulative degradation and starts working at time 0. Let  $X(t)$  be the degradation level of the system at time  $t$  with  $X(0) = 0$ . We assume that  $\{X(t), t \geq 0\}$  follows a homogeneous gamma process whose increment  $X(t) - X(s)$  between two times  $s$  and  $t$ ,  $0 \leq s < t$ , follows a Gamma density function with shape parameter  $\alpha \cdot (t - s)$  and scale parameter  $\beta$

$$f_{\alpha \cdot (t-s), \beta}(x) = \frac{1}{\Gamma(\alpha \cdot (t-s))} \beta^{\alpha \cdot (t-s)} x^{\alpha \cdot (t-s)-1} e^{-\beta x}, \quad x \geq 0,$$

where  $\Gamma$  denotes the gamma function given by

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du.$$

We assume that the system fails when its degradation level exceeds the value  $L$ . Let  $\sigma_L$  be the hitting time of the threshold  $L$  (i.e., failure time due to degradation),  $\bar{F}_{\sigma_L}$  and  $f_{\sigma_L}$  be its survival function and its density function respectively, then

$$\bar{F}_{\sigma_L}(t) = P(\sigma_L \geq t) = P(X(t) \leq L) = 1 - \frac{\Gamma(\alpha t, L\beta)}{\Gamma(\alpha t)}, \quad t \geq 0, \quad (2)$$

$$f_{\sigma_L}(t) = \frac{\alpha}{\Gamma(\alpha t)} \int_{L\beta}^\infty \{\log(z) - \psi(\alpha t)\} z^{\alpha t-1} e^{-z} dz, \quad (3)$$

where  $\Gamma(\alpha, x) = \int_x^\infty z^{\alpha-1} e^{-z} dz$  and  $\psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$  are respectively the Gamma function and Digamma function.

2. The system also fails whenever a shock occurs. We assume that the shock arrival times are modeled by a non-homogeneous Poisson process  $\{N_s(t), t \geq 0\}$  with stochastic increasing intensity depending on both degradation level  $X(t)$  and working time  $t$

$$r(t, X(t)) = r_1(t) 1_{\{X(t) \leq M_s\}} + r_2(t) 1_{\{X(t) > M_s\}}, \quad (4)$$

where  $r_1(t)$  and  $r_2(t)$  denote two non-decreasing failure rates and  $r_1(t) \leq r_2(t)$  for  $t \geq 0$ . The quantity  $M_s$  represents a fixed deterioration threshold. The expression 4 means that the

degradation evolution affects to the occurrence of traumatic shocks in the sense that the system is more prone to shocks when the degradation increases and exceeds a given level. Let  $\bar{F}_1(t)$  and  $\bar{F}_2(t)$  be the survival functions associated to the failure rates  $r_1$  and  $r_2$ , that is,

$$\bar{F}_i(t) = \exp \left\{ - \int_0^t r_i(z) dz \right\}, \quad t \geq 0, \quad i = 1, 2. \quad (5)$$

3. In order to monitor the system health, a periodic inspection scheme with period  $T$  is planned at a cost rate  $C_i > 0$  to detect the degradation level. At each inspection, if the system is still running, we estimate the MRL from the detected degradation level, and we make a preventive maintenance decision based on this indicator. If the value of MRL is less than a threshold  $m$ , the system is preventively replaced with a cost  $C_p > C_i$ ; otherwise, nothing is done. But if the system is detected to be in a failure state (i.e., due to degradation or shock), a corrective replacement is performed with a cost  $C_c > C_p$ . In this case, because of the system inactivity after failure, an additional cost is incurred from the failure time until the replacement time at a cost rate  $C_d > 0$ . We assume that all the maintenance operations are perfect and take negligible time.

According to the above descriptions, constructing the MRL and choosing the optimal values of  $T$  and  $m$  are very important deciding the performance of the maintenance strategy for the considered system. Two next sections deal with the mathematical models to construct and optimize these quantities.

### 3. MEAN RESIDUAL LIFE FUNCTION

Generally, the mean residual life function at time  $t \geq 0$  can be defined as [1]

$$E [\sigma_f - t \mid \sigma_f > t, \mathbf{Z}(t)],$$

where  $\sigma_f$  denotes the time to failure and  $\mathbf{Z}(t)$  denotes the covariates vector related to the system condition obtained at time  $t$ . In the model presented in this paper, we assume that the data collected in the inspection times can perfectly reflect the degradation level of the system at time  $t$ ,  $\mathbf{Z}(t) = X(t)$ . Thus, given the degradation level at time  $t$   $X(t) = z$  and known the system is still working at time  $t$ ,  $\sigma_f > t$ , the MRL function is given by

$$m_z(t) = E [\sigma_f - t \mid \sigma_f > t, X(t) = z] = \int_t^\infty R(u \mid X(t) = z) du, \quad (6)$$

where  $R(u \mid X(t) = z)$  denotes the conditional reliability of the system at time  $u$ ,  $u \geq t$ , given  $X(t) = z$ .

For the considered system, where the system fails whenever degradation level exceeds a fixed threshold  $L$  or a shock occurs,  $R(u \mid X(t) = z)$  is hence computed by:

$$R(u \mid X(t) = z) = P(X(u) < L, N_s(u) = 0 \mid X(t) = z) = R_1 1_{\{z \leq M_s\}} + R_2 1_{\{z > M_s\}}, \quad (7)$$

where 1 denotes the indicator function which equals 1 if the argument is true and 0 otherwise,  $R_1$  and  $R_2$  are given respectively by:

$$R_1 = \bar{F}_{\sigma_{M_s-z}}(u-t) \frac{\bar{F}_1(u)}{\bar{F}_1(t)} + \int_t^u \bar{F}_{\sigma_L-\sigma_{M_s}}(u-v) \frac{\bar{F}_1(v) \bar{F}_2(u)}{\bar{F}_1(t) \bar{F}_2(v)} f_{\sigma_{M_s-z}}(v-t) dv,$$

$$R_2 = \bar{F}_{\sigma_{L-z}}(u-t) \frac{\bar{F}_2(u)}{\bar{F}_2(t)},$$

where  $\bar{F}_{\sigma_{M_s-z}}$ ,  $\bar{F}_{\sigma_{L-z}}$ ,  $f_{\sigma_{M_s-z}}$ ,  $\bar{F}_1$  and  $\bar{F}_2$  is computed from equations 2, 3 and 5.  $\bar{F}_{\sigma_L-\sigma_{M_s}}$  denotes the survival function of the random variable  $\sigma_L - \sigma_{M_s}$  given by [12]

$$\bar{F}_{\sigma_L-\sigma_{M_s}}(t) = \bar{F}_{(\sigma_{M_s}, \sigma_L)}(t, x, y) 1_{\{y > x > 0\}} + \bar{F}_d(t, x, y) 1_{\{y = x > 0\}}, \quad (8)$$

where

$$\bar{F}_{(\sigma_{M_s}, \sigma_L)}(t, x, y) = - \int \int_{M_s < x < L, 0 < x+y < L, 0 < y} \left( \int_0^\infty f_{\alpha u, \beta}(x) du \right) \frac{\partial f_{\alpha t, \beta}(y)}{\partial t} dx dy,$$

and

$$\bar{F}_d(t, x, y) = \int_t^\infty \int_0^{y-t} \alpha \int_0^{M_s} f_{\alpha x, \beta}(w) dw \left( \int_{L-y}^\infty \frac{e^{-\beta z}}{z} dz \right) dx dy.$$

Consequently, the MRL function of the considered system is computed by equations 6 and 7.

Analyzing the monotonicity of the MRL function is important to compute the probabilities of the different maintenance actions. Lemma 1 shows the monotonicity of the MRL function. In order to fulfill the space requirements of the conference, we omit its proof in the present paper; interested readers can refer to [13].

**Lemma 1** *For a fixed degradation level  $z$ ,  $0 \leq z \leq L$ , the MRL function,  $m_z(t)$ , is a non-decreasing function of the time  $t$ . For a fixed time  $t$ ,  $0 \leq t$ , the MRL function,  $m_z(t)$ , is a non-decreasing function of the degradation level  $z$ .*

To illustrate the property of monotonicity of the MRL function, Figure 1 shows the MRL surface as a function of  $t$  and  $z$  when the shocks intensities  $r_1(t)$  and  $r_2(t)$  are linear in time  $t$ . Figure 1 shows that

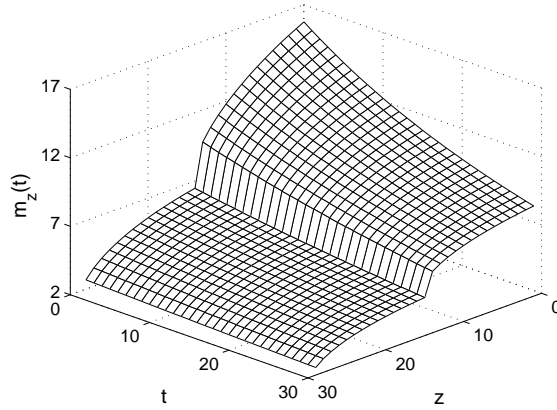


Figure 1: MRL function -  $\alpha = 0.1$ ,  $\beta = 0.1$ ,  $L = 30$ ,  $M_s = 15$ ,  $r_1(t) = 0.0025t + 0.01$  and  $r_2(t) = 0.0025t + 0.1$ .

the MRL function,  $m_z(t)$ , depends on both  $z$  and  $t$ , so it contains more information on the system compared to the degradation level. This is why the MRL function is more significant for maintenance decision-making than degradation level, and it is then interesting to estimate and integrate the MRL in maintenance planning.

#### 4. MAINTENANCE COST MODEL

In this section, the maintenance cost model of the considered strategy based on the MRL is analyzed. By applying equation 1, the long-run expected maintenance cost rate is computed by the following formula:

$$C_\infty(T, m) = \frac{C_p P_p(T, m) + C_c P_c(T, m) + C_i E[N_i(T, m)] + C_d E[W_d(T, m)]}{E[S(T, m)]}, \quad (9)$$

where  $E[S(T, m)]$  is the expected length of a replacement cycle under the considered policy,  $P_p(T, m)$ ,  $P_c(T, m)$ ,  $E[N_i(T, m)]$  and  $E[W_d(T, m)]$  are, respectively, the probability of a preventive replacement, the probability of a corrective replacement, the expected number of inspections and the expected

down time of the system in a replacement cycle. Computing these quantities is complicated but quite classical. The idea is, firstly, to represent all the possible events of MRL related to each quantity, then, to transform these events into the ones of degradation levels thanks to the monotony property of the MRL function, and finally, to compute the associated probabilities. In this way, we must define a degradation level corresponding to a given MRL value. For a fixed working time  $t \geq 0$ , let  $z_{t,m}$  be the minimum value of degradation level  $z$  such that the mean residual life is less than  $m \geq 0$ , that is,

$$z_{t,m} = \inf \{z \geq 0, m_z(t) \leq m\} \quad (10)$$

(if  $z_{t,m} > L$ , then we impose  $z_{t,m} \equiv L$ ). Furthermore, using the monotonicity of the MRL function, for  $t_1 \leq t_2$ ,  $m_z(t_1) \geq m_z(t_2)$ ; hence  $z_{t_1,m} \geq z_{t_2,m}$  for  $t_1 \leq t_2$ . The remainder of this section reports the detailed mathematical results of the quantities of interest.

#### 4.1. Preventive maintenance probability

Under the considered MRL-based maintenance policy, a preventive maintenance is performed at inspection time if the system is still working and the MRL calculated from the detected degradation level of the system is less than the value  $m$ . As such the probability of the preventive replacement in a renewal cycle is calculated by

$$P_p(T, m) = \sum_{k=0}^{\infty} P_p((k+1)T), \quad (11)$$

where,  $P_p((k+1)T)$  is the probability of a preventive replacement at time  $(k+1)T$ ,  $k = 0, 1, 2, \dots$ , given by:

$$\begin{aligned} P_p((k+1)T) &= P\left(m_{z_{kT}}(kT) > m \geq m_{z_{(k+1)T}}((k+1)T), (k+1)T < \sigma_L, \sigma_s > (k+1)T\right) \\ &= P\left(X(kT) < z_{kT,m}, z_{(k+1)T,m} \leq X((k+1)T), (k+1)T < \sigma_L, \sigma_s > (k+1)T\right) \\ &= P\left(kT < \sigma_{z_{kT,m}}, \sigma_{z_{(k+1)T,m}} \leq (k+1)T < \sigma_L, \sigma_s > (k+1)T\right), \end{aligned}$$

where  $\sigma_{z_{t,m}}$  is the hitting time of the degradation level  $z_{t,m}$  and  $\sigma_s$  is the time to the first shock. To explicite the formula of  $P_p((k+1)T)$ , we consider three exclusive events:  $\{M_s \leq z_{(k+1)T,m}\}$ ,  $\{z_{(k+1)T,m} < M_s \leq z_{kT,m}\}$  and  $\{z_{kT,m} < M_s\}$ . Before going ahead with the calculation, we define, for  $0 \leq u \leq v \leq w$  and  $0 \leq z_1 \leq z_2 \leq z_3$  (i.e.,  $0 = \sigma_0 \leq \sigma_{z_1} \leq \sigma_{z_2} \leq \sigma_{z_3}$ ), the functions  $G(u, v)$  and  $H_{\sigma_{z_1}, \sigma_{z_2}, \sigma_{z_3}}(u, v, w)$  by

$$G(u, v) = P(N_s(v) = 0 \mid \sigma_{M_s} = u) = \bar{F}_1(u) \frac{\bar{F}_2(v)}{\bar{F}_2(u)}, \quad (12)$$

$$H_{\sigma_{z_1}, \sigma_{z_2}, \sigma_{z_3}}(u, v, w) = f_{\sigma_{z_2} - \sigma_{z_1}}(v - u) \bar{F}_{\sigma_{z_3} - \sigma_{z_2}}(w - v), \quad (13)$$

where  $\bar{F}_1$  and  $\bar{F}_2$  is given from equation 5,  $\bar{F}_{\sigma_{z_3} - \sigma_{z_2}}$  and  $f_{\sigma_{z_2} - \sigma_{z_1}}$  is computed and derived respectively from equation 8.

When  $M_s \leq z_{(k+1)T,m}$ , the probability  $P_p((k+1)\Delta T)$  is computed as

$$\begin{aligned} P_{p,1} &= \int_0^{kT} \int_u^{kT} G(u, (k+1)T) H_{\sigma_{M_s}, \sigma_{z_{(k+1)T,m}}, \sigma_{z_{kT,m}}}(u, v, (k+1)T) f_{\sigma_{M_s}}(u) dv du \\ &+ \int_0^{kT} \int_{kT}^{(k+1)T} G(u, (k+1)T) H_{\sigma_{M_s}, \sigma_{z_{(k+1)T,m}}, \sigma_L}(u, v, (k+1)T) f_{\sigma_{M_s}}(u) dv du \\ &+ \int_{kT}^{(k+1)T} \int_u^{(k+1)T} G(u, (k+1)T) H_{\sigma_{M_s}, \sigma_{z_{(k+1)T,m}}, \sigma_L}(u, v, (k+1)T) f_{\sigma_{M_s}}(u) dv du \\ &+ \int_0^{kT} \int_u^{kT} \int_{kT}^{(k+1)T} -G(u, (k+1)T) \frac{\partial H_{\sigma_0, \sigma_{M_s}, \sigma_{z_{(k+1)\Delta T, m}}}(0, u, v)}{\partial v} \times \\ &\quad H_{\sigma_{z_{(k+1)T,m}}, \sigma_{z_{kT,m}}, \sigma_L}(v, w, (k+1)T) dw dv du. \end{aligned} \quad (14)$$

When  $z_{(k+1)T,m} < M_s \leq z_{kT,m}$ , the probability  $P_p((k+1)T)$  is rewritten as

$$\begin{aligned}
P_{p,2} &= \int_0^{kT} \int_u^{kT} G(v, (k+1)T) H_{\sigma_{z_{(k+1)T,m}}, \sigma_{M_s}, \sigma_{z_{kT,m}}} (u, v, (k+1)T) f_{\sigma_{z_{(k+1)T,m}}}(u) dv du \\
&+ \int_0^{kT} \int_{kT}^{(k+1)T} G(v, (k+1)T) H_{\sigma_{z_{(k+1)T,m}}, \sigma_{M_s}, \sigma_L} (u, v, (k+1)T) f_{\sigma_{z_{(k+1)T,m}}}(u) dv du \\
&+ \bar{F}_1((k+1)T) \int_0^{(k+1)T} H_{\sigma_0, \sigma_{z_{(k+1)T,m}}, \sigma_{M_s}} (0, u, (k+1)T) du \\
&+ \int_{kT}^{(k+1)T} \int_u^{(k+1)T} G(v, (k+1)T) H_{\sigma_{z_{(k+1)T,m}}, \sigma_{M_s}, \sigma_L} (u, v, (k+1)T) f_{\sigma_{z_{(k+1)T,m}}}(u) dv du \\
&+ \int_0^{kT} \int_u^{kT} \int_{kT}^{(k+1)T} -G(v, (k+1)T) \frac{\partial H_{\sigma_0, \sigma_{z_{(k+1)T,m}}, \sigma_{M_s}}(0, u, v)}{\partial v} \times \\
&\quad H_{\sigma_{M_s}, \sigma_{z_{kT,m}}, \sigma_L}(v, w, (k+1)T) dw dv du. \tag{15}
\end{aligned}$$

And when  $z_{kT,m} < M_s$ , the probability  $P_p((k+1)T)$  is given by

$$\begin{aligned}
P_{p,3} &= \bar{F}_1((k+1)T) \int_0^{kT} \int_{kT}^{(k+1)T} H_{\sigma_{z_{(k+1)T,m}}, \sigma_{z_{kT,m}}, \sigma_{M_s}} (u, v, (k+1)T) f_{\sigma_{z_{(k+1)T,m}}}(u) dudv \\
&+ \bar{F}_1((k+1)T) \int_0^{kT} H_{\sigma_0, \sigma_{z_{(k+1)T,m}}, \sigma_{z_{kT,m}}} (0, u, (k+1)T) du \\
&+ \bar{F}_1((k+1)T) \int_{kT}^{(k+1)T} H_{0, \sigma_{z_{(k+1)T,m}}, \sigma_{M_s}} (0, u, (k+1)T) du \\
&+ \int_{kT}^{(k+1)T} \int_u^{(k+1)T} G(v, (k+1)T) H_{\sigma_{z_{(k+1)T,m}}, \sigma_{M_s}, \sigma_L} (u, v, (k+1)T) f_{\sigma_{z_{(k+1)T,m}}}(u) dudv \\
&+ \int_0^{kT} \int_{kT}^{(k+1)T} \int_v^{(k+1)T} -G(w, (k+1)T) \frac{\partial H_{\sigma_0, \sigma_{z_{(k+1)T,m}}, \sigma_{z_{kT,m}}}(0, u, v)}{\partial v} \times \\
&\quad H_{\sigma_{z_{kT,m}}, \sigma_{M_s}, \sigma_L}(v, w, (k+1)T) dw dv du. \tag{16}
\end{aligned}$$

Consequently, the probability  $P_p((k+1)T)$  is given by:

$$P_p((k+1)T) = P_{p,1} 1_{\{M_s \leq z_{(k+1)T,m}\}} + P_{p,2} 1_{\{z_{(k+1)T,m} < M_s \leq z_{kT,m}\}} + P_{p,3} 1_{\{z_{kT,m} < M_s\}}. \tag{17}$$

where  $P_{p,1}$ ,  $P_{p,2}$  and  $P_{p,3}$  are given by 14, 15 and 16 respectively.

## 4.2. Corrective maintenance probability

Similarly to the above analysis, the probability of a corrective replacement under the considered MRL-based maintenance policy in a renewal cycle is calculated by

$$P_c(T, m) = \sum_{k=0}^{\infty} P_c((k+1)T), \tag{18}$$

where,  $P_c((k+1)T)$  is the probability of a corrective replacement at time  $(k+1)T$ ,  $k = 0, 1, 2, \dots$ , given by:

$$\begin{aligned}
P_c((k+1)T) &= P(kT < \sigma_{z_{kT,m}} \leq \sigma_L \leq (k+1)T, \sigma_L < \sigma_s) + P(kT < \sigma_s \leq (k+1)T, \sigma_s < \sigma_{z_{kT,m}}) \\
&+ P(kT < \sigma_{z_{kT,m}} \leq \sigma_s \leq (k+1)T, \sigma_s < \sigma_L).
\end{aligned}$$

When  $M_s \leq z_{kT,m}$ , the probability  $P_c((k+1)T)$  is given by

$$P_{c,1} = \int_0^{kT} f_{\sigma_{M_s}}(u) \int_{kT}^{(k+1)T} \int_v^{(k+1)T} -\frac{\partial}{\partial w} \left( H_{\sigma_{M_s}, \sigma_{z_{kT,m}}, \sigma_L}(u, v, w) G(u, w) \right) dw dv du$$

$$\begin{aligned}
& + \int_{kT}^{(k+1)T} \int_u^{(k+1)T} -\frac{\partial}{\partial v} \left( \bar{F}_{\sigma_L - \sigma_{M_s}}(v-u) G(u, v) \right) f_{\sigma_{M_s}}(u) dv du \\
& + \int_0^{kT} \int_{kT}^{(k+1)T} -\frac{\partial}{\partial v} (G(u, v)) H_{\sigma_0, \sigma_{M_s}, \sigma_{z_{kT, m}}}(0, u, v) dv du \\
& + \int_{kT}^{(k+1)T} f_1(u) \bar{F}_{\sigma_{M_s}}(u) du.
\end{aligned} \tag{19}$$

When  $z_{kT, m} < M_s$ , the probability  $P_c((k+1)T)$  is computed by

$$\begin{aligned}
P_{c,2} & = \int_{kT}^{(k+1)T} \int_u^{(k+1)T} \int_v^{(k+1)T} -\frac{\partial}{\partial w} \left( H_{\sigma_{z_{kT, m}}, \sigma_{M_s}, \sigma_L}(u, v, w) G(u, w) \right) f_{\sigma_{z_{kT, m}}}(u) dw dv du \\
& + \int_{kT}^{(k+1)T} \int_u^{(k+1)T} H_{\sigma_0, \sigma_{z_{kT, m}}, \sigma_{M_s}}(0, u, v) f_1(v) dv du \\
& + \int_{kT}^{(k+1)T} f_1(u) \bar{F}_{\sigma_{z_{kT, m}}}(u) du.
\end{aligned} \tag{20}$$

Hence, the probability  $P_c((k+1)T)$  is given by

$$P_c((k+1)T) = P_{c,1} \mathbf{1}_{\{M_s \leq z_{kT, m}\}} + P_{c,2} \mathbf{1}_{\{M_s > z_{kT, m}\}}, \tag{21}$$

where  $P_{c,1}$  and  $P_{c,2}$  are given by 19 and 20 respectively.

### 4.3. Expected downtime in a replacement cycle

The system downtime corresponds to the time interval from a failure to the next replacement. Assuming  $(k+1)T$  is the failure time, the downtime of the considered system is expressed by

$$\begin{aligned}
W_d(T, m) & = \sum_{k=0}^{\infty} ((k+1)T - \sigma_L) \mathbf{1}_{\{kT \leq \sigma_{z_{kT, m}}, kT \leq \sigma_L \leq (k+1)T, \sigma_L < \sigma_s\}} \\
& + \sum_{k=0}^{\infty} ((k+1)T - \sigma_s) \mathbf{1}_{\{kT \leq \sigma_{z_{kT, m}}, kT \leq \sigma_s \leq (k+1)T, \sigma_s < \sigma_L\}}.
\end{aligned} \tag{22}$$

As such, the expected downtime is given by

$$E[W_d(T, m)] = \sum_{k=0}^{\infty} d_1(k, T) \mathbf{1}_{\{M_s < z_{kT, m}\}} + \sum_{k=0}^{\infty} d_2(k, T) \mathbf{1}_{\{M_s \geq z_{kT, m}\}}, \tag{23}$$

where  $d_1(k, T)$  is given by

$$\begin{aligned}
d_1(k, T) & = \int_0^{kT} \int_{kT}^{(k+1)T} \int_v^{(k+1)T} -((k+1)T - w) \frac{\partial}{\partial w} \left( H_{\sigma_{M_s}, \sigma_{z_{kT, m}}, \sigma_L}(u, v, w) G(u, w) \right) dw dv du \\
& + \int_{kT}^{(k+1)T} \int_u^{(k+1)T} -((k+1)T - v) \frac{\partial}{\partial v} \left( \bar{F}_{\sigma_L - \sigma_{M_s}}(v-u) G(u, v) \right) f_{\sigma_{M_s}}(u) dv du \\
& + \int_0^{kT} \int_{kT}^{(k+1)T} -((k+1)T - v) \frac{\partial}{\partial v} (G(u, v)) H_{\sigma_0, \sigma_{M_s}, \sigma_{z_{kT, m}}}(0, u, v) dv du \\
& + \int_{kT}^{(k+1)T} ((k+1)T - u) f_1(u) \bar{F}_{\sigma_{M_s}}(u) du,
\end{aligned} \tag{24}$$

and  $d_2(k, T)$  is given by

$$\begin{aligned}
d_2(k, T) & = \int_{kT}^{(k+1)T} \int_u^{(k+1)T} \int_v^{(k+1)T} -((k+1)T - w) \frac{\partial}{\partial w} \left( H_{\sigma_{z_{kT, m}}, \sigma_{M_s}, \sigma_L}(u, v, w) G(v, w) \right) \times \\
& \quad f_{\sigma_{z_{kT, m}}}(u) dw dv du \\
& + \int_{kT}^{(k+1)T} \int_u^{(k+1)T} ((k+1)T - v) H_{\sigma_0, \sigma_{z_{kT, m}}, \sigma_{M_s}}(0, u, v) f_1(v) dv du \\
& + \int_{kT}^{(k+1)T} ((k+1)T - u) f_1(u) \bar{F}_{\sigma_{z_{kT, c}}}(u) du.
\end{aligned} \tag{25}$$



#### 4.4. Expected replacement time and expected number of inspections

Under the considered MRL-based maintenance policy, the system can be replaced either preventively or correctively. So the expected time to a system replacement is given by

$$E[S(T, m)] = \sum_{k=0}^{\infty} (k+1)T(P_p((k+1)T) + P_c((k+1)T)), \quad (26)$$

where  $P_p((k+1)T)$  and  $P_c((k+1)T)$  are given by 17 and 21 respectively. And the expected number of inspections in a replacement cycle is given by

$$E[N_i(T, m)] = \frac{E[S(T, m)]}{T}. \quad (27)$$

where  $E[S(T, m)]$  is given by 26.

### 5. MAINTENANCE OPTIMIZATION AND NUMERICAL EXAMPLE

As indicated in the previous sections, the inspection period  $T$  and the preventive replacement threshold  $m$  decide the performance of the considered maintenance strategy. Thus, optimizing the strategy is reduced to find the optimal values  $T_{opt}$  and  $m_{opt}$  that minimize the long-run expected maintenance cost rate  $C_{\infty}(T, m)$  given by equation 9, that is

$$C_{\infty}(T_{opt}, m_{opt}) = \min_{T, m} \{C_{\infty}(T, m), T > 0, 0 \leq m \leq MTTF\}, \quad (28)$$

where  $MTTF$  denotes the mean time to failure of system, which is computed by

$$MTTF = \int_0^{\infty} \bar{F}_1(u) \bar{F}_{\sigma_{M_s}}(u) du + \int_0^{\infty} \int_0^{\infty} \bar{F}_{\sigma_L - \sigma_{M_s}}(v) \frac{\bar{F}_1(u) \bar{F}_2(v+u)}{\bar{F}_2(u)} f_{\sigma_{M_s}}(u) dv du. \quad (29)$$

Due to the complexity of the cost function  $C_{\infty}(T, m)$ , obtaining the analytical solution of  $T_{opt}$  and  $m_{opt}$  is almost impossible; numerical methods (e.g., policy iteration algorithm [11]) are used to overcome the problem.

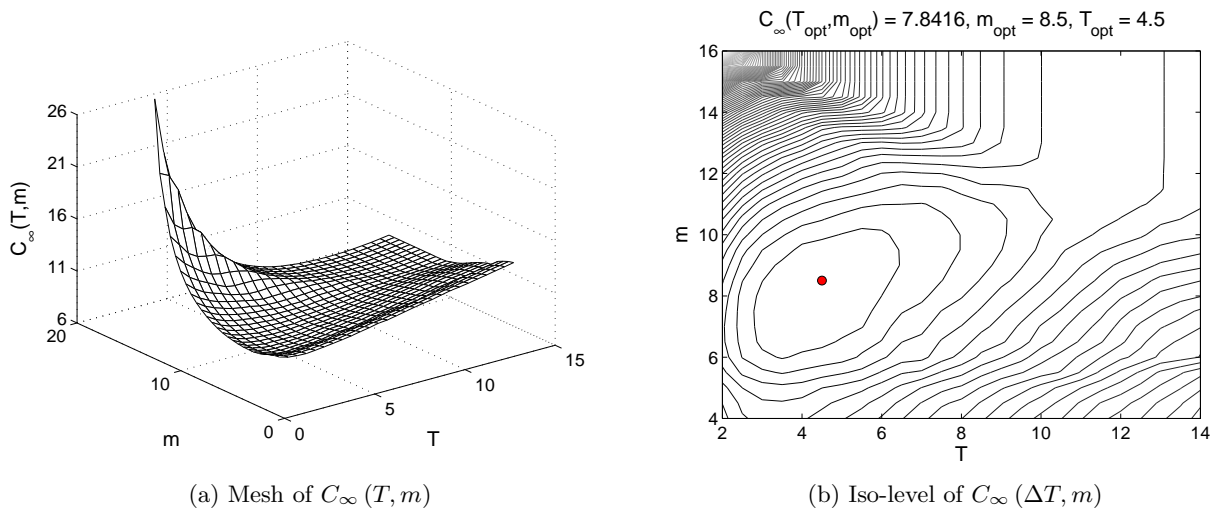


Figure 2: Long-run expected maintenance cost rate  $C_{\infty}(T, m)$

To illustrate the cost surface and the optimum solution of the considered MRL-based maintenance strategy, we use the following data set:  $\alpha = 0.5$ ,  $\beta = 0.5$ ,  $M_s = 15$ ,  $L = 30$ ,  $r_1(t) = 0.0025t + 0.01$ ,

$r_2(t) = 0.0025t + 0.1$ ,  $C_i = 5$ ,  $C_p = 50$ ,  $C_c = 100$  and  $C_d = 25$ . The numerical example of the strategy is shown in Figure 2. Sub-figures 2a and 2b illustrate respectively the shape and the iso-level cost curves of the expected maintenance cost rate when the inspection period  $T$  and the preventive replacement threshold  $m$  vary. The convexity of the cost surface shows the existence of an optimal setting of parameters  $T$  and  $m$ . In fact, for the chosen data set, optimal values are  $T_{opt} = 4.5$  and  $m_{opt} = 8.5$ , which correspond to the optimal cost rate given by  $C_\infty(4.5, 8.5) = 7.8416$ .

## 6. CONCLUSION

The present paper introduces the cost model of a predictive maintenance strategy for a system subject to the computing failure modes due to degradation and shocks. The system is described by an extension of DTS model where the shock intensity is assumed to be dependent on the system degradation level. We have shown that, under the assumptions of the model, the MRL function of the system is a decreasing function of the degradation level and working time. The preventive maintenance decisions of the proposed strategy is based on the MRL function calculated at different inspection times. Due to the monotonicity of the MRL function, the degradation levels at different inspection times for the same value of MRL are different. Hence, the considered MRL-based maintenance strategy is equivalent to a degradation-based maintenance strategy with multiple preventive degradation thresholds. Since these thresholds can finely adapt to the health state of system, this strategy is better than the one based on single degradation threshold. Furthermore, for our new strategy, multiple preventive thresholds are integrated in an unique MRL indicator, so it is generally more simple and more complete than degradation-based maintenance strategies.

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