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VARIATION OF MIXED HODGE STRUCTURES

PATRICK BROSNAN AND FOUAD EL ZEIN

Abstract. Variation of mixed Hodge structures(VMHS), introduced by P. Deligne, is a linear structure reflecting the geometry on cohomology of the fibers of an algebraic family, generalizing variation of Hodge structures for smooth proper families, introduced by P. Griffiths. Hence, it is a strong tool to study the variation of the geometric structure of fibers of a morphism. We describe here the degenerating properties of a VMHS of geometric origin and the existence of a relative monodromy filtration, as well the definition and properties of abstract admissible VMHS.

Introduction

The object of the paper is to discuss the definition of admissible variations of mixed Hodge structure (VMHS), the results in [18] and applications to the proof of algebraicity of the locus of Hodge cycles [3], [4]. Since we wish to present an expository article we did choose to accompany the evolution of the ideas from the geometric properties of algebraic families with their singularities, to their representation by VMHS degenerating at the discriminant locus of the family when the fibres acquire singularities. In the last section we present a summary of the results mentioned above.

The study of morphisms in algebraic geometry is at the origin of the theory of VMHS. To begin with smooth proper morphisms of smooth varieties $f : X \to V$, the underlying differentiable structure of the various fibers does not vary, by Ehresman's theorem; the fibers near a point of the parameter space V are diffeomorphic in this case, but the algebraic or analytic structure on the fibers do vary.

From another point of view, locally, near a point on the parameter space, we may think of a morphism as being given by a fixed differentiable manifold and a family of analytic structures parameterized by the neighborhood of the point.

So the cohomology of the fibers does not vary, but, in general the Hodge structure, which is sensitive to the analytic structure, does. In this case, the cohomology groups of the fibers form a local system $\mathcal{L}_{\mathbb{Z}}$. So we start by the study of the structure of local systems and its relation to flat connections corresponding to the study of linear differential equations on manifolds. In the geometric case, the local system of cohomology of the fibres define the Gauss-Manin connection.

The theory of variation of Hodge Structure (VHS) adds to the local system the Hodge structures on the cohomology of the fibers, and transforms geometric problems concerning smooth proper morphisms into linear algebra problems involving the Hodge filtration by complex subspaces of cohomology vector spaces of the fibers. The data of VHS denoted by $(\mathcal{L}, F)$ has three levels of definition: the local system

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of groups $\mathcal{L}_{\mathbb{Z}}$, the Hodge filtration F varying holomorphically with the fibers defined as a filtration by sub-bundles of $\mathcal{L}_V := \mathcal{O}_V \otimes \mathcal{L}_{\mathbb{Z}}$ on the base V, while the Hodge decomposition is on the differentiable bundle $\mathcal{L}_\infty := \mathcal{C}_V^\infty \otimes \mathcal{L}_{\mathbb{Z}} = \oplus_{p+q=i}\mathcal{L}^{p,q}$.

In general, a morphism onto the smooth variety V is smooth outside a divisor D called its discriminant, in which case the above description apply on the complement $V - D$, then the VHS is said to degenerate along D which means it acquires singularities. The study of the singularities may be carried in two ways, either by introducing the theory of mixed Hodge structure (MHS)on the singular fibers, or by the study of the asymptotic behavior of the VHS in the neighborhood of D, but in this case we may blow-up closed subvarieties in D without modifying the family on $V - D$, hence we may suppose D a normal crossing divisor (NCD) by Hironaka's results. We may also suppose the parameter space reduced to a disc and D to a point, since many arguments are carried over an embedded disc in V with center a point in D. In this setting, Grothendieck proved first in positive characteristic that the local monodromy around points in D of a local system of geometric origin is quasi-unipotent. Deligne explains a set of arguments to deduce geometric results on varieties over a field in characteristic zero from the case of positive characteristic [2]. Direct proofs exist using desingularization and spectral sequences [6], [22].

Technically it is easier to write this expository article if we suppose the local monodromy unipotent, although this does not change basically the results.
For a local system with unipotent local monodromy, we need to introduce Deligne's canonical extension of the analytic flat vector bundle $\mathcal{L}_{V-D}$ into a bundle $\mathcal{L}_V$ on V characterized by the fact that the extended connection has logarithmic singularities with nilpotent residues. It is on $\mathcal{L}_V$ that the Hodge filtrations F will extend as a filtration by subbundles, but they do not define anymore a Hodge filtration on the fibres of the bundle over points in D. Instead, combined with the local monodromy around the components of D through a point of D, a new structure called the limit mixed Hodge structure (MHS)and the companion results on the Nilpotent orbit and $SL(2)-$orbit [27] describe in the best way the asymptotic behavior of the VHS near a point of D.

The above summary is the background needed to understand the motivations, the definitions and the problems raised in the theory. That is why we recall in the first section, the relations between local systems and linear differential equations as well Thom-Whitney's results on the topological properties of morphisms of algebraic varieties. The section ends with the definition of a VMHS on a smooth variety.

After introducing the theory of mixed Hodge structure (MHS) on the cohomology of algebraic varieties, Deligne proposed to study the variation of such linear MHS structure (VMHS) reflecting the variation of the geometry on cohomology of families of algebraic varieties ([11], Pb. 1.8.15). In the second section we study the properties of degenerating geometric VMHS.

In the last section we give the definition and properties of admissible VMHS and describe important local results of Kashiwara [18]. In this setting we recall the definition of normal functions and we explain recent results on the algebraicity of the zero set of normal functions to answer a question raised by Griffiths and Green.

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1. Variation of mixed Hodge structures

The classical theory of linear differential equations on an open subset of $\mathbb{C}$ has developed into the theory of connections on manifolds, while the monodromy of the solutions developed into representation theory of the fundamental group of a space. With the development of sheaf theory, a third definition of local system as locally constant sheaves, appeared to be a powerful tool to study the cohomology of families of algebraic varieties. In his modern lecture notes [8] with a defiant classical title, on linear differential equations with regular singular points, Deligne proved the equivalence between these three notions and studied their singularities. The applications in the study of singularities of morphisms lead to the problems on degeneration of VMHS.

1.1. **Local systems and representations of the fundamental group.** We refer to [8] for this section; the notion of local system coincides with the theory of representations of the fundamental group of a topological space.

In this section, we suppose the topological space M *locally path connected and locally simply connected* (each point has a basis of connected neighborhoods $(U_i)_{i\in I}$ with trivial fundamental groups i.e $\pi_0(U_i) = e$ and $\pi_1(U_i) = e$). In particular, on complex algebraic varieties, we refer to the transcendental topology and not the Zariski topology to define local systems.

Definition 1.1. Let Λ be a ring. A local system of Λ-modules on a topological space X is a sheaf $\mathcal{L}$ of Λ_X-modules on X such that, for each $x \in X$, there is a neighborhood U and a non-negative integer n such that $\mathcal{L}_{|U} \cong \Lambda_U^n$. A local system of Λ-modules is said to be *constant* if it is isomorphic on X to Λ_X^r for some fixed r.

Definition 1.2. Let L be a finitely generated $\mathbb{Z}-$module. A representation of a group G is a homomorphism of groups

$$G \xrightarrow{\rho} Aut_{\mathbb{Z}}(L)$$

from G to the group of $\mathbb{Z}-$linear automorphisms of L, or equivalently a linear action of G on L.

We will also use the definition for $\mathbb{Q}-$vector spaces instead of $\mathbb{Z}$-modules.

1.1.1. *Monodromy.* If $\mathcal{L}$ is a local system of Λ-modules on a topological space M and $f : N \to M$ is a continuous map, then $f^{-1}(\mathcal{L})$ is a local system of Λ-modules on N.

Lemma 1.3. *A local system $\mathcal{L}$ on the interval $[0,1]$ is necessarily constant.*

Proof. Let $\mathcal{L}^{\text{ét}}$ denote the étalé space of $\mathcal{L}$. Since $\mathcal{L}$ is locally constant, $\mathcal{L}^{\text{ét}} \to [0,1]$ is a covering space. Since $[0,1]$ is contractible, $\mathcal{L}^{\text{ét}}$ is a product. This implies that $\mathcal{L}$ is constant. □

Let $\gamma : [0,1] \to M$ be a loop in M with origin a point v and let $\mathcal{L}$ be a $\mathbb{Z}-$local system on M with fiber L at v. The inverse image $\gamma^{-1}(\mathcal{L})$ of the local system is isomorphic to the constant sheaf defined by L on $[0,1]$: $\gamma^{-1}\mathcal{L} \simeq L_{[0,1]}$, hence we deduce from this property the notion of monodromy.

Definition 1.4 (Monodromy)**.** The composition of the linear isomorphisms

$$L = \mathcal{L}_v = \mathcal{L}_{\gamma(0)} \simeq \Gamma([0,1], \mathcal{L}) \simeq \mathcal{L}_{\gamma(1)} = \mathcal{L}_v = L$$

is denoted by T and called the monodromy along γ. It depends only on the homotopy class of γ.

The monodromy of a local system $\mathcal{L}$ defines a representation of the fundamental group $\pi_1(M,v)$ of a topological space M on the stalk at v, $\mathcal{L}_v = L$

$$\pi_1(M,v) \xrightarrow{\rho} Aut_{\mathbb{Z}}(\mathcal{L}_v)$$

which characterizes local systems on connected spaces in the following sense

Proposition 1.5. *Let M be a connected topological space. The above correspondence is an equivalence between the following categories*
i) $\mathbb{Z}-$local systems with finitely generated $\mathbb{Z}-$modules L on M
ii) Representations of the fundamental group $\pi_1(M,v)$ by linear automorphisms of finitely generated $\mathbb{Z}-$modules L.

1.2. **Connections and Local Systems.** The concept of connections on analytic manifolds (resp. smooth complex algebraic variety) is a generalization of the concept of system of $n-$linear first order differential equations.

Definition 1.6. Let $\mathcal{F}$ be a locally free holomorphic $\mathcal{O}_X-$module on a complex analytic manifold X (resp. smooth algebraic complex variety). A connection on $\mathcal{F}$ is a $\mathbb{C}_X-$linear map

$$\nabla : \mathcal{F} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{F}$$

satisfying the following condition for all sections f of $\mathcal{F}$ and φ of $\mathcal{O}_X$:

$$\nabla(\varphi f) = d\varphi \otimes f + \varphi \nabla f$$

known as Leibnitz condition.

We define a morphism of connections as a morphism of $\mathcal{O}_X-$modules which commutes with ∇.

1.2.1. The definition of ∇ extends to differential forms in degree p as a $\mathbb{C}-$linear map

$$\nabla^p : \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \Omega^{p+1}_X \otimes_{\mathcal{O}_X} \mathcal{F} \text{ s.t. } \nabla^p(\omega \otimes f) = d\omega \otimes f + (-1)^p \omega \wedge \nabla f$$

The connection is said to be integrable if its curvature $\nabla^1 \circ \nabla^0 : F \to \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{F}$ vanishes ($\nabla = \nabla^0$, and the curvature is a linear morphism).

Then it follows that the composition of maps $\nabla^{i+1} \circ \nabla^i = 0$ vanishes for all $i \in \mathbb{N}$ for an integrable connection. In this case a de Rham complex is associated to ∇

$$(\Omega^*_X \otimes_{\mathcal{O}_X} \mathcal{F}, \nabla) \colon = \mathcal{F} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{F} \cdots \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{F} \overset{\nabla^p}{\to} \cdots \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{F}$$

Proposition 1.7. *The horizontal sections $\mathcal{F}^\nabla$ of a connection ∇ on a module $\mathcal{F}$ on an analytic (resp. algebraic) smooth variety X , are defined as the solutions of the differential equation on X (resp. on the analytic associated manifold X^h)*

$$\mathcal{F}^\nabla = \{f : \nabla(f) = 0\}.$$

When the connection is integrable, $\mathcal{F}^\nabla$ is a local system of dimension $dim\, \mathcal{F}$.

Proof. This result is based on the relation between differential equations and connections. Locally, we consider a small open subset $U \subset X$ isomorphic to an open set of $\mathbb{C}^n$ s.t. $\mathcal{F}_{|U}$ is isomorphic to $\mathcal{O}^m_U$. This isomorphism is defined by the choice of a basis of sections $(e_i)_{i\in[1,m]}$ of $\mathcal{F}$ on U and extends to the tensor product of $\mathcal{F}$ with the module of differential forms: $\Omega^1_U \otimes \mathcal{F} \simeq (\Omega^1_U)^m$.
In terms of the basis $e = (e_1, \cdots, e_m)$ of $F_{|U}$, a section s is written as
$s = \sum_{i\in[1,m]} y_i e_i$ and $\nabla s = \sum_{i\in[1,m]} dy_i \otimes e_i + \sum_{i\in[1,m]} y_i \nabla e_i$ where
$\nabla e_i = \sum_{j\in[1,m]} \omega_{ij} \otimes e_j$.
The connection matrix Ω_U is the matrix of differential forms $(\omega_{ij})_{i,j\in[1,m]}$, sections of Ω^1_U; its $i-$th column is the transpose of the line image of $\nabla(e_i)$ in $(\Omega^1_U)^m$. Then the restriction of ∇ to U corresponds to a connection on $\mathcal{O}^m_U$ denoted ∇_U and defined on sections $y = (y_1, \cdots, y_m)$ of $\mathcal{O}^m_U$ on U, written in column as $\nabla_U {}^t y = d({}^t y) + \Omega_U {}^t y$ or

$$\nabla_U \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} dy_1 \\ \vdots \\ dy_m \end{pmatrix} + \Omega_U \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

the equation is in $End(T, \mathcal{F})_{|U} \simeq (\Omega^1 \otimes \mathcal{F})_{|U}$ where T is the tangent bundle to X. Let $(x_1, \cdots, x_n)$ denotes the coordinates of $\mathbb{C}^n$, then ω_{ij} decompose as

$$\omega_{ij} = \sum_{k\in[1,n]} \Gamma^k_{ij}(x)\, dx_k$$

so that the equation of the coordinates of horizontal sections is given by linear partial differential equations for $i \in [1, m]$ and $k \in [1, n]$

$$\frac{\partial y_i}{\partial x_k} + \sum_{j\in[1,m]} \Gamma^k_{ij}(x)\, y_j = 0$$

The solutions form a local system of dimension m, since the Frobenius condition is satisfied by the integrability hypothesis on ∇. □

The connection appears as a global version of linear differential equations, independent of the choice of local coordinates on X.

Remark 1.8. The natural morphism for a complex local system $\mathcal{L} \to (\Omega^*_X \otimes_{\mathbb{C}} \mathcal{L}, \nabla)$ defines a resolution of $\mathcal{L}$ by coherent modules, hence induces isomorphisms on cohomology

$$H^i(X, \mathcal{L}) \simeq H^i(R\Gamma(X, (\Omega^*_X \otimes_{\mathbb{C}} \mathcal{L}, \nabla)))$$

where we take hypercohomology on the right. On a smooth differentiable manifold X, the natural morphism $\mathcal{L} \to (\mathcal{E}^*_X \otimes_{\mathbb{C}} \mathcal{L}, \nabla)$ defines a soft resolution of $\mathcal{L}$ and induces isomorphisms on cohomology

$$H^i(X, \mathcal{L}) \simeq H^i(\Gamma(X, (\mathcal{E}^*_X \otimes_{\mathbb{C}} \mathcal{L}, \nabla))$$

Theorem 1.9 (Deligne)**.** [8] *The functor* $(\mathcal{F}, \nabla) \mapsto \mathcal{F}^{\nabla}$ *is an equivalence between the category of integrable connections on an analytic manifold* X *and the category of complex local systems on* X *with quasi-inverse defined by* $\mathcal{L} \mapsto \mathcal{L}_X$*.*

1.2.2. *Local system of geometric origin.* The structure of local system appears naturally on the cohomology of a smooth and proper family of varieties.

Theorem 1.10 (differentiable fibrations)**.** *Let* $f : M \to N$ *be a proper differentiable submersive morphism of manifolds. For each point* $v \in N$ *there exists an open neighbourhood* U_v *of* v *such that the differentiable structure of the inverse image* $M_{U_v} = f^{-1}(U_v)$ *decomposes as a product of a fibre at* v *with* U_v*:*

$$f^{-1}(U_v) \xrightarrow{\varphi}_{\simeq} U_v \times M_v \quad s.t. \quad pr_1 \circ \varphi = f_{|U_v}$$

The proof follows from the existence of a tubular neighbourhood of the submanifold M_v ([32] thm. 9.3).

Corollary 1.11 (Locally constant cohomology)**.** *In each degree* i*, the cohomology sheaf of the fibers* $R^i f_* \mathbb{Z}$ *is constant on a small neighbourhood* U_v *of any point* v *of fiber* $H^i(M_v, \mathbb{Z})$ *i.e there exists an isomorphism between the restriction* $(R^i f_* \mathbb{Z})_{|U_v}$ *with the constant sheaf* $H^i_{U_v}$ *defined on* U_v *by the vector space* $H^i = H^i(M_v, \mathbb{Z})$*.*

Proof. Let U_v be isomorphic to a ball in $\mathbb{R}^n$ over which f is trivial, then for any small ball B_ρ included in U_v, the restriction $H^i(M_{U_v}, \mathbb{Z}) \to H^i(M_{B_\rho}, \mathbb{Z})$ is an isomorphism since M_{B_ρ} is a deformation retract of M_{U_v}. □

Remark 1.12 (Algebraic family of complex varieties)*.* Let $f : X \to V$ be a smooth proper morphism of complex algebraic varieties , then f defines a differentiable locally trivial fiber bundle on V. We still denotes by f the differentiable morphism $X^{dif} \to V^{dif}$ associated to f, then the complex of real differential forms $\mathcal{E}^*_X$ is a fine resolution of the constant sheaf $\mathbb{R}$ and $R^i f_* \mathbb{R} \simeq \mathcal{H}^i(f_* \mathcal{E}^*_X)$ is a local system called of geometric origin. Such local systems carry additional structures and have specific properties which are the subject of study in this article.

We give now the abstract definition of VMHS, ([11],1.8.14) and then explain how the geometric situation leads to such structure.

Definition 1.13. A VMHS on an analytic manifold X consists of
1) A local system $\mathcal{L}_{\mathbb{Z}}$ of $\mathbb{Z}-$modules of finite type,
2) A finite increasing filtration $\mathcal{W}$ of $\mathcal{L}_{\mathbb{Q}} := \mathcal{L}_{\mathbb{Z}} \otimes \mathbb{Q}$ by sublocal systems of rational vector spaces,
3) A finite decreasing filtration $\mathcal{F}$ by locally free analytic subsheaves of $\mathcal{L}_X := \mathcal{L}_{\mathbb{Z}} \otimes \mathcal{O}_X$ whose sections on X satisfy the infinitesimal (Griffiths) transversality relation with respect to the connection ∇ defined by the structure of local system on $\mathcal{L}_{\mathbb{C}} := \mathcal{L}_{\mathbb{Z}} \otimes \mathbb{C}$ on $\mathcal{L}_X$

$$\nabla(\mathcal{F}^p) \subset \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{F}^{p-1}$$

such that $\mathcal{W}$ and $\mathcal{F}$ define a MHS on each fiber $(\mathcal{L}_X(t), \mathcal{W}(t), \mathcal{F}(t))$ at t of the bundle $\mathcal{L}_X$.

The definition of VHS is obtained in the particular pure case when the weight filtration is constant but for one index. The induced filtration by $\mathcal{F}$ on the graded objects $Gr^{\mathcal{W}}_m \mathcal{L}$ form a VHS. A morphism of VMHS is a morphism of local systems compatible with the filtrations.

Definition 1.14. The VMHS is graded polarizable if the graded objects $Gr_m^{\mathcal{W}}\mathcal{L}$ are polarizable variation of Hodge structure.

1.3. **VMHS of geometric origin.** In the above situation of smooth algebraic morphisms, the cohomology of the fibers carry a Hodge structure (HS) which leads to the theory of variation of Hodge structure (VHS) on an underlying local system and which is the subject of another course. We describe here structural theorems of algebraic morphisms and as a consequence the variation of mixed Hodge structure (VMHS) they define on the cohomology of the fibers over strata of the parameter space. The asymptotic properties of a VMHS near the boundary of a strata is studied under the terminology of degenerating VMHS and will be discussed here.

The study of the whole data, including many strata has developed in the last twenty years after the introduction of perverse sheaves. In the projective case, a remarkable decomposition result is proved in [2]. In the transcendental case, this result apply for Hodge differential modules [26]. These results are beyond the scope of this article. Instead we discuss preliminary results needed to understand such theory.

1.3.1. *Background on morphisms of algebraic varieties and local systems.* We describe here structural theorems of algebraic morphisms in order to deduce later VMHS on various strata of the parameter space.

Stratification theory on a variety consists of the decomposition of a variety into the disjoint union of smooth locally closed algebraic (or analytic) subvarieties called strata (a strata is smooth but the variety may be singular along a strata). By construction, the closure of a strata is a union of additional strata of lower dimensions. A Whitney stratification satisfies two more conditions named after H. Whitney. We are interested here in their consequence, after the work of J. Mather: the local topological trivial property at any point of a strata that will be useful in the study of local cohomology. Thom described in addition the topology of the singularities of algebraic morphisms. Next, we summarize these results.

1.3.2. *Thom-Whitney's stratifications.* Let $f : X \to V$ be an algebraic morphism. There exist finite Whitney stratifications $\mathfrak{X}$ of X and $\mathcal{S} = \{S_l\}_{l\leq d}$ of V (dim. $S_l = l$, dim. $V = d$) such that for each connected component S of an $\mathcal{S}$ stratum S_l of V
i)$f^{-1}S$ is a topological fibre bundle over S, union of connected components of strata of $\mathfrak{X}$, each mapped submersively to S.
ii) Local topological triviality: for all $v \in S$, there exist an open neighborhood $U(v)$ in S and a stratum preserving homeomorphism $h : f^{-1}(U) \simeq f^{-1}(v) \times U$ s.t. $f|U = p_U \circ h$ where p_U is the projection on U.
This statement can be found in an article by D. T. Lê and B. Teissier [23] and [15] by Goretsky and MacPherson. Since the restriction f/S to a stratum S is a locally trivial topological bundle, we deduce

Corollary 1.15. *For each integer i, the higher direct cohomology sheaf $(R^i f_*\mathbb{Z}_X)/S$ is locally constant on each stratum S of V.*

Then we say that $R^i f_*\mathbb{Z}_X$ is constructible on V and $Rf_*\mathbb{Z}_X$ is cohomologically constructible on V.

1.3.3. *Geometric Variation of Mixed Hodge Structures.* The abstract definition of VMHS above summarizes in fact properties of the variation of MHS defined on the cohomology of the fibers over strata of the parameter space. We suppose next

the parameter space a complex disc D that we can suppose small enough to have a topological fibration on the punctured disc D^*. Hence the cohomology groups $(H^i(X_t, \mathbb{Q}))$ form a local system $(R^i f_* \mathbb{Q}_X)$ on D^* with an associated flat analytic bundle $(R^i f_* \mathbb{C}_X)/D^* \otimes \mathcal{O}_{D^*}$ endowed with an analytic connection ∇ whose flat sections form a local system isomorphic to $R^i f_* \mathbb{C}_X)/D^*$.

Suppose the fibers of f are algebraic varieties, then a MHS exists on the cohomology groups $(H^i(X_t, \mathbb{Z}), W, F)$ of the fibers X_t. The following proposition describes properties of the weight and Hodge filtrations: the variation of the weight filtration W is locally constant in t and the variation of the Hodge filtration F is analytic in t.

Proposition 1.16. *Suppose the fibers of the above morphism $f : X \to D$ are algebraic and the radius of D small enough*
i) For all integers $i \in \mathbb{N}$, the restriction to D^* of the higher direct image cohomology $(R^i f_* \mathbb{Z}_X)/D^*$ (resp. $R^i f_* \mathbb{Z}_X/\mathrm{Torsion})/D^*$ are local systems of free $\mathbb{Z}_{D^*}-$modules of finite type.
ii) The weight filtration W on the cohomology $H^i(X_t, \mathbb{Q})$ of a fiber X_t at $t \in D^*$ define a filtration $\mathcal{W}$ by sublocal systems of $(R^i f_* \mathbb{Q}_X)/D^*$.
iii) The Hodge filtration F on the cohomology $H^i(X_t, \mathbb{C})$ define a filtration $\mathcal{F}$ by analytic sub-bundles of $(R^i f_* \mathbb{C}_X)/D^* \otimes \mathcal{O}_{D^*}$ whose locally free sheaf of sections on D^* satisfy the infinitesimal Griffiths transversality with respect to the (Gauss Manin)connection ∇

$$\nabla \mathcal{F}^p \subset \Omega^1_{D^*} \otimes_{\mathcal{O}_{D^*}} \mathcal{F}^{p-1}.$$

iv) If we suppose f projective, then the induced filtration by $\mathcal{F}$ on the graded objects $Gr^{\mathcal{W}}_m (R^i f_* \mathbb{C}_X)/D^*$ are polarizable variation of Hodge structure (VHS).

The proposition is a generalization to the non proper case of results in the smooth proper case. The proof follows the historical developments of the theory, and it is in two steps. In the first step we summarize the results for VHS and in the second step we use the technique introduced by Deligne by covering X by simplicial smooth varieties [10].

VHS defined by a smooth proper morphism. The main point is to prove that the variation of the Hodge filtration is analytic. The original proof by Griffiths is based on the description of the Hodge filtration F as a map to the classifying space of all filtrations of the cohomology vector space of a fiber at a reference point. Here the Hodge filtration of the cohomology at a fiber y are transported horizontally to the reference point.

We summarize here a proof based on the use of relative connections by Deligne ([8], 2.20.3), and Katz-Oda [21]. Let $f : X \to V$ denotes a smooth morphism of analytic varieties, $f^{-1}\mathcal{F}$ the sheaf theoretic inverse of a sheaf $\mathcal{F}$ on V, and let $\Omega^1_{X/V}$ denotes the sheaf of relative differential forms.

Definition 1.17. i) A relative connection on a coherent sheaf of modules on X

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{X/V}$$

is defined by an $f^{-1}\mathcal{O}_V$ linear map ∇ satisfying for all local sections

$$f \in \mathcal{O}_X, v \in \mathcal{V} : \quad \nabla(fv) = f.\nabla v + df.v$$

ii) A relative local system on X is a sheaf $\mathcal{L}$ with a structure of $f^{-1}\mathcal{O}_V-$module, locally isomorphic on X to the inverse image of a coherent sheaf on a subset of V

$\forall x \in X$, $\exists x \in U \subset X, W \subset S : f_{|U} : U \to W$, $\exists\, \mathcal{F}_W$ a coherent sheaf : $\mathcal{L}_{|U} \simeq (f^{-1}\mathcal{F})_{|U}$.

Then the theory is similar to the absolute case and there is an equivalence between relative local systems and flat relative connections. Moreover, the relative de Rham complex $\Omega^*_{X/V}$ is defined in the flat case and the flat sections of $\mathcal{V}$ form a relative local system.

Example. Let $\mathcal{L}_{\mathbb{Z}}$ be a local system on X, then $\mathcal{L}_{\text{rel}} := f^{-1}\mathcal{O}_V \otimes \mathcal{L}_{\mathbb{Z}}$ is a relative local system and $\Omega^*_{X/V} \otimes \mathcal{L}_{\mathbb{Z}}$ is its de Rham resolution. In particular, we have ([8], 2.27.2):

$$\mathcal{O}_V \otimes R^i f_* \mathcal{L}_{\mathbb{Z}} \simeq R^i f_*(\Omega^*_{X/V} \otimes \mathcal{L}_{\mathbb{Z}})$$

To prove that the Hodge filtration vary holomorphically with the paramaters and that the connection satisfy the infinitesimal transversality, we consider the exact sequence of differential forms

$$0 \to f^*\Omega^1_V \to \Omega^1_X \to \Omega^1_{X/Y} \to 0,$$

and the exact sequence of complexes

$$0 \to f^*\Omega^1_V \otimes F^{p-1}\Omega^{*-1}_X \to F^p\Omega^*_X \to F^p\Omega^*_{X/Y} \to 0$$

taking the higher direct image, Katz-Oda [21] prove that the associated connecting morphism

$$R^i f_* F^p \Omega^*_{X/Y} \to \Omega^1_V \otimes F^{p-1} R^i f_* \Omega^*_{X/Y}$$

coincide with the connection. More generally, there exists a filtration $L^r\Omega^i_X := f^*\Omega^r_V \otimes \Omega^{i-r}_X \subset \Omega^i_X$ of the complex Ω^*_X.

General case. The proof of the proposition is deduced from the smooth proper case via Deligne's simplicial resolutions [10]: there exists a smooth simplicial variety X_* defined by a family $\{X_n\}_{n\in\mathbb{N}}$ over X defining a cohomological hyperresolution of X, which gives in particular an isomorphism between the cohomology groups of the simplicial variety X_* and X. Applying the structural theorem to the morphisms $f_i : X_i \to D$, we deduce that the restriction of f_i to D^* is smooth so that we can use the relative de Rham complex of each X_i over D^* as in the above case.
We do not give the details of the proof here, since we will treat a similar case later to study the asymptotic behavior of the VMHS near the origin of the disc, in presence of singularities of f on the fiber at the origin. Precisely, there are two different cases. In the first case of a proper morphism $f : X \to D$, the varieties are proper over D. In the second case when f is not proper, but with algebraic fibers, the construction of MHS is based on the completion of X by a divisor Z over which the varieties X_i of X_* are completed by normal crossing divisors Z_i, so we can use the de Rham complex with logarithmic singularities along Z_i. The key point here is that the NCD form a family of relative NCD over D^* for D small enough, for a finite number of indices which determine the cohomology.

Corollary 1.18. *For each integer i and each stratum $S \subset V$ of a Thom-Whitney stratification of $f : X \to V$, the restriction of the higher direct cohomology sheaf $(R^i f_* \mathbb{Z}_X)/S$ underly a VMHS on S.*

Proof. By definition of S, the restriction of the higher direct image is a local system $\mathcal{L}_i$. By the above argument via simplicial coverings, for any embedded disc in S with center a point $v \in S$, we know only that the restriction to D^* is a VMHS. We prove that this VMHS extends across v.
Since the local system extends to the whole disc, the local monodromy is trivial at v, but we don't know yet about the extension of the weight and the Hodge filtrations. Since the local monodromy on the weight local subsystems $\mathcal{W}_r$ is induced from $\mathcal{L}_i$, it is trivial at v and $\mathcal{W}_r$ extend at v.

The proof of the extension of the Hodge filtration to an analytic bundle over D can be deduced from the study of the asymptotic behavior of the Hodge filtration in the next section. The extension of the Hodge filtration has been proved for the proper smooth case [28], [27].

In the next section we will define the limit MHS of the VMHS on D^*, and a comparison theorem via the natural morphism from the MHS on the fiber X_v to this limit MHS, is stated as the basic local invariant cycle theorem ([16],VI). Since the local monodromy is trivial in our case, both MHS coincide above the point v, hence the Hodge filtration extends also by analytic bundles over D since it is isomorphic to the limit Hodge filtration.

1.4. **Singularities of local systems.** In general the local system $(R^i f_* \mathbb{Z}_X)/S$ over a strata S does not extend as a local system to the boundary of S.

The study of the singularities at the boundary may be carried through the study of the singularities of the associated connection. It is important to distinguish in the geometric case between the data over the closure $\partial S := \overline{S} - S$ of S and the data that can be extracted from the asymptotic behavior on S, which are linked by the local invariant cycle theorem.

The degeneration can be studied locally at points of ∂S or globally, in which case we suppose the boundary of S a NCD, since we are often reduced to such case by the desingularization theorem of Hironaka.

We discuss in this section, the quasi-unipotent property of the monodromy and Deligne's canonical extension of the connection.

1.4.1. *Local monodromy.* To study the local properties of $R^i f_* \mathbb{Z}_X$ at a point $v \in V$, we consider an embedding of a small disc D in V with center v. Then, we reduce the study to the case of a proper analytic morphism $f : X \to D$ defined on an analytic space to a complex disc D. The inverse image $f^{-1}(D^*)$ of the punctured disc D^*, for D small enough, is a topological fiber bundle ([7], Exp. 14, (1.3.5)). It follows that a monodromy homeomorphism is defined on the fiber X_t at a point $t \in D^*$ by restricting X to a closed path $\gamma : [0,1] \to D^*, \gamma(u) = exp(2i\pi u)t$. The inverse fiber bundle is trivial over the interval: $\gamma^* X \simeq [0,1] \times X_t$ and the monodromy on X_t is defined by the path γ as follows

$$T : X_t \simeq (\gamma^* X)_0 \simeq (\gamma^* X)_1 \simeq X_t$$

The monodromy is independent of the choice of the trivilization, up to homotopy, and can be achieved for singular X via the integration of a special type of vector fields compatible with a Thom stratification of X [30].

Remark 1.19. The following construction, suggested in the introduction of ([7], Exp.13, Introduction) shows how X can be recovered as a topological space from the monodromy. There exists a retraction $r_t : X_t \to X_0$ of the general fiber X_t

to the special fiber X_0 at 0, satisfying $r_t \circ T = r_t$, then starting with the system (X_t, X_0, T, r_t) we define
i) X' and $f : X' \to S^1$ by gluing the boundaries of $X_t \times [0,1]$, $X_t \times 0$ and $X_t \times 1$ via T. A map $r' : X' \to X_0$ is deduced from r_t.
ii) then $f : X \to D$ is defined as the cone of f'

$$X \simeq X' \times [0,1]/(X' \times 0 \xrightarrow{r'} X_0) \to S^1 \times [0,1]/(S^1 \times 0 \to 0) \simeq D$$

1.4.2. *Quasi-unipotent monodromy.* If we suppose $D^* \subset S$, then the monodromy of the local system $(R^i f_* \mathbb{Z}_X)/D^*$ along a generator of D^*, as a linear operator on the cohomology $H^i(X_t, \mathbb{Q})$ may have only roots of unity as eigenvalues. This condition discovered first by Grothendieck for algebraic varieties over a field of positive characteristic, is also true over $\mathbb{C}$ [17].

Proposition 1.20. *The monodromy T of the local system $(R^i f_* \mathbb{Z}_X)/D^*$ defined by an algebraic (resp. proper analytic) morphism $f : X \to D$ is quasi-unipotent*

$$\exists\, a, b \in \mathbb{N} \quad \text{s.t.} \quad (T^a - Id)^b = 0$$

A proof in the analytic setting is explained in [6] and [22]. It is valid for an abstract VHS ([27], 4.5) where the definition of the local system over $\mathbb{Z}$ is used in the proof; hence it is true for VMHS. Finally, we remark that the theorem is true also for the local system defined by the Milnor fiber.

1.4.3. *Universal fiber.* A canonical way to study the general fiber, independently of the choice of $t \in D^*$, is to introduce the universal covering $\widetilde{D}^*$ of D^* which can be defined by Poincaré half plane $\mathbb{H} = \{u \in \mathbb{C} : \operatorname{Im} u > 0\}$, $\pi : \mathbb{H} \to D^* : u \mapsto exp\, 2i\pi u$. The inverse image $\widetilde{X}^* := \mathbb{H} \times_D X$ is a topological fibre bundle trivial over $\mathbb{H}$ with fibre homeomorphic to X_t. Let $H : \mathbb{H} \times X_t \to \widetilde{X}^*$ denotes a trivialization of the fiber bundle with fiber X_t at t, then the translation $u \to u+1$ extends to $\mathbb{H} \times X_t$ and induces via H, a transformation T of $\widetilde{X}^*$ s.t. the following diagram commutes

$$\begin{array}{ccc} X_t & \xrightarrow{I_0} & \widetilde{X}^* \\ \downarrow T_t & & \downarrow T \\ X_t & \xrightarrow{I_1} & \widetilde{X}^* \end{array}$$

where I_0 is defined by the choice of a point $u_0 \in \mathbb{H}$ satisfying $e^{2i\pi u_0} = t$ s.t. $I_0 : x \mapsto H(u_0, x)$, then $I_1 : x \mapsto H(u_0 + 1, x)$ and T_t is the monodromy on X_t. Hence T acts on $\widetilde{X}^*$ as a universal monodromy operator.

1.4.4. *Canonical extension with logarithmic singularities.* For the global study of the asymptotic behavior of the local system on a strata S near ∂S, we introduce a general construction by Deligne [8] for an abstract local system.

Hypothesis. Let Y be a normal crossing divisor (NCD) with smooth irreducible components $Y = \cup_{i \in I} Y_i$ in a smooth analytic variety X, $j : X^* := X - Y \to X$ the open embedding, and $\Omega^*_X(logY)$ the complex of sheaves of differential forms with logarithmic poles along Y. It is a complex of subsheaves of $j_* \Omega^*_{X-Y}$. There exists a global residue morphism $Res_i : \Omega^1_X(logY) \to \mathcal{O}_{Y_i}$ defined locally as follows: given a point $y \in Y_i$ and z_i an analytic local coordinate equation of Y_i at y, a differential $\omega = \alpha \wedge dz_i/z_i + \omega'$ where ω' does not contain dz_i and α is regular along Y_i, then $Res_i(\omega) = \alpha_{|Y_i}$.

Definition 1.21. Let F be a vector bundle on X. An analytic connection on F has logarithmic poles along Y if the entries of the connection matrix are one forms in $\Omega^1_X(LogY)$, hence define a $\mathbb{C}-$ linear map satisfying Leibnitz condition

$$\nabla : F \to \Omega^1_X(LogY) \otimes_{\mathcal{O}_X} F$$

Then the definition of ∇ extends to

$$\nabla^i : \Omega^i_X(LogY) \otimes_{\mathcal{O}_X} F \to \Omega^{i+1}_X(LogY) \otimes_{\mathcal{O}_X} F$$

it is integrable if $\nabla^1 \circ \nabla = 0$, then a logarithmic complex $\Omega^*_X(LogY)(F) := (\Omega^*_X(LogY) \otimes_{\mathcal{O}_X} F, \nabla)$ is defined in this case.
The composition of ∇ with the residue map:

$$(Res_i \otimes Id) \circ \nabla : F \to \Omega^1_X(LogY) \otimes_{\mathcal{O}_X} F \to \mathcal{O}_{Y_i} \otimes F$$

vanishes on the product $\mathcal{I}_{Y_i}F$ of F where $\mathcal{I}_{Y_i}$ is the ideal of definition of Y_i. It induces a linear map:

Lemma 1.22. *([8],3.8.3) The residue of the connection is a linear endomorphism of analytic bundles on Y_i*

$$Res_i(\nabla) : F \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_i} \to F \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_i}.$$

Theorem 1.23 (Logarithmic extension)**.** *([8], 5.2) Let $\mathcal{L}$ be a complex local system on the complement of the NCD: Y in X with locally unipotent monodromy along the components $Y_i, i \in I$ of Y. Then there exists a locally free module $\mathcal{L}_X$ on X which extends $\mathcal{L}_{X^*} := \mathcal{L} \otimes \mathcal{O}_{X^*}$, moreover the extension is unique if the connection ∇ has logarithmic poles with respect to $\mathcal{L}_X$*

$$\nabla \colon \mathcal{L}_X \to \Omega^1_X(LogY) \otimes \mathcal{L}_X$$

with nilpotent residues along Y_i.

Proof. The local system $\mathcal{L}$ is *locally unipotent along Y if at any point $y \in Y$ all T_j are unipotent*, in which case the extension we describe is *canonical*. The construction has two steps, the first describes a local extension of the bundle, the second consists to show that local coordinates patching
of the bundle over X^* extends to a local coordinates patching of the bundle over X. The property of flat bundles is important here since it is not known how to extend any analytic bundle on X^*. We rely on a detailed exposition of Malgrange [24]. We describe explicitly the first step, since it will be useful in applications.
Let y be a point in Y, and let $U(y)$ be a polydisc D^n with center y and $\mathcal{L} \to U(y)^* := (D^*)^p \times D^{n-p}$ the restriction of the local system. The universal covering $\widetilde{U}(y)^*$ of $(D^*)^p \times D^{n-p}$ is defined by

$$\mathbb{H}^{n-p} \times D^p = \{t = (t_1, \cdots, t_n) \in \mathbb{C}^n : \forall i \leq p, \ \operatorname{Im} t_i > 0 \ \text{ and } \ \forall i > p, |t_i| < \varepsilon\}$$

where the covering map is

$$\pi : \mathbb{H}^{n-p} \times D^p \to (D^*)^{n-p} \times D^p : t \to (e^{2i\pi t_1}, \cdots, e^{2i\pi t_p}, t_{p+1}, \cdots, t_n).$$

We fix a reference point $t_0 \in U(y)^*$ so that the local system is determined by a vector space L with the action of the various monodromy T_j for $j \leq p$ corresponding to the generators γ_j around Y_j of the fundamental group of $U(y)^*$.

The inverse image $\widetilde{\mathcal{L}} := \pi^*(\mathcal{L}_{|U(y)})$ on $\widetilde{U}(y)^*$ is trivial with global sections a vector space $\widetilde{L}$ isomorphic to L. The action of the monodromy T_j for $j \leq p$ on $\widetilde{L}$ is defined by the formula:

$$T_j\, v(t) = v(t_1, \cdots, t_j + 1, \cdots, t_n), \quad \text{for a section } v \in \widetilde{L}.$$

We define $N_j = -\frac{1}{2i\pi} Log T_j = \frac{1}{2i\pi} \sum_{k>0} (I - T_j)^k / k$ as a nilpotent endomorphism of $\widetilde{L}$. We consider the following embeddedding of the vector space $\widetilde{L}$ of multivalued sections of $\mathcal{L}$ into the subspace of analytic sections of the sheaf $\mathcal{L}_{\widetilde{U}(y)^*}$ on $\widetilde{U}(y)^*$ by the formula:

$$\widetilde{L} \to \mathcal{L}_{\widetilde{U}(y)^*} : v \mapsto \widetilde{v} = (exp(2i\pi \Sigma_{i \leq p} t_i N_i)).v$$

Notice that the exponential is a linear sum of multiples of $Id - T_j$ with analytic coefficients, hence its action defines an analytic section.

Lemma 1.24. *Let $j_y : U(y)^* \to U(y)$, then the analytic section $\widetilde{v} \in \mathcal{L}_{\widetilde{U}(y)^*}$ descends to an analytic section of $j_{y,*}\mathcal{L}_{U(y)^*}$.*

We show $\widetilde{v}(t + e_j) = \widetilde{v}(t)$ for all $t \in \widetilde{U}(y)^*$ and all vectors $e_j = (0, \cdots, 1_j, \cdots, 0)$. We have
$\widetilde{v}(t + e_j) = ([exp(2i\pi \Sigma_{i \leq p} t_i N_i] exp(2i\pi N_j).v)(t + e_j)$
where
$(exp(2i\pi N_j).v)(t + e_j) = (T_j^{-1}.v)(t + e_j) = (T_j^{-1} T_j.v)(t) = v(t)$
since $(T_j.v)(t) = v(t + e_j)$.

The bundle $\mathcal{L}_X$ is defined by the locally free subsheaf of $j_*\mathcal{L}_{X^*}$ with fibre $\mathcal{L}_{X,y}$ generated as an $\mathcal{O}_{X,y}-$module by the the sections $\widetilde{v}$ for $v \in \widetilde{L}$.

In terms of the local coordinates $z_i, i \in [1, n]$ of $U(y)$, the analytic function on $U(y)^*$ defined by a vector $v \in L$ is given by the formula

$$\widetilde{v}(z) = (exp(\Sigma_{i \leq p} (log z_i) N_i)).v$$

where $2i\pi t_i = log z_i$ is a determination of the logarithm, moreover $\nabla \widetilde{v} = \Sigma_{i \leq p} \widetilde{N_i.v} \otimes \frac{dz_i}{z_i}$. □

Remark 1.25. i) The residue along a component Y_i of the logarithmic connection ∇ is a nilpotent linear endomorphism of the analytic bundle $\mathcal{L}_{Y_i} := \mathcal{L}_X \otimes \mathcal{O}_{Y_i}$

$$\mathcal{N}_i := Res_i(\nabla) : \mathcal{L}_{Y_i} \to \mathcal{L}_{Y_i},$$

ii) Let $Y_{i,j} := Y_i \cap Y_j$, then the restrictions of $\mathcal{N}_{Y_i}$ and $\mathcal{N}_{Y_j}$ commute ([8] 3.10).
iii) Let $Y_i^* := Y_i - \cup_{k \in I-i} Y_i \cap Y_k$. There is no global local system underlying $\mathcal{L}_{Y_i^*}$. Locally, at any point $y \in Y_i$, a section $s_i : Y_i \cap U(y)^* \to U(y)^*$ may be defined by the hyperplane parallel to $Y_i \cap U(y)^*$ through the reference point t_0. Then $\mathcal{L}_{Y_i}$ is isomorphic to the canonical extension of the inverse local system $s_i^*\mathcal{L}$ ([8], 3.9.b).

Equivalently, the formula: $T_i = exp(-2i\pi Res_i(\nabla))$ ([8], 1.17, 3.11) holds on the sheaf $\Psi_{z_i}\mathcal{L}$ of nearby cycles.

1.4.5. *Relative monodromy filtration.* In the above formula the endomorphism $T^a - Id$ is nilpotent. This was one of the starting point to study the degeneration, when Deligne introduced a monodromy filtration satisfying some kind of degenerating Lefschetz formula and representing the Jordan form of a nilpotent endomorphism by a canonical filtration.

Let V be a vector space, W a finite increasing filtration of V by subspaces and N a nilpotent endomorphism of V compatible with W, then Deligne proves in ([11], prop. 1.6.13)

Lemma 1.26. *There exists at most a unique increasing filtration M of V satisfying $NM_i \subset M_{i-2}$ s.t. for all integers k and $l \geq 0$*

$$N^k : Gr^M_{k+l} V Gr^W_l V \xrightarrow{\sim} Gr^M_{l-k} Gr^W_l V$$

Remark 1.27. The lemma is true for an object V of an abelian category $\mathbb{A}$, a finite increasing filtation W of V by subobjects of V in $\mathbb{A}$ and a nilpotent endomorphism N. Such generalization is particularly interesting when applied to the abelian category of perverse sheaves on a complex variety.

Definition 1.28. When it exists, such filtration is called the relative monodromy filtration of N with respect to W and denoted by $M(N, W)$.

Lemma 1.29. *If the filtration W is trivial of weight a, the relative monodromy filtration M exists on the vector space V and satisfy*

$$NM_i \subset M_{i-2}; \quad N^k : Gr^M_{a+k} V \xrightarrow{\sim} Gr^M_{a-k} V$$

The above definition has a striking similarity with Hard Lefschetz theorem.

Example. Let $V = \oplus_i H^i(X, \mathbb{Q})$ denotes the direct sum of cohomology spaces of a smooth projective complex variety X of dimension n and $N = \cup c_1(H)$ the nilpotent endomorphism defined by the cup product with the cohomology class of an hyperplane section H of X. Consider the increasing filtration of V

$$M_i(V) = \oplus_{j \geq n-i} H^j(X, \mathbb{Q})$$

By hard Lefschetz theorem, the repeated action $N^i : Gr^M_i V \simeq H^{n-i}(X, \mathbb{Q}) \to Gr^M_{-i} V \simeq H^{n+i}(X, \mathbb{Q})$ is an isomorphism. Hence M_i coincides with the the monodromy filtration defined by N and centered at 0. Following this example, the property of the relative monodromy filtration appears as a degenerate form of Lefschetz result on cohomology.

The monodromy filtration centered at 0. Let N be nilpotent on V s.t. $N^{l+1} = 0$. This is the case W is trivial s.t. $W_0 = V$ and $W_{-1} = 0$, then M is constructed as follows. Let $M_l := V$, $M_{l-1} := \mathrm{Ker} N^l$ and $M_{-l} := N^l(V)$, $M_{-l-1} := 0$ then $N^l : Gr^l V \simeq Gr^{-l} V$ and the induced morphism N' by N on the quotient space V/M_{-l} satisfy $(N')^l = 0$ so that the definition of M is by induction on the index of nilpotency. The primitive part of $Gr^M_i V$ is defined for $i \geq 0$ as

$$P_i := \mathrm{Ker} N^{i+1} : Gr^M_i \to Gr^M_{-i-2}, \quad Gr^M_i V \simeq \oplus_{k \geq 0} N^k P_{i+2k}$$

The decomposition at right follows from this definition, and the proof is similar to the existence of a primitive decomposition following the hard Lefschetz theorem on compact kähler manifolds. In this case the filtration M gives a description of the Jordan form of the nilpotent endomorphism N, independent of the choice of a Jordan basis as follows. For all $i \geq 0$, we have the following properties:
- $N^i : M_i \to M_{-i}$ is surjective,
- $\mathrm{Ker} N^{i+1} \subset M_i$ and
- $\mathrm{Ker} N^{i+1}$ projects surjectively onto the primitive subspace $P_i \subset Gr^M_i$.

Let $(e^j_i)_{j \in I_i}$ denotes a subset of elements in $\mathrm{Ker} N^{i+1}$ which lift a basis of $P_i \subset Gr^M_i V$, then the various elements $N^k(e^j_i)_{j \in I_i, 0 \geq 0 k \leq i}$ define a Jordan basis of V for N. In particular, each element $(e^j_i)_j$ for fixed i gives rise to a Jordan block of length $i+1$ in the matrix of N.

1.4.6. *Limit Hodge filtration.* Let $(\mathcal{L}, F)$ be an abstract polarized VHS on a punctured disc D^*, where the local system is defined by a unipotent endomorphism T on a $\mathbb{Z}-$module L; then W. Schmid [17] showed that such VHS is asymptotic to a " Nilpotent orbit" defined by a filtration F called limit or asymptotic such that for $N = LogT$ the nilpotent logarithm of T, and $W(N)$ the monodromy filtration, the data $(L, W(N), F)$ form a MHS.
This positive answer to a question of Deligne was one of the starting point of the linear aspect of degeneration theory developed here, but the main development occurred with the discovery of Intersection cohomology.

There is no such limit filtration F in general for a VMHS.

2. Degeneration of VMHS

Families of algebraic varieties parameterized by a non singular algebraic curve, acquire in general singularities changing their topology at a finite number of points of the curve. If we center a disc D at one of these points, we are in the above case of a family over D^* which extends over the origin in an algebraic family over D. The fiber at the origin may be changed by modification along a subvariety, which do not change the family over D^*.
The study of the degeneration follows the same pattern as the definition of the VMHS. The main results have been established first for the degeneration of abstract VHS [27], then a geometric construction has been given in the case of degeneration of smooth families [28]. These results will be assumed since we concentrate our attention on the singular case here.

The degeneration of families of singular varieties is reduced to the case of smooth families by the technique of simplicial coverings already mentioned. Such covering by simplicial smooth algebraic varieties with NCD above the origin and satisfying the descent cohomological property, induce a covering of the fibers over D^* which is fit to study the degeneration of the cohomology of the fibers.

In the case of open families, we use the fact that we can complete algebraic varieties with a NCD at infinity Z, which moreover can be supposed a relative NCD over the punctured disc.

2.1. **Diagonal degeneration of geometric VMHS.** The term diagonal refers to a type of construction of the weight as diagonal with respect to a simplicial covering. The next results describes the cohomological degeneration of the MHS of an algebraic family over a disc. Here $\mathbb{Z}(b)$ will denote the MHS on the group $(2i\pi)^b\mathbb{Z} \subset \mathbb{C}$ of type $(-b, -b)$ and its tensor product with a MHS on a group V is denoted $V(b) := V \otimes \mathbb{Z}(b)$ and called twisted MHS on V.
Hypothesis. Let $f : X \to D$ be a proper analytic morphism defined on an analytic manifold X to a complex disc D, Z a closed analytic subspace of X and suppose the fibers of X and Z over D algebraic, then for D small enough:
the weight filtration on the family $H^n(X_t - Z_t, \mathbb{Q})$ define a filtration by sub-local systems $\mathcal{W}$ of $R^n f_*\mathbb{Q}_{|D^*}$ on D^*. The graded objects $Gr_i^{\mathcal{W}} R^n f_*\mathbb{Q}_{|D^*}$ underly a variation of Hodge structures (VHS) on D^* defining a limit MHS at the origin [27], [28]. The construction below gives back this limit MHS in the VHS case for $Z = \emptyset$ and is deduced from this case by the diagonalization process for a simplicial family of varieties.
Let $\widetilde{X}^* := X \times_D \widetilde{D}^*$ (resp. $\widetilde{Z}^* := Z \times_D \widetilde{D}^*$) be the inverse image on the universal covering $\widetilde{D}^*$ of D^*, then the inverse image $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ on $\widetilde{D}^*$ is trivial and defines

a filtration W^f by subspaces of $H^n(\tilde{X}^* - \widetilde{Z}^*, \mathbb{Q})$, called here the finite weight filtration.

Theorem 2.1. *There exists a MHS on the cohomology $H^n(\widetilde{X}^* - \widetilde{Z}^*, \mathbb{Z})$ with weight filtration W defined over $\mathbb{Q}$ and Hodge filtration defined over $\mathbb{C}$ satisfying*
i) the finite filtration W^f is a filtration by sub-MHS of $H^n(\widetilde{X}^* - \widetilde{Z}^*, \mathbb{Q})$.
ii) the induced MHS on $Gr_i^{W^f} H^n(\widetilde{X}^* - \widetilde{Z}^*, \mathbb{Q})$ coincides with the limit MHS defined by the VHS on the family $Gr_i^W H^n(X_t - Z_t, \mathbb{Q})$.
iii) Suppose moreover that f is quasi-projective, then for all integers a and b, the logarithm of the unipotent part T^u of the monodromy $N = \frac{1}{2i\pi} Log T^u$ induces an isomorphism for $b \geq 0$

$$GrN^b : Gr_{a+b}^W Gr_a^{W^f} H^n(\widetilde{X}^* - \widetilde{Z}^*, \mathbb{Q}) \simeq Gr_{a-b}^W Gr_a^{W^f} H^n(\widetilde{X}^* - \widetilde{Z}^*, \mathbb{Q})(-b)$$

Remark 2.2 (Category of limit MHS). The above MHS will be called the limit MHS of the VMHS defined by the fibers of f. In general we define a structure called limit MHS, by the following data: (V, W^f, W, F, N) where (V, W, F) form a MHS, W^f is an increasing filtration by sub-MHS and N is a nilpotent endomorphism of MHS: $(V, W, F) \to (V, W, F)(-1)$ compatible with W^f such that W is the relative weight filtration of (V, W^f, N).

Limit MHS form an additive category with kernel and cokernel but which is not abelian since W^f is not necessarily strict.

The assertion iii) characterizes the weight W as the monodromy filtration of N relative to W^f, which proves its existence in the case of geometric variations. The proof will occupy this section and is based on the reduction to the smooth case, via a simplicial hypercovering resolution of X, followed by a diagonalization process of the weight of each term of the hypercovering [5], as in the case of the weight in the MHS of a singular variety [4],[5].
Plan of the proof. Precisely, let $Y = f^{-1}(0)$ and consider a smooth hypercovering $\pi : X_* \to X$ over X where each term X_i is smooth and proper over X, such that $Z_* := \pi^{-1}(Z)$, $Y_* := \pi^{-1}(Y)$ and $Z_* \cup Y_*$ are NCD in X_* with no common irreducible component in Y_* and Z_*. Let $V := X - Z$, $V_i := X_i - Z_i$, then $\pi|V_* : V_* \to V$ is an hypercovering. Notice that only a finite number of terms X_i (resp. V_i) of the hypercovering are needed to compute the cohomology of X (resp. V). Then by Thom-Whitney theorems, for D small enough, X_i and Z_i are topological fibre bundle over D^* as well Z_i^* is a relative NCD in X_i^* for a large number of indices i, and for each $t \in D^*$, $(X_*)_t$ (resp. $(V_*)_t$) is an hypercovering of X_t (resp. V_t). Then for each index i, X_i and the various intersections of the irreducible components of Z_i, are proper and smooth families over D^*, so that we can apply in this situation the results of J. Steenbrink on the degeneration process for a geometric family of HS [28]. The method consists first to compute the hypercohomology of the sheaf of the nearby cycles $\Psi_f(\mathbb{C}_V)$ on $Y - Y \cap Z$ as the hypercohomology of a simplicial nearby cycles $\Psi_{f\circ\pi}(\mathbb{C}_{V_*})$

$$\mathbb{H}^n(Y - Y \cap Z, \Psi_f(\mathbb{C}_V)) \simeq H^n(\widetilde{V}^*, \mathbb{C}) \simeq H^n(\widetilde{V}_*^*, \mathbb{C}) \simeq \mathbb{H}^n(Y_* - Y_* \cap Z_*, \Psi_f(\mathbb{C}_{V_*}))$$

where we denote by tilde the inverse image of a space over D^* to the universal cover $\widetilde{D}^*$ and where the third term is the cohomology of the simplicial space $\widetilde{V}_*^*$.

We restrict the construction to the unipotent cohomology, denoted by an index u (that is the subspace where the action of the monodromy is unipotent) although the theorem is true without this condition. The cohomology is computed as the

hypercohomology of the simplicial variety Y_* with value in some sheaf denoted $\Psi_{f\circ\pi}(LogZ_*)$ that we *define here*. Such complex is a logarithmic version of the nearby cycle complex of sheaves on Y_* satisfying a simplicial cohomological mixed Hodge complex data, such that

$$H^n(\widetilde{V}^*,\mathbb{C})^u \simeq \mathbb{H}^n(Y_*,\Psi_{f\circ\pi}(LogZ_*))$$

Then the assertion ii) refers to the case of geometric VHS, which is first generalized on each X_i to the non proper case and then applied to each term X_i.
With this in mind, the method of proof use general results on *simplicial trifiltered complexes* that we develop now; still we need later to describe explicitly the complexes involved.

2.2. **Filtered mixed Hodge complex(FMHC).** The proof involves abstract results concerning FMHC, with three filtrations W^f, W and F, where W^f induces on cohomology a filtration by sub-MHS defined by W and F. We define first the category of complexes with three filtrations.

2.2.1. Let $\mathbb{A}$ be an abelian category, $F_3\mathbb{A}$ the category of three filtered objects of $\mathbb{A}$ with finite filtrations, and $K^+F_3\mathbb{A}$ the category of three filtered complexes of objects of $\mathbb{A}$ bounded at left, with morphisms defined up to homotopy respecting the filtrations.

Definition 2.3. A morphism $f:(K,F_1,F_2,F_3)\to(K',F_1',F_2',F_3')$ in $K^+F_3\mathbb{A}$ where F_1,F_1' are increasing, is called a quasi-isomorphism if the following morphisms f^i_j are bi-filtered quasi - isomorphisms for all $i\geq j$

$$f^i_j:(F_1^iK/F_1^jK,F_2,F_3)\to(F_1'^iK/F_1'^jK,F_2',F_3')$$

The category $D^+F_3\mathbb{A}$ is obtained from $K^+F_3\mathbb{A}$ by inverting the above quasi-isomorphisms; the objects in $D^+F_3\mathbb{A}$ are trifiltered complexes but the group of morphisms $Hom(K,K')$ of complexes change, since we add to a quasi-isomorphism f in $Hom(K,K')$ an inverse element in $Hom(K',K)$ denoted $1/f$ s.t. $f\circ(1/f)=Id$ (resp. $(1/f)\circ f=Id$), where equal to the identity means homotopic to the identity of K' (resp. K). In fact this changes completely the category since different objects, not initially isomorphic, may become isomorphic in the new category.

Definition 2.4 (Filtered mixed Hodge complex (FMHC))**.** A FMHC is given by
i) A complex $K_\mathbb{Z}\in ObD^+(\mathbb{Z})$ s.t. $H^k(K_\mathbb{Z})$ is a $\mathbb{Z}-$module of finite type for all k.
ii) A bi-filtered complex $(K_\mathbb{Q},W^f,W)\in ObD^+F_2(\mathbb{Q})$ and an isomorphism $K_\mathbb{Q}\simeq K_\mathbb{Z}\otimes\mathbb{Q}$ in $D^+(\mathbb{Q})$ where W (resp. W^f) is an increasing filtration by weight (resp. finite weight).
iii) A tri-filtered complex $(K_\mathbb{C},W^f,W,F)\in ObD^+F_3(\mathbb{C})$ and an isomorphism $(K_\mathbb{C},W^f,W)\simeq(K_\mathbb{Q},W^f,W)\otimes\mathbb{C}$ in $D^+F_2(\mathbb{C})$ where F is a decreasing filtration called Hodge filtration.
The following axiom is satisfied: for all $j\leq i$, the following system is a MHC

$$(W^f_iK_\mathbb{Q}/W^f_jK_\mathbb{Q},W),\ (W^f_iK_\mathbb{C}/W^f_jK_\mathbb{C},W,F)$$
$$\alpha^j_i:(W^f_iK_\mathbb{Q}/W^f_jK_\mathbb{Q},W)\otimes\mathbb{C}\simeq(W^f_iK_\mathbb{C}/W^f_jK_\mathbb{C},W)$$

2.2.2. We define as well a sheaf version as a cohomological FMHC on a topological space X,

$$K_{\mathbb{Z}} \in ObD^+(X,\mathbb{Z}),\ (K_{\mathbb{Q}}, W^f, W) \in ObD^+F_2(X,\mathbb{Q})$$
$$(K_{\mathbb{C}}, W^f, W, F) \in ObD^+F_3(X,\mathbb{C});\ \alpha : (K_{\mathbb{Q}}, W^f, W) \otimes \mathbb{C} \simeq (K_{\mathbb{C}}, W^f, W)$$

s.t. W_i^f/W_j^f is a cohomological MHC on X.
The global section functor Γ on X can be filtered derived using acyclic tri-filtered canonical resolutions such as Godement flabby resolutions.

Lemma 2.5. *The derived global section functor $R\Gamma$ of a cohomological FMHC on X is a FMHC.*

Proposition 2.6. *Let (K, W^f, W, F) denotes a FMHC, then*
i) The filtrations $W[n]$ and F define a MHS on the cohomology $H^n(K)$.
ii) The terms of the spectral sequence defined by the filtration W^f on K, with induced weight W and Hodge F filtrations

$$({}_{W^f}E_r^{p,q}, W, F) = Gr_{-p}^{W^f} H^{p+q}(W^f_{-p+r-1}K/W^f_{-p-r}K), W, F)$$

form a MHS and the differentials d_r are morphisms of MHS for $r \geq 1$.
iii) The filtration W^f is a filtration by sub-MHS and we have

$$(Gr_{-p}^{W^f} H^{p+q}(K), W[p+q], F) \simeq ({}_{W^f}E_\infty^{p,q}, W, F).$$

The proof is in ([13], thm 2.8). The formula for ${}_{W^f}E_r^{p,q}$ above coincides with Deligne's definition for the spectral sequence. On this formula, the MHS on ${}_{W^f}E_r^{p,q}$ is defined as on the cohomology of any MHC. We prove that the differential

$${}_{W^f}E_r^{p,q} \to_{W^f} E_r^{p+r,q-r+1} = Gr_{-p-r}^{W^f} H^{p+q+1}(W^f_{-p-1}K/W^f_{-p-2r}K)$$

is compatible with MHS. It is deduced from the connection morphism ∂ defined by the exact sequence of complexes

$$0 \to W^f_{-p-r}K/W^f_{-p-2r}K \to W^f_{-p+r-1}K/W^f_{-p-2r}K \to W^f_{-p+r-1}K/W^f_{-p-r}K \to 0$$

Let $\varphi : H^{p+q+1}(W^f_{-p-r}K/W^f_{-p-2r}K) \to H^{p+q+1}(W^f_{-p-1}K/W^f_{-p-2r}K)$ denotes the morphism induced by the natural embedding $W^f_{-p-r}K \to W^f_{-p-1}K$ and $\pi\circ(\varphi|) : W^f_{-p}H^{p+q+1}(W^f_{-p-r}K/W^f_{-p-2r}K) \to Gr_{-p-r}^{W^f} H^{p+q+1}(W^f_{-p-1}K/W^f_{-p-2r}K)$ the composition of the restriction $\varphi|$ of φ to W^f_{-p} with the projection π onto ${}_{W^f}E_r^{p+r,q-r+1}$ and $W^f_{-p}H^{p+q}(W^f_{-p+r-1}K/W^f_{-p-r}K) \xrightarrow{\partial|} W^f_{-p}H^{p+q+1}(W^f_{-p-r}K/W^f_{-p-2r}K)$ the restriction of ∂ to the subspace W^f_{-p} then $d_r = \pi\circ\varphi|(\circ\partial|)$. Since the connection ∂ is compatible with MHS, as is $\pi\circ(\varphi|)$, so is d_r.
The isomorphism $H^{p,q}({}_{W^f}E_r^{*,*}, d_r) \to {}_{W^f}E_{r+1}^{p,q}$ is induced by the embedding $W^f_{-p+r-1}K \to W^f_{-p+r}K$, hence it is also compatible with W and F. We deduce that the recurrent filtrations on $E_{r+1}^{p,q}$ induced by W and F on $E_r^{p,q}$ coincide with W and F on ${}_{W^f}E_{r+1}^{p,q}$.

Definition 2.7 (Limit mixed Hodge complex (LMHC))**.** i) A LMHC: (K, W^f, W, F) in $D^+F_3\mathbb{A}$ is given by the above data i) to iii) of a FMHC, satisfying the following:
1) The sub-complexes $(W_i^f K, W, F)$ are MHC for all indices i.
2) For all $n \in \mathbb{Z}$, we have induced MHC

$$(Gr_n^{W^f} K_{\mathbb{Q}}, W), (Gr_n^{W^f} K_{\mathbb{C}}, W, F), Gr_n\alpha : (Gr_n^{W^f} K_{\mathbb{Q}}, W) \otimes \mathbb{C} \simeq (Gr_n^{W^f} K_{\mathbb{C}}, W).$$

3) The spectral sequence of $(K_{\mathbb{Q}}, W^f)$ degenerates at rank 2:
$E_2(K_{\mathbb{Q}}, W^f) \simeq E_\infty(K_{\mathbb{Q}}, W^f)$.
ii) A cohomological LMHC on a space X is given by a sheaf version of the data i) to iii) s.t. $R\Gamma(X, K, W^f, W, F)$ is a LMHC.

Proposition 2.8. *Let (K, W^f, W, F) denotes a LMHC, then*
i) The filtrations $W[n]$ and F define a MHS on the cohomology $H^n(K)$, and W^f induces a filtration by sub-MHS.
ii) The MHS deduced from i) on $E_\infty^{p,q}(K, W^f) = Gr_{-p}^{W^f} H^{p+q}(K)$ coincide with the MHS on the terms of the spectral sequence ${}_{W^f}E_2^{p,q}$ deduced from the MHS on the terms ${}_{W^f}E_1^{p,q} = H^{p+q}(Gr_{-p}^{W^f} K)$.

The proof, similar to the above case of FMHC, is in ([13], thm. 2.13).

2.3. **Diagonal direct image of a simplicial cohomological FMHC.** We define naturally the direct image $(R\pi_* K, W, W^f, F)$ of a simplicial cohomological FMHC K on a simplicial space $\pi : X_* \to X$ over X [10]. The important point here is that the weight W is in fact a diagonal sum $\delta((\pi)_* W, L)$ with respect to a filtration L . This operation is of the same nature as the mixed cone over a morphism of MHC where the sum of the weight in the cone is also diagonal.

Definition 2.9. A simplicial cohomological FMHC on a simplicial (resp. simplicial strict) space X_* is given by a complex $K_{\mathbb{Z}}$, a bi-filtered complex $(K_{\mathbb{Q}}, W^f, W)$, an isomorphism $K_{\mathbb{Q}} \simeq K_{\mathbb{Z}} \otimes \mathbb{Q}$ and a tri-filtered complex $(K_{\mathbb{C}}, W^f, W, F)$ on X_* with an isomorphism $(K_{\mathbb{C}}, W^f, W) \simeq (K_{\mathbb{Q}}, W^f, W) \otimes \mathbb{C}$
such that the following axiom is satisfied: the restriction K_p of K to each X_p is a cohomological FMHC

2.3.1. *Differential graded cohomological FMHC defined by a simplicial cohomological FMHC.* Let $\pi : X_* \to X$ be a simplicial space over X. We define $R\pi_* K$ as a cosimplicial cohomological FMHC by deriving first π_i on each space X_i, on which we deduce an intermediary structure called a differential graded cohomological FMHC on X as follows. Let I_*^* be an injective or $\pi-$acyclic resolution of K on X_*, that is a resolution I_i^* on X_i varying functorially with the index i, then $\pi_* I_*^*$ is a cosimplicial complex of abelian sheaves on X with double indices where p on $\pi_* I_i^p$ is the complex degree and i the cosimplicial degree. It is endowed by the structure of a double complex with the differential $\sum_i (-1)^i \delta_i x^{pq}$ deduced from the cosimplicial structure, as in [10]. Such structure is known as a cohomological differential graded DG^+-complex. If we do this operation on the various levels, rational and complex we obtain the following structure.
Differential graded cohomological FMHC. A differential graded DG^+-complex C_*^* is a bounded below complex of graded objects, with two degrees, the first defined by the complex and the second by the gradings. It is endowed with two differentials and viewed as a double complex.
A differential graded cohomological MHC is defined by a system of a DG^+-complex, a filtered and a bifiltered complex with compatibility isomorphisms

$$C_*^*, (C_*^*, W), (C_*^*, W, F)$$

s.t. for each degree n of the grading, the component (C_n^*, W, F) is a CMHC.

2.3.2. *The higher direct image of a simplicial cohomological FMHC.* It is defined by the simple complex associated to the double complex $(\pi_{i,*}I_i^*)$ with total differential involving the face maps δ_i of the simplicial structure and the differentials d_q on I_q^*:

$$s(\pi_{q,*}I_q^*)^n_{q\in\mathbb{N}} := \oplus_{p+q=n}\pi_{q,*}I_q^p; \quad d(x_q^p) = d_q(x_q^p) + \Sigma_i(-1)^i\delta_i x_q^p$$

The filtration L with respect to the second degree will be useful

$$L^r(s(\pi_{q,*}I_q^*)_{q\in\mathbb{N}}) = s(\pi_{q,*}I_q^*)_{q\geq r}$$

so we can deduce a cohomological FMHC on X by summing into a simple complex

$$R\pi_*K := s(R\pi_{q,*}K|X_q)_{q\in\mathbb{N}} := s(\pi_{q,*}I_q^*)_{q\in\mathbb{N}}$$

Definition 2.10. The diagonal direct image $(R\pi_*K, W, W^f, F)$ of K on X_* is defined as follows
i) The weight diagonal filtration W (resp. the finite weight diagonal filtration W^f) is

$$\delta(W,L)_n R\pi_*K := \oplus_p W_{n+p}\pi_{p,*}I_p^*, \quad \delta(W^f,L)_n R\pi_*K := \oplus_p W^f_{n+p}\pi_{p,*}I_p^*$$

ii) The simple Hodge filtration F is $F^n R\pi_*K := \oplus_p F^n\pi_{p,*}I_p^*$.

Lemma 2.11. *The complex $(R\pi_*K, W^f, W, F)$ is a cohomological FMHC and we have*

$$(W_i^f/W_j^f)(R\pi_*K, W, F) \simeq \oplus_p((W^f_{i+p}/W^f_{j+p})\pi_{p,*}I_p^*, \delta(W,L), F),$$
$$(Gr_i^W(W_i^f/W_j^f)R\pi_*K, F) \simeq \oplus_p(Gr^W_{i+p}(W^f_{i+p}/W^f_{j+p})\pi_{p,*}I_p^*[-p], F).$$

2.4. **Construction of a Limit MHS on the unipotent nearby cycles.** We illustrate the above theory by an explicit construction of a LMHC on the nearby cycles that is applied to define the limit MHS of a geometric VMHS. Let $f : V \to D$ be a quasi-projective morphism to a disc. If D is small enough, the morphism is a topological bundle on D^*, hence the higher direct images $R^i f_*\mathbb{C}$ underly a variation of the MHS defined on the cohomology of the fibers. In order to obtain at the limit a canonical structure not depending on the choice of the general fiber at a point $t \in D^*$, we introduce what we call here the universal fiber to define the *nearby cycle complex of sheaves* $\Psi_f(\mathbb{C})$, of which we recall the definition in the complex analytic setting. Let $V_0 = f^{-1}(0)$, $V^* = V - V_0$, $p : \widetilde{D}^* \to D^*$ a universal cover of the punctured unit disc D^*, and consider the following diagram

$$\begin{array}{ccccccc} \widetilde{V}^* & \xrightarrow{\widetilde{p}} & V^* & \xrightarrow{j} & V & \xleftarrow{i} & V_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widetilde{D}^* & \xrightarrow{p} & D^* & \to & D & \hookleftarrow & \{0\} \end{array}$$

where $\widetilde{V}^* := V^* \times_D \widetilde{D}^*$. For each complex of sheaves $\mathcal{F}$ of abelian groups on V^*, the nearby cycle complex of sheaves $\Psi_f(\mathcal{F})$ is defined as:

$$\Psi_f(\mathcal{F}) := i^* R\, j_* R\, \widetilde{p}_* \widetilde{p}^*(\mathcal{F}).$$

Let $\widetilde{D}^* = \{u = x + iy \in \mathbb{C} : x < 0\}$ and the exponential map $p(u) = \exp u$. The translation $u \to u + 2i\pi$ on $\widetilde{D}^*$ lifts to an action on $\widetilde{V}^*$, inducing an action on $R\,\widetilde{p}_*\widetilde{p}^*(\mathcal{F})$ and finally a monodromy action T on $\Psi_f(\mathcal{F})$.
The method to construct the limit MHS is to explicit a structure of mixed Hodge complex on nearby cycles $R\Gamma(Y, \Psi_f(\mathbb{C})) = R\Gamma(\widetilde{V}^*, \mathbb{C})$. The technique used here puts such structure on the complex of subsheaves $\Psi_f^u(\mathbb{C})$ where the action of T

is unipotent, then the problem may be reduced to this case. In view of recent development, the existence of the weight filtration with rational coefficients becomes clear in the frame of the abelian category of perverse sheaves since the weight filtration is exactly the monodromy filtration defined by the nilpotent action of the logarithm of T on the perverse sheaf $\Psi_f^u(\mathbb{Q})$, up to a shift in indices. Hence we will concentrate here on the construction of the weight filtration on the complex $\Psi_f^u(\mathbb{C})$. The construction is carried first for a smooth morphism, then applied to each space of a smooth simplicial covering of V.

2.5. **Smooth morphism.** For a smooth morphism $f : V \to D$, the work of Deligne [7] and the smooth proper case [28] suggests to construct the limit MHS on the universal fiber $\widetilde{V}^*$, however the MHS depends on the properties at infinity of the fibers, so we need to introduce a compactification of the morphism f. Then, we suppose there exists a proper morphism called also $f : X \to D$ with algebraic fibres which induces the given morphism on V. This will apply to a quasi-projective morphism in which case we may suppose the morphism $f : X \to D$ projective. Moreover, we suppose the divisor at infinity $Z = X - V$, the special fiber $Y = f^{-1}(0)$ and their union $Z \cup Y$ normal crossing divisors in X. To study the case of the smooth morphism on $V = X - Z$, we still cannot use the logarithmic complex $\Omega^*_{\widetilde{X}^*}(Log\widetilde{Z}^*)$ since $\widetilde{X}^*$ is analytic in nature (it is defined via the exponential map). So we need to introduce a sub-complex of sheaves of $\Omega^*_{\widetilde{X}^*}(Log\widetilde{Z}^*)$, essentially described in [7], which underly the structure of cohomological FMHC.

Construction of a FMHC. We may start with the following result. Let c be a generator of the cohomology $H^1(D^*, \mathbb{Q})$ and denote by $\eta_{\mathbb{Q}} = \cup f^*(c)$ the cup product with the inverse image $f^*(c) \in H^1(X^*, \mathbb{Q})$. Locally at a point $y \in Y$, a neighborhood X_y is a product of discs and $X_y^* := X_y - (Y \cap X_y)$ is homotopic to a product of n punctured discs, hence $H^i(X_y^*, \mathbb{Q}) \simeq \wedge^i H^1(X_y^*, \mathbb{Q}) \simeq \wedge^i(\mathbb{Q}^n)$. The morphism $\eta_{\mathbb{Q}}$

$$H^i(X_y^*, \mathbb{Q}) \xrightarrow{\eta_{\mathbb{Q}}} H^{i+1}(X_y^*, \mathbb{Q})$$

is the differential of an acyclic complex $(H^i(X_y^*, \mathbb{Q})_{i\geq 0}, d = \eta_{\mathbb{Q}})$. A truncation $H^{*\geq i+1}(X_y^*, \mathbb{Q})$ of this complex defines a resolution ([7],lecture 14, lemma 4.18.4)

$$H^i(\widetilde{X}_y^*, \mathbb{Q})^u \to H^{i+1}(X_y^*, \mathbb{Q}) \to \cdots \to H^p(X_y^*, \mathbb{Q}) \xrightarrow{\eta_{\mathbb{Q}}} H^{p+1}(X_y^*, \mathbb{Q}) \to \cdots$$

of the cohomology in degree i of the space $\widetilde{X}_y^* = X_y \times_{D^*} \widetilde{D}^*$ which is homotopic to a Milnor fiber. Dually, we have an isomorphism $H^i({X_y}^*, \mathbb{Q})/\eta_{\mathbb{Q}} H^{i-1}({X_y}^*, \mathbb{Q}) \simeq H^i(\widetilde{X}_y^*, \mathbb{Q})^u$.

This construction can be lifted to the complex level, to produce in our case a cohomological FMHC on Y computing the cohomology of the space $\widetilde{X}^* = X \times_{D^*} \widetilde{D}^*$ homotopic to a general fiber as follows.

We use first the logarithmic complex $\Omega^*_X(Log(Y \cup Z))$ to compute $\mathrm{R\,j}_*\mathbb{C}_{X-(Y\cup Z)}$ $(j : X - (Y \cup Z) \to X)$. On the level of differential forms, df/f represents the class $2i\pi f^*(c)$, since $\int_{|z|=1} dz/z = 2i\pi$, and $\wedge df/f$ realizes the cup product as a morphism (of degree 1)

$$i_Y^*(\Omega^*_X(LogY \cup Z) \xrightarrow{\eta := \wedge df/f} i_Y^*(\Omega^*_X(LogY \cup Z)[1]$$

satisfying $\eta^2 = 0$ so to get a double complex. *By the above local result, the simple associated complex is quasi-isomorphic to the sub-sheaf of unipotent nearby cycles*

Ψ^u_f ([7], ([13], thm 2.6)).
Since Y and Z are NCD, we can write the logarithmic complex as $\Omega^*_X(LogY)(LogZ)$ so to introduce *the weight filtration W^Y (resp. W^Z) with respect to Y (resp. Z) in addition to the weight filtration $W^{Y\cup Z}$*. The simple complex of interest to us is realized as a sub-complex of $\Omega^*_{\widetilde{X}^*}(Log\widetilde{Z}^*)$ generated by $\Omega^*_X(LogY)(LogZ)$ and the variable u on $\widetilde{X}^*$. It is the image of the complex $\mathbb{C}[U]\otimes\Omega^*_X(LogY)(LogZ)$ by the embedding I defined by

$$I(U^p) := (-1)^{p-1}(p-1)!u^{-p}, \quad d(u\otimes 1) = 1\otimes\tfrac{df}{f}$$

since $u = Logf$ on $\widetilde{X}^*$. The monodromy acts as $T(u^p\otimes\omega) = (u+2i\pi)^p\otimes\omega$. Now if we put $N = \frac{1}{2i\pi}LogT$, then for $p\geq 0$, $N((Logf)^{-p}) = -p(Logf)^{-(p+1)}$([7], examples 4.6). If we define an action ν on $\mathbb{C}[U]\otimes\Omega^*_X(LogY)(LogZ)$ by $\nu(U^p) = U^{p+1}$, the embedding I satisfy $I(\nu(U^p)) = N(I(U^{p+1}))$, that is the action ν corresponds to N (we may use as Kashiwara the variable N instead of U to emphasize that the action of N is induced by the multiplication by the variable N).

Remark 2.12. If we use $\widetilde{D}^* = \mathbb{H}$ with $p(u) = e^{2i\pi u}\in D^*$ for covering space, then $u = \frac{1}{2i\pi}Logf$ on $\widetilde{X}^*$ and $d(u\otimes 1) = 1\otimes\frac{1}{2i\pi}\frac{df}{f}$ while the monodromy is given as $T(u^p\otimes\omega) = (u+1)^p\otimes\omega$. From the embedding we deduce the differential as $\cup f^*(c)$, and we need to define $N = LogT$ to get for $p\geq 0$, $N((Logf)^{-p}) = -p(Logf)^{-(p+1)}$.

Still to get regular filtrations we need to work on a finite complex deduced as a quotient modulo an acyclic sub-complex, hence we construct the following trifiltered complex on which the filtrations are regular

$$(\Psi^u_Y(LogZ), W^f, W, F)$$

as follows

$$\begin{aligned}
&(\Psi^u_Y)^r(LogZ) := \oplus_{p\geq 0, q\geq 0, p+q=r}\Omega^{p+q+1}_X(LogY\cup Z)/W^Y_p\\
&W_i(\Psi^u_Y)^r(LogZ) := \oplus_{p+q=r}W^{Y\cup Z}_{i+2p+1}\Omega^{p+q+1}_X(LogY\cup Z)/W^Y_p,\\
&W^f_i = \oplus_{p+q=r}W^Z_i, W^Y_i = \oplus_{p+q=r}W^Y_{i+p+1}, F^i = \oplus_{p+q=r}F^{i+p+1}
\end{aligned}$$

It is the simple complex associated to the double complex

$$(\Psi^u_Y)^{p,q}(LogZ) := \Omega^{p+q+1}_X(LogY\cup Z)/W^Y_p, d, \eta), p\geq 0, q\geq 0$$

with the usual differential d of forms for fixed p and the differential $\wedge df/f$ for fixed q, hence the total differential is $D\omega = d\omega + (df/f)\wedge\omega$.
The projection map $(\Psi^u_Y)^{p,q}(LogZ)\to(\Psi^u_Y)^{p+1,q-1}(LogZ)$ is the action of an endomorphism on the term of degree $(p+q)$ of the complex commuting with the differential, hence an endomorphism $\nu:\Psi^u_Y\to\Psi^u_Y$ of the complex. The study of such complex is reduced to the smooth proper case applied to X and the intersections of the components of Z via the residue Res_Z on $Gr^{W^f}_*\Psi^u_Y(LogZ)$.

Remark 2.13. By construction, the differentials are compatible with the above embedding. We take the quotient by various submodules W^Y_p which form an acyclic sub-complex, hence we have an isomorphism $\mathbb{H}^*(Y,\Psi^u_Y(LogZ))\simeq H^*(\widetilde{V}^*,\mathbb{C})^u$ s.t. the action of ν induces $N = \frac{1}{2i\pi}LogT$.

Theorem 2.14. *The trifiltered complex $(\Psi^u_Y(LogZ), W^f, W, F)$ is a cohomological limit mixed Hodge complex (LMHC) which endows the cohomology $H^*(\widetilde{V}^*,\mathbb{C})^u$ with*

a limit MHS such that the the weight filtration W is equal to the monodromy weight filtration relative to W^f.

The theorem results from the following proposition where $Z = \cup_{i\in I_1} Z_i$ denotes a decomposition into components of Z, $Z^J = Z_{i_1} \cap \ldots \cap Z_{i_r}$ for $J = \{i_1, \cdots, i_r\} \subset I_1$, $Z^r := \coprod_{J\subset I_1, |J|=r} Z^J$.

Proposition 2.15. *i) Let $k_Z : Y - (Z \cap Y) \to Y$, and $j_Z : (X^* - Z^*) \to X^*$. There exists a natural quasi-isomorphism*

$$\Psi^u_Y(LogZ) \xrightarrow{\approx} Rk_{Z,*}(k^*_Z \Psi^u_f(\mathbb{C})) \xrightarrow{\approx} \Psi^u_f(Rj_{Z,*}\mathbb{C}_{X^*-Z^*})$$

ii) The graded part for W^f is expressed with the LMHC for the various proper smooth maps $Z^{-p} \to D$ for $p < 0$, with singularities along the NCD: $Z^{-p} \cap Y$

$$Res_Z : (Gr^{W^f}_{-p}(\Psi^u_Y)(LogZ), W, F) \simeq (\Psi^u_{Z^{-p}\cap Y}[p], W[-p], F[p])$$

and the spectral sequence with respect to W^f is given by the limit MHS of the unipotent cohomology of $(\widetilde{Z}^{-p})^* = Z^{-p} \times_D \widetilde{D}^*$ twisted by (p)

$$({}_{W^f}E^{p,q}_1, W, F) \simeq \mathbb{H}^{p+q}(Z^{-p} \cap Y, (\Psi^u_{Z^{-p}\cap Y}[p], W[-p], F[p])) \simeq$$
$$(\mathbb{H}^{2p+q}((\widetilde{Z}^{-p})^*, \mathbb{C})^u, W[-2p], F[p]) \simeq (\mathbb{H}^{2p+q}((\widetilde{Z}^{-p})^*, \mathbb{C})^u, W, F)(p)$$

iii) The endomorphism ν shift the weight by -2: $\nu(W_i\Psi^u_Y) \subset W_{i-2}\Psi^u_Y$ and preserves W^f. It induces an isomorphism

$$\nu^i : Gr^W_{a+i} Gr^{W^f}_a \Psi^u_Y(LogZ) \xrightarrow{\sim} Gr^W_{a-i} Gr^{W^f}_a \Psi^u_Y(LogZ)$$

Moreover, the action of ν corresponds to the logarithm of the monodromy on the cohomology $H^*(\widetilde{V}^*, \mathbb{C})^u$
iv) The induced monodromy action $\bar{\nu}$ defines an isomorphism

$$\bar{\nu}^i : Gr^W_{a+i} Gr^{W^f}_a H^n(Y, \Psi^u_Y(LogZ)) \xrightarrow{\sim} Gr^W_{a-i} Gr^{W^f}_a H^n(Y, \Psi^u_Y(LogZ))$$

Corollary 2.16. *The weight filtration induced by W on the cohomology $H^*(\widetilde{V}^*, \mathbb{C})^u$ satisfies the characteristic property of the monodromy weight filtration relative to the weight filtration W^f.*

The main argument consists to deduce iv) from the corresponding isomorphism on the complex level in iii) after a reduction to the proper case. We remark also ([13], prop. 3.5) that the spectral sequence ${}_{W^f}E^{p,q}_r$ is isomorphic to the weight spectral sequence of any fiber $X_t - Z_t$ for $t \in D^*$ and degenerates at rank 2.

Proof of the proposition in the proper smooth case (VHS). The complex $\Psi^u_Y(LogZ)$ for $Z = \emptyset$ coincides with Steenbrink's complex Ψ^u_Y on Y in $X = V$ [28]. In this case W^f is trivial, $W^{Y\cup Z} = W^Y$ on $\Omega^*_X(LogY)$ and (Ψ^u_Y, W, F) is a MHC since its graded object is expressed in terms of the Hodge complexes defined by the embedding of the various s intersections of components Y_i of $Y = \cup_{i\in I} Y_i$ denoted as $a_s : Y^s := \coprod_{J\subset I, |J|=s} Y^J \to X$ where $Y^J = Y_{i_1} \cap \ldots \cap Y_{i_s}$ for $J = \{i_1, \cdots, i_s\}$. The residue

$$(Gr^W_r \Psi^u_Y, F) \xrightarrow{Res} \oplus_{p\geq sup\,(0,-r)} a_{r+2p+1,*}(\Omega^*_{Y^{r+2p+1}}[-r-2p], F[-p-r])$$

defines an isomorphism with the HC of weight r at right, then the assertion i) of the proposition for $Z = \emptyset$ reduces to the quasi-isomorphism

$$\Psi^u_Y \xrightarrow{\approx} \Psi^u_f(\mathbb{C})$$

Locally, the cohomology $H^i(\Psi^u_Y(\mathbb{C})_y)$ of the stalk at y is equal to the unipotent cohomology of the universal Milnor fiber $\widetilde{X}^*_y$ of f at y, hence the quasi-isomorphism above follows by construction of $\Psi^u_{Y,y}$ since

$$H^i(\Psi^u_f(\mathbb{C})_y) \simeq H^i(\widetilde{X}^*_y, \mathbb{C})^u \simeq H^i(\Psi^u_{Y,y})$$

To prove this local isomorphism, we use the spectral sequence of Ψ^u_Y with respect to the columns of the underlying double complex. Since the $p-$th column is isomorphic to $(Rj_*\mathbb{C}/W^Y_p)[p+1]$, the stalk at y is : $E^{-p,q}_{1,y} \simeq H^{p+q+1}(X^*_y, \mathbb{C})$, for $p \geq 0, q \geq 0$ and 0 otherwise, with differential $d = \eta$; hence the term $E^{-p,q}_{2,y}$ is equal to 0 for $p > 0$ and equal to $H^{p+q}(\widetilde{X}^*_y, \mathbb{C})$ for $p = 0$ where $(E^{-p,q}_{1,y}, \eta)$ is the resolution of the cohomology of Milnor fiber mentioned earlier, then the global isomorphism follows

$$\mathbb{H}^i(Y, \Psi^u_Y) \simeq \mathbb{H}^i(\widetilde{X}^*, \mathbb{C})^u$$

In the assertion ii) we use the residue to define an isomorphism on the first terms of the spectral sequence with respect to W with the HS defined by Y^i after a twist

$${}_W E^{p,q}_1 = \mathbb{H}^{p+q}(Y, Gr^W_{-p}\Psi^u_Y, F) = \oplus_{q' \geq sup\,(0,p)} H^{2p-2q'+q}(Y^{2q'-p+1}, \mathbb{C}), F)(p-q')$$

The assertion iii) reduces to an isomorphism

$$Gr^W_i \Psi^u_Y \xrightarrow{\nu^i} Gr^W_{-i}\Psi^u_Y$$

which can be checked easily since

$$W_i(\Psi^u_Y)^{p,q} := W_{i+2p+1}\Omega^{p+q+1}_X(LogY)/W^Y_p = W_{i-2}(\Psi^u_Y)^{p+1,q-1} = \cdots = W_{-i}(\Psi^u_Y)^{p+i,q-i}$$

while the two conditions $(p+i \geq 0, p \geq 0)$ for $W_i(\Psi^u_Y)^{p,q}$, become successively $(p+i)-i = p \geq 0, p+i \geq 0$ for $W_{-i}(\Psi^u_Y)^{p+i,q-i}$, hence they are interchanged. We end the proof in the next section. □

2.6. **Polarized Hodge-Lefschetz structure.** The first correct proof of the assertion iv) is given in [26] in the more general setting of polarized Hodge-Lefschetz modules. We follow [25] for an easy exposition in our case.

2.6.1. *Hodge Lefschetz structure.* Two endomorphisms on a finite dimensional bigraded real vector space $L = \oplus_{i,j\in\mathbb{Z}} L^{i,j}$, $l_1 : L^{i,j} \to L^{i+2,j}$ and $l_2 : L^{i,j} \to L^{i,j+2}$, define a Lefschetz structure if they commute and if moreover the morphisms obtained by composition

$$l^i_1 : L^{-i,j} \to L^{i,j}, i > 0 \text{ and } l^j_2 : L^{i,-j} \to L^{i,j}, j > 0$$

are isomorphisms.
It is classical to deduce from the classical representation theory, as in hard Lefschetz theorem, that such structure corresponds to a finite dimensional representation of the group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$; then a primitive decomposition follows

$$L^{i,j} = \oplus_{r,s\geq 0} l^r_1 l^s_2 L^{i-2r,j-2s}_0, \text{ where } L^{-i,-j}_0 = L^{-i,-j} \cap Ker\, l^{i+1}_1 \cap Ker\, l^{j+1}_2, i \geq 0, j \geq 0$$

A Lefschetz structure is called *Hodge - Lefschetz structure* if in addition $L^{i,j}$ underly real Hodge structures and l_1, l_2 are compatible with such structures.
A polarization of L is defined by a real bigraded bilinear form $S : L \otimes L \to \mathbb{R}$ compatible with HS, s.t.

$$S(l_i x, y) + S(x, l_i y) = 0,\ i = 1, 2$$

It extends into a complex Hermitian form to $L \otimes \mathbb{C}$ such that the induced form $S(x, Cl_1^i l_2^j y)$ is symmetric positive definite on $L_0^{-i,-j}$, where C is the Weil operator defined by the HS.
A differential $d : L \to L$ is a morphism compatible with H.S satisfying:

$$d : L^{i,j} \to L^{i+1,j+1}, i, j \in \mathbb{Z},\ d^2 = 0,\ [d, l_i] = 0, i = 1, 2,$$
$$S(dx, y) = S(x, dy), x, y \in L.$$

Theorem 2.17 ([26], [25])**.** *Let (L, l_1, l_2, S, d) be a bigraded Hodge-Lefschetz structure with a differential d and polarization S, then the cohomology $(H^*(L, d), l_1, l_2, S)$ is a polarized Hodge-Lefschetz structure.*

We assume the theorem and that we apply to the weight spectral sequence, where $n = \dim.\ X$, as follows.
Let $K_{\mathbb{C}}^{i,j,k} = H^{i+j-2k+n}(Y^{2k-i+1}, \mathbb{C})(i-k)$, for $k \geq sup\,(0, i)$, and $K_{\mathbb{C}}^{i,j,k} = 0$ otherwise. Then the residue induces an isomorphism of $K_{\mathbb{C}}^{i,j} = \oplus_{k \geq sup\,(0,i)} K_{\mathbb{C}}^{i,j,k}$ with the terms of the spectral sequence above: ${}_W E_1^{r,q-r} \simeq K_{\mathbb{C}}^{r,q-n}$. Since the special fiber Y is projective, the cup-product with an hyperplane section class defines a morphism $l_1 = \smile c$ satisfying hard Lefschetz theorem on the various smooth proper intersections Y^i of the components of Y, while l_2 is defined by the action of N on E_1 deduced from the action of ν on the complex Ψ_Y^u. The differential d is defined on the terms of the spectral sequence which are naturally polarized as cohomology of smooth projective varieties. Then, all the conditions to apply the above result on differential polarized bigraded Hodge-Lefschetz structures are satisfied, so we can deduce

Corollary 2.18. *For all $q, r \geq 0$ the endomorphism N induces an isomorphism of HS*

$$N^r : Gr_{q+r}^W H^q(\widetilde{X}^*, \mathbb{Q})^u \to Gr_{q-r}^W H^q(\widetilde{X}^*, \mathbb{Q})^u(-r)$$

This ends the proof in the smooth proper case.

Remark 2.19 (Normal crossing divisor case). Let $X = \cup_{i \in I} X_i$ be embedded as a NCD with smooth irreducible components X_i, in a smooth variety V projective over the disc D, such that the fiber Y at 0 and its union with X is a NCD in V. Then the restriction of f to the intersections $X_J = \cap_{i \in J} X_i$, $J \subset I$, is a NCD $Y_J \subset X_J$, and the limit MHC $\Psi_{Y_J}^u$ for various $J, \emptyset \neq J \subset I$, form a simplicial cohomological MHC on the semi-simplicial variety X_* defined by X. In this case the finite filtration W^f on the direct image, coincides with the increasing filtration associated by change of indices to the canonical decreasing filtration L on the simplicial complex, that is $W_i^f = L^{-i}$, so that we can apply the general theory to obtain a cohomological LMHC defining the LMHS on the cohomology $H^*(\widetilde{X}^*, \mathbb{Q})$. This is an example of the general singular case.
If we add $X_\emptyset = V$ to the simplicial variety X_* we obtain the cohomology with compact support of the general fiber of $V - X$ which is Poincaré dual to the cohomology of $V - X$, the complement in V of the NCD. This remark explain the parallel (in fact dual) between the logarithmic complex case and the simplicial case.

Proof of the proposition in the open smooth case. We consider the maps $(\widetilde{X}^* - \widetilde{Z}^*) \xrightarrow{\widetilde{j}_Z} \widetilde{X}^* \xrightarrow{\widetilde{j}_Y} X$, then the assertion i) follows from the isomorphisms

$$(\Psi^u_Y(LogZ), W^f) \simeq i^*_Y R\widetilde{j}_{Y,*}(\Omega^*_{\widetilde{X}^*}(Log\widetilde{Z}^*), W^{\widetilde{Z}^*})$$
$$\simeq i^*_Y R\widetilde{j}_{Y,*}(R\widetilde{j}_{Z,*}\mathbb{C}_{\widetilde{X}^*-\widetilde{Z}^*}, \tau) \simeq \Psi^u_f(Rj_{Z,*}\mathbb{C}_{X^*-Z^*}, \tau)$$

Let $I_1 \subset I$ denotes the set of indices of the components of Z, Z^i the union of the intersections Z^J for $J \subset I_1, |J| = i$ and $a_{Z^i} : Z^i \to X$. The assertion ii) follows from the corresponding bifiltered isomorphism residue along Z:

$$(Gr_i^{W^Z}\Omega^*_X(LogY)(Log(Z), W, F) \simeq a_{Z^i,*}(\Omega^*_{Z^i}(LogY \cap Z^i), W[i], F[-i]).$$

More generally, we have residue isomorphisms Res_Z and Res_Y ([13], 3.3.2)

$$Gr^W_m(W^f_b/W^f_a)(\Psi^u_Y(LogZ), F) \xrightarrow{Res_Z} Gr^W_{m-j}(\Psi^u_{Z^j\cap Y}[-j], F[-j])$$
$$\xrightarrow{Res_Y} \oplus_{j\leq m+p, j\in[a,b], p\geq 0}(\Omega^*_{Z^j\cap Y^{m-j+2p+1}}[-m-2p], F[-m-p])$$

where $Z^i \cap Y$ is the union of $Z^J \cap Y$, so we can deduce the structure of LMHC from the proper case.
The isomorphism of complexes in the assertion iii), can be easily checked. While the assertion iv) for a smooth proper $X \to D$, is deduced via the above Res_Z from the proper smooth projective case $Z^j \to D$ for various j as follows. The monodromy ν induces on the spectral sequence the isomorphism for $p \leq 0$

$$(Gr^W_{i+b}({}_{W^f}E_1^{p,i}), d_1) \xrightarrow{\nu^b} (Gr^W_{i-b}({}_{W^f}E_1^{p,i}), d_1)$$

which commutes with the differential d_1 equal to a Gysin morphism alternating with respect to the embeddings of components of Z^{-p} into Z^{-p-1}. Since the isomorphism

$$\nu^b : (Gr^W_{2p+i+b}(H^{2p+i}(Z^{-p}, \mathbb{C})_{p\in\mathbb{Z}}, Gysin) \xrightarrow{\sim} Gr^W_{2p+i-b}(H^{2p+i}(Z^{-p}, \mathbb{C})_{p\in\mathbb{Z}}, Gysin)$$

has been checked in the proper case Z^{-p}, we deduce then iv)

$$Gr^W_{i+b}Gr_i^{W^f}H^n(Y, \Psi^u_Y(LogZ)) \xrightarrow{\nu^b} Gr^W_{i-b}Gr_i^{W^f}H^n(Y, \Psi^u_Y(LogZ))$$

□

2.7. **Quasi-projective case.** Let $f : V \to D$ be quasi-projective. There exists a simplicial smooth hypercovering of V of the following type. First, we consider an extension into a projective morphism $f : X \to D$ by completing with $Z = X - V$, then we consider a simplicial smooth hypercovering $\pi : X_* \to X$ with $\pi_i := \pi_{|X_i}$ s.t. $Z_i := \pi_i^{-1}(Z)$ consists of a NCD in X_i. Let $f_i := f \circ \pi_i : X_i \to D$; we may suppose Y_i and $Y_i \cup Z_i$ NCD in X_i so to consider the simplicial cohomological limit MHC $(\Psi^u_{Y_*}(LogZ_*), W^f, W, F)$ and its direct image $R\pi_*(\Psi^u_{Y_*}(LogZ_*), W^f, W, F)$ on X, then the theorem results from the following proposition ([13], 3.26, 3.29)

Proposition 2.20. *The tri-filtered complex*

$$R\pi_*(\Psi^u_{Y_*}(LogZ_*), W^f, W, F)$$

satisfy the following properties
i) Let $k_Z : Y - (Z \cap Y) \to Y, j_Z : X^* - Z^* \to X^*$, then there exists natural quasi-isomorphisms

$$R\pi_*\Psi^u_{Y_*}(LogZ_*) \xrightarrow{\sim} Rk_{Z,*}(k^*_Z\Psi^u_f(\mathbb{C})) \xrightarrow{\sim} \Psi^u_f(Rj_{Z,*}\mathbb{C}_{X^*-Z^*})$$

ii) The graded part for W^f is expressed in terms of the cohomological limit MHC for the various smooth maps $(X_i - Z_i) \to D$

$$Gr^{W^f}_{-p} R\pi_*(\Psi^u_{Y_*}(LogZ_*), W, F) \simeq \oplus_i R\pi_{i,*}(Gr^{W^{Z_i}}_{i-p} \Psi^u_{X_i}(LogZ_i)[-i], W[-i], F) \simeq \\ \oplus_i R\pi_{i,*}(\Psi^u_{Z_i^{i-p}\cap Y_i}[p-2i], W[-p], F[p-i])$$

The spectral sequence with respect to W^f is given by the twisted LMHS on the cohomology of $(\widetilde{Z}_i^{i-p})^* = Z_i^{i-p} \times_D \widetilde{D}^*$

$$\begin{aligned} &{}_{W^f}E_1^{p,q}(R\Gamma(Y, R\pi_*(\Psi^u_{Y_*}(LogZ_*), W^f, W, F)) \simeq \\ &\oplus_i \mathbb{H}^{p+q}(Z_i^{i-p} \cap Y_i, (\Psi^u_{Z_i^{i-p}\cap Y_i}[p-2i], W[-p], F[p-i])) \simeq \\ &\oplus_i (H^{2p+q-2i}((\widetilde{Z}_i^{i-p})^*, \mathbb{C})^u, W, F)(p-i) \end{aligned}$$

iii) The monodromy ν shift the weight by -2: $\nu(W_i\Psi^u_Y) \subset W_{i-2}\Psi^u_Y$ and preserves W^f. It induces an isomorphism

$$\nu^i : Gr^W_{a+i} Gr^{W^f}_a R\pi_* \Psi^u_{Y_*}(LogZ_*) \xrightarrow{\sim} Gr^W_{a-i} Gr^{W^f}_a R\pi_* \Psi^u_{Y_*}(LogZ_*)$$

iv) The induced iterated monodromy action ν^i defines an isomorphism

$$\nu^i : Gr^W_{a+i} Gr^{W^f}_a H^n(Y, R\pi_* \Psi^u_{Y_*}(LogZ_*)) \xrightarrow{\sim} Gr^W_{a-i} Gr^{W^f}_a H^n(Y, R\pi_*(\Psi^u_{Y_*}(LogZ_*))$$

The proof is by reduction to the smooth proper case, namely the various intersections Z_i^j of the components of the NCD Z_i in X_i as in the smooth open case. The spectral sequence is expressed as a double complex as in the case of the diagonal direct image in general. In particular the differential $d_1 : {}_{W^f}E_1^{n-i-1,i} \to {}_{W^f}E_1^{n-i,i}$ is written in terms of alternating Gysin maps associated to $(\widetilde{Z}_j^{i+j-n})^* \to (\widetilde{Z}_j^{i+j-n-1})^*$ and simplicial maps d' associated to $(\widetilde{Z}_j^{i+j-n})^* \to (\widetilde{Z}_{j-1}^{i+j-n})^*$ in the double complex ([10], 8.1.19) and ([13],3.30.1) written as

$$\begin{array}{ccc} H^{2n-2j-i}((\widetilde{Z}_j^{i+j-n})^*, \mathbb{C})^u & \xrightarrow{\partial = Gysin} & H^{2n+2-2j-i}((\widetilde{Z}_j^{i+j-n-1})^*, \mathbb{C})^u \\ \uparrow d' & & \uparrow d' \\ H^{2n-2j-i}((\widetilde{Z}_{j-1}^{i+j-n+1})^*, \mathbb{C})^u & \xrightarrow{\partial = Gysin} & H^{2n+2-2j-i}((\widetilde{Z}_{j-1}^{i+j-n-1})^*, \mathbb{C})^u \end{array}$$

The isomorphism iii) follows from the same property on each X_i while the isomorphism iv) is deduced from the smooth case above.

2.8. **Alternative construction, existence and uniqueness.** We deduce the limit structure on cohomology of a quasi projective family from the case of a relative open NCD in a projective smooth family. Instead of general simplicial variety the result follows from the simplicial variety defined by this special case.

2.8.1. *Hypothesis.* Let $f : X \to D$ be a projective family, $i_Z : Z \to X$ a closed embedding and $i_X : X \to P_D$ a closed embedding in a smooth family h of projective spaces over a disc D s.t. $h \circ i_X = f$. By Hironaka desingularization we construct

diagrams

$$\begin{array}{ccccc} Z''_0 & \to & X''_0 & \to & P''_0 \\ \downarrow & & \downarrow & & \downarrow \\ Z'_0 & \to & X'_0 & \to & P'_0 \\ \downarrow & & \downarrow & & \downarrow \\ Z_0 & \to & X_0 & \to & P_0 \\ & \searrow & \downarrow & \swarrow & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} Z'' & \to & X'' & \to & P''_D \\ \downarrow & & \downarrow & & \downarrow \\ Z' & \to & X' & \to & P'_D \\ \downarrow & & \downarrow & & \downarrow \\ Z & \to & X & \to & P_D \\ & \searrow & \downarrow & \swarrow & \\ & & D & & \end{array}$$

first by blowing up centers over Z so to obtain a smooth space $p : P'_D \to P_D$ such that $P'_0 := p^{-1}(P_0)$, $Z' := p^{-1}(Z)$ and $P'_0 \cup Z'$ are all NCD; set $X' := p^{-1}(X)$, then

$$p| : X' - Z' \xrightarrow{\sim} X - Z, \quad p| : P'_D - Z' \xrightarrow{\sim} P_D - Z$$

are isomorphisms since the modifications are all over Z. Next, by blowing up centers over X' we obtain a smooth space $q : P''_D \to P'_D$ such that $X'' := q^{-1}(X')$, $P''_0 := q^{-1}(P'_0)$, $Z'' := q^{-1}(Z')$ and $P''_0 \cup X''$ are all NCD, and $q| : P''_D - X'' \xrightarrow{\sim} P'_D - X'$ is an isomorphism. For D small enough, X'', Z'' and Z' are relative NCD over D^*. Hence we deduce the diagrams

$$\begin{array}{ccccc} X'' - Z'' & \xrightarrow{i''_X} & P''_D - Z'' & \xleftarrow{j''} & P''_D - X'' \\ q_X \downarrow & & q \downarrow & & q \downarrow \wr \\ X' - Z' & \xrightarrow{i'_X} & P'_D - Z' & \xleftarrow{j'} & P'_D - X' \end{array}$$

Since all modifications are above X', we still have an isomorphism induced by q at right. For dim.$P_D = d$ and all integers i, the morphism $q^* : H_c^{2d-i}(P''_D - Z'', \mathbb{Q}) \to H_c^{2d-i}(P'_D - Z', \mathbb{Q})$ is well defined on cohomology with compact support since q is proper; its Poincaré dual is called the trace morphism $Trq : H^i(P''_D - Z'', \mathbb{Q}) \to H^i(P'_D - Z', \mathbb{Q})$ and satisfy the relation $Trq \circ q^* = Id$. Moreover, the trace morphism is defined as a morphism of sheaves $q_* \mathbb{Z}_{P''_D - Z''} \to \mathbb{Z}_{P'_D - Z'}$ [31] , hence an induced trace morphism $(Trq)|(X'' - Z'') : H^i(X'' - Z'', \mathbb{Q}) \to H^i(X' - Z', \mathbb{Q})$ is well defined. Taking the inverse image on a universal covering $\widetilde{D}^*$, we get a diagram of universal fibers

$$\begin{array}{ccccc} (\widetilde{X}'' - \widetilde{Z}'')^* & \xrightarrow{i''_X} & (\widetilde{P}''_D - \widetilde{Z}'')^* & \xleftarrow{j''} & (\widetilde{P}''_D - \widetilde{X}'')^* \\ q_X \downarrow & & q \downarrow & & q \downarrow \wr \\ (\widetilde{X}' - \widetilde{Z}')^* & \xrightarrow{i'_X} & (\widetilde{P}'_D - \widetilde{Z}')^* & \xleftarrow{j'} & (\widetilde{P}'_D - \widetilde{X}')^* \end{array}$$

Proposition 2.21. *With the notations of the above diagram, we have short exact sequences*

$$0 \to H^i((\widetilde{P}''_D - \widetilde{Z}'')^*, \mathbb{Q}) \xrightarrow{(i''_X)^* - Trq} H^i(\widetilde{X}'' - \widetilde{Z}'')^*, \mathbb{Q}) \oplus H^i(\widetilde{P}'_D - \widetilde{Z}')^*, \mathbb{Q})$$
$$\xrightarrow{(i'_X)^* - (Trq)|(X'' - Z'')} H^i(\widetilde{X}' - \widetilde{Z}')^*, \mathbb{Q}) \to 0$$

Since we have a vertical isomorphism q at right of the above diagram, we deduce a long exact sequence of cohomology spaces containing the sequences of the proposition; the injectivity of $(i''_X)^* - Trq$ and the surjectivity of $(i'_X)^* - (Trq)|X'' - Z''$ are deduced from $Trq \circ q^* = Id$ and $(Trq)|X'' - Z'' \circ q^*|X' - Z' = Id$, hence the long exact sequence of cohomology deduced from the diagram splits into short exact sequences.

Corollary 2.22. *The cohomology $H^i((\widetilde{X}-\widetilde{Z})^*,\mathbb{Z})$, is isomorphic to $H^i((\widetilde{X}'-\widetilde{Z}')^*,\mathbb{Z})$ since $X-Z\simeq X'-Z'$, carry the limit MHS isomorphic to the cokernel of $(i''_X)^*-Trq$ acting as a morphism of limit MHS.*

The cokernel is defined in the additive category of limit MHS. We remark here that the exact sequence is strict not only for the weight W, but also for W^f since it is isomorphic to a similar exact sequence for each fiber at a point $t\in D^*$, where W^f is identified with the weight filtration on the respective cohomology groups over the fiber at t. Hence the sequence remains exact after taking the graded part $Gr^W_*Gr^{W^f}_*$ of each term. The left term carry a limit MHS as the special case of the complementary of the NCD: $Z''\to D$ over D into the smooth proper variety $P''_D\to D$, while the middle term is the complementary of the intersection of the NCD: $Z''\to D$ over D with the NCD: $X''\to D$ over D. Both cases can be treated by the above special cases without the general theory of simplicial varieties. Hence we deduce a the limit MHS at right as a quotient. This shows that the limit structure is uniquely defined by the construction on NCD and dually the logarithmic case for smooth families.

3. Admissible variation of mixed Hodge structure

The degeneration properties of VMHS of geometric origin on a punctured disc are not necessarily satisfied for general VMHS as it has been the case for VHS with the results of Schmid. The notion of admissible VMHS introduced in [29] over a disc, assume all the degeneration properties of the geometric case satisfied by definition. Such definition has been extended in [18] to analytic spaces and is satisfactory for natural operations such as the direct image by a projective morphism of varieties [26]. We mention here the main local properties of admissible VMHS over the complement of a normal crossing divisor (DCN) proved by Kashiwara in [18].

As an application of this concept we describe a natural MHS on the cohomology of an admissible VMHS. In this setting we recall the definition of normal functions and we explain recent results on the algebraicity of the zero set of normal functions to answer a question raised by Griffiths and Green.
The results apply in general for a VMHS with quasi-unipotent local monodromy at the points of degeneration of the NCD , however we assume the local monodromy unipotent, to simplify the exposition and the proofs.

3.1. **Definition and results.** We consider a VMHS $(\mathcal{L},W,F)$ on the complement X^* of a NCD in an analytic manifold X with unipotent local monodromy and we denote by $(\mathcal{L}_X,\nabla)$ Deligne's canonical extension of $\mathcal{L}\otimes\mathcal{O}_{X^*}$ into an analytic vector bundle on X with a flat connection having logarithmic singularities [8]. The filtration by sub-local systems W define a filtration by canonical extensions of $W\otimes\mathcal{O}_{X^*}$, sub-bundles of $\mathcal{L}_X$, denoted $\mathcal{W}_X$. The graded object $Gr^{\mathcal{W}_X}_k\mathcal{L}_X$ is the canonical extension of $Gr^W_k\mathcal{L}$ and we know that the Hodge filtration by sub-bundles extends on $Gr^{\mathcal{W}_X}_k\mathcal{L}_X$ by Scmid's result [27].

Definition 3.1. ([29], 3.13) A graded-polarizable variation of mixed Hodge structure $(\mathcal{L},W,\mathcal{F}_{D^*})$ over the punctured unit disc D^* with local monodromy T, is called *pre-admissible* if
i) The Hodge filtration $\mathcal{F}_{D^*}\subset\mathcal{L}_{D^*}$ extends to a filtration $\mathcal{F}_D$ of $\mathcal{L}_D$ by sub-bundles inducing for each k on $Gr^{\mathcal{W}_D}_k\mathcal{L}_D$, Schmid's extension of the Hodge filtration.

ii) Let $W^0 := \mathcal{W}_D(0)$, $F_0 := \mathcal{F}_D(0)$ denote the filtrations of the fibre $L_0 := \mathcal{L}_X(0)$ at $0 \in D$, T the local monodromy at 0, $N = \log T$, then the following coonditions are satisfied: $NF_0^p \subset F_0^{p-1}$ for all $p \in \mathbb{Z}$ and the weight filtration $M(N, W^0)$ relative to W^0 exists.

Notice that the extension of the filtration $\mathcal{F}_{D^*}$ to $\mathcal{L}_D$ cannot be deduced in general from the various Schmid's extensions to $Gr_k^{\mathcal{W}_D}\mathcal{L}_D$.

We remark that the filtrations $M := M(N, W^0)$ and F_0 at the origin define a MHS:

Lemma 3.2 (Deligne)**.** *The data (L_0, M, F_0) defined at the origin by the pre-admissible VMHS: $(\mathcal{L}, W, \mathcal{F})$ over D^* in D, is a MHS.*

The endomorphism N is compatible with the MHS of type $(-1, -1)$.

The proof due to Deligne is stated in the appendix to [29]. The result follows from the following properties:
i) (L_0, W^0, F_0, N) satisfy $NF_0^p \subset F_0^{p-1}$ and $NW_k^0 \subset W_k^0$,
ii) the relative filtration $M(N, W^0)$ exists,
iii) for each k, $(Gr_k^{W^0} L_0, M, F_0)$ is a MHS.

The admissibility property in the next definition by Kashiwara coincide over D^* with the above definition in the unipotent case (but not the quasi-unipotent case) as proved in [18].

Definition 3.3. ([18],1.9) Let X be a complex analytic space and $U \subset X$ a non singular open subset, complement of a closed analytic subset. A graded polarizable variation of mixed Hodge structure $(\mathcal{L}, W, \mathcal{F}_U)$ on U is called *admissible* if for every analytic morphism $f : D \to X$ on a disc which maps D^* to U, the inverse $(f_{|D^*})^*(\mathcal{L}, W, \mathcal{F}_U)$ is a pre-admissible variation on D^*.

In the case of locally unipotent admissible VMHS, Kashiwara notice that pre-admissible VMHS in the disc are necessarily admissible.

The following criteria in [18] states that admissibility can be tested in codimension one:

Theorem 3.4. *(*[18]*, 4.5.2) Let X be a complex manifold, $U \subset X$ the complement of a NCD and let Z be a closed analytic subset of codimension ≥ 2 in X. An admissible VMHS: $(\mathcal{L}, W, \mathcal{F}_U)$ on U whose restriction to $U - Z$ is admissible in $X - Z$, is necessarily admissible in X.*

In particular the existence of the relative weight filtration at a point $y \in Y_2$ follows from its existence at the nearby points on $Y - Y_2$. Such result is stated and checked locally in terms of nilpotent orbits localized at y.
Next, we cite the following fundamental result for admissible variations of MHS

Theorem 3.5. *Let $\mathcal{L}$ be an admissible graded polarized VMHS: $(\mathcal{L}, W, \mathcal{F})_{X-Y}$ on the complement of a NCD: Y in a complex compact smooth algebraic variety X, $j : X - Y \to X$, Z a sub-NCD of Y, $U := X - Z$, then for all degrees, the cohomology groups $\mathbb{H}^k(U, j_{!*}\mathcal{L})$ of the intermediate extension, carry a canonical mixed Hodge structure.*

This result follows from M. Saito's general theory of mixed Hodge modules [26] but it is obtained here directly via the logarithmic complex. In both cases it relies heavily on the local study of VMHS by Kashiwara in [18] that is highlighted in the

next section. We use also the purity of the intersection cohomology in [19], [5]. The curve case is treated in [29].

3.1.1. *Properties.* 1) We describe below a logarithmic de Rham complex $\Omega^*_X(LogY)\otimes \mathcal{L}_X$ with coefficients in $\mathcal{L}_X$ on which the weight filtration is defined in terms of the local study in [18] while the Hodge filtration is easily defined and compatible with the result in [20].
2) We realize the cohomology of $U = X - Z$ as the cohomology of a cohomological mixed Hodge complex (MHC): $\Omega^*(\mathcal{L}, Z)$ subcomplex of $\Omega^*_X(LogY) \otimes \mathcal{L}_X$ containing the intermediate extension $j_{!*}\mathcal{L}$ of $\mathcal{L}$ as a sub-MHC: $IC(X, \mathcal{L})$ such that the quotient complex define a strucyure of MHC on $i^!_Z j_{!*}\mathcal{L}[1]$.
3) If the weights of $j_{!*}\mathcal{L}$ are $\geq a$, then the weights on $\mathbb{H}^i(U, j_{!*}\mathcal{L})$ are $\geq a + i$.
4) The MHS is defined dually on $\mathbb{H}^i_c(U, j_{!*}\mathcal{L})$ (resp. $\mathbb{H}^i(Z, j_{!*}\mathcal{L})$) of weights $\leq a + i$ if the weights of $j_{!*}\mathcal{L}\mathcal{L}$ are $\leq a$.
5) Let H be a smooth hypersurface intersecting transversally $Y \cup Z$ such that $H \cup Y \cup Z$ is a NCD, then the Gysin isomorphism $i^*_H\Omega^*(\mathcal{L}, Z) \simeq i^!_H\Omega^*(\mathcal{L}, Z)[2]$ is defined as an isomorphism of MHC with a shift in indices.

3.2. **Local study of Infinitesimal Mixed Hodge structures after Kashiwara.** The two global results above are determined by the study of the local properties of VMHS. We state here the local version of the definitions and results in [18], but we skip the proofs, as they are technically complex, although based on invariants in linear algebra, as in the general case of VMHS.
An extensive study of infinitesimal mixed Hodge structures (IMHS) is needed to state and check locally the decomposition property of the graded complex for the weight filtration of the logarithmic complex.

3.2.1. *Infinitesimal Mixed Hodge structure (IMHS).* It is convenient in analysis to consider complex MHS $(L, W, F, \overline{F})$ where we do not need W to be rational but we suppose the three filtrations $W, F, \overline{F}$ opposed [9]. In particular a complex HS of weight k is given as $(L, F, \overline{F})$ satisfying $L \simeq \oplus_{p+q=k} L^{pq}$, where $L^{pq} = F^p \cap \overline{F}^q$. In the case of a MHS with underlying rational structure on W and L, $\overline{F}$ on L is just the conjugate of F with respect to the rational structure.
To define polarization, we recall that the conjugate space $\overline{L}$ of a complex vector space L, is the same group L with a different complex structure, such that the identity map on the group L defines a real linear map $\sigma : L \to \overline{L}$ and the product by scalars satisfy the relation $\forall \lambda \in \mathbb{C}, v \in L,\ \lambda \times_{\overline{L}} \sigma(v) := \sigma(\overline{\lambda} \times_L v)$, then the complex structure on $\overline{L}$ is unique. A morphism $f : V \to V'$ defines a morphism $\overline{f} : \overline{V} \to \overline{V}'$ satisfying $\overline{f}(\sigma(v)) = \sigma(f(v))$.

3.2.2. *Hypothesis.* We consider a complex vector space L of finite dimension, two filtrations $F, \overline{F}$ of L by complex sub-vector spaces, an integer k and a non-degenerate linear map $S : L \otimes \overline{L} \to \mathbb{C}$ satisfying

$$S(x, \sigma(y)) = (-1)^k \overline{S(y, \sigma(x))} \text{ for } x, y \in L \quad \text{and} \quad S(F^p, \sigma(\overline{F}^q)) = 0 \text{ for } p + q > k.$$

Let $(N_1, \ldots, N_l)$ be a set of mutually commuting nilpotent endomorphisms of L s.t.

$$S(N_j x, y) + S(x, N_j y) = 0 \quad \text{and} \quad N_j F^p \subset F^{p-1},\ N_j \overline{F}^p \subset \overline{F}^{p-1}.$$

Recall that by definition, a MHS is of weight w if the HS on Gr^W_k is of weight $w+k$.

Definition 3.6 (Nilpotent Orbit)**.** The above data is called a (polarized) nilpotent orbit of weight w, if the following equivalent conditions are satisfied [18]
1) There exists a real number c s.t. $(L, (e^{i\sum t_j N_j})F, , (e^{-i\sum t_j N_j})\overline{F})$ is a H.S of weight w polarized by S for all $t_j > c$.
2) The weight filtration W of $N = \sum_j t_j N_j$ with $t_j > 0$ for all j, does not depend on the various t_j; (W, F) define a MHS on L of weight w and the bilinear form S_k s.t. $S_k(x, y) = S(x, N^k y)$ polarizes the induced HS of weight $w+k$ on the primitive subspace $P_k = Ker(N^{k+1} : Gr_k^W \to Gr_{-k-2}^W)$.

Henceforth, all nilpotent orbits are polarized.
We consider now a filtered version of the above data $(L; W; F; \overline{F}; N_1, \ldots, N_l)$ with an increasing filtration W s.t. $N_j W_k \subset W_k$ but without any given fixed bilinear product S.

Definition 3.7 (Mixed nilpotent orbit)**.** The above data $(L; W, F; \overline{F}; N_1, \ldots, N_l)$ is called a mixed nilpotent orbit (graded polarized) if for each integer i, $(Gr_i^W L; F_|; \overline{F}_|; (N_1)_|, \ldots, (N_l)_|)$, with the restricted structures, is a nilpotent orbit of weight i for some polarization S_i.

This structure is called pre-infinitesimal mixed Hodge module in ([18], 4.2). A pre-admissible VMHS: $(f^*\mathcal{L}, W, \mathcal{F})_{|D^*}$ on D^* defines such structure at $0 \in D$.

Definition 3.8 (IMHS)**.** ([18], 4.3) A mixed nilpotent orbit $(L; W, F; \overline{F}; N_1, \ldots, N_l)$ is called an infinitesimal mixed Hodge structure (IMHS) if the following conditions are satisfied:
i) For each $J \subset I = \{1, \ldots, l\}$, the monodromy filtration $M(J)$ of $\sum_{j\in J} N_j$ relative to W exists and satisfy $N_j M_i(J) \subset M_{i-2}(J)$ for all $j \in J$ and $i \in \mathbb{Z}$.
ii) The filtrations $M(I), F, \overline{F}$ define a graded polarized MHS. The filtrations W and $M(J)$ are compatible with the MHS as well the morphisms N_i are of type $(-1, -1)$.

IMHS are called IMHM in [18]; Deligne remarked, the fact that if the relative monodromy filtration $M(\sum_{i\in I} N_i, W)$ exists in the case of a mixed nilpotent orbit, then it is necessarily the weight filtration of a MHS.
The following criteria is the infinitesimal statement which corresponds to the result that admissibility may be checked in codim.1.

Theorem 3.9. *(*[18]*, 4.4.1) A mixed nilpotent orbit $(L; W, F; \overline{F}; N_1, \ldots, N_l)$ is an IMHS if the monodromy filtration of N_j relative to W exists for any $j = 1, \ldots, l$.*

This result is not used here and it is directly satisfied in most applications. Its proof is embedded in surprisingly important properties of IMHS valuable for their own sake.

3.2.3. *Properties of IMHS.* We describe now fundamental properties frequently needed in various constructions in mixed Hodge theory with degenerating coefficients.
We start with an important property of a relative weight filtration, used in various proofs.

Theorem 3.10. *(*[18]*, 3.2.9) Let (L, W, N) be a filtered space with a nilpotent endomorphism with a relative monodromy filtration $M(N, W)$, then for each l, there exists a canonical decomposition*

$$Gr_l^M L \simeq \oplus_k Gr_k^W Gr_l^M L$$

In the proof, Kashiwara describes a natural subspace of $Gr_l^M L$ isomorphic to $Gr_k^W Gr_l^M L$ in terms of W and N.

In the case of an IMHS as above, $(Gr_l^M Gr_k^W L, F_|, \overline{F}_|)$ and $(Gr_k^W Gr_l^M L, F_|, \overline{F}_|)$ are endowed with induced H.S of weight l; the isomorphism in Zassenhaus lemma between the two groups is compatible with H.S. in this case, and $(Gr_l^M L, F_|, \overline{F}_|)$ is a direct sum of H.S. of weight l for various k. Deligne's remark that the relative weight filtration $M(\sum_{i\in I} N_i, W)$ is the weight filtration of a MHS, may be deduced from this result. Another application is the proof of

Proposition 3.11. *([18],5.2.5) Let $(L; W, F; \overline{F}; N_1, \dots, N_l)$ be an IMHS and for $J \subset \{1, \dots, l\}$ set $M(J)$ the relative weight of $N \in C(J) = \{\Sigma_{j\in J} t_j N_j, t_j > 0\}$. Then, for $J_1, J_2 \subset \{1, \dots, l\}$, $N_1 \in C(J_1)$, $M(J_1 \cup J_2)$ is the weight filtration of N_1 relative to $M(J_2)$.*

The geometric interpretation of this result in the complement of a NCD $Y_1 \cup Y_2$ is that the degeneration to Y_1 followed by the degeneration along Y_1 to a point $x \in Y_1 \cap Y_2$ yields the same limit MHS as the degeneration to Y_2 then along Y_2 as well the direct degeneration along a curve in the complement of $Y_1 \cup Y_2$.

3.2.4. *Abelian category of IMHS.* The morphisms of two mixed nilpotent orbit (resp. IMHS) $(L; W, F; \overline{F}; N_1, \dots, N_l)$ and $(L'; W', F'; \overline{F}'; N'_1, \dots, N'_l)$ are defined to be compatible with both, the filtrations and the nilpotent endomorphisms.

Proposition 3.12. *i) The category of mixed nilpotent orbits is abelian.*
ii) The category of IMHS is abelian.
For all $J \subset I$, $M(J)$ and W are filtrations by sub-MHS of the MHS defined by $M(I)$ and F and the various functors defined by the filtrations $W_j, Gr_j^W, M_j(J), Gr_j^{M(J)}$, are exact functors.

We define a corresponding mixed nilpotent orbit structure Hom on the vector space $Hom(L, L')$ with the following classically defined filtrations: $Hom(W, W')$, $Hom(F, F')$ and $Hom(\overline{F}, \overline{F}')$, and the natural endomorphisms denoted: $Hom(N_1, N'_1), \dots, Hom(N_l, N'_l)$ on $Hom(L, L')$. Similarly a structure called the tensor product is defined.

Remark 3.13. Among the specific properties of the filtrations of IMHS, we mention the distributivity used in various proofs. In general three subspaces A, B, C of a vector space do not satisfy the following distributivity property: $(A + B) \cap C = (A \cap C) + (B \cap C)$. A family of filtrations $F_1, \dots, F_n$ of a vector space L is said to be distributive if for all $p, q, r \in \mathbb{Z}$, have

$$(F_i^p + F_j^q) \cap F_k^r = (F_i^p \cap F_k^r) + (F_j^q \cap F_k^r)$$

In the case of an IMHS $(W, F, N_i, i \in I)$, for $J_1 \subset \cdots \subset J_k \subset I$, the family of filtrations $\{W, F, M(J_1), \dots, M(J_k)\}$ is distributive ([18], prop. 5.2.4).

3.3. **Deligne-Hodge theory on the cohomology of a smooth variety.** We describe now a weight filtration on the logarithmic complex with coefficients in the canonical extension of an admissible VMHS on the complement of a NCD, based on the local study in [18].

Hypothesis. Let X be a smooth and compact complex algebraic variety, $Y = \cup_{i\in I} Y_i$ a NCD in X with smooth irreducible components Y_i, and $(\mathcal{L}, W, F)$ a graded polarized VMHS on $U := X - Y$ *admissible* on X with unipotent local monodromy along Y.

Notations. We denote by $(\mathcal{L}_X, \nabla)$ Deligne's canonical extension of $\mathcal{L} \otimes \mathcal{O}_U$ into an analytic vector bundle on X with a flat connection having logarithmic singularities. The filtration by sub-local systems W of $\mathcal{L}$ define a filtration by canonical extensions denoted $\mathcal{W}_X \subset \mathcal{L}_X$, while by definition of admissibility the filtration $\mathcal{F}_U$ extends by sub-bundles $\mathcal{F}_X \subset \mathcal{L}_X$.

The aim of this section is to deduce from the local study in [18] the following global result

Theorem 3.14. *There exists a weight filtration W and a filtration F on the logarithmic complex with coefficients in $\mathcal{L}_X$ such that the bi-filtered complex*

$$(\Omega^*_X(LogY) \otimes \mathcal{L}_X, W, F)$$

underlies a structure of mixed Hodge complex and induces a canonical MHS on the cohomology groups $H^i(U, \mathcal{L})$ of $U := X - Y$.

The filtration F is classically deduced on the logarithmic complex from the above bundles $\mathcal{F}_X$:

$$F^p = 0 \to F^p\mathcal{L}_X \cdots \to \Omega^i_X(LogY) \otimes F^{p-i}\mathcal{L}_X \to \cdots\cdots$$

Before giving a proof, we need to describe the weight filtration W.

3.3.1. *Local definition of the weight W on the logarithmic complex.* For $J \subset I$, let $Y_J := \cap_{i\in J} Y_i$, $Y^*_J := Y_J - \cup_{i\notin J}(Y_i \cap Y_J)$ ($Y^*_\emptyset := X - Y$) and let the various $j : Y^*_J \to X$ denote uniformly the embeddings. Let $X(y) \simeq D^{m+l}$ be a neighborhood of a point y in Y, and $U(y) = X(y) \cap U \simeq (D^*)^m \times D^l$ where D is a complex disc, denoted with a star when the origin is deleted. The fundamental group $\pi_1(U(y))$ is a free abelian group generated by n elements representing classes of closed paths around the origin, one for each D^* in the various axis with one dimensional coordinate z_i (the hypersurface Y_i is defined locally by the equation $z_i = 0$). Then the local system $\mathcal{L}$ corresponds to a representation of $\pi_1(U(y))$ in a vector space L defined by the action of commuting unipotent automorphisms T_i for $i \in [1, m]$ indexed by the local components Y_i of Y and called monodromy action around Y_i.

Classically L is viewed as the fibre of $\mathcal{L}$ at the base point of the fundamental group $\pi_1(X^*)$, however to represent the fibre of Deligne's extended bundle at y, we view L *as the vector space of multivalued sections of* $\mathcal{L}$ (that is the sections of the inverse of $\mathcal{L}$ on a universal covering of $U(y)$).

The logarithm of the unipotent monodromy $N_i := -\frac{1}{2i\pi} LogT_i$ is a nilpotent endomorphism. Recall the embedding $L \to \mathcal{L}_X(y) : v \to \widetilde{v}$ defined by the formulas

$$\widetilde{v}(z) = (exp(\Sigma_{j\in J}(logz_j)N_j)).v, \quad \nabla\widetilde{v} = \Sigma_{j\in J}\widetilde{N_j.v} \otimes \frac{dz_j}{z_j} \tag{3.1}$$

where a basis of L is sent on a basis of $\mathcal{L}_{X,y}$ and the action of N_i on L is determined by the residue of the connection.

Local description of $Rj_*\mathcal{L}$.

The fibre at y of the complex $\Omega^*_X(LogY) \otimes \mathcal{L}_X$ is quasi-isomorphic to a Koszul

complex as follows. We associate to $(L, N_i), i \in [1, m]$ a strict simplicial vector space such that for all sequences $(i.) = (i_1 < \cdots < i_p)$

$$L(i.) = L, \qquad N_{i_j} : L(i. - i_j) \to L(i.)$$

Another notation is $s(L(J), N.)_{J\subset[1,m]}$ where J is identified with the strictly increasing sequence of its elements and where $L(J) = L$.

Definition 3.15. The simple complex defined by the simplicial vector space above is the Koszul complex (or the exterior algebra) defined by (L, N_i) and denoted by $\Omega(L, N.)$.

We remark that $\Omega(L, N.)$ is quasi-isomorphic to the Koszul complex $\Omega(L, Id-T.)$ defined by $(L, Id - T_i), i \in [1, m]$.

Lemma 3.16. *For $M \subset I$ and $y \in Y^*_M$, the above correspondence $v \mapsto \tilde{v}$, from L to $\mathcal{L}_{X,y}$, extended to $L(i_1, \ldots, i_j) \to (\Omega^*_X(LogY) \otimes \mathcal{L}_X)_y$ by $v \mapsto \tilde{v}\frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_j}}{z_{i_j}}$, induces quasi-isomorphisms*

$$(\Omega^*_X(LogY) \otimes \mathcal{L}_X)_y \cong \Omega(L, N_j, j \in M) \cong s(L(J), N.)_{J\subset[1,m]} \tag{3.2}$$

hence for $j : U \to X$ we have $(Rj_*\mathcal{L})_y \cong \Omega(L, N_j, j \in M)$.

This description of $(Rj_*\mathcal{L})_y$ is the model for the description of the next various perverse sheaves.
The intermediate extension $j_{!*}\mathcal{L}$.
Let $N_J = \prod_{j\in J} N_j$ denotes a composition of endomorphisms of L, we consider the strict simplicial sub-complex of the de Rham logarithmic complex defined by $N_J L := Im N_J$ in $L(J) = L$.

Definition 3.17. The simple complex defined by the above simplicial sub-vector space is the intersection complex of L denoted by

$$IC(L) := s(N_J L, N.)_{J\subset M}, \ N_J L := N_{i_1} N_{i_2} \ldots N_{i_j} L, J = \{i_1, \ldots, i_j\} \tag{3.3}$$

Locally the germ of the intermediate extension $j_{!*}\mathcal{L}$ of $\mathcal{L}$ at a point $y \in Y^*_M$ is quasi-isomorphic to the above complex [5], [19]

$$j_{!*}(\mathcal{L})_y \simeq IC(L) \simeq s(N_J L, N.)_{J\subset M} \tag{3.4}$$

3.3.2. *Definition of $N * W$.* Let (L, W, N) denotes an increasing filtration W on a vector space L with a nilpotent endomorphism N compatible with W s.t. the relative monodromy filtration $M(N, W)$ exists, then a new filtration $N * W$ of L is defined ([18], 3.4) by the formula

$$(N * W)_k := NW_{k+1} + M_k(N, W) \cap W_k = NW_{k+1} + M_k(N, W) \cap W_{k+1} \tag{3.5}$$

where the last equality follows from ([18], Prop 3.4.1).
For each index k, the endomorphism $N : L \to L$ induces a morphism $N : W_k \to (N * W)_{k-1}$ and the identity I on L induces a morphism $I : (N * W)_{k-1} \to W_k$.
We remark two important properties of $N * W$ ([18], lemma 3.4.2):
i) The relative weight filtration exists and: $M(N, N * W) = M(N, W)$.
ii) We have the decomposition property

$$\begin{aligned} Gr_k^{N*W} &\simeq \mathrm{Im}\,(N : Gr^W_{k+1}L \to Gr_k^{N*W})L) \oplus \mathrm{Ker}\,(I : Gr_k^{N*W}L \to Gr^W_{k+1}L) \\ Im\,(N : Gr^W_{k+1}L \to Gr_k^{N*W}L) &\simeq Im\,(N : Gr^W_{k+1}L \to Gr^W_{k+1}L). \end{aligned} \tag{3.6}$$

referred to, as W and $N * W$ form a graded distinguished pair.

3.3.3. *The filtration W^J on an IMHS L.* The fiber $\mathcal{L}(y)$ at a point $y \in Y_M^*$ defines an IMHS: $(L, W, F, N_i, i \in M)$; in particular the relative weight filtration $M(J, W)$ of $N \in C(J) = \{\Sigma_{j\in J} t_j N_j, t_j > 0\}$ exists for all $J \subset M$. A basic lemma [18], cor. 5.5.4)asserts that:

Lemma 3.18. *$(L, N_1 * W, F, N_i, i \in M)$ and $(M(N_j, W), F, N_i, i \in M - \{j\})$ are IMHS.*

In particular, an increasing filtration W^J of L may be defined recursively by the star operation

$$W^J := N_{i_1} * (\dots (N_{i_j} * W) \dots) \text{ for } J = \{i_1, \dots, i_j\} \tag{3.7}$$

(denoted $\Psi_J * W$ in [18], (5.8.2), see also [1].) It describes the fibre of the proposed weight filtration on $\mathcal{L}(y)$ for $y \in Y_J^*, J \subset M$. The filtration W^J does not depend on the order of composition of the respective transformations $N_{i_k}*$ since in the case of an IMHS : $N_{i_p} * (N_{i_q} * W) = N_{i_q} * (N_{i_p} * W)$ for all $i_p, i_q \in J$ according to ([18], Prop 5.5.5). The star operation has the following properties for all $J_1, J_2 \subset M$ and $J \subset K \subset M$:

$$M(J_1, W^{J_2}) = M(J_1, W)^{J_2}, \quad M(K, W^J) = M(K, W).$$

Definition 3.19. The filtration W on the de Rham complex associated to an IMHS $(L, W, F, N_i, i \in M)$ is defined as the Koszul complex

$$W_k(\Omega(L, N.)) := s(W^J_{k-|J|}, N.)_{J\subset M}$$

where for each index $i \in M - J$, the endomorphism $N_i : L \to L$ induces a morphism $N_i : W^J_k \to (N_i * W^J)_{k-1}$ (careful to the same notation for W on L and on $\Omega(L, N.)$). It is important to add the canonical inclusion $I : (N_i * W^J)_{k-1} \to W^J_k$ to the above data. For example, for $|M| = 2$, the data with alternating differentials N_i, is written as follows:

$$\begin{array}{ccccccc}
L & \underset{\xleftarrow{I}}{\xrightarrow{N_1}} & L & W_k & \underset{\xleftarrow{I}}{\xrightarrow{N_1}} & (N_1 * W)_{k-1} \\
I \uparrow\downarrow N_2 & & I \uparrow\downarrow N_2 & I \uparrow\downarrow N_2 & & I \uparrow\downarrow N_2 \\
L & \underset{\xleftarrow{I}}{\xrightarrow{N_1}} & L & (N_2 * W)_{k-1} & \underset{\xleftarrow{I}}{\xrightarrow{N_1}} & (N_1 * N_2 * W)_{k-2}
\end{array}$$

3.3.4. *Local decomposition of W.* The proof of the decomposition involves a general description of perverse sheaves in the normally crossing divisor case ([18], section 2), in terms of the following de Rham data $DR(L)$ deduced from the IMHS:

$$DR(L) := \{L_J, W^J, F^J, N_{K,J} = L_J \to L_K, I_{J,K} = L_K \to L_J\}_{K\subset J\subset M}$$

where for all $J \subset M$, $L_J := L$, $F_J := F$, W^J is the filtration defined above, $N_{K,J} := \prod_{i\in J-K} N_i$ and $I_{J,K} := Id : L \to L$. A set of properties stated by Kashiwara ([18], 5.6) are satisfied. In particular :
For each $J \subset M$, the data: $(L_J, W^J, F_J, \{N_j, j \in M\})$, is an IMHS
which is essentially proved in ([18](5.8.5) and (5.8.6)).
The following result is satisfied by the data and it is a basic step to check the structure of MHC on the logarithmic de Rham complex

Lemma 3.20. *Let $(L, F, N_i, i \in M)$ be an IMHS, and for $K \subset J \subset M$, let $A := J - K$, $N_A = W_i^K L \to W_{i-|A|}^J L$ denotes the composition of the linear endomorphisms $N_j, j \in A$ and $I_A : W_{i-|A|}^J L \hookrightarrow W_i^K L$ the inclusion, then we have*

$$Gr_{i-|A|}^{W^J} L \simeq Im(Gr_i N_A : Gr_i^{W^K} L \to Gr_{i-|A|}^{W^J} L) \oplus Ker(Gr_i I_A : Gr_{i-|A|}^{W^J} L \to Gr_i^{W^K} L).$$

The proof by induction on the length $|A|$ of A ([18] 5.6.7 and lemma 5.6.5), is based at each step for $a \in J - K$, on the decomposition:

$$Gr_{i-1}^{W^{K\cup a}} L \simeq \mathrm{Im}\,(Gr_i N_a : Gr_i^{W^K} L \to Gr_{i-1}^{W^{K\cup a}}) L \oplus \mathrm{Ker}\,(Gr_i I_a : Gr_{i-1}^{W^{K\cup a}} L \to Gr_i^{W^K} L).$$

Corollary 3.21 ([18], 5.6.10)**.** *i) Set*

$$P_k^J(L) := \cap_{K\subset J, K\neq J} Ker\,(I_{J,K} : Gr_k^{W^J} \to Gr_{k+|J-K|}^{W^K}) \subset Gr_k^{W^J}$$

then $P_k^J(L)$ has pure weight k with respect to the weight filtration $M(\sum_{j\in J} N_j, L)$.
ii)$P_k^J(L)$ with the action of N_j for $j \in J$, is an infinitesimal VHS.
iii) We have: $Gr_k^{W^J} L \simeq \oplus_{K\subset J} N_{J-K} P_{k-|J-K|}^K(L)$.

This corollary is proved in [18]; the statement (iii) is proved in ([18], 2.3.1).

Lemma 3.22 ([18] prop. 2.3.1)**.** *The graded vector space of the filtration $W_k, k \in \mathbb{Z}$ on $\Omega(L, N.)$ satisfy the decomposition property into a direct sum of Intersection complexes*

$$Gr_k^W(\Omega(L, N.)) \simeq \oplus IC(P_{k-|J|}^J(L)[-|J|))_{J\subset M} \tag{3.8}$$

The lemma follows from the corollary. It is the local statement of the structure of MHC on the logarithmic de Rham complex.

3.3.5. *Global definition and properties of the weight W.* The local study ended with the local decomposition into Intersection complexes. We develop now the corresponding global results. Taking the residue of the connection, we define nilpotent analytic linear endomorphisms of $\mathcal{L}_{Y_i}$ compatible with the filtration $\mathcal{W}_{Y_i}$ by sub-analytic bundles:

$$\mathcal{N}_i := Res_i(\nabla) : \mathcal{L}_{Y_i} \to \mathcal{L}_{Y_i}, \quad \mathcal{N}_J = \mathcal{N}_{i_1} \cdots \mathcal{N}_{i_j} : \mathcal{L}_{Y_J} \to \mathcal{L}_{Y_J}$$

The pure Intersection complex of a polarized VHS.
We introduce the global Intersection complex $IC(X, \mathcal{L})$ as the sub-complex of $\Omega_X^*(LogY) \otimes \mathcal{L}_X$ whose terms in each degree are $\mathcal{O}_X-$modules with singularities along the strata Y_J of Y, defined in terms of the analytic nilpotent endomorphisms $\mathcal{N}_i$ and $\mathcal{N}_J$ for subsets $J \subset I$ of the set I of indices of the components of Y:

Definition 3.23. The Intersection complex $IC(X, \mathcal{L})$ is the sub-analytic complex of $\Omega_X^*(LogY) \otimes \mathcal{L}_X$ whose fibre at a point $y \in Y_M^*$ is defined in terms of a set of coordinates $z_i, i \in M$, defining equations of Y_M, as an $\Omega_{X,y}^*$ sub-module, generated by the sections $\tilde{v} \wedge_{j\in J} \frac{dz_j}{z_j}$ for $\tilde{v} \in N_J\mathcal{L}(y)$ and $J \subset M$ ($N_\emptyset = Id$).

This definition is independent of the choice of coordinates; moreover the restriction of the section is still defined in the sub-complex near y, since $N_J L \subset N_{J-i} L$ for all $i \in J$.
For example, for $M = \{1, 2\}$ at $y \in Y_M^*$, the sections in $(\Omega_X^2(LogY) \otimes \mathcal{L}_X)_y \cap IC(X, \mathcal{L})_y$ are generated by $\tilde{v}\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$ for $\tilde{v} \in N_J\mathcal{L}(y)$, $\tilde{v}\frac{dz_1}{z_1} \wedge dz_2$ for $\tilde{v} \in N_1\mathcal{L}(y)$, $\tilde{v}dz_1 \wedge \frac{dz_2}{z_2}$ for $\tilde{v} \in N_2\mathcal{L}(y)$ and $\tilde{v}dz_1 \wedge dz_2$ for $\tilde{v} \in \mathcal{L}(y)$. We deduce from the local result

Lemma 3.24. *The intersection complex $IC(X,\mathcal{L})[n]$ shifted by $n :=$dim X, where $\mathcal{L}$ is locally unipotent polarized on $X-Y$, is quasi-isomorphic to the unique auto-dual complex on X, intermediate extension $j_{!*}\mathcal{L}[n]$ satisfying $RHom(j_{!*}\mathcal{L}[n],\mathbb{Q}_X[2n]) \simeq j_{!*}\mathcal{L}[n]$.*

The shift by n is needed for the compatibility with the definitions in [2]. The next theorem is proved in [19] and [5]

Theorem 3.25. *Let $(\mathcal{L},F)$ be a polarized VHS of weight a, then the sub-complex $(IC(X,\mathcal{L}),F)$ of the logarithmic complex with induced filtration F is a Hodge complex which defines a pure HS of weight $a+i$ on the Intersection cohomology $IH^i(X,\mathcal{L})$.*

The proof is in terms of L^2-cohomology or square integrable forms with coefficients in Deligne's extension $\mathcal{L}_X$ and an adequate metrics. The filtration F on $IC(X,\mathcal{L})$ defined in an algebro-geometric way yields the same Hodge filtration as in L^2-cohomology as proved elegantly in [20] using the auto-duality of the Intersection cohomology.
The global filtration W^J. The relative monodromy weight filtrations $\mathcal{M}(J,\mathcal{W}_{Y_J}) := \mathcal{M}(\sum_{i\in J}\mathcal{N}_i,\mathcal{W}_{Y_J})$ of $\sum_{i\in J}\mathcal{N}_i$ with respect to the restriction $\mathcal{W}_{Y_J}$ of $\mathcal{W}_X$ on $\mathcal{L}_X$ to Y_J, exist for all $J\subset I$, so that we can define the global filtrations

$$(\mathcal{N}_i*\mathcal{W}_{Y_i})_k := \mathcal{N}_i\mathcal{W}_{Y_i,k+1}+\mathcal{M}_k(\mathcal{N}_i,\mathcal{W}_{Y_i})\cap\mathcal{W}_k = \mathcal{N}_i\mathcal{W}_{Y_i,k+1}+\mathcal{M}_k(\mathcal{N}_i,\mathcal{W}_{Y_i})\cap\mathcal{W}_{k+1}$$

and for all $J\in I$ an increasing filtration $\mathcal{W}^J$ of $\mathcal{L}_{Y_J}$ recursively by the star operation

$$\mathcal{W}^J := \mathcal{N}_{i_1}*(\ldots(\mathcal{N}_{i_j}*\mathcal{W})\ldots) \text{ for } J=\{i_1,\ldots,i_j\}$$

Definition 3.26. The filtration W on the de Rham complex with coefficients in the canonical extension $\mathcal{L}_X$ defined by $\mathcal{L}$ is constructed by a decreasing induction on the dimension of the strata Y_J^* as follows:
i) On $U := X-Y$, the sub-complex $(W_r)_{|U}$ coincides with $\Omega_U^*\otimes(\mathcal{W}_r)_{|U}\subset\Omega_U^*\otimes\mathcal{L}_U$.
ii) We suppose W_r defined on the complement of the closure of the union of strata $\cup_{|M|=m}Y_M$, then for each point $y\in Y_M^*$ we define W_r locally in a neighborhood of $y\in Y_M^*$, on $(\Omega_X^*(LogY)\otimes\mathcal{L}_X)_y$, in terms of the IMHS $(L,W,F,N.)$ at y and a set of coordinates z_i for $i\in M$, defining a set of local equations of Y_M at y:
W_r is generated as an $\Omega_{X,y}^*-$ sub-module by the germs of the sections $\wedge_{j\in J}\frac{dz_j}{z_j}\otimes\tilde{v}$ for $v\in W^J_{r-|J|}L$ where $\tilde{v}$ is the corresponding germ of $(\mathcal{L}_X)_y$.

The definition of W above is independent of the choice of coordinates on a neighborhood $U(y)$, since if we choose a different coordinate $z_i'=fz_i$ instead of z_i with f invertible holomorphic at y, we check first that the submodule $W^J_{r-|J|}(\mathcal{L}_X)_y$ of $\mathcal{L}_{X,y}$ defined by the image of $W^J_{r-|J|}L$ is independent of the coordinates as in the construction of the canonical extension. Then we check that for a fixed $\alpha\in W^J_{r-|J|}\mathcal{L}_{X,y}$, since the difference $\frac{dz_i'}{z_i'}-\frac{dz_i}{z_i}=\frac{df}{f}$ is holomorphic at y, the difference of the sections $\wedge_{j\in J}\frac{dz_j'}{z_j'}\otimes\alpha-\wedge_{j\in J}\frac{dz_j}{z_j}\otimes\alpha$ is still a section of the $\Omega_{X,y}^*-$sub-module generated by the germs of the sections $\wedge_{j\in(J-i)}\frac{dz_j}{z_j}\otimes W^J_{r-|J|}(\mathcal{L}_X)_y$.
Finally, we remark that the sections defined by induction at y restrict to sections already defined by induction on $(U(y)-Y_M^*\cap U(y))$.
The bundles $\mathcal{P}_k^J(\mathcal{L}_{Y_J})$. Given a subset $J\subset I$, the filtration $\mathcal{W}^J$ induces a filtration by sub-analytic bundles of $\mathcal{L}_{Y_J}$, then we introduce the following analytic bundles

$$\mathcal{P}_k^J(\mathcal{L}_{Y_J}) := \cap_{K\subset J,K\neq J}\mathrm{Ker}\,(I_{J,K}: Gr_k^{\mathcal{W}^J}\mathcal{L}_{Y_J}\to Gr_{k+|J-K|}^{\mathcal{W}^K})\mathcal{L}_{Y_J}\subset Gr_k^{\mathcal{W}^J}\mathcal{L}_{Y_J}$$

where $I_{J,K}$ is induced by the natural inclusion $\mathcal{W}_k^J\mathcal{L}_{Y_J} \subset \mathcal{W}_{k+|J-K|}^K\mathcal{L}_{Y_J}$. In particular $\mathcal{P}_k^{\emptyset}(\mathcal{L}_X) = Gr_k^{\mathcal{W}}\mathcal{L}_X$ and $\mathcal{P}_k^J(\mathcal{L}_{Y_J}) = 0$ if $Y_J^* = \emptyset$.

Proposition 3.27. *i) The weight $W[n]$ shifted by $n :=$dim.X is a filtration by perverse sheaves defined over $\mathbb{Q}$, sub-complexes of $Rj_*\mathcal{L}[n]$.*
ii) The bundles $\mathcal{P}_k^J(\mathcal{L}_{Y_J})$ are Deligne's extensions of local systems $\mathcal{P}_k^J(\mathcal{L})$ on Y_J^*.
iii) The graded perverse sheaves for the weight filtration, satisfy the decomposition property into intermediate extensions for all k

$$Gr_k^W(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[n] \simeq \oplus_{J\subset I}(i_{Y_J})_* j_{!*}\mathcal{P}_{k-|J|}^J(\mathcal{L})[n-|J|].$$

where j denotes uniformly the inclusion of Y_J^* into Y_J for each $J \subset I$, $\mathcal{P}_p^J(\mathcal{L})$ on Y_J^* is a polarized VHS pure with respect to the weight induced by $\mathcal{M}(\sum_{j\in J}\mathcal{N}_j, \mathcal{L}_{Y_J^*})$.

The proof is essentially based on the local study above which makes sense over $\mathbb{Q}$ as $\mathcal{L}$ and W are defined over $\mathbb{Q}$. In particular, we deduce that the various graded complexes $Gr_k^W(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[n]$ are Intersections complexes over $\mathbb{C}$ from which we deduce that the extended filtration W_k on the de Rham complex satisfy the condition of support of perverse sheaves with respect to the stratification defined by Y. Similarly, the proof apply to the Verdier dual of W_k as the complexes Gr_k^W are auto-dual.
We need to prove that the local rational structure of the complexes W_k glue into a global rational structure, as perverse sheaves may be glued as the usual sheaves, although they are not concentrated in a unique degree. Since the total complex $Rj_*\mathcal{L}$ is defined over $\mathbb{Q}$, the gluing isomorphisms induced on the various extended W_k are also defined over $\mathbb{Q}$. Another proof of the existence of the rational structure is based on Verdier's specialization [14]. The next result is compatible with [11], cor 3.3.5).

Corollary 3.28. *The de Rham logarithmic mixed Hodge complex of an admissible VMHS of weight $\omega \geq a$ induces on the cohomolgy $\mathbb{H}^i(X-Y,\mathcal{L})$ a MHS of weight $\omega \geq a+i$.*

Indeed, the weight W_k on the logarithmic complex vanishes for $k \leq a$.

Corollary 3.29. *The Intersection complex $(IC(X,\mathcal{L})[n], W, F)$ of an admissible VMHS, with induced filtration as an embedded sub-complex of the de Rham logarithmic mixed Hodge complex, is a mixed Hodge complex satisfying for all k:*

$$Gr_k^W IC(X,\mathcal{L}) = IC(X, Gr_k^W\mathcal{L}).$$

The existence of relative filtrations is important since in general the intersection complex of an extension of two local systems, is not the extension of their intersection complex. We need to check, for each $J \subset I$ of length j, the following property of the induced filtration $\mathcal{W}^J \cap \mathcal{N}_J\mathcal{L}_{Y^J}$ on $\mathcal{N}_J\mathcal{L}_{Y^J}$:

$$Gr_{k-j}^{\mathcal{W}^J}(\mathcal{N}_J\mathcal{L}_{Y^J}) \simeq \mathrm{Im}\,(\mathcal{N}_J : Gr_k^{\mathcal{W}}\mathcal{L}_{Y^J} \to Gr_k^{\mathcal{W}}\mathcal{L}_{Y^J}).$$

The problem is local. We prove the following statement by induction on the length j of J: For each J of length $j > 0$, we have a split exact sequence:

$$0 \to Gr_{k-j}^{W^J}(N_J L) \to Gr_{k-j}^{W^J} L \to Gr_{k-j}^{W^J}(L/N_J L) \to 0$$

and an isomorphism: $Gr_{k-j}^{W^J}(N_J L) \simeq (\mathrm{Im}\, N_J : Gr_k^W L \to Gr_k^W L)$.
To prove the step of the induction, we need to deduce for all N_i with $i \notin J$ a split

exact sequence for $J \cup i$:

$$0 \to Gr^{W^{J\cup i}}_{k-j-1}(N_i N_J L) \to Gr^{W^{J\cup i}}_{k-j-1} L \to Gr^{W^{J\cup i}}_{k-j-1}(L/N_i N_J L) \to 0$$

and moreover, isomorphisms:

$$Gr^{W^{J\cup i}}_{k-j-1}(N_i N_J L) \simeq (\mathrm{Im}\, N_i : Gr^{W^J}_{k-j}(N_J L) \to Gr^{W^J}_{k-j}(N_J L)) \simeq \\ (\mathrm{Im}\, N_i N_J : Gr^W_k L \to Gr^W_k L).$$

To this end we apply the following lemma:

Lemma 3.30 (Graded split sequence)**.** *Let (L, W) be a filtered vector space and $M := M(N, W)$, then the filtrations induced by $N * W$ on the terms of the exact sequence : $0 \to NL \to L \to L/NL \to 0$ satisfy the following properties:*

(3.9) $Gr^{N*W}_k NL \simeq Im\,(N : Gr^W_{k+1} L \to Gr^W_{k+1} L)$, $Gr^{N*W}_k (L/NL) \simeq Gr^M_k (L/NL)$

Moreover, the associated graded exact sequence

$$0 \to Gr^{N*W}_k (NL) \to Gr^{N*W}_k L \to Gr^{N*W}_k (L/NL) \to 0 \tag{3.10}$$

is split with the splitting defined by the isomorphism

$$Ker\,(I : Gr^{N*W}_k L \to Gr^W_{k+1} L) \simeq Gr^M_k (L/NL).$$

The assertion is deduced from the graded distinguished pair decomposition of $Gr^{N*W}_* L$, and the following isomorphisms proved in ([18], cor 3.4.3):
$Gr^W_{k+1} Gr^{N*W}_k L \simeq (\mathrm{Im}\, N : Gr^W_{k+1} L \to Gr^W_{k+1} L)$, and for $a \leq k$: $Gr^W_a Gr^{N*W}_k L \simeq Gr^W_a (\mathrm{Ker}\, I : Gr^{N*W}_k L \to Gr^W_{k+1} L) \simeq (\mathrm{Coker}\, N : Gr^W_a Gr^M_{k+2} L \to Gr^W_a Gr^M_k L)$.
from which we deduce the isomorphism:

$$\mathrm{Ker}\, I : Gr^{N*W}_k L \to Gr^W_{k+1} L \simeq \mathrm{Coker} N : W_k Gr^M_{k+2} L \to W_k Gr^M_k L \simeq W_k Gr^M_k L/NL$$

The quotient filtration $Q_k := ((N * W)_k + NL)/NL$ is isomorphic to $M_k(L/NL)$, since $Q_k/Q_{k-1} \simeq Gr^M_k (L/NL)$ is isomorphic to $W_k Gr^M_k (L/NL) \simeq Gr^M_k (L/NL)$, as the morphism $Gr^W_{a+2} Gr^M_k L \xrightarrow{N} Gr^W_a Gr^M_k L$ is surjective for $a > k$.

3.3.6. *MHS on cohomology groups of the Intersection complex.* Let I_1 be a finite subset of I and let $Z := \cup_{i \in I_1} Y_i$ be a sub-NCD of Y. We describe next a MHS on the hypercohomology $\mathbb{H}^*(X - Z, j_{!*}\mathcal{L})$.
Let $j' : (X - Y) \to (X - Z)$, $j'' : (X - Y_1) \to X$ s.t. $j = j'' \circ j'$. The fibre at a point $y \in Z$ of the logarithmic de Rham complex is isomorphic to the Koszul complex $\Omega(L, N_i, i \in M)$ for some subset M of I. To describe the fibre of the complex $Rj''_*(j'_{!*}\mathcal{L})_y$ as a sub-complex, we consider $M_1 := M \cap I_1$ and $M_2 := M - M_1$, and for $J \subset M$: $J_1 = J \cap M_1$ and $J_2 = J \cap M_2$.

Definition 3.31. With the above notations, we define a sub-analytic complex $\Omega^*(\mathcal{L}, Z)$ of the logarithmic de Rham complex $\Omega^*_X(LogY) \otimes \mathcal{L}_X$ locally at a point $y \in Y^*_M$ in terms of a set of coordinates $z_i, i \in M$, equations of Y_M: The fiber $\Omega^*(\mathcal{L}, Z)$ is generated as an $\Omega^*_{X,y}$ sub-module, by the sections $\tilde{v} \wedge_{j \in J} \frac{dz_j}{z_j}$ for each $J \subset M$ and $v \in N_{J_2} L$.

Lemma 3.32. *We have:* $(Rj''_*(j'_{!*}\mathcal{L})_y \simeq (\Omega^*(\mathcal{L}, Z)_y$

The intersection of a neighborhood of y with $X - Z$ is homeomorphic to $U = U_1^* \times U_2 := (D^*)^{n_1} \times D^{n_2}$. At a point $z = (z_1, z_2) \in U$, the fiber of the Intersection complex is isomorphic with a Koszul complex: $(j'_{!*}\mathcal{L})_z \simeq IC(U_2, L) := s(N_{J_2}L, N_i, i \in M_2)_{J_2 \subset M_2}$ on which $N_i, i \in M_1$, acts. By comparison with the logarithmic de Rham complex along Z, we consider the Koszul double complex: $s(IC(U_2, L), N_i)_{i \in M_1}$ which is quasi-isomorphic to the fibre $(Rj''_*(j'_{!*}\mathcal{L}))_y$, since the terms of its classical spectral sequence are $E_2^{p,q} \simeq R^p j''_* H^q((j'_{!*}\mathcal{L})_y)$.

Example. In the 3−dimensional space, let Y_3 be defined by $z_3 = 0$ and D a small disc at the origin of $\mathbb{C}$, then $(Rj''_*(j'_{!*}\mathcal{L}))_y = \mathbb{R}\Gamma(D^3 - (D^3 \cap Y_3), \mathcal{L})$ is defined by the diagram, with differentials defined by N_i with a + or - sign:

$$\begin{array}{ccccc} L & \xrightarrow{N_1,N_2} & L \oplus L & \xrightarrow{N_1,N_2} & L \\ \downarrow N_3 & & \downarrow N_3 & & \downarrow N_3 \\ N_3L & \xrightarrow{N_1,N_2} & N_3L \oplus N_3L & \xrightarrow{N_1,N_2} & N_3L \end{array}$$

Proposition 3.33. *i) The filtrations W and F of the logarithmic de Rham complex induce on $\Omega^*(\mathcal{L}, Z)$ a structure of MHC defining a MHS on the hypercohomology $\mathbb{H}^*(X - Z, j_{!*}\mathcal{L})$ such that the graded perverse sheaves for the weight filtration, satisfy the decomposition property into intermediate extensions for all k*

$$Gr_k^W \Omega^*(\mathcal{L}, Z)[n] \simeq \oplus_{J_1 \subset I_1} (i_{Y_{J_1}})_* j_{!*} \mathcal{P}^{J_1}_{k-|J_1|}(\mathcal{L})[n - |J_1|].$$

where j denotes uniformly the inclusion of $Y^*_{J_1}$ into Y_{J_1} for each $J_1 \subset I_1$ where $I_1 \subset I$ and $Z := \cup_{i \in I_1} Y_i$. In particular, for $J_1 = \emptyset$, we have $j_{!*} Gr_k^W(\mathcal{L})[n]$, otherwise the summands are supported by Z.

The proof is local and based on the properties of relative filtrations (it is wrong otherwise). If the complex is written at the point y as a double complex

$$s(s(N_{J_2}L, N_i, i \in M_1)_{J_1 \subset M_1})_{J_2 \subset M_2} = s(\Omega(N_{J_2}L, N_i, i \in M_1)_{J_2 \subset M_2}$$

for each term of index $J = (J_1, J_2)$, the filtration on $N_{J_2}L$ is induced by $W^{(J_1,J_2)}$ on L, hence:

$$Gr^{W^{(J_1,J_2)}}_{k-|J_1|-|J_2|} N_{J_2}L \simeq N_{J_2} Gr^{W^{J_1}}_{k-|J_1|} L \simeq \oplus_{K \supset J_1} N_{J_2} N_{K-J_1} P^K_{k-|K|}(L)$$

where the first isomorphism is obtained by iterating the formula for J of length 1 in the lemma on the graded split sequence, and the second isomorphism follows the decomposition of the second term. Then, we can write Gr_k^W of the double complex as: $\oplus_{J_1 \subset M_1} IC(P^K_{k-|K|}(L)[-|J_1|]$.

Remark 3.34. 1) We may always suppose that $\mathcal{L}$ is a VMHS on $X - (Y \cup Z)$ (that is to enlarge Y) and consider Z as a subspace of Y equal to a union of components of Y.
2) If Z is a union of intersections of components of Y, these techniques should apply to construct a sub-complex of the logarithmic de Rham complex endowed with the structure of MHC with the induced filtrations and hypercohomology $\mathbb{H}^*(X - Z, j_{!*}\mathcal{L})$; for example, we give below the fibre of the complex at the intersection of two lines in the plane, first when $Z = Y_1$, then for $Z = Y_1 \cap Y_2$:

$$\begin{array}{ccc} L & \xrightarrow{N_1} & L \\ \downarrow N_2 & & \downarrow N_2 \\ N_2L & \xrightarrow{N_1} & N_2L \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{N_1} & L \\ \downarrow N_2 & & \downarrow N_2 \\ N_2L & \xrightarrow{N_1} & N_1L \cap N_2L \end{array}$$

3.3.7. *Thom-Gysin isomorphism.* Let H be a smooth hypersurface intersecting transversally Y such that $H \cup Y$ is a NCD, then $i_H^* j_{!*}\mathcal{L}$ is isomorphic to the intermediate extension $((j_{Y\cap H})_{!*}^*(i_H^*\mathcal{L})$ of the restriction of $\mathcal{L}$ to H and the residue with respect to H induces an isomorphism $R_H : i_H^*(\Omega^*(\mathcal{L}, H)/j_{!*}\mathcal{L}) \simeq i_H^* j_{!*}\mathcal{L}[-1]$ inverse to Thom-Gysin isomorphism $i_H^* j_{!*}\mathcal{L}[-1] \simeq i_H^! j_{!*}\mathcal{L}[1] \simeq i_H^*(\Omega^*(\mathcal{L}, H)/j_{!*}\mathcal{L})$. Moreover, if H intersects transversally $Y \cup Z$ such that $H \cup Y \cup Z$ is a NCD, then we have a triangle

$$(i_H)_* i_H^! \Omega^*(\mathcal{L}, Z) \to \Omega^*(\mathcal{L}, Z) \to \Omega^*(\mathcal{L}, Z \cup H) \stackrel{[1]}{\to}$$

hence the isomorphism of the quotient complex with the cohomology with support: $(\Omega^*(\mathcal{L}, Z \cup H)/\Omega^*(\mathcal{L}, Z)) \simeq i_H^! \Omega^*(\mathcal{L}, Z)[1]$ induced by the connection, the isomorphism of the restriction $i_H^*\Omega^*(\mathcal{L}, Z)$ with the complex $\Omega^*(i_H^*\mathcal{L}, Z \cap H)$ constructed directly on H, the residue with respect to H: $\Omega^*(\mathcal{L}, Z\cup H) \to i_{H,*}\Omega^*(i_H^*\mathcal{L}, Z\cap H)[1]$ vanishing on $\Omega^*(\mathcal{L}, Z)$ and inducing an inverse to the Thom-Gysin isomorphism $i_H^*\Omega^*(\mathcal{L}, Z) \simeq i_H^!\Omega^*(\mathcal{L}, Z)[2]$ are all compatible with the filtrations up to a shift in degrees.

3.3.8. *Duality and Cohomology with compact support.* We recall first, Verdier's dual of a bifiltered complex. Let (K, W, F) be a complex with two filtration on a smooth compact Käler or complex algebraic variety X, and $\omega_X := \mathbb{Q}_X[2n](n)$ a dualizing complex with the trivial filtration and a Tate twist of the filtration by dim.$X = n$ and a degree shift by $2n$ (so that the weight remains 0on a complex on a smooth compact Käler or complex algebraic variety X. We denote by $\mathcal{D}(K)$ the complex dual to K with filtrations:

$$\mathcal{D}(K) := RHom(K, \mathbb{Q}_X[2n](n)),\ W_{-i}\mathcal{D}(K) := \mathcal{D}(K/W_{i-1}),\ F^{-i}\mathcal{D}(K) := \mathcal{D}(K/F^{i+1})$$

then we have: $\mathcal{D}Gr_i^W K \simeq Gr_{-i}^W \mathcal{D}K$ and $\mathcal{D}Gr_F^i K \simeq Gr_F^{-i}\mathcal{D}K$. The dual of a mixed Hodge complex is a MHC.
In the case of $K = \Omega_X^*(LogY) \otimes \mathcal{L}_X[n] = Rj_*\mathcal{L}[n]$, the dual $\mathcal{D}K = j_!\mathcal{L}[n]$ with the dual structure of MHC defines the MHS on cohomology with compact support.

Corollary 3.35. *i) An admissible VMHS $\mathcal{L}$ of weight $\omega \leq a$ induces on the cohomolgy with compact support $\mathbb{H}_c^i(X - Y, \mathcal{L})$ a MHS of weight $\omega \leq a + i$.*
ii) The cohomology $\mathbb{H}^i(Y, j_!\mathcal{L})$ carry a MHS of weight $\omega \leq a + i$.

This result is compatible with [11], thm 3.3.1).
i) The dual admissible VMHS $\mathcal{L}^*$ of weight $\omega \geq -a$, and its de Rham logarithmic mixed Hodge complex is of weight $\omega \geq -a$, with dual quasi-isomorphic to $j_!\mathcal{L}[2n](n)$ and weight $\omega \leq a$. It induces on the cohomolgy $\mathbb{H}_c^i(X - Y, \mathcal{L})$ a MHS of weight $\omega \leq a + i$.
ii) The weights on $\mathbb{H}_Y^i(X, j_{!*}\mathcal{L})$ satisfy $\omega \geq a + i$ and by duality: $\mathbb{H}^i(Y, j_{!*}\mathcal{L})$, has weights $\omega \leq a + i$.

3.3.9. *The dual filtration $N!W$.* We introduce the filtration ([18], 3.4.2)

$$(N!W)_k := W_{k-1} + M_k(N, W) \cap N^{-1}W_{k-1}.$$

The following morphisms are induced by Id (resp. N) on L:
$I : W_{k-1} \to (N!W)_k$ and $N : (N!W)_k \to W_{k-1}$ satisfying $N \circ I = N$ and $I \circ N = N$ on L. Now we prove the duality with $N * W$.

Lemma 3.36. *Let W^* denotes the filtration on the vector space $L^* := Hom(L, \mathbb{Q})$ dual to a filtration W on L, then for all a*

$$(N^*!W^*)_a = (N * W)^*_a \subset L^*.$$

Let $M_i := M_i(N, W)$, it is auto-dual as a filtration of the vector space L: $M_i^* = M_i(N^*, W^*)$. To prove the inclusion of the left term into the right term for $a = -k$, we must write an element $\varphi \in L^*$ vanishing on $NW_k + M_{k-1} \cap W_{k-1} = NW_k + M_{k-1} \cap W_k$ as a sum of elements $\varphi = \gamma + \delta$ where $\delta \in W^*_{-k-1}$ vanish on W_k and $\gamma \in M^*_{-k} \cap (N^*)^{-1} W^*_{-k-1}$ vanish on $NW_k + M_{k-1}$. We construct γ such that $\gamma(NW_k + M_{k-1}) = 0$ and $\gamma_{|W_k} = \varphi_{|W_k}$ which is possible as $\varphi(M_{k-1} \cap W_k) = 0$; then we put $\delta = \varphi - \gamma$. The opposite inclusion is clear.
Now we deduce from the decomposition orthogonal to the case of $N * W$:

Corollary 3.37. *We have the decomposition:*

$$Gr_k^{N!W} \simeq Im(I : Gr_{k-1}^W \to Gr_k^{N!W}) \oplus (Ker(N : Gr_k^{N!W} \to Gr_{k-1}^W).$$

Properties of the iterated filtrations $\overline{W}$.
We associate to each IMHS with nilpotent endomorphisms N_i for $i \in M$ and to each subset $J \subset M$, an increasing filtration $\overline{W}^J$ of L recursively by the ! operation

$$\overline{W}^J := N_{i_1}!(\dots(N_{i_j}!W)\dots) \text{ for } J = \{i_1, \dots, i_j\}$$

since for all i and j, $N_i!N_jW = N_j!N_i!W$. This family satisfy the data in ([18], prop. 2.3.1) determined by the following morphisms, defined for $K \subset J$:
$I_{K,J} : \overline{W}_k^K \to \overline{W}_{k+|J-K|}^J$ and $N_{J,K} : \overline{W}_k^J \to \overline{W}_{k-|J-K|}^K$, so we deduce

Lemma 3.38. *i) Set for each $J \subset M$,*

$$Q_k^J(L) := \cap_{K \subset J, K \neq J} Ker(N_{J,K} : Gr_k^{\overline{W}^J} \to Gr_{k-|J-K|}^{\overline{W}^K}),$$

then $Q_k^J(L)$ has pure weight with respect to the weight $M(\sum_{j \in J} N_j, L)$.
ii) We have: $Gr_k^{\overline{W}^J} \simeq \oplus_{K \subset J} I_{K,J}(Q_{k-|J-K|}^K(L))$.
iii) Let (L^*, W^*) be dual to (L, W), then $Q_k^J(L^*) \simeq (P_{-k}^J(L))^*$.

Remark 3.39. By local duality at a point $x \in Y$, $\mathcal{D}_x i_x^* R j_* \mathcal{L} \simeq i_x^! \mathcal{D} R j_* \mathcal{L} \simeq i_x^! j_! \mathcal{D}\mathcal{L}$. Hence the filtration $\overline{W^*}$ is on the cohomology of $j_!\mathcal{L}^*$ at the point x.

3.3.10. *Complementary results.* We consider from now on a pure Hodge complex of weight $\omega(K) = a$; its dual $\mathcal{D}K$ is pure of weight $\omega(\mathcal{D}K) = -a$. For a pure polarized VHS $\mathcal{L}$ on $X - Y$ of weight $\omega = b$, hence $\mathcal{L}[n]$ of weight $\omega = a = b + n$, the polarization defines an isomorphism $\mathcal{L}(a)[n] \simeq \mathbb{R}Hom(\mathcal{L}[n], \mathbb{Q}_{X-Y}(n)[2n]) \simeq \mathcal{L}^*[n]$, then Verdier's auto-duality of the intersection complex reads as follows:

$$j_{!*}\mathcal{L}[n] \simeq (\mathcal{D}(j_{!*}\mathcal{L}[n])(-a)$$
$$\mathbb{H}^{-i}(X, j_{!*}\mathcal{L}[n])^* = H^i(Hom(\mathbb{R}\Gamma(X, j_{!*}\mathcal{L}[n]), \mathbb{C}) \simeq H^i(X, \mathcal{D}(j_{!*}\mathcal{L}[n]) \simeq$$
$$H^i(X, j_{!*}\mathcal{D}(\mathcal{L}[n]) \simeq H^i(X, j_{!*}(\mathcal{L}^*[n]) \simeq H^i(X, j_{!*}(\mathcal{L}[n])(a)$$

For $K = \mathcal{L}[n]$, the exact sequence :$i^! j_{!*}K \to j_{!*}K \to j_*K$ yields an isomorphism $i^! j_{!*}K[1] \simeq j_*K/j_{!*}K$, hence

$$\mathcal{D}(i^* j_* K / i^* j_{!*} K) \simeq \mathcal{D}(i^! j_{!*} K[1]) \simeq i^* j_{!*} \mathcal{D}K[-1]$$

For K pure of weight $\omega(K) = a$, $W_a j_* K = j_{!*}K$, $\omega(\mathcal{D}K) = -a$, and for $r > a$

$$W_{-r} i^* j_{!*} \mathcal{D}K[-1] \simeq \mathcal{D}(i^* j_* K / W_{r-1}).$$

We deduce from the polarization: $\mathcal{D}K \simeq K(a)$ where (a) drops the weight by $-2a$, a definition of the weight on $i^* j_{!*} K$ for $r > a$

$$W_{-r+2a} i^* j_{!*} K \simeq \mathcal{D}(i^* j_* K / W_{r-1})[1], \quad GR^W_{-r+2a} i^* j_{!*} K \simeq \mathcal{D} Gr^W_r (i^* j_* K)[1].$$

Corollary 3.40. *i) The complex $(i^* j_{!*} K[-1], F)$ with restricted Hodge filtration and the weight filtration for $k > 0$*

$$W_{a-k} i^* j_{!*} K[-1] \simeq \mathcal{D}(i^* j_* K / W_{a+k-1}),$$

satisfying $Gr^W_{a-k} i^* j_{!*} K[-1] \simeq \mathcal{D} Gr^W_{a+k}(i^* j_* K / i^* j_{!*} K)$ has the structure of a MHC. (In particular, the cohomology $\mathbb{H}^i(Y, j_{!*}K)$ carry a MHS of weight $\omega \leq a + i$).

We remark: $W_{a-1} i^* j_{!*} K[-1] \simeq i^* j_{!*} K[-1] = \mathcal{D}(i^* j_* K / j_{!*} K)$, hence:
$\mathbb{H}^i(Y, j_{!*}K) = \mathbb{H}^{i+1}(Y, j_{!*}K[-1])$ has weight $\omega \leq a - 1 + i + 1$.
By the above corollary, we have

$$GR^W_{-r+2a} i^* j_{!*} K \simeq \mathcal{D}(Gr^W_r (\Omega^*_X(LogY) \otimes \mathcal{L}_X)[n])[1] \simeq$$
$$\oplus_{J \subset I} j_{!*}(\mathcal{P}^J_{r-|J|}(\mathcal{L}))^*[n + 1 - |J|].$$

Example. For a polarized unipotent VHS on a punctured disc D^*, $i^* j_{!*} K[-1]$ is the complex $(L \xrightarrow{N} NL)$ in degree 0 and 1, quasi-isomorphic to $\ker N$, while $(i^* j_* K / j_{!*} K \simeq L/NL$ and $\mathcal{D}(i^* j_* K / j_{!*} K) \simeq (L/NL)^*$. The isomorphism is induced by the polarization Q as follows

$$\begin{array}{ccccc} (L/NL)^* & \simeq & (L^* & \xrightarrow{N^*} & (NL)^*) \\ \uparrow\simeq & & \uparrow\simeq & & \uparrow\simeq \\ \ker N & \simeq & (L & \xrightarrow{-N} & NL \end{array}$$

where $-N$ corresponds to N^* since $Q(Na, b) + Q(a, Nb) = 0$. The isomorphism we use is $\ker N \simeq (L/NL)^*$ in degree 0; we set for $k > 0$:
$M(N)_{a-k+1}(\ker N) := W_{a-k}(L \to NL) \simeq \mathcal{D}(i^* j_* K / W_{a+k-1}) = (L/(NL + M_{a+k-2}))^*$.

3.3.11. *Final remark on the decomposition theorem.* We do not cover here the ultimate results in Hodge theory with coefficients in an IMHS (see [2] and [26]).

We explain here basic points in the theory. In the presence of a projective morphism $f : X \to V$ and a polarized VHS of weight a on the smooth open subset $X - Y$ with intermediate extension $j_{!*}\mathcal{L}$ on X, a theory of perverse filtration ${}^p\tau_i(Rf_* j_{!*}\mathcal{L})$ on the direct image on V has been developed in [2]. It defines on the cohomology of the inverse of an open algebraic set $U \subset V$, a perverse filtration ${}^p\tau_i \mathbb{H}^r(f^{-1}(U), j_{!*}\mathcal{L})$.
We suppose X smooth, Y a NCD and we consider open sets complement to some NCD Z sub-divisor of Y, since we may reduce the problem to such case by desingularizing X.
1) If $X_U := f^{-1}(U)$ is the complement of a sub-NCD of Y, the theory asserts that the subspaces ${}^p\tau_i$ are sub-MHS of $\mathbb{H}^r(f^{-1}(U), j_{!*}\mathcal{L})$.
2) Let $v \in V$ be a point in the zero-dimensional strata of a Thom-Whitney stratification compatible with f and $Rf_* j_{!*}\mathcal{L}$ and suppose $X_v := f^{-1}(v)$ a sub-NCD of Y, then the local-purity theorem in [12] corresponds to the following semi- purity property of the weights ω on the cohomology of a tubular neighborhood T_v of

$f^{-1}(v)$, inverse image $f^{-1}(B_v)$ of a neighborhood B_v of v:
i) $\omega > a + r$ on ${}^p\tau_{\leq r}\mathbb{H}^r(T_v - X_v, j_{!*}\mathcal{L})$, and dually
ii) $\omega \leq a + r$ on $\mathbb{H}^r(T_v - X_v, j_{!*}\mathcal{L})/{}^p\tau_{\leq r}\mathbb{H}^r(T_v - X_v, j_{!*}\mathcal{L})$.
3) The corresponding decomposition theorem on $B_v - v$ states the isomorphism with the cohomology of a perverse cohomology:

$$Gr_i^{{}^p\tau}\mathbb{H}^r(T_v - X_v, j_{!*}\mathcal{L}) \simeq \mathbb{H}^{r-i}(B_v - v, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L})).$$

4) By iterating the cup-product with he class η of a relative hyperplane section, we have Lefschetz type isomorphisms of perverse cohomology sheaves

$${}^p\mathcal{H}^{-i}(Rf_*j_{!*}\mathcal{L}[n]) \xrightarrow{\eta^i} {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}[n]).$$

In [26], the proof is carried via the techniques of differential modules and is based on extensive use of Hodge theory on the perverse sheaves of near-by and vanishing cycles. A direct proof may be obtained by induction on the strata on V. If we suppose the decomposition theorem and Lefschetz types isomorphisms on $V - v$, one may prove directly that the perverse filtration on the cohomology $\mathbb{H}^r(T_v - X_v, j_{!*}\mathcal{L})$ is compatible with MHS, following the use of the monodromy filtration $W(\eta)$ of the nilpotent endomorphism defined by cup-product with the class η of a relative hyperplane section of type $(1,1)$ on the total cohomology $\oplus_i\mathbb{H}^r(T_v - X_v, j_{!*}\mathcal{L})$. This filtration $W(\eta)$ is compatibile with MHS and the graded part $Gr_i^{W(\eta)}$ coincide with $Gr_{-i}^{{}^p\tau}$. Then it makes sense to prove the semi-purity property, from which we deduce the extension of the decomposition over the point v and check the Lefschetz isomorphism, which complete the inductive step (for a general strata we intersect with a transversal section so to reduce to the case of a zero dimensional strata).

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