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# How to model marine reserves ?

Patrice Loisel \*Pierre Cartigny †

Abstract: The safeguarding of resources is one of the principal subjects of halieutics studies. Among the solutions proposed to avert the disappearance of species, the setting in place of no take reserves is often mentioned. Most work on this subject, theoretical as well as applied, was undertaken in recent years. In this paper, we seek to compare two different models presented in existing literature by highlighting their underlying assumptions. Both models were derived from what is often referred to as the "model of Schaefer-Clark" (reference to the work of the last author on *Mathematical Bioeconomics* : Clark [7]). We show that various variations of this model lead to properties that can be very different.

Keywords: dynamical system; calculus of variation; infinite horizon; marine reserve; bioeconomic model

## 1 Introduction

It is now well known and largely accepted within the scientific community that the exploitation of halieutics resources has reached a critical threshold and solutions must be found urgently to conserve marine biodiversity – and indeed the existence of certain species. Beyond the application of quota policies, other strategies have been proposed such as the creation of marine reserves. The study of the role of reserves in fishery management has been

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the subject of renewed interest in recent years [2, 5, 8, 10, 11, 14]. The International Conference on the Economics of Marine Protected Areas (MPA) held in July 2000 in Vancouver was one of the starting points in the development of this new paradigm: the use of MPAs as an instrument in the management of fisheries.

In economics literature, Sanchirico and Wilen [18] seem to have been the first to suggest that MPAs could be beneficial not only from an ecological but also from an economic point of view. In their dynamic and spatial model of a Marine Reserve Creation, they analyze whether the transfer of biomass from the reserve to areas where catch is allowed could create economic profit from the MPA since its creation could help to improve a depleted biomass and increase catch outside of the reserve. They call this a double-payoff because in this case the MPA would increase both biomass and economic profits from the fishery.

Both theoretical and applied aspects of the subject are well documented in the literature. From a theoretical point of view, numerous types of models have been proposed based on differential equations, mixing an inter temporal dynamic that corresponds to the population growth under consideration with a spatial distribution of this dynamic over diverse zones [9, 15, 18].

Sanchirico and Wilen construct a model where "the population structure is characterized in a manner consistent with modern biological ideas that stress patchiness, heterogeneity and interconnections among and between patches" [17]. In this model, independent growth dynamics thus are associated with different patches.

Another type of model has been used in the literature to account for an analogue structure [19] where a population develops different characteristics in sub-zones. In these models, the population of the entire zone under consideration follows a given dynamic evolution and the diverse sub-zones have dynamics such that by aggregating them together one may rediscover the global dynamic.

The question that one then must ask is whether these two approaches may be used interchangeably. Few studies have focused on comparing these different model types. We would like to demonstrate that the choice between a patch model and a global model is not a neutral one, and highlight char-

acteristics of these two models that often are not specified in the literature.

The two models that we will compare are both Clark type (Schaefer, Gordon,...) [7] whose dynamic is logistic (Verhulst) and therefore widely used in halieutics dynamics. We use them to study the consequences of setting up a marine reserve from both an economic (inter temporal revenue) and a biologic point of view (population stocks).

The first, the patch model, assumes a relative autonomy between reserve and non-reserve zones. This type of model is fairly widely used. The second model, known as the global model, assumes for its part a greater interaction between zones.

The rest of the paper is organized as follows. In Section 3, we introduce and examine the two models we wish to study. We then compare results obtained, particularly using numerical simulation, in Section 4. Section 5 concludes and is followed by a series of annexes that present the demonstration of various results.

## 2 Two variations on the Clark model

We present two modelizations for a protected area in a given zone. These two modelizations derive from the well known fishery model studied for instance by Clark [7] among others.

### 2.1 The first variation: patches model

#### 2.1.1 The model

We consider a fish population that lives in a zone characterised by a carrying capacity  $K = 1$ . We assume that this zone splits in two sub-zones, with capacity respectively equal to  $\alpha$  and  $1 - \alpha$ ; we denote the stock of the corresponding sub-populations by  $x_1$  and  $x_2$ .

These two populations follow two independent evolutions laws and this is the reason why we use the “patches-concept”. These evolutions are given by:

$$\frac{dx_1}{dt}(t) := \dot{x}_1(t) = F_1(x_1(t))$$

$$\frac{dx_2}{dt}(t) := \dot{x}_2(t) = F_2(x_2(t)).$$

The standard reference for the evolutions law is the logistic law:

$$F_1(x_1) := r_1 x_1 \left(1 - \frac{x_1}{\alpha}\right)$$

$$F_2(x_2) := r_2 x_2 \left(1 - \frac{x_2}{1 - \alpha}\right).$$

But for our purpose it is enough to assume that the  $F_i$  are strictly concave,  $C^1$  functions defined on  $[0, \alpha]$  respectively on  $[0, 1 - \alpha]$  and the  $F_i$  satisfy  $F_i(0) = 0, F_1(\alpha) = F_2(1 - \alpha) = 0$ .

We now assume that some exchange exists between these two patches and that this can be represented in terms of the density of these populations. More precisely we assume that the existence of some diffusion between these two patches can be captured by the following:

$$\lambda \left( \frac{x_2}{1 - \alpha} - \frac{x_1}{\alpha} \right)$$

where  $\lambda \geq 0$  represents a diffusion coefficient ( $\frac{x_2}{1 - \alpha}, \frac{x_1}{\alpha}$  being the density of the populations). The value of the diffusion coefficient depends on the location of the protected area.

From now on, we decide that the first zone with capacity  $\alpha$  is a protected area where no catch is allowed, whereas in the second zone fishing is allowed. The “normal” situation, i.e. the protected area acts like a source of biomass, corresponds to the case where the density inside the protected area is bigger than outside, i.e.

$$\frac{x_2}{1 - \alpha} \leq \frac{x_1}{\alpha}.$$

The growth of the two sub-populations are governed respectively by the following dynamics:

$$\begin{aligned} \dot{x}_1(t) &= F_1(x_1(t)) + \lambda(t) \left( \frac{x_2(t)}{1 - \alpha} - \frac{x_1(t)}{\alpha} \right) \\ \dot{x}_2(t) &= F_2(x_2(t)) - \lambda(t) \left( \frac{x_2(t)}{1 - \alpha} - \frac{x_1(t)}{\alpha} \right) - h(t) \end{aligned} \tag{1}$$

where  $h(t)$  is the capture rate at time  $t$ .

We note, from the positiveness of  $\lambda$  and of the functions  $F_i(\cdot)$ , that if the system (1) possesses an equilibrium, it has to be necessarily normal.

As it is generally assumed the catch is proportional to the fishing effort  $E$ , and to the density of the population [4], therefore given by:

$$h(t) = qE(t) \frac{x_2(t)}{1 - \alpha}$$

where  $q$ , the catchability coefficient, represents the fishing death rate when the density of the population is equal to one. We assume

$$0 \leq E(t) \leq E_M \quad q > 0$$

The catch is sold on a market. In order to simplify we assume a constant price,  $p$ , over time and a constant cost,  $c$ , proportional to the effort. Therefore the revenue at  $t$  time is

$$ph(t) - cE(t) = (pq \frac{x_2(t)}{1 - \alpha} - c)E(t)$$

We then consider the discounted total revenue on an infinite horizon is given by

$$J(E(\cdot), \lambda(\cdot)) := \int_0^\infty e^{-\delta t} (pq \frac{x_2(t)}{1 - \alpha} - c)E(t) dt \quad (2)$$

where  $\delta > 0$  is an actualisation factor.

We assume the existence of a manager whose goal is the maximisation of this total revenue. Moreover we assume that this manager can act on the fishing effort  $E$  and on some characteristics of the reserve (closure, location) that are captured by  $\lambda$ . Then the manager faces the following control problem:

$$\begin{aligned} \max_{E(\cdot), \lambda(\cdot)} & J(E(\cdot), \lambda(\cdot)) \\ \text{s.t.} & (1) \\ & 0 \leq E(t) \leq E_M \end{aligned} \quad (3)$$

**Remark** In many papers the states variables stand for the densities of the populations and not for the amount of the biomass. The link with the present model is obtained in setting:

$$X_1 = \frac{x_1}{\alpha}, \quad X_2 = \frac{x_2}{1-\alpha}$$

The two dynamical equations that give the evolution of the populations in the logistic case, then become:

$$\begin{aligned} \dot{X}_1 &= r_1 X_1 (1 - X_1) + \frac{\lambda}{\alpha} (X_2 - X_1) \\ \dot{X}_2 &= r_2 X_2 (1 - X_2) - \frac{\lambda}{1-\alpha} (X_2 - X_1) - Q E X_2 \end{aligned} \quad (4)$$

with  $Q = \frac{q}{1-\alpha}$ .

This model with patches could be considered as a more general one than the [1] paper which corresponds to  $\alpha = \frac{1}{2}$ .

### 2.1.2 Analysis of the solutions

We will study the previous optimal control problem by the help of the calculus of variations theory .

From the dynamic (1), we deduce the expression of the effort in terms of the state variables:

$$E(t) = \frac{1-\alpha}{q x_2(t)} (F_1(x_1(t)) + F_2(x_2(t)) - \dot{x}_1(t) - \dot{x}_2(t))$$

and then we obtain the new form of the objective. Thus the optimisation problem becomes:

$$\max_{X \in C} \int_0^\infty e^{-\delta t} \left( p - \frac{c(1-\alpha)}{q x_2} \right) [F_1(x_1) + F_2(x_2) - \dot{x}_1 - \dot{x}_2] dt$$

$C$  being the set of admissible curves:

$$C = \{x(\cdot) = (x_1(\cdot), x_2(\cdot)) \mid x_i(\cdot) \in BC^1([0, \infty[), x_i(0) \text{ given}, \\ G(x_1, x_2) - q E_M \frac{x_2}{1-\alpha} \leq \dot{x}_1 + \dot{x}_2 \leq G(x_1, x_2)\} \quad (5)$$

with  $G(x_1, x_2) := F_1(x_1) + F_2(x_2)$  and  $BC^1$  stands for the bounded with bounded derivative functions defined on the interval  $[0, \infty[$ .

It is known that on  $BC^1$ , the first order optimality conditions given by the Euler-Lagrange equations, apply (see [3]). We suppose that  $x(\cdot)$  stands for an interior solution and then  $x(\cdot)$  has to satisfy:

$$l_{x_i}(x_1(t), x_2(t)) - \frac{d}{dt}l_{\dot{x}_i}(x_1(t), x_2(t)) + \delta l_{\dot{x}_i}(x_1(t), x_2(t)) = 0$$

$l(\cdot, \cdot)$  being the non actualised Lagrangian of the calculus of variations problem,  $l_{x_i}(\cdot, \cdot)$  stands for the derivative with respect to  $x_i$  and  $l_{\dot{x}_i}(\cdot, \cdot)$  stands for the derivative with respect to  $\dot{x}_i$ .

The Euler-Lagrange equations becomes then

$$\begin{aligned} \dot{x}_1 &= x_2 \left( \frac{pqx_2}{c(1-\alpha)} - 1 \right) (F_2'(x_2) - \delta) + F_1(x_1) + F_2(x_2) \\ \dot{x}_2 &= x_2 \left( \frac{pqx_2}{c(1-\alpha)} - 1 \right) (\delta - F_1'(x_1)). \end{aligned} \quad (6)$$

We first are interested by the non trivial equilibria ,  $(x_1^*, x_2^*)$ , of (6) i.e. such that  $F_i(x_i^*) \neq 0$ , i.e.  $x_1^* \neq 0, \alpha$  and  $x_2^* \neq 0, 1 - \alpha$ . It is easy to establish that such equilibria have to satisfy (Appendix 1)

$$F_1'(x_1^*) = \delta.$$

We now assume that  $r_1 > \delta$ . Then in the logistic case, from the strict concavity of  $F_1(\cdot)$ , we immediatly deduce the existence of a unique  $x_1^* \in ]0, \alpha/2[$ . Therefore we obtain the following result whose proof is postponed in Appendix 1.

**Lemma 1** *In the logistic case with  $r_1 > \delta$ , there is a unique positive non trivial solution,  $(x_1^*, x_2^*)$ , of the Euler-Lagrange equations (6). This solution is characterised by*

$$x_1^* = \frac{\alpha(r_1 - \delta)}{2r_1}$$

and  $x_2^*$  given by

$$x_2 \left[ \frac{2r_2pq}{c(1-\alpha)^2} x_2^2 - \left( \frac{pq}{c(1-\alpha)} (r_2 - \delta) + \frac{r_2}{1-\alpha} \right) x_2 - \delta \right] = \alpha \frac{(r_1 - \delta)(r_1 + \delta)}{4r_1}. \quad (7)$$



Clearly  $x_1^* \in ]0, \alpha[$ . It remains to show that  $x_2^* \in [0, 1 - \alpha]$ . This can be done straightforwardly but we prefer to use the following approach. We recall that if (1) possesses a non-zero equilibrium  $(x_1^*, x_2^*)$  then it is necessarily normal that is to say  $\frac{x_2^*}{1-\alpha} \leq \frac{x_1^*}{\alpha}$  and we first observe that if this last condition holds then  $x_2^* \leq \frac{x_1^*}{\alpha}(1 - \alpha) \leq 1 - \alpha$ .

We prove now that this condition holds under conditions on the ratio  $\frac{pq}{c}$  and on the size of the different zones given by  $\alpha$ .

Let us introduce  $T(\cdot)$  defined from the left hand member in (7)

$$T(z) = z[2r_2\theta z^2 - (\theta(r_2 - \delta) + r_2)z - \delta]$$

where  $\theta := \frac{pq}{c}$ . Then:

$$T\left(\frac{x_2^*}{1-\alpha}\right) = \frac{\alpha}{1-\alpha} \frac{(r_1 - \delta)(r_1 + \delta)}{4r_1}.$$

As we assume  $r_1 > \delta$ , then  $T\left(\frac{x_2^*}{1-\alpha}\right) > 0$ . From the graph of  $T(\cdot)$  we deduce that the inequality

$$\frac{x_2^*}{1-\alpha} \leq \frac{x_1^*}{\alpha}$$

holds if and only if

$$T\left(\frac{x_2^*}{1-\alpha}\right) \leq T\left(\frac{x_1^*}{\alpha}\right).$$

Let us introduce when  $r_1 \neq r_2$

$$\theta_0 := \left(\frac{2r_1}{r_1 - \delta} + \frac{r_2}{\delta}\right) \frac{r_1}{r_1 - r_2}. \quad (8)$$

The following result is detailed in Appendix 2.

## Lemma 2

*In the logistic case,*

1. *if  $r_1 > r_2$ , for each  $\theta > \theta_0$  if  $\alpha$  satisfies*

$$\alpha(r_1 + \delta + \frac{r_1 - \delta}{r_1}(\theta\delta\frac{r_1 - r_2}{r_1} - r_2) - 2\delta) \leq \frac{r_1 - \delta}{r_1}(\theta\delta\frac{r_1 - r_2}{r_1} - r_2) - 2\delta \quad (9)$$

then the solution  $(x_1^*, x_2^*)$ , of the Euler Lagrange equation (6) given in Lemma 1, is normal i.e. satisfies:

$$\frac{x_2^*}{1 - \alpha} \leq \frac{x_1^*}{\alpha}$$

2. if  $r_1 \leq r_2$ ,  $(x_1^*, x_2^*)$  is never normal i.e.

$$\frac{x_2^*}{1 - \alpha} > \frac{x_1^*}{\alpha}.$$

Then it remains to prove that  $(x_1^*, x_2^*)$  is a candidate to the optimisation problem (3), thus that the constraints are satisfied.

If the conditions in the Lemma 1 and Lemma 2.1 hold, then  $F_1(x_1^*) > 0$  and  $\frac{x_2^*}{1 - \alpha} < \frac{x_1^*}{\alpha}$ . Therefore to this corresponds a unique coefficient of diffusion  $\lambda^* > 0$ .

From (1) we deduce that

$$E^* = \frac{1 - \alpha}{qx_2^*}(F_1(x_1^*) + F_2(x_2^*)) > 0. \quad (10)$$

Moreover the expression of the total revenue is given by

$$J^* = (pq \frac{x_2^*}{1 - \alpha} - c) \frac{E^*}{\delta}. \quad (11)$$

This revenue is positive if the fishery profit is positive at this equilibrium, that is to say if

$$\frac{x_2^*}{1 - \alpha} > \frac{c}{pq} = \frac{1}{\theta}.$$

But this inequality holds because we have

$$T(\frac{x_2^*}{1 - \alpha}) > T(\frac{1}{\theta}) = (1 - \theta) \frac{r_2}{\theta^2}$$

and due to the fact that  $\theta \geq \theta_0 > 1$ , this last term is nonpositive .

**Proposition 1**

In the logistic case if  $r_1 > \delta$  the problem (3) possesses at most a non trivial and positive optimal stationary solution characterised by

$$x_1^* = \frac{\alpha(r_1 - \delta)}{2r_1}$$

and  $x_2^*$  given by (7). The corresponding effort, diffusion coefficient and total revenue are given respectively by (10), (1), (11).

**Remarks**

- 1) When  $\alpha = \frac{1}{2}$  the value of  $\theta_0$  coincides with the value  $\tilde{p}_m$  given in [1].
- 2) From the expression of  $\theta_0$  in (8), we observe that  $r_1$  can't be closed to  $r_2$ . If this is not the case, then the value of the dimensionless ratio  $\theta$  has to be very high. But this can be unrealistic because the value of  $\theta$  is given by the economic environment.
- 3) From (8) with a given value for  $\theta$  we can precise a bound for  $r_2$  expressed in terms of  $r_1, \delta$ :

$$r_2 \leq \frac{\theta - \frac{2r_1}{r_1 - \delta}}{\theta + \frac{r_1}{\delta}} r_1.$$

## 2.2 The second variation: the splitting of a unique zone

In this second model we start with an unique zone with capacity  $K$  that we normalise to one,  $K = 1$ . Let us assume that the population follows a standard evolution law:

$$\dot{z}(t) = \phi(z(t)) \quad (\text{for instance } \phi(z) = rz(t)(1 - z))$$

$\phi(\cdot)$  being a  $C^1$  concave function defined on  $[0, 1]$ , with  $\phi(0) = \phi(1) = 0$  and  $\phi'(1) < 0$ . We assume that this zone splits first in a part that is a reserved area where no fishing is allowed and a complementary part that is open to harvest. We assume that these two parts have respectively  $\alpha$  and  $1 - \alpha$  as a carrying capacity.

The main difference with the previous model is that the two populations,

which stocks are respectively  $x_1$  and  $x_2$  follow the evolution laws given by:

$$\begin{aligned}\dot{x}_1(t) &= F(x_1(t), x_1(t) + x_2(t)) \\ \dot{x}_2(t) &= F(x_2(t), x_1(t) + x_2(t))\end{aligned}$$

where  $F(.,.)$  satisfies the standard assumption of regularity with  $F(x, z) = 0$  if and only if  $x = 0$  or  $z = 1$  and where  $F(.)$  and  $\phi(.)$  satisfy

$$F(x_1, x_1 + x_2) + F(x_2, x_1 + x_2) = \phi(x_1 + x_2).$$

For instance  $F$  can be a logistic function

$$F(x, z) = rx(1 - z) = rx(1 - (x_1 + x_2)).$$

As in the previous model, there is some diffusion between the two zones which can be represented by:

$$\lambda\left(\frac{x_2}{1 - \alpha} - \frac{x_1}{\alpha}\right).$$

Then the two populations evolve following the dynamics:

$$\begin{aligned}\dot{x}_1(t) &= F(x_1(t), x_1(t) + x_2(t)) + \lambda\left(\frac{x_2(t)}{1 - \alpha} - \frac{x_1(t)}{\alpha}\right) \\ \dot{x}_2(t) &= F(x_2(t), x_1(t) + x_2(t)) - \lambda\left(\frac{x_2(t)}{1 - \alpha} - \frac{x_1(t)}{\alpha}\right).\end{aligned}$$

We want to stress on the fact that this new model is consistent in the sense that the sum of the two dynamics is exactly the evolution law of the total population.

Now taking into account the catch in the zone where fishing is allowed, we derive the final dynamics of the populations

$$\begin{aligned}\dot{x}_1(t) &= F(x_1(t), x_1(t) + x_2(t)) + \lambda\left(\frac{x_2(t)}{1 - \alpha} - \frac{x_1(t)}{\alpha}\right) \\ \dot{x}_2(t) &= F(x_2(t), x_1(t) + x_2(t)) - \lambda\left(\frac{x_2(t)}{1 - \alpha} - \frac{x_1(t)}{\alpha}\right) - qE\frac{x_2(t)}{1 - \alpha}\end{aligned}\tag{12}$$

where  $E \in [0, E_M]$  stands for the fishing effort and  $q > 0$  is the catchability coefficient.

In order to compare with the previous model, we assume that a manager has as an objective to maximise the actualised total revenue as presented

before. To do so, he has to act on two controls, the fishing effort and the location of the reserve area given by  $\lambda$ . Therefore the manager faces the following program of optimisation

$$\begin{aligned} \max_{E(\cdot), \lambda(\cdot)} \quad & \int_0^\infty e^{-\delta t} (pq \frac{x_2(t)}{1-\alpha} - c) E(t) dt \\ \text{s.c.} \quad & (12) \\ & 0 \leq E(t) \leq E_M \end{aligned} \tag{13}$$

### 2.2.1 Analysis of the solutions

From the dynamic equations (12) we can derive the expression of the effort

$$E(t) = \frac{1-\alpha}{qx_2(t)} (\phi(x_1(t) + x_2(t)) - \dot{x}_1(t) - \dot{x}_2(t)) \tag{14}$$

and then we obtain the equivalent problem to (13) as a calculus of variations problem:

$$\max_{X \in C} \int_0^\infty e^{-\delta t} (p - \frac{c(1-\alpha)}{qx_2}) [\phi(x_1 + x_2) - \dot{x}_1 - \dot{x}_2] dt$$

where  $C$  stands for the set of feasible curves defined by:

$$C = \{x(\cdot) = (x_1(\cdot), x_2(\cdot)) \quad x_i(\cdot) \in BC^1([0, \infty[), x_i(0) \text{ given}, \\ \phi(x_1 + x_2) - qE_M \frac{x_2}{1-\alpha} \leq \dot{x}_1 + \dot{x}_2 \leq \phi(x_1 + x_2)\}.$$

We know that in this framework a necessary optimality condition for an interior solution  $x(\cdot)$  is given by the Euler-Lagrange equations that are:

$$\begin{aligned} \dot{x}_1 &= x_2 \left( \frac{pqx_2}{c(1-\alpha)} - 1 \right) (\phi'(x_1 + x_2) - \delta) + \phi(x_1 + x_2) \\ \dot{x}_2 &= x_2 \left( \frac{pqx_2}{c(1-\alpha)} - 1 \right) (\delta - \phi'(x_1 + x_2)). \end{aligned}$$

For now, we will stick to the logistic case. The Euler-Lagrange equations are then:

$$\begin{aligned} \dot{x}_1 &= x_2 \left( \frac{pqx_2}{c(1-\alpha)} - 1 \right) (r - 2r(x_1 + x_2) - \delta) + r(x_1 + x_2)(1 - (x_1 + x_2)) \\ \dot{x}_2 &= x_2 \left( \frac{pqx_2}{c(1-\alpha)} - 1 \right) (\delta - r + 2r(x_1 + x_2)). \end{aligned} \tag{15}$$

In order to derive the non trivial positive equilibria, denoted by  $(x_1^*, x_2^*)$ , we first consider the second equation in (15) with the condition

$$x_1^* + x_2^* = \frac{r - \delta}{2r}.$$

This implies as a result

$$0 = r(x_1^* + x_2^*)(1 - (x_1^* + x_2^*)) = \frac{(r - \delta)(r + \delta)}{4r}.$$

A contradiction if  $r > \delta$ . In the case where  $r = \delta$ , then we obtain the trivial solution  $x_1^* = x_2^* = 0$ .

Then we deduce that an equilibrium has to necessarily satisfy

$$x_2^* = \frac{c(1 - \alpha)}{pq}.$$

With the help of first equation in (15) we find that either  $x_1^* + x_2^* = 0$  or  $x_1^* + x_2^* = 1$ .

Finding a non trivial equilibrium implies to exclude the first condition. Therefore we have proved the following result

**Lemma 3** *In the logistic case there is a unique non trivial and positive solution for the Euler-Lagrange solutions (15) given by:*

$$(x_1^*, x_2^*) = \left(1 - \frac{c(1 - \alpha)}{pq}, \frac{c(1 - \alpha)}{pq}\right).$$

In order to examine whether this candidate solution of the problem (13) can be optimal or not, we have to derive the corresponding effort and diffusion coefficient. From the expression of the effort (14), we obtain  $\phi(x_1^* + x_2^*) = \phi(1) = 0$  that is to say

$$E^* = 0.$$

We also deduce that

$$\lambda^* = 0$$

except if  $\theta = 1$ . Finally at this equilibrium the intertemporal revenue is null too. We have established the following proposition

**Proposition 2**

*In the logistic case the problem (13) possesses at most a non trivial and positive stationary solution given by*

$$(x_1^*, x_2^*) = \left(1 - \frac{c(1 - \alpha)}{pq}, \frac{c(1 - \alpha)}{pq}\right).$$

*The corresponding effort, diffusion coefficient and total revenue are null.*

**Remarks**

- 1) It is easy to obtain that this equilibrium is normal, i.e.

$$\frac{x_1^*}{\alpha} > \frac{x_2^*}{1 - \alpha}$$

if the fishery is profitable, that is to say if

$$pq - c > 0.$$

- 2) An adaptation of the model given in Gomez et al. [6] to our case of a no take zone is:

$$\begin{aligned} \dot{x}_1 &= \alpha r(x_1 + x_2)(1 - (x_1 + x_2)) + \lambda\left(\frac{x_2}{1 - \alpha} - \frac{x_1}{\alpha}\right) \\ \dot{x}_2 &= (1 - \alpha)r(x_1 + x_2)(1 - (x_1 + x_2)) - \lambda\left(\frac{x_2}{1 - \alpha} - \frac{x_1}{\alpha}\right) - q_2 E_2 \frac{x_2}{1 - \alpha}. \end{aligned} \quad (16)$$

We can get the same results as those given earlier. In Gomez et al. [6], fishing is allowed in the so called artisanal zone (corresponding to the protected area in our case) and the objective to be maximised is somewhat different (it takes into account the revenues of the artisanal and industrial fisheries). Here also it has been proved that a unique solution exists but with a non null effort and a non null revenue.

### 3 Comparison, Numerical application

In this section we underline the differences between the results we obtained in the previous sections for both the patches case and the global model. From their expressions given in the Propositions 1 and Proposition 2, we can make the following remarks for the equilibria  $(x_1^*, x_2^*)$ :

- In the model with patches the first component  $x_1^*$  doesn't depend explicitly on the ratio  $\frac{c}{pq}$ , whereas it does in the global model.
- In the global model the second component is given by  $x_2^* = \frac{c(1-\alpha)}{pq}$ , whilst the patches model doesn't possess any equilibrium with such a component as the second, cf. Appendix 1.

Thus the expressions of the equilibria are different in the two models.

Now we established that the optimal effort and the corresponding total revenue at  $(x_1^*, x_2^*)$  was null for the global model. This doesn't seem to be the case for the patches model, we will later show with simulations that optimal effort and total revenue are not significantly close to zero.

In order to continue the comparison, let's arbitrarily fix the parameters  $p, q, c, \alpha, \delta$ . Thus the models depend only on the instantaneous growth rates  $r_1, r_2$  and  $r$  respectively.

If we let  $r_1 = r_2 = r$ , in Lemma 2 we established that the equilibrium  $(x_1^*, x_2^*)$  was never normal in the patches case, while it is always normal for the global model (Remark 1 in section 3.2).

Now to compare our models with  $r_1$  and  $r_2$  only near  $r$ , we noted in the Remark 2 of section 3.1 that this situation wasn't a realistic one.

Then comparing these two models is not an easy task. Our first conclusion is: the role of the instantaneous growth rates of the biomasses are crucial to choose such or such model. An assumption that is not underlined in general.

Now let's come back to the comparison of the optimal efforts and revenues by using simulations. The main issue is to determine significant growth rates that are not equal.

But this choice shouldn't depend on our particular models with preserving



areas. It should be the same for a wide class of models. For instance for models that correspond to a situation where fishing is allowed in the the two areas ([13]). We consider thus:

$$\begin{aligned}\dot{x}_1 &= F_1(x_1) + \lambda\left(\frac{x_2}{1-\alpha} - \frac{x_1}{\alpha}\right) - qEx_1 \\ \dot{x}_2 &= F_2(x_2) - \lambda\left(\frac{x_2}{1-\alpha} - \frac{x_1}{\alpha}\right) - qEx_2\end{aligned}\tag{17}$$

and

$$\begin{aligned}\dot{x}_1 &= F(x_1, x_1 + x_2) + \lambda\left(\frac{x_2}{1-\alpha} - \frac{x_1}{\alpha}\right) - qEx_1 \\ \dot{x}_2 &= F(x_2, x_1 + x_2) - \lambda\left(\frac{x_2}{1-\alpha} - \frac{x_1}{\alpha}\right) - qEx_2\end{aligned}\tag{18}$$

with the same assumptions as before. In order to compare numerically (17) and (18) we will face the same issue to determine significant growth rates. We propose to use this new situation in order to fix values for  $r_1, r_2, r$ . The new problem we consider now is to maximize the same objective as before

$$J(E) = \int_0^\infty (pq(x_1 + x_2) - c)Ee^{-\delta t} dt$$

but with (17) and (18). We observe that (18) corresponds to the classic Clark model, it is enough to let  $z = x_1 + x_2$  to obtain that the dynamic is  $\dot{z} = \phi(z) - qEz$ .

We can assume that a manager has no reason to use one model rather than another. Then the two models can be considered as equivalent in the sense that they provide the same optimal effort.

Hence, we propose the following procedure to determine a system of growth rates: Let's set an arbitrary choice of values for  $r_1$  and  $r_2$ . From the first order optimality conditions, given here by the Pontryagin principle, we can derive the optimal value of the corresponding effort for problem (17). We hand-over this value in the first order optimality conditions of the second problem (18) from which we derive the value of the growth rate  $r$ .

To follow this procedure we set:

$$\alpha = .5, \delta = .05, c = .15, q = 2., \lambda = 20$$

and we obtained for

$$r_1 = 0.4, r_2 = 0.05$$

that

$$\bar{E} = .0566 \quad \text{and} \quad r = 0.28739.$$

Let's now go back to our models with protected areas from where we take the previous values for the parameters and where we set  $r_1 = .4, r_2 = .05$  and  $r = 0.28739$ .

Then for the model with patches we found that:

- the optimal effort is  $E^* = .0457$ , and the biomass values are respectively  $x_1^* = .21875, x_2^* = .0302$

and for the global model

- $E^* = 0$  et  $x_1^* = .875, x_2^* = .125$ .

We observe that the optimal efforts corresponding to the patches case,  $E^* = .0457$ , and the Clark model (18),  $\bar{E} = 0.0566$ , have similar sizes. We know that the optimal value of the effort in this last model can't be considered as null. Therefore we can deduce that in the first model with patches the effort is not null.

Then the two models have different qualitative behaviour: total revenues and optimal effort are totally different.

## 4 Conclusion

In this article, we have shown that different models have been proposed and used in the literature for the same MPA problematic. We focused on demonstrating the importance of the hypotheses underlying two types of models – the patch model and sub-zone model – particularly the crucial role played by the growth functions (rate and form), and on studying the different results produced by them.

A manager who wishes to study the role of an MPA in a given zone must first know if the entire zone is artificially divided or if it can be broken down into patches (entities with their own dynamics). Without taking this precaution, and in obtaining the very different results that we have seen, the manager risks taking erroneous decisions.

The two preceding models of resource dynamics are adapted to the case where control instruments are independent of the size of the no take reserve. If the manager must take size into account in his decisions, the modelling of the dynamic has to be changed. For instance, it is necessary to consider a depending of  $\alpha$  diffusion coefficient. A justification is given in Appendix 3. In this Appendix, we also underline that this coefficient can be given by formula

$$\lambda(\alpha) = \lambda_0 \alpha (1 - \alpha)$$

which is the expression considered by Boncoeur (cf. [4]).

## 5 Appendix

### 5.1 Appendix 1

We determine the non trivial equilibria of the Euler-Lagrange equations (6)

$$\begin{aligned} \dot{x}_1 &= x_2 \left( \frac{pqx_2}{c(1-\alpha)} - 1 \right) (F_2'(x_2) - \delta) + F_1(x_1) + F_2(x_2) \\ \dot{x}_2 &= x_2 \left( \frac{pqx_2}{c(1-\alpha)} - 1 \right) (\delta - F_1'(x_1)). \end{aligned}$$

If we assume that  $x_2^* = \frac{c(1-\alpha)}{pq}$ , from the first equation:

$$F_1(x_1) + F_2(x_2^*) = 0. \quad (19)$$

From the assumption of the non triviality of the equilibria, we have that  $c \neq 0, \alpha \neq 1, c \neq pq$  and then  $F_2(x_2^*) > 0$ . Therefore we can't find any  $x_1 \in [0, \alpha]$  such that (19) holds. Thus there is no non trivial equilibrium with  $x_2^* = \frac{c(1-\alpha)}{pq}$ .

Therefore in order for a non trivial equilibrium to exist it is necessary that

$$F_1'(x_1^*) = \delta.$$

In the logistic case it is easy to compute that

$$x_1^* = \frac{\alpha(r_1 - \delta)}{2r_1}.$$

Reporting this value in (6) we get the following equation for  $x_2^*$

$$x_2 \left( \frac{pqx_2}{c(1-\alpha)} - 1 \right) (F_2'(x_2) - \delta) + F_2(x_2) = -F_1(x_1^*)$$

which is (7) in the logistic case

$$x_2 \left[ \frac{2r_2pq}{c(1-\alpha)^2} x_2^2 - \left( \frac{pq}{c(1-\alpha)} (r_2 - \delta) + \frac{r_2}{1-\alpha} \right) x_2 - \delta \right] = \alpha \frac{(r_1 - \delta)(r_1 + \delta)}{4r_1}.$$

The graphs of the functions defined by the left and right hand members are curves that cross in a single  $x_2^*$  if  $r_1 \geq \delta$ . But this last inequality holds because from our assumption we have

$$F_1(x_1^*) = \alpha \frac{(r_1 - \delta)(r_1 + \delta)}{4r_1} \geq 0.$$

This ends the proof of Lemma 1.

## 5.2 Appendix 2

In order to find conditions for the inequality  $\frac{x_2^*}{1-\alpha} \leq \frac{x_1^*}{\alpha}$  to be true, we know that it is equivalent to consider the inequality  $T(\frac{x_2^*}{1-\alpha}) \leq T(\frac{x_1^*}{\alpha})$ . This last inequality becomes

$$\frac{\alpha}{1-\alpha} \frac{(r_1 - \delta)(r_1 + \delta)}{4r_1} \leq T\left(\frac{x_1^*}{\alpha}\right) = \frac{r_1 - \delta}{2r_1} \left[ \frac{2r_2pq}{c} \left(\frac{r_1 - \delta}{2r_1}\right)^2 - \left(\frac{pq}{c}(r_2 - \delta) + r_2\right) \frac{r_1 - \delta}{2r_1} - \delta \right]$$

that is equivalent to

$$\alpha(r_1 + \delta) \leq (1 - \alpha) \left( \frac{r_1 - \delta}{r_1} \left( \theta \delta \frac{r_1 - r_2}{r_1} - r_2 \right) - 2\delta \right).$$

1) If  $r_1 > r_2$  the right hand member has to be positive, this implies the following condition on  $\theta$ :

$$\theta > \theta_0 := \left( \frac{2r_1}{r_1 - \delta} + \frac{r_2}{\delta} \right) \frac{r_1}{r_1 - r_2}.$$

Now if this condition on  $\theta$  holds, from the previous inequality we should deduce (9).

2) If  $r_1 \leq r_2$ , the right hand member is always negative and therefore  $(x_1^*, x_2^*)$  can't be normal.

This ends the proof of Lemma 2.

### 5.3 Appendix 3

We consider the case where the manager has the size of the preserved area as control. We will first prove that the diffusion coefficient has to depend on this size.

We start with the dynamics and the objective given in the second variation (§ 3.2). We suppose that the manager has to maximise his objective by using the fishing effort  $E$  and the size of the preserved area that is captured by  $\alpha$ . We always denote by  $z$  the stock of the total population and the two sub-populations stocks by  $x_1$  and  $x_2$  respectively. Then the densities in the two regions are  $d_1 = \frac{x_1}{\alpha}$  and  $d_2 = \frac{x_2}{1-\alpha}$ .

When  $\alpha = 0$  we can only find a single zone and thus  $x_1 = 0$  and  $x_2 = z$ . In this case, it is natural to set for the densities:  $d_1 = 0$  and  $d_2 = z$  respectively. Now if  $\alpha = 1$ , it is natural to set:  $d_1 = z$   $d_2 = 0$ .

As we have done before, we assume that some diffusion exists between the two zones and that it is proportional to the difference of the two densities. Therefore in order to respect our previous remark, we have to set

$$\lambda(\alpha)\left(\frac{x_2}{1-\alpha} - \frac{x_1}{\alpha}\right) \quad (20)$$

where the diffusion coefficient depends on  $\alpha$ . Indeed if  $\alpha = 0$ , from  $d_1 = 0$  we deduce that  $\lambda(0)\frac{x_2}{1-\alpha} = \lambda(0)z = 0$  because in this case we can only find a single zone, and thus  $\lambda(0) = 0$ . From a similar argument, we deduce that for  $\alpha = 1$ , we have  $\lambda(1) = 0$ . Now for  $\alpha \in ]0, 1[$  the coefficient  $\lambda(\alpha)$  is certainly not null.

For instance we can let

$$\lambda(\alpha) = \lambda_0\alpha(1-\alpha) \quad (21)$$

and in this case the diffusion is modelised by

$$\lambda_0\alpha(1-\alpha)\left(\frac{x_2}{1-\alpha} - \frac{x_1}{\alpha}\right). \quad (22)$$

This expression is the one proposed by Boncoeur in [4].

Then the problem of the manager becomes in this setting

$$\begin{aligned} \max_{E(\cdot), \alpha} \quad & \int_0^\infty e^{-\delta t} (pq\frac{x_2(t)}{1-\alpha} - c)E(t) dt \\ \text{s.c.} \quad & (12) \end{aligned} \quad (23)$$

where the diffusion coefficient in (12) is given by (21).

The solution of this problem is straightforward using the Pontryagin maximum principle. We won't mention it in this paper.

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