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"A note on the Entropy Solutions of the Hydrodynamic Model of Traffic Flow" revisited

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This note revisits a paper from Velan and Florian [Velan and Florian(2002)] dealing with the entropy condition in traffic flow models. It aims to clarify the application of this condition for non-differentiable fundamental diagrams and then to correct some misunderstandings that appear in the above-mentioned paper. Notably, this note clearly exhibits that the non-smoothness of the fundamental diagram does not change the properties of the LWR solutions: (i) existence of a unique entropy solution and (ii) non-uniqueness of weak solutions. These precisions are important because piecewise linear fundamental diagrams appear to accurately fit with experimental observations and cannot be disproved on an alleged mathematical basis.

1. Introduction

[Velan and Florian(2002)] propose a complete description of the entropy condition for the solutions of the LWR model. This condition aims to select a unique among all the weak solutions. Entropy conditions were first studied independently by [Oleinik(1957)] and [Lax(1957)] and their existence and uniqueness proven by [Kružkov(1970)]. Several authors have then reformulated this condition to emphasize its physical meaning. For instance, [Ansorge(1990)] proved that the entropy condition is equivalent to what he calls the driver's ride impulse, i.e. drivers either (i) tend to instantaneously decelerate when crossing a discontinuity corresponding to an increase in the density profile or (ii) smooth their speed increase in the reverse case. It is worth noticing that this equivalence has only been demonstrated for a strictly concave and continuously differentiable fundamental diagram. [Lebacque(1996)] shows that the entropy condition can be rephrased by stating that the flow should always be locally maximized.

A central question in [Velan and Florian(2002)] is the influence of a non differentiable fundamental diagram on the solutions of the LWR model. This question is crucial because experimental observations put credit on piecewise linear fundamental diagrams (PLFD) [Leclercq(2005), Chiabaut et al.(2009)Chiabaut, Buisson, and Leclercq] and especially on triangular ones. In [Velan and Florian(2002)] it is claimed that, with the latter diagrams, the solution of the LWR model is unique but non-entropic. This note aims to invalidate this result. Considering a triangular fundamental diagram, we will demonstrate (i) that the weak solutions of the LWR are not unique and (ii) that the solution which is usually retained (and claimed to be unique) is in fact the unique weak entropy solution in the sense of Kruskov [Kružkov(1970)]. This means that PLFD cannot be disproved on an alleged mathematical basis, since contrary to what is claimed in [Velan and Florian(2002)], the entropy criterion is indeed respected by its solutions.

2. Entropy solutions for Lipschitz-continuous fluxes

For simplicity, we restrict ourselves to the case where Q depends only on k, and not on the variables x and t (this is also the case in [Velan and Florian(2002), Section 6]). Consider the LWR model:

$$\begin{aligned} &\partial_t k + \partial_x(Q(k(x,t)) = 0, \\ &k(x,0) = k_0(x), x \in \mathbb{R}, \end{aligned} \tag{1}$$

under the following assumptions:

$$Q \text{ is a Lipschitz-continuous function from } \mathbb{R} \text{ to } \mathbb{R}, \tag{2a}$$
$$k_0 \in L^\infty(\mathbb{R}) \tag{2b}$$

Note that the assumption (2a) is satisfied by any piecewise linear continuous flux function, and thus by the PLFD. The proper mathematical formulation of this problem, along with the existence and uniqueness theory are now wellknown, see e.g. [Godlewski and Raviart(1991), Smoller(1982), Dafermos(2000), Serre(1999), Bressan(2000)]. In particular, it is wellknown (e.g. [Dafermos(2000), Theorem 6.2.1 p. 86]) that the system (2) admits a unique entropy weak solution, which is defined as follows.

Definition 2.1 (Entropy weak solution) *Under assumptions* (2),

- *a* ***weak solution*** *to* (1) *is a function* $k \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ *such that*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (k\partial_t\varphi + Q(k)\partial_x\varphi)\,dxdt + \int_{\mathbb{R}} k_0(x)\varphi(x,0)dx = 0, \forall\varphi \in C^1_c(\mathbb{R}\times\mathbb{R},\mathbb{R}). \tag{3}$$

- *an* ***entropy weak solution*** *to* (1) *is a function* $k \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ *such that, for any convex function* $\eta \in C^2(\mathbb{R})$ *(the entropy) and any associated entropy flux, i.e. any function* Φ *such that* $\Phi' = Q'\eta'$ *a.e. (that is almost everywhere in the sense of the Lebesgue measure).*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (\eta(k)\partial_t\varphi + \Phi(k)\partial_x\varphi)\,dxdt + \int_{\mathbb{R}} \eta(k_0(x))\varphi(x,0)dx \geq 0, \forall\varphi \in C^1_c(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}_+). \tag{4}$$

Let us now consider the so-called Riemann problem associated to (1), that is Problem (1) with the following initial condition:

$$k_0 = \begin{cases} k_\ell \text{ for } x<0, \\ k_r \text{ for } x>0, \end{cases} \quad \text{with } (k_\ell,k_r)\in\mathbb{R}^2. \tag{5}$$

Let $\sigma \in \mathbb{R}$, and let $\tilde{k} \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ be the function defined a.e. by:

$$\tilde{k}(x,t) = \begin{cases} k_\ell \text{ for } x<\sigma t, \\ k_r \text{ for } x>\sigma t. \end{cases} \tag{6}$$

We then give the celebrated Oleinik condition [Oleinik(1957)] for the characterization of the entropy weak solution, which gives a handy way [Dafermos(1972)] to check whether a discontinuous solution is entropic or not. We shall use this wellknown condition in Section 3 to construct the entropy solutions for the Riemann problem in the case of the piecewise linear diagram flux function.

Corollary 2.1 *Under assumptions* (2)*, let* k_ℓ *and* $k_r \in \mathbb{R}$*,* $k_\ell \neq k_r$*. Let* $I(k_\ell,k_r) = \{\theta k_\ell + (1-\theta)k_r, \theta \in [0,1]\}$ *be the interval with end points* k_ℓ *and* k_r*. Let* C *be the affine function defined by* $k \in I(k_\ell,k_r) \mapsto C(k) = Q(k_\ell) + \sigma(k-k_\ell)$*, with* $\sigma = \dfrac{Q(k_\ell)-Q(k_r)}{k_\ell-k_r}$ *(see Figure 1). Then the function* $\tilde{k}$ *defined by* (6) *is the unique weak entropy solution to* (1)*,* (5) *if and only if the following condition is satisfied:*

$$\forall k \in I(k_\ell,k_r),\ (k_\ell-k_r)\,(Q(k)-C(k)) \leq 0. \tag{7}$$

which can also be written:

$$\begin{cases} Q(k) \geq C(k), \forall k \in [k_\ell,k_r] & \text{if } k_\ell < k_r, \\ Q(k) \leq C(k), \forall k \in [k_\ell,k_r] & \text{if } k_\ell > k_r. \end{cases} \tag{8}$$

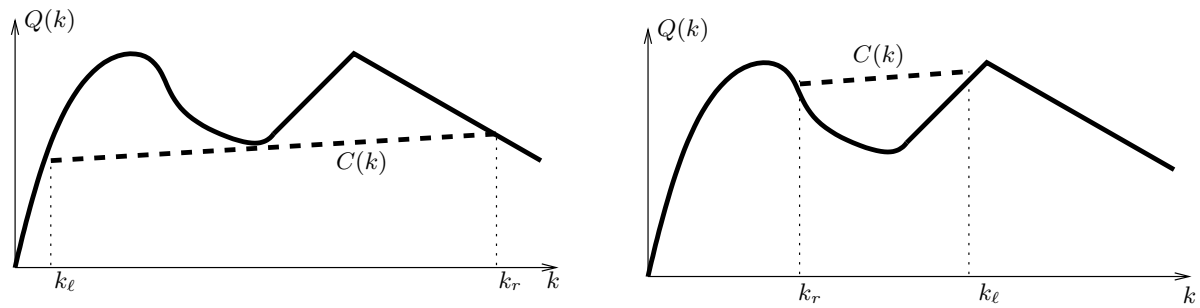

Figure 1 **Admissible discontinuities in an entropy weak solution: if $k_\ell < k_r$ the flux is above its chord; if $k_\ell > k_r$, the flux is under its chord.**

Another well-known condition is the Lax entropy condition [Lax(1957), Lax(1973)], which states that if the function Q is continuously differentiable, then the function $\tilde{k}$ is an entropy weak condition if and only if

$$Q'(k_\ell) > \sigma > Q'(k_r).$$

An important point here is that this condition is no longer valid if Q is not continuously differentiable. Therefore, the Lax condition does not hold in the case of a piecewise linear diagram. Note that if the flux is differentiable and concave or convex, then the Lax condition and the Oleinik condition are equivalent.

We stated the Oleinik and Lax conditions for a solution consisting of two constant states separated by one line of discontinuity. Of course, the same conditions hold for a weak solution to be an entropy weak solution if the considered weak solution to the Riemann problem consists of several constant states separated by discontinuities and satisfies the Rankine Hugoniot for each discontinuity. In fact, a usual way to solve the Riemann problem (1),(5) is to first construct weak solutions which are self-similar (that is such that if $k(x_1,t_1) = k(x_2,t_2)$ then $\frac{x_1}{t_1} = \frac{x_2}{t_2}$), and which consist of constant states that are separated either by discontinuities or by regular zones (usually refered to as rarefaction waves or fans). For each discontinuity, one then checks whether the entropy condition is satisfied or not. Discontinuities may be either shocks, in this case the entropy inequality is strict, or contact discontinuities, in which case the entropy inequality is an equality. If the entropy condition is satisfied, we are assured by the uniqueness result that this piecewise regular or constant solution thus constructed is the unique entropy weak condition to (1),(5). If it is not,then one should construct another weak solution of this type: again by the existence, we are assured that there exists at least one.

3. Weak and entropy weak solutions of the Riemann problem for the piecewise linear diagram

Let k_c and k_m be the so-called critical and maximum densities, $0 < k_c < k_m$. Let u and $w \in \mathbb{R}_+$ be such that $uk_c = w(k_c - k_m)$. A linear piecewise diagram Q is a function from $[0, k_m]$ to $\mathbb{R}_+$ defined by:

$$Q(k) = \begin{cases} uk \text{ for } 0 \le k \le k_c, \\ w(k_m - k) \text{ for } k_c \le k \le k_m. \end{cases} \tag{9}$$

Let us give the weak and entropy weak solutions of the Riemann problem for this flux. Consider first the (easy) cases $k_\ell < k_r \le k_c$ (or $k_r < k_\ell \le k_c$) and $k_c \le k_\ell < k_r$ (or $k_c \le k_r < k_\ell$). In both cases, the values k_ℓ and k_r of the initial data lie on one side of k_c, so that solving the Riemann problem amounts to solving a linear transport equation, namely $k_t + (uk)_x = 0$ if $k_\ell < k_r \le k_c$ and $k_t + (w(k_m - k))_x = 0$ if $k_c \le k_\ell < k_r$. For both cases, the weak solution is unique: it is in fact the entropy weak solution and all entropy inequalities are equalities. These cases are described in Figure 2. For clarity, give the solutions in both (x,t) and (t,x) referentials.

Next we turn to the case $k_\ell < k_c < k_r$, depicted in Figure 3. The weak solution with constant states k_ℓ and k_r is also an entropy weak solution since it satisfies the Oleinik condition (7): the chord is under the flux function. Hence this is a case of uniqueness of the weak solution.

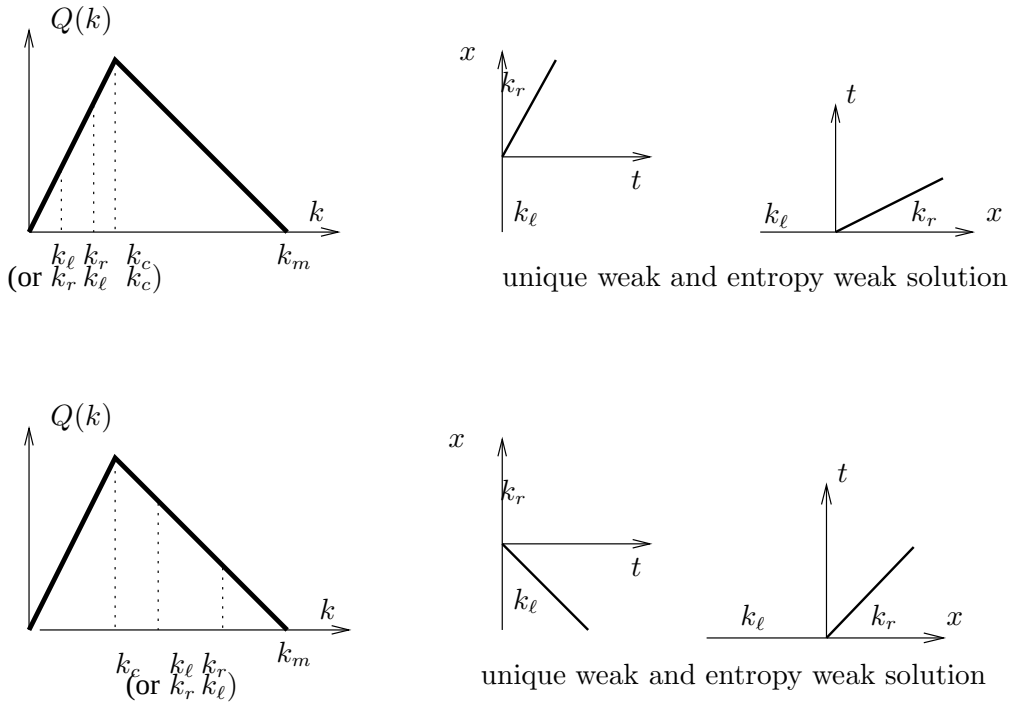

Figure 2 **Weak and entropy weak solution to the PLD Riemann problem: the "linear" cases**

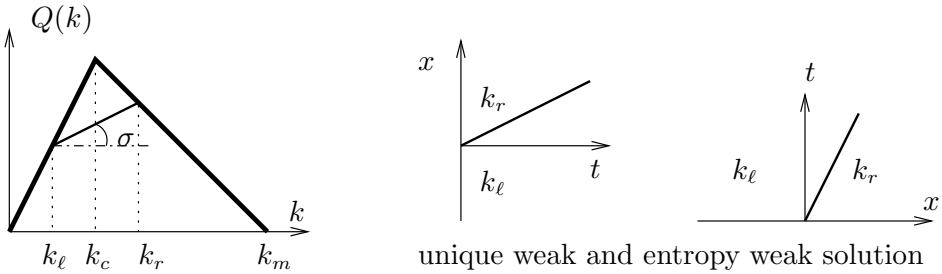

Figure 3 **Weak and entropy weak solution to the PLD Riemann problem: case of one weak (and entropy weak) solution**

Finally, we consider the case $k_r < k_c < k_\ell$ in Figure 4. The weak solution with constant states k_ℓ and k_r is no longer an entropy weak solution since it does not satisfy the Oleinik condition (7) (the chord is under the flux function, whereas it should be above). However, one obtains an entropy weak function by considering an intermediate constant state k_c. Both discontinuities $k_r|k_c$ and $k_c|k_\ell$ satisfy the Oleinik condition (the chord is actually the flux function itself), so that the piecewise constant function depicted on the bottom diagram of Figure 4 is the unique entropy weak solution. Note that this entropy weak solution is the one usually retained in traffic flow.

4. Discussion

This note clearly shows that the non-smoothness of the fundamental diagram does not change the properties of the LWR solutions: (i) there exists a unique entropy weak solution of the LWR model and (ii) weak solutions are not unique when $k_\ell > k_c > k_r$. However, it is important to note that the driver's ride impulse rule exhibited by Ansorge [Ansorge(1990)] to identify entropic solutions does not hold for non-smooth fundamental diagrams. As Velan and Florian [Velan and Florian(2002)] based their proof in section 6 on this argument, this explains the misunderstandings (see p. 443, column 2, line 8 ; p. 443, column 2, line 41, p. 444, column 2, line 1). Indeed, the Lax condition which was used in [Ansorge(1990)] is not valid for non differentiable functions, and therefore we used here the Oleinik condition to find all entropy weak solutions of the LPD Riemann problem.

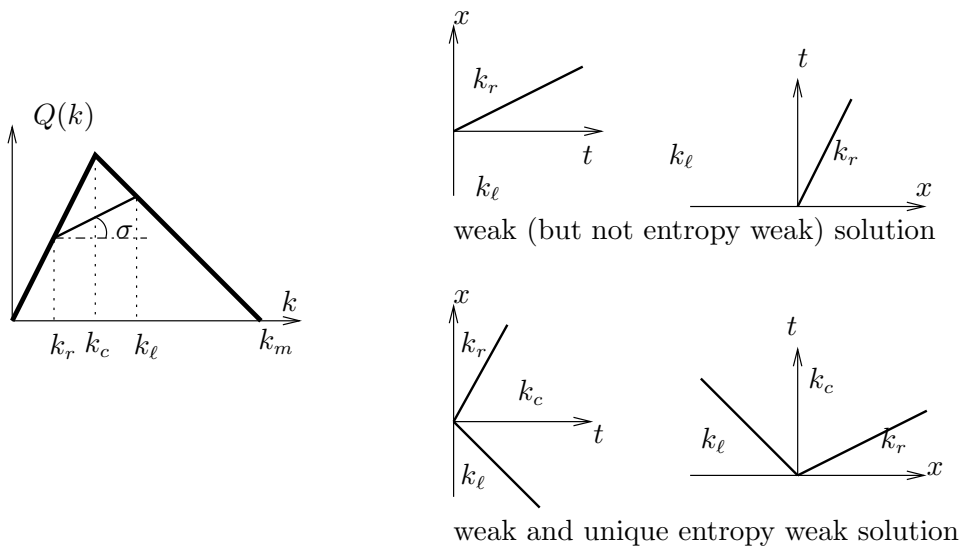

Figure 4 Weak and entropy weak solution to the PLD Riemann problem: case of two weak solutions (and one entropy weak solution)

Thus, non-smooth fundamental diagrams and especially piecewise linear ones provide solutions that perfectly respect the entropy condition. Such diagrams cannot then be disproved on a mathematical basis. In fact, the main difference between piecewise linear fundamental diagrams and smoother ones is platoon behavior during acceleration phase. In the first case, platoons remain stable and successive vehicles accelerate at the same rate. In the latter case, some platoon dispersion is observed, i.e. front vehicles drive faster than middle ones. Experimental evidences appear to favor the first situation (see for example the NGSIM dataset) but this question mainly remains open in the traffic flow community.

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