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Abstract: This work concerns robust static output feedback controller (SOFC) design for uncertain and disturbed Takagi-Sugeno (TS) systems using an H-infinity criterion. The main result is based on a descriptor formulation of the closed-loop dynamics. The proposed approach allows avoiding appearance of crossing terms between the controller's and the TS system's input matrices leading to easier LMI formulation than existing studies in the literature. Moreover, the proposed SOFC design conditions don't require any restrictions on the output equation and allow dealing with unmeasurable premise variables. Indeed, taking advantage of the uncertain TS modeling, nonlinearities associated to unmeasurable premises variables can be reported from the nominal part to the uncertainties. To provide LMIs of less conservatism, the results are conducted in the non-quadratic framework. Finally, two numerical examples and a benchmark of a crane system are proposed to illustrate the efficiency of the SOFC design methodology.

Keywords: Takagi-Sugeno models, Robust Static Output Feedback Controller, Descriptor Redundancy, Non-Quadratic Stabilization, LMI, H∞ criterion, Unmeasurable premise variables.

1. Introduction

Takagi-Sugeno fuzzy systems (Takagi and Sugeno, 1985) have shown their interests since they allow extending some of the linear control concepts to the nonlinear cases (Takagi and Wang, 2001). Indeed, a TS fuzzy model is a collection of linear time invariant systems blended together with nonlinear membership functions. Therefore, convenient control approaches for such systems have been proposed through the concept of Parallel Distributed Compensation (PDC) (Takagi and Wang, 2001; Sala *et al.*, 2005). PDC controllers design has been studied using a quadratic Lyapunov functions, see (Takagi and Wang, 2001; Takagi *et al.*, 1996; Sala *et al.*, 2005) and references therein. These approaches remain conservative since they need to find common Lyapunov matrices for a set of linear matrix inequalities (LMI) constraints. Thus, many ways have been proposed to relax these conditions. For instance, relaxation schemes have been developed based on rewriting the closed-loop interconnection structure of the considered control plant (Xiaodong and Qingling, 2003; Tuan *et al.*, 2001). Other works have considered piecewise Lyapunov functions (Johansson *et al.*, 1999) and, more accurately with the fuzzy aggregation of TS models, through a non-quadratic approach (also called fuzzy Lyapunov approach) (Tanaka *et al.*, 2003; Guerra and Vermeiren, 2004; Rhee and Won, 2006; Feng, 2006; Lam and Leung, 2007; Chang and Yang, 2010; Guerra *et al.*, 2012). For a review of relaxation issues, one can refer to (Sala, 2009).

Among control theory, regarding to output stabilization of TS fuzzy models, many works have been done for observer-based controller (OBC) design (Tanaka *et al.*, 1998; Ma *et al.*, 1998; Yoneyama *et al.*, 2000; Yoneyama *et al.*, 2001; Guerra *et al.*, 2006; Mansouri *et al.*, 2009), dynamic output feedback controller (DOFC) design in both the quadratic and the non-quadratic case (Li *et al.*, 2000; Assawinchaichote *et al.*, 2004; Nguang and Shi, 2006; Zerar *et al.*, 2008; Guelton *et al.*, 2008; Guelton *et al.*, 2009). Static Output Feedback Controllers (SOFC) are of some interests for practical applications since they only require available signals from the plant to be controlled and doesn't need any online differential equation solving, so reducing online computational cost (Syrmos *et al.*, 1997). In (Huang and Nguang, 2006) and (Huang and Nguang, 2007), SOFC design methodologies for TS fuzzy systems are proposed in terms of Bilinear Matrix Inequalities instead of LMI. In (Chadli *et al.*, 2002), LMI conditions have been proposed. However, in these studies, some restrictive modeling assumptions have been made such that the output equation must be linear (full column rank matrix) and without direct transfers from the inputs to the outputs. Note that these restrictions are due to the occurrence of crossing terms in the closed-loop formulation (Zerar *et al.*, 2008; Syrmos *et al.*, 1997; Zhou and Doyle, 1996). Moreover, the results proposed in these previous studies being quadratic, they may suffer from conservatism. Note also that, unmeasurable premises constitute an important problem in static output feedback since it is somewhat difficult to estimate unmeasured variables without introducing a dynamical equation (observer) in the feedback and so reduce the applicative interest of SOFC. This problem has been seldom treated in the literature dealing with SOFC for TS systems and has been the topic of a very recent work where premises selection in SOFC fuzzy structure has been proposed (Tognetti *et al.*, 2012).

In (Tanaka *et al.*, 2007), a descriptor redundancy formulation has been employed to derive new non-quadratic LMI stability conditions for state feedback PDC controllers. One of the interests of such approach is that it allows decoupling crossing terms in the closed-loop dynamics formulation (Bouarar *et al.*, 2010). Therefore, based on this property, LMI based non-quadratic robust DOFC design has been proposed in (Guelton *et al.*, 2008; Guelton *et al.*, 2009). Following this way, one of our preliminary studies has dealt with SOF controller design (Bouarar *et al.*, 2009). In the latter, in aid of the redundancy property, non-quadratic LMI based SOFC design using a fuzzy Lyapunov approach for nominal TS systems has been proposed. However, this preliminary result isn't relevant in the case of unmeasurable premises. Note that the above described studies dealing with continuous time TS models, complementary works has been recently done in the discrete time framework (which is reputed more favorable than the continuous time case for LMI purposes). Indeed, the descriptor redundancy has been recently used for OBC or SOFC design for discrete-time fuzzy models (Chang and Yang, 2011; Chadli and Guerra, 2012).

The goal of this paper is to propose a descriptor redundancy approach leading to strict LMI based robust non-quadratic SOFC design for the class of continuous time uncertain and disturbed TS systems without any restrictions on the output equations. Taking advantage of the uncertain plant, it will be emphasis through an example how this modeling approach is relevant to deals with unmeasured premise variables for SOFC design. Moreover, the non-quadratic framework will be considered with recent conservatism improvements (Mozelli *et al.*, 2009b).

This paper is organized as follows. First, convenient notations and lemma will be described. Then, in section 3, the problem statement of robust SOFC design for TS fuzzy models is presented with highlights on the descriptor redundancy approach and unmeasurable premises. Afterward, relaxed non-quadratic LMI based robust SOFC design conditions are proposed for the considered class of uncertain TS fuzzy models. Then, an extension to uncertain TS models subject to external disturbances is proposed by the use of a H_∞ criterion. Finally, two numerical examples are provided to show the effectiveness of the proposed approach and an engineering example of a crane system is proposed as a benchmark in simulation.

2. Notations and Lemma

In the sequel, when there is no ambiguity, the time t in a time varying variable will be omitted for space convenience. As usual, in a matrix, $(*)$ indicates a symmetrical transpose quantity. Moreover I denotes an identity matrix with appropriate dimension. Let us consider $h_i(z)$ as scalar convex functions, the matrices Y_i and T_{ij} for, $i \in \{1,\dots,r\}$ and $j \in \{1,\dots,l\}$, with appropriate dimensions, we will denote $Y_h = \sum_{i=1}^{r} h_i(z) Y_i$, $T_{hh} = \sum_{k=1}^{r}\sum_{i=1}^{r} h_k(z) h_i(z) T_{ik}$. Finally, one denotes $\dot{X}_h = \frac{d\left(\sum_{i=1}^{r} h_i(z) X_i\right)}{dt}$ and

$$\overset{\cdot}{\overbrace{\left(X_h\right)^{-1}}} = \frac{d\left(\left(\sum_{i=1}^{r} h_i\left(z\right)X_i\right)^{-1}\right)}{dt}.$$

The following lemmas will be useful in the following sections to lead to and to relax LMI conditions.

Lemma 1 (Zhou and Khargonekar, 1988):

For real matrices X and Y with appropriate dimensions and a positive scalar ε, we have:

$$X^T Y + Y^T X \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y \tag{1}$$

Lemma 2 (Tuan *et al.*, 2001): Consider the Parameterized Matrix Inequality (PMI) given by:

$$\sum_{k=1}^{r}\sum_{i=1}^{r} h_k\left(z\right) h_i\left(z\right) \Omega_{ik} < 0 \tag{2}$$

with $h_i\left(z\right) \geq 0$ and $\sum_{i=1}^{r} h_i\left(z\right) = 1$.

The PMI (2) is verified if the matrix inequalities (3) and (4) hold:

$$\Omega_{ii} < 0 \text{, for all } i, j = 1, 2, ..., r \tag{3}$$

$$\frac{1}{r-1}\Omega_{ii} + \frac{1}{2}\left(\Omega_{ij} + \Omega_{ji}\right) < 0 \text{, for } 1 \leq i \neq j \leq r, \tag{4}$$

Note that there exist numerous relaxation schemes in the literature which can be used without affecting the mathematical development of the results proposed below, for a review on these techniques, see (Sala, 2009). Nevertheless, the one presented in lemma 2 (Tuan *et al.*, 2001) constitutes a good compromise between conservatism and computational cost since it doesn't require additional slack variables.

3. Problem statement

Let us consider the class of uncertain and disturbed TS fuzzy systems described by:

$$\begin{cases} \dot{x}(t)=\sum_{i=1}^{r}h_i\left(z(t)\right)\left(\left(A_i+\Delta A_i(t)\right)x(t)+\left(B_i+\Delta B_i(t)\right)u(t)+F_i\varphi(t)\right) \\ y(t)=\sum_{i=1}^{r}h_i\left(z(t)\right)\left(\left(C_i+\Delta C_i(t)\right)x(t)+\left(D_i+\Delta D_i(t)\right)u(t)+G_i\varphi(t)\right) \end{cases} \tag{5}$$

where r represents the number of fuzzy rules. $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^q$, $z(t)\in\mathbb{R}^p$ and $\varphi(t)\in\mathbb{R}^{d\le n}$ represent respectively the state, the input, the output, the premisses and the external disturbances vectors. $h_i\left(z(t)\right)$ are positive membership functions satisfying the convex sum proprieties $h_i\left(z(t)\right)\ge 0$ and $\sum_{i=1}^{r}h_i\left(z(t)\right)=1$. $A_i\in\mathbb{R}^{n\times n}$, $B_i\in\mathbb{R}^{n\times m}$, $C_i\in\mathbb{R}^{q\times n}$, $D_i\in\mathbb{R}^{q\times m}$, $F_i\in\mathbb{R}^{d\times n}$, $G_i\in\mathbb{R}^{d\times q}$ are real matrices. $\Delta A_i(t)\in\mathbb{R}^{n\times n}$, $\Delta B_i(t)\in\mathbb{R}^{n\times m}$, $\Delta C_i(t)\in\mathbb{R}^{q\times n}$ and $\Delta D_i(t)\in\mathbb{R}^{q\times m}$ are Lebesgue measurable uncertainties which can be rewritten as, for $S\equiv A,B,C$ or D, $\Delta S_i(t)=H_i^s f_i^s(t)N_i^s$ (Zhou and Khargonekar, 1988). In that case, for the subscript $s\equiv a,b,c$ or d one has H_i^s, N_i^s constant matrices with appropriate dimensions and $f_i^s(t)$ uncertain matrices bounded such that $\left(f_i^s(t)\right)^T f_i^s(t)\le I$.

Let us consider the following non-PDC SOFC:

$$u(t)=\left(\sum_{i=1}^{r}h_i(z(t))L_i\right)\left(\sum_{i=1}^{r}h_i(z(t))W_i^5\right)^{-1}y(t) \tag{6}$$

where $L_1\in\mathbb{R}^{m\times q}$ and $W_i^5\in\mathbb{R}^{q\times q}$ are real gain matrices to be synthesized.

Remark 1: To be coherent with the aim of SOFC which is to control a dynamical system from the only knowledge of measurable signals, the premises vector $z(t)$ is required to depend only on the inputs $u(t)$, the outputs $y(t)$ and, eventually on measurable state variables. Therefore, in the case where unmeasurable variables $\bar{z}(t)$ exists in the nonlinear model to be controlled, it is still possible to report these nonlinearities from the nominal part to the uncertainties of the relevant uncertain TS model (5) to design a robust non-PDC SOFC (6) composed of membership functions $h_i(z(t))$ which are strictly independent of $\bar{z}(t)$. Therefore, the robust static output feedback controller design proposed in the sequel cope with unmeasurable premises variables. The benefit of uncertain TS modeling in the case of unmeasured variables will be illustrated through the example of a crane system provided in section 6.3.

The classical way to write a closed-loop dynamics consists on substituting (6) into (5) leading, with the above defined notations, to:

$$\begin{aligned}\dot{x}(t)=&\left(\bar{A}_h+\bar{B}_hL_h\left(W_h^5\right)^{-1}\left(I-\bar{D}_hL_h\left(W_h^5\right)^{-1}\right)^{-1}\bar{C}_h\right)x(t)\\&+\bar{B}_hL_h\left(W_h^5\right)^{-1}\left(I-\bar{D}_hL_h\left(W_h^5\right)^{-1}\right)^{-1}G_h\varphi(t)+F_h\varphi(t)\end{aligned} \tag{7}$$

where $\bar{A}_h=A_h+\Delta A_h(t)$, $\bar{B}_h=B_h+\Delta B_h(t)$, $\bar{C}_h=C_h+\Delta C_h(t)$ and $\bar{D}_h=D_h+\Delta D_h(t)y(t)$.

Hence, the closed-loop dynamics (7) involves numerous crossing terms between the gains $L_h\left(W_h^5\right)^{-1}$ and the system's matrices C_h, ΔC_h, B_h, ΔB_h, D_h and ΔD_h. This leads to a strong difficulty to obtain convenient LMI conditions for the design of SOFC (6). Indeed, previous studies on SOFC design have reduced the problem by making restrictive assumptions such as C common and column full rank matrices, $D_h = 0$ and without uncertainties in the output equation (Huang and Nguang, 2006; Huang and Nguang, 2007; Chadli *et al.*, 2002). In the sequel, we will see how this problem of crossing terms in the closed-loop dynamics may be overcame.

In (Guelton *et al.*, 2008; Guelton *et al.*, 2009), LMI based design for DOFC controllers has been proposed in aid of a descriptor approach. These ones are based on an interesting property called the descriptor redundancy (Tanaka *et al.*, 2007; Bouarar *et al.*, 2010; Chen, 2004). In the present study, taking advantage of a redundancy formulation for decoupling crossing terms occurring in (7), equations (5) and (6) can be easily rewritten as:

$$\begin{cases} \dot{x}(t) = \left(A_h + \Delta A_h(t)\right)x(t) + \left(B_h + \Delta B_h(t)\right)u(t) + F_h\varphi(t) \\ 0\dot{y}(t) = -y(t) + \left(C_h + \Delta C_h(t)\right)x(t) + \left(D_h + \Delta D_h(t)\right)u(t) + G_h\varphi(t) \end{cases} \quad (8)$$

and

$$0\dot{u}(t) = -u(t) + L_h\left(W_h^5\right)^{-1} y(t) \quad (9)$$

Note that the redundancy consists on introducing virtual dynamics in the output equations of (8) and in the control law (9). Then, considering the extended state vector $\tilde{x}(t)=\left[x^T(t) \quad y^T(t) \quad u^T(t)\right]^T$, the closed-loop dynamics can be expressed by the following descriptor:

$$\tilde{E}\dot{\tilde{x}}(t)=\left(\tilde{A}_{hh}+\Delta\tilde{A}_h(t)\right)\tilde{x}(t)+\tilde{F}_h\varphi(t) \tag{10}$$

with $\tilde{E}=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\tilde{A}_{hh}=\begin{bmatrix} A_h & 0 & B_h \\ C_h & -I & D_h \\ 0 & L_h\left(W_h^5\right)^{-1} & -I \end{bmatrix}$, $\Delta\tilde{A}_h(t)=\begin{bmatrix} \Delta A_h(t) & 0 & \Delta B_h(t) \\ \Delta C_h(t) & 0 & \Delta D_h(t) \\ 0 & 0 & 0 \end{bmatrix}$ and $\tilde{F}_h=\begin{bmatrix} F_h \\ G_h \\ 0 \end{bmatrix}$.

Therefore, (5) is stabilized via the control law (6) if (10) is stable. Thus, the goal is now to provide sufficient LMI stability conditions allowing to find the matrices L_h and W_h^5 ensuring the stability of (10) . Unlike previous studies on static output feedbackfor continuous time TS models (Huang and Nguang, 2006; Huang and Nguang, 2007; Chadli *et al.*, 2002) where the quadratic stability conditions are not strictly LMI, writing the closed-loop system (10) by the use of descriptor redundancy allows to avoid appearance of crossing terms between the state space matrices and the controller's ones. Therefore, the benefit of this descriptor formulation will be emphasized in the following section since it makes easier to obtain strict LMI based non-quadratic stability conditions without restrictions on the output equation of (5).

4. LMI based conditions for SOFC design

The main result is presented in this section. Let us first focus on the non-quadratic stabilization of uncertain TS systems (5) but without external disturbances ($\varphi(t)=0$).

Theorem 1: Consider, for all $k=1,2,...,r$, ϕ_k the lower bounds of $\dot{h}_k(z)$. The TS fuzzy model (5) (with $\varphi(t)=0$) is asymptotically stable via the non-PDC SOFC (6) if there exist, for all combinations of $i=1,2,...,r$, $1\le i\ne j\le r$ and $k=1,2,...,r$, the matrices $W_j^1=\left(W_j^1\right)^T>0$, W_j^5, W_j^7, W_j^8, W_j^9, L_i, R_{ij} and the scalars ε_{ij}^{1a}, ε_{ij}^{1c}, ε_{ij}^{7b}, ε_{ij}^{7d}, ε_{ij}^{8b}, ε_{ij}^{8d}, ε_{ij}^{9b} and ε_{ij}^{9d} such that the following LMI conditions are satisfied:

$$\Gamma_{ii}<0 \tag{11}$$

$$\frac{1}{r-1}\Gamma_{ii}+\frac{1}{2}\left(\Gamma_{ij}+\Gamma_{ji}\right)<0 \tag{12}$$

$$W_k^1+R_{ij}\ge 0 \tag{13}$$

where $\Gamma_{ij}=\begin{bmatrix}\tilde{\Upsilon}_{ij}+\tilde{H}_{ij} & \tilde{Z}_{ij}^T\\ \tilde{Z}_{ij} & -\tilde{P}_{ij}\end{bmatrix}$,

$$\tilde{\Upsilon}_{ij}=\begin{bmatrix}A_iW_j^1+W_j^1A_i^T+B_iW_j^7+\left(W_j^7\right)^TB_i^T-\sum_{k=1}^{r}\phi_k\left(W_1^k+R_{ij}\right) & (*) & (*)\\ \left(W_j^8\right)^TB_i^T+C_iW_j^1+D_iW_j^7 & -W_j^5-\left(W_j^5\right)^T+D_iW_j^8+\left(W_j^8\right)^TD_i^T & (*)\\ \left(W_j^9\right)^TB_i^T-W_j^7 & \left(W_j^9\right)^TD_i^T+L_i-W_j^8 & -W_j^9-\left(W_j^9\right)^T\end{bmatrix}$$

,

$$\tilde{H}_{ij}=\begin{bmatrix}\varepsilon_{ij}^{1a}H_i^a\left(H_i^a\right)^T+\left(\varepsilon_{ij}^{7b}+\varepsilon_{ij}^{8b}+\varepsilon_{ij}^{9b}\right)H_i^b\left(H_i^b\right)^T+\varepsilon_{ij}^{1c}H_i^c\left(H_i^c\right)^T+\varepsilon_{ij}^{7d}H_i^d\left(H_i^d\right)^T & 0 & 0\\ 0 & \left(\varepsilon_{ij}^{8d}+\varepsilon_{ij}^{9d}\right)H_i^d\left(H_i^d\right)^T & 0\\ 0 & 0 & 0\end{bmatrix}$$

,

$$\tilde{Z}_{ij} = \begin{bmatrix} N_i^a W_j^1 & 0 & 0 \\ N_i^b W_j^7 & 0 & 0 \\ 0 & N_i^c W_j^1 & 0 \\ 0 & N_i^b W_j^8 & 0 \\ 0 & N_i^d W_j^7 & 0 \\ 0 & N_i^d W_j^8 & 0 \\ 0 & 0 & N_i^b W_j^9 \\ 0 & 0 & N_i^d W_j^9 \end{bmatrix} \text{ and } \tilde{P}_{ij} = diag\begin{bmatrix} \varepsilon_{ij}^{1a} I & \varepsilon_{ij}^{7b} I & \varepsilon_{ij}^{1c} I & \varepsilon_{ij}^{8b} I & \varepsilon_{ij}^{7d} I & \varepsilon_{ij}^{8d} I & \varepsilon_{ij}^{9b} I & \varepsilon_{ij}^{9d} I \end{bmatrix}.$$

Proof: Let us consider the non-quadratic candidate fuzzy Lyapunov function given by:

$$v\left(x(t)\right) = \tilde{x}^T(t)\tilde{E}\left(\tilde{W}_h\right)^{-1}\tilde{x}(t) \tag{14}$$

For (16) being a Lyapunov function one needs $v\left(x(t)\right) > 0$, $\dot{v}\left(x(t)\right) < 0$. Thus, classically for descriptor systems, see e.g. (Taniguchi *et al.*, 2000; Bouarar *et al.*, 2007), one needs:

$$\tilde{E}\left(\tilde{W}_h\right)^{-1} = \left(\tilde{W}_h\right)^{-T}\tilde{E} \tag{15}$$

Therefore, let us consider $\tilde{W}_h = \begin{bmatrix} W_h^1 & W_h^2 & W_h^3 \\ W_h^4 & W_h^5 & W_h^6 \\ W_h^7 & W_h^8 & W_h^9 \end{bmatrix}$. Multiplying (15), left by $\tilde{W}_h^T$ and right by $\tilde{W}_h$, one has $\tilde{W}_h^T\tilde{E} = \tilde{E}\tilde{W}_h$ which leads to $W_h^1 = \left(W_h^1\right)^T > 0$ (ensuring $v\left(x(t)\right) > 0$), $W_h^2 = 0$ and $W_h^3 = 0$. Then, the closed-loop system (10) is stable if:

$$\dot{v}\left(x(t)\right) = \dot{\tilde{x}}^T(t)\tilde{E}\left(\tilde{W}_h\right)^{-1}\tilde{x}(t) + \tilde{x}^T(t)\tilde{E}\left(\tilde{W}_h\right)^{-1}\dot{\tilde{x}}(t) + \tilde{x}^T(t)\tilde{E}\overset{\bullet}{\overbrace{\left(\tilde{W}_h\right)^{-1}}}\tilde{x}(t) < 0 \tag{16}$$

Considering (10), (16) is obviously satisfied if:

$$\left(\tilde{A}_{hh}^{T} + \Delta\tilde{A}_{h}^{T}(t)\right)\left(\tilde{W}_{h}\right)^{-1} + \left(\tilde{W}_{h}\right)^{-T}\left(\tilde{A}_{hh} + \Delta\tilde{A}_{h}(t)\right) + \tilde{E}\,\dot{\overline{\left(\tilde{W}_{h}\right)^{-1}}} < 0 \tag{17}$$

Multiplying left by $\tilde{W}_h^T$ and right by $\tilde{W}_h$ and since $\tilde{W}_h^T\tilde{E} = \tilde{E}\tilde{W}_h > 0$, (17) yields:

$$\tilde{W}_{h}^{T}\left(\tilde{A}_{hh}^{T} + \Delta\tilde{A}_{h}^{T}(t)\right) + \left(\tilde{A}_{hh} + \Delta\tilde{A}_{h}(t)\right)\tilde{W}_{h} + \tilde{E}\tilde{W}_{h}\,\dot{\overline{\left(\tilde{W}_{h}\right)^{-1}}}\,\tilde{W}_{h} < 0 \tag{18}$$

It is well-known that $\tilde{W}_h\dot{\overline{\left(\tilde{W}_h\right)^{-1}}}\tilde{W}_h = -\dot{\tilde{W}}_h$, see e.g. (Bouarar *et al.*, 2008). Thus (18) can be rewritten as:

$$\Psi_{hhh} + \Delta\Psi_{hh}(t) - \tilde{E}\dot{\tilde{W}}_{h} < 0 \tag{19}$$

with $\Psi_{hhh} = \tilde{W}_h^T\tilde{A}_{hh}^T + \tilde{A}_{hh}\tilde{W}_h$ and $\Delta\Psi_{hh}(t) = \tilde{W}_h^T\Delta\tilde{A}_h^T(t) + \Delta\tilde{A}_h(t)\tilde{W}_h$.

Extending Ψ_{hhh} with matrices defined in (10), it yields:

$$\Psi_{hhh} = \begin{bmatrix} A_hW_h^1 + W_h^1A_h^T + B_hW_h^7 + \left(W_h^7\right)^TB_h^T & (*) & (*) \\ \left(W_h^8\right)^TB_h^T + C_hW_h^1 - W_h^4 + D_hW_h^7 & -W_h^5 - \left(W_h^5\right)^T + D_hW_h^8 + \left(W_h^8\right)^TD_h^T & (*) \\ \left(W_h^9\right)^TB_h^T + L_h\left(W_h^5\right)^{-1}W_h^4 - W_h^7 & -\left(W_h^6\right)^T + \left(W_h^9\right)^TD_h^T + L_h - W_h^8 & \Psi_{hhh}^{(3,3)} \end{bmatrix} \tag{20}$$

where $\Psi_{hhh}^{(3,3)} = L_h \left(W_h^5\right)^{-1} W_h^6 + \left(W_h^6\right)^T \left(\left(W_h^5\right)^{-1}\right)^T L_h^T - W_h^9 - \left(W_h^9\right)^T$

Let us recall that, due to the nature of the candidate Lyapunov function (14), $W_h^4, W_h^5, ..., W_h^9$ are slack decision matrices that are free of choice. A way to run to LMI conditions is to choose $W_4^h = 0$ and $W_6^h = 0$. Thus, in order to guarantee the invertibility of $\tilde{W}_h = \begin{bmatrix} W_h^1 & 0 & 0 \\ 0 & W_h^5 & 0 \\ W_h^7 & W_h^8 & W_h^9 \end{bmatrix}$ the matrices W_h^5 and W_h^9 must be non-singular and $W_h^1 > 0$. Consequently, (20) becomes:

$$\bar{\Psi}_{hh} = \begin{bmatrix} A_h W_h^1 + W_h^1 A_h^T + B_h W_h^7 + \left(W_h^7\right)^T B_h^T & (*) & (*) \\ \left(W_h^8\right)^T B_h^T + C_h W_h^1 + D_h W_h^7 & -W_h^5 - \left(W_h^5\right)^T + D_h W_h^8 + \left(W_h^8\right)^T D_h^T & (*) \\ \left(W_h^9\right)^T B_h^T - W_h^7 & \left(W_h^9\right)^T D_h^T + L_h - W_h^8 & -W_h^9 - \left(W_h^9\right)^T \end{bmatrix} \tag{21}$$

Now, extending $\Delta\Psi_{hh}(t)$, it yields:

$$\Delta\Psi_{hh}(t) = \begin{bmatrix} \Delta\Psi_{hh}^{(1,1)}(t) & (*) & (*) \\ \Delta\Psi_{hh}^{(2,1)}(t) & \left(W_h^8\right)^T \left(N_h^d\right)^T \left(f_h^d(t)\right)^T \left(H_h^d\right)^T + H_h^d f_h^d(t) N_h^d W_h^8 & (*) \\ \left(W_h^9\right)^T \left(N_h^b\right)^T \left(f_h^b(t)\right)^T \left(H_h^b\right)^T & \left(W_h^9\right)^T \left(N_h^d\right)^T \left(f_h^d(t)\right)^T \left(H_h^d\right)^T & 0 \end{bmatrix} \tag{22}$$

with

$\Delta\Psi_{hh}^{(1,1)}(t) = W_h^1 \left(N_h^a\right)^T \left(f_h^a(t)\right)^T \left(H_h^a\right)^T + H_h^a f_h^a(t) N_h^a W_h^1 + \left(W_h^7\right)^T \left(N_h^b\right)^T \left(f_h^b(t)\right)^T \left(H_h^b\right)^T + H_h^b f_h^b(t) N_h^b W_h^7$

and $\Delta\Psi_{hh}^{(2,1)}(t) = \left(W_h^8\right)^T \left(N_h^b\right)^T \left(f_h^b(t)\right)^T \left(H_h^b\right)^T + H_h^c f_h^c(t) N_h^c W_h^1 + H_h^d f_h^d(t) N_h^d W_h^7$

Expression (22) can be bounded using lemma 1 such that:

$$\Delta\Psi_{hh}\left(t\right) \le \Delta\bar{\Psi}_{hh} = \begin{bmatrix} \Delta\bar{\Psi}_{hh}^{(1,1)} & (*) & (*) \\ 0 & \Delta\bar{\Psi}_{hh}^{(2,2)} & (*) \\ 0 & 0 & \left(\varepsilon_{hh}^{9b}\right)^{-1}\left(W_h^9\right)^T\left(N_h^b\right)^T N_h^b W_h^9 + \left(\varepsilon_{hh}^{9d}\right)^{-1}\left(W_h^9\right)^T\left(N_h^d\right)^T N_h^d W_h^9 \end{bmatrix} \tag{23}$$

with:

$$\begin{aligned}\Delta\bar{\Psi}_{hh}^{(1,1)} = {} & \varepsilon_{hh}^{1a} H_h^a \left(H_h^a\right)^T + \left(\varepsilon_{hh}^{7b} + \varepsilon_{hh}^{8b} + \varepsilon_{hh}^{9b}\right) H_h^b \left(H_h^b\right)^T + \varepsilon_{hh}^{1c} H_h^c \left(H_h^c\right)^T + \varepsilon_{hh}^{7d} H_h^d \left(H_h^d\right)^T \\ & + \left(\varepsilon_{hh}^{1a}\right)^{-1} W_h^1 \left(N_h^a\right)^T N_h^a W_h^1 + \left(\varepsilon_{hh}^{7b}\right)^{-1}\left(W_h^7\right)^T\left(N_h^b\right)^T N_h^b W_h^7\end{aligned}$$

and

$$\begin{aligned}\Delta\bar{\Psi}_{hh}^{(2,2)} = {} & \left(\varepsilon_{hh}^{8d} + \varepsilon_{hh}^{9d}\right) H_h^d \left(H_h^d\right)^T + \left(\varepsilon_{hh}^{1c}\right)^{-1}\left(W_h^1\right)^T\left(N_h^c\right)^T N_h^c W_h^1 + \left(\varepsilon_{hh}^{8b}\right)^{-1}\left(W_h^8\right)^T\left(N_h^b\right)^T N_h^b W_h^8 \\ & + \left(\varepsilon_{hh}^{7d}\right)^{-1}\left(W_h^7\right)^T\left(N_h^d\right)^T N_h^d W_h^7 + \left(\varepsilon_{hh}^{8d}\right)^{-1}\left(W_h^8\right)^T\left(N_h^d\right)^T N_h^d W_h^8\end{aligned}$$

Note that (23) can be rewritten as:

$$\Delta\bar{\Psi}_{hh} = \tilde{H}_{hh} + \tilde{Z}_{hh}^T\left(\tilde{P}_{hh}\right)^{-1}\tilde{Z}_{hh} \tag{24}$$

with

$$\tilde{H}_{hh} = \begin{bmatrix} \varepsilon_{hh}^{1a} H_h^a \left(H_h^a\right)^T + \left(\varepsilon_{hh}^{7b} + \varepsilon_{hh}^{8b} + \varepsilon_{hh}^{9b}\right) H_h^b \left(H_h^b\right)^T + \varepsilon_{hh}^{1c} H_h^c \left(H_h^c\right)^T + \varepsilon_{hh}^{7d} H_h^d \left(H_h^d\right)^T & 0 & 0 \\ 0 & \left(\varepsilon_{hh}^{8d} + \varepsilon_{hh}^{9d}\right) H_h^d \left(H_h^d\right)^T & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{Z}_{hh} = \begin{bmatrix} N_h^a W_h^1 & 0 & 0 \\ N_h^b W_h^7 & 0 & 0 \\ 0 & N_h^c W_h^1 & 0 \\ 0 & N_h^b W_h^8 & 0 \\ 0 & N_h^d W_h^7 & 0 \\ 0 & N_h^d W_h^8 & 0 \\ 0 & 0 & N_h^b W_h^9 \\ 0 & 0 & N_h^d W_h^9 \end{bmatrix} \text{ and } \tilde{P}_{hh} = diag\left[\varepsilon_{hh}^{1a} I \quad \varepsilon_{hh}^{7b} I \quad \varepsilon_{hh}^{1c} I \quad \varepsilon_{hh}^{8b} I \quad \varepsilon_{hh}^{7d} I \quad \varepsilon_{hh}^{8d} I \quad \varepsilon_{hh}^{9b} I \quad \varepsilon_{hh}^{9d} I \right].$$

Therefore, from (21) and (24), (19) is verified if the following condition holds:

$$\bar{\Psi}_{hh} + \tilde{H}_{hh} + \tilde{Z}_{hh}^T \left(\tilde{P}_{hh} \right)^{-1} \tilde{Z}_{hh} - \tilde{E}\dot{\tilde{W}}_h < 0 \tag{25}$$

Applying the Schur complement on (25), one obtains:

$$\begin{bmatrix} \bar{\Psi}_{hh} + \tilde{H}_{hh} - \tilde{E}\dot{\tilde{W}}_h & \tilde{Z}_{hh}^T \\ \tilde{Z}_{hh} & -\tilde{P}_{hh} \end{bmatrix} < 0 \tag{26}$$

Let us now focus on $\dot{W}_h^1 = \sum_{k=1}^{r} \dot{h}_k(z) W_k^1$ included in the term $-\tilde{E}\dot{\tilde{W}}_h$. From the convex property of the membership functions $h_k(z)$ one has $\sum_{k=1}^{r} h_k(z) = 1$, thus $\sum_{k=1}^{r} \dot{h}_k(z) = 0$ and so, for any fuzzy matrices $\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z) h_j(z) R_{ij}$, one has $\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(z) h_j(z) \dot{h}_k(z) R_{ij} = 0$. Therefore, one can write:

$$\dot{W}_h^1 = \sum_{k=1}^{r} \dot{h}_k(z) \left(W_k^1 + R_{ij} \right) \geq \sum_{k=1}^{r} \phi_k \left(W_k^1 + R_{ij} \right) \tag{27}$$

with, for $k=1,...,r$, $W_k^1+R_{ij}\geq 0$ and where ϕ_k are the lower bounds of $\dot{h}_k\left(z\right)$.

Finally, from (26) and (27), after applying lemma 2, (26) is satisfied if the conditions proposed in theorem 1 hold. That ends the proof. ■

Remark 2: Equation (27) improve the proposed relaxation for non-quadratic TS based stability conditions proposed in (Mozelli *et al.*, 2009b). Indeed, in the latter study a common slack decision variable R has been introduced instead of fuzzy distributed ones R_{ij}, for $i=1,...,r$ and $j=1,...,r$. Moreover, it also improve the SOFC design conditions proposed in (Bouarar *et al.*, 2009) for TS systems without uncertainties where a particular case of the present relaxation is considered with $R_{ij}=-W_1^r$. Note also that it can be argue that introducing fuzzy distributed slack variables may increase the number of decision variables and so the computational cost. Nevertheless, with the growing devices computational capabilities, this concern may be considered as less of a drawback and, if a solution cannot be found from theorem 1 due to computational crashes, it is still possible to check if a solution exists by setting $R_{ij}=R$ common.

5. H∞ controller design

This section aims at extending the previous results to the case of TS fuzzy systems with external disturbances. Hence, considering $\varphi\left(t\right)\neq 0$ and using a H_∞ criterion, the objective is now to stabilize (5) such that the influence of the external disturbance $\varphi\left(t\right)$ on the output behavior is minimized. Let us consider the following H_∞ criterion (Takagi and Wang, 2001):

$$\int_0^{\infty}\left(y^T(t)y(t)-\lambda^2\varphi^T(t)\varphi(t)\right)dt \le 0 \tag{28}$$

Recall that $\tilde{x}(t)=\begin{bmatrix}x^T(t) & y^T(t) & u^T(t)\end{bmatrix}^T$, thus (28) can be rewritten as:

$$\int_0^{\infty}\left(\tilde{x}^T(t)\tilde{Q}\tilde{x}(t)-\lambda^2\varphi^T(t)\varphi(t)\right)dt \le 0 \tag{29}$$

with $\tilde{Q}=\begin{bmatrix}0 & 0 & 0\\ 0 & I & 0\\ 0 & 0 & 0\end{bmatrix}$.

In that case, the stability of the closed-loop system (10) is guaranteed under the constraint (29) if the LMI conditions summarized in the following theorem hold.

Theorem 2: Consider, for all $k=1,2,...,r$, ϕ_k the lower bounds of $\dot{h}_k(z)$. The TS fuzzy model (5) is asymptotically stabilized via the non-PDC SOFC (6) and guarantees the attenuation level $\lambda=\sqrt{\eta}$ if there exist, for all combinations of $i=1,2,...,r$, $1\le i\ne j\le r$ and $k=1,2,...,r$, the matrices $W_j^1=\left(W_j^1\right)^T>0$, W_j^5, W_j^7, W_j^8, W_j^9, L_i, R_{ij} and the positive scalars ε_{ij}^{1a}, ε_{ij}^{1c}, ε_{ij}^{7b}, ε_{ij}^{7d}, ε_{ij}^{8b}, ε_{ij}^{8d}, ε_{ij}^{9b} and ε_{ij}^{9d} such that the following LMI conditions are satisfied.

Minimize $\eta>0$ such that:

$$\Theta_{ii}<0 \tag{30}$$

$$\frac{1}{r-1}\Theta_{ii}+\frac{1}{2}\left(\Theta_{ij}+\Theta_{ji}\right)<0 \tag{31}$$

$$W_k^1+R_{ij}\geq 0 \tag{32}$$

where $\Theta_{ij}=\left[\begin{array}{ccc|c|cc} & & & & 0 & (*) \\ & \tilde{\Upsilon}_{ij}+\tilde{H}_{ij} & & (*) & (*) & (*) \\ & & & & 0 & 0 \\ \hline & \tilde{Z}_{ij} & & -\tilde{P}_{ij} & 0 & 0 \\ \hline 0 & W_j^5 & 0 & 0 & -I & 0 \\ F_i^T & G_i^T & 0 & 0 & 0 & -\eta I \end{array}\right]$ and with the matrices $\tilde{\Upsilon}_{ij}$, $\tilde{H}_{ij}$, $\tilde{Z}_{ij}$ and $\tilde{P}_{ij}$ defined in theorem 1.

Proof: The stability of the closed-loop system (10) is guaranteed, under the constraint (29), if:

$$\dot{v}\left(x(t)\right)+\tilde{x}^T(t)\tilde{Q}\tilde{x}(t)-\lambda^2\varphi^T(t)\varphi(t)<0 \tag{33}$$

That is to say if:

$$\begin{aligned}\tilde{x}^T(t)\Big(\left(\tilde{A}_{hh}^T+\Delta\tilde{A}_h^T(t)\right)\tilde{W}_h^{-1}+\left(\tilde{W}_h^{-1}\right)^T\left(\tilde{A}_{hh}+\Delta\tilde{A}_h(t)\right)+\tilde{E}\dot{\overline{\tilde{W}_h^{-1}}}+\tilde{Q}\Big)\tilde{x}(t)\\ +\varphi^T(t)\tilde{F}_h^T\tilde{W}_h^{-1}\tilde{x}(t)+\tilde{x}^T(t)\left(\tilde{W}_h^{-1}\right)^T\tilde{F}_h\varphi(t)-\lambda^2\varphi^T(t)\varphi(t)<0\end{aligned} \tag{34}$$

which is obviously satisfied if:

$$\begin{bmatrix} \left(\tilde{A}_{hh}^T + \Delta\tilde{A}_h^T(t)\right)\tilde{W}_h^{-1} + \left(\tilde{W}_h^{-1}\right)^T\left(\tilde{A}_{hh} + \Delta\tilde{A}_h(t)\right) + \tilde{E}\dot{\overline{\tilde{W}_h^{-1}}} + \tilde{Q} & (*) \\ \tilde{F}_h^T\tilde{W}_h^{-1} & -\lambda^2 I \end{bmatrix} < 0 \tag{35}$$

Multiplying left by $\begin{bmatrix} \tilde{W}_h^T & 0 \\ 0 & I \end{bmatrix}$ and right by $\begin{bmatrix} \tilde{W}_h & 0 \\ 0 & I \end{bmatrix}$, one obtains:

$$\begin{bmatrix} W_h^T\left(\tilde{A}_{hh}^T + \Delta\tilde{A}_h^T(t)\right) + \left(\tilde{A}_{hh} + \Delta\tilde{A}_h(t)\right)W_h + \tilde{E}W_h\dot{\overline{\tilde{W}_h^{-1}}}W_h + W_h{}^T\tilde{Q}W_h & (*) \\ \tilde{F}_h^T & -\lambda^2 I \end{bmatrix} < 0 \tag{36}$$

Following the same way as for the proof of theorem 1, (36) is satisfied if (32) holds as well as:

$$\begin{bmatrix} \Upsilon_{hh} + \tilde{H}_{hh} + \tilde{Z}_{hh}^T\left(\tilde{P}_{hh}\right)^{-1}\tilde{Z}_{hh} + W_h^T\tilde{Q}W_h & (*) \\ \tilde{F}_h^T & -\lambda^2 I \end{bmatrix} < 0 \tag{37}$$

Note that $\tilde{W}_h^T\tilde{Q}\tilde{W}_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \left(W_h^5\right)^T W_h^5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, using the Schur complement and lemma 2, (30) and (31) yield.

That ends the proof. ■

Discussion on non-quadratic approaches limits:

The LMI conditions proposed in theorems 1 and 2 depend on the lower bounds of $\dot{h}_k(z)$ for $k = 1,...,r$. Even if it is often pointed out as a criticism to fuzzy Lyapunov approaches since these parameters may be difficult to choose, a way to obtain these bounds has been proposed in (Tanaka *et al.*, 2003) in some special cases. Moreover, let us recall that this approach remains one of the least conservative in terms of

LMI based design (Mozelli *et al.*, 2009b). In (Tanaka *et al.*, 2007; Guerra *et al.*, 2007; Guelton *et al.*, 2009), a fuzzy Lyapunov candidate function has been reduced leading to relaxed quadratic stability. Indeed, some elements in the Lyapunov matrix can be set common in order to make the LMIs free of membership function's lower bounds. In the present study, this remains on setting W_1 common matrices in the previous theorems. However, following the latter way, the 'price' to pay for more practical applicability is obviously an increase of the conservatism. An elegant way has also been recently proposed to overcome the knowledge of the membership function derivative bounds in (Bernal and Guerra, 2010; Guerra *et al.*, 2012). However, in these studies, the design goal has been reduced to a local view point and lead to complex LMI formulation which are, at this time only available for standard stability analysis and stabilization. Another approach based on line integral Lyapunov functions, which is not investigated in this paper, has been proposed to avoid appearance of membership function derivatives in non-quadratic stability conditions (Rhee and Won, 2006; Guelton *et al.*, 2010, Mozelli 2009a). Therefore, some further research efforts will have to be done to extend these approaches to SOFC design. However, Let us point out that the goal of the present study is not to reach the difficulties of non-quadratic approaches but to overcome and derive LMI based SOFC design for a general class of TS systems (5) without constraining assumptions on output equation and including the most commonly used and effective non-quadratic approach for conservatism reduction.

6. Numerical examples

In this section, in order to show the efficiency and the applicability of the proposed fuzzy approaches, three examples are considered. The aim of the first one is to compare the conservatism of the approach proposed in theorem 2 without uncertainties, the quadratic stability conditions discussed in the above section (W_1 set as a common matrix in theorem 1 and 2) and the non-quadratic conditions proposed in our preliminary study (Bouarar *et al.*, 2009) through a numerical example. Then, a second example is devoted to show the effectiveness of the SOFC based design on a 4^{th} order and 4 rules numerical uncertain and disturbed TS system containing nonlinearities in the output equation. Finally, the benchmark of a crane system is considered in simulation as a third example to illustrate the validity of the proposed approach on an engineering application with unmeasurable premises variables reported as uncertainties.

6.1. Example 1:

Let us consider the following uncertain and disturbed TS fuzzy model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t) + F_i \varphi(t)\right) \\ y(t) = \sum_{i=1}^{2} h_i\left(z(t)\right)\left(C_i x(t) + D_i u(t) + G_i \varphi(t)\right) \end{cases} \tag{38}$$

with $A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -2 & -4 \\ 10 & -2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $C_1 = \begin{bmatrix} 2 & -10 \\ 5 & -1 \end{bmatrix}$, $C_2 = \begin{bmatrix} -3 & 20 \\ -7 & -2 \end{bmatrix}$,

$D_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $D_2 = \begin{bmatrix} -2 \\ 0.5 \end{bmatrix}$, $F_1 = F_2 = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$, $G_1 = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$,

$y(t) = \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}^T$, $z(t) = y_1(t)$ and $h_1\left(z(t)\right) = \cos^2\left(y_1(t)\right) = 1 - h_2\left(z(t)\right)$.

Using the Matlab LMI Toolbox, the attenuation level value corresponding to the quadratic approach (W_1 set as common matrix and $R_{ij} = -W_1$ in theorem 2) is $\lambda = 1.2758$. This solution can be improved since the non-quadratic LMI conditions are reputed of less conservatism. Nevertheless, in that case, the respectively lower bounds ϕ_1 and ϕ_2 of $\dot{h}_1(z)$ and $\dot{h}_2(z)$, which are difficult to choose in practice, are required. For the sake of generality, one proposes to study the influence of these bounds on the conservatism of the proposed LMI conditions regarding to the above discussed quadratic result and the non-quadratic conditions proposed in our preliminary study (Bouarar *et al.*, 2009). Thus, the attenuation level has been computed from theorem 2 of the present study and theorem 2 in (Bouarar *et al.*, 2009) for $\phi_2 = -1$ and $\phi_1 \in [-4 \quad -0.5]$. These results are summarized in Fig. 1. From the latter, one can conclude that, to stabilize (38) without uncertainties, the proposed non-quadratic approach (theorem 2) leads to the lower conservatism results since the H_∞ performances are improved.

6.2. Example 2:

Let us consider the following 4[th] order-4 rules uncertain and disturbed TS system with nonlinear output equation:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{4} h_i(z(t))\left((A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))u(t) + F_i\varphi(t)\right) \\ y(t) = \sum_{i=1}^{4} h_i(z(t))\left((C_i + \Delta C_i(t))x(t) + (D_i + \Delta D_i(t))u(t) + G_i\varphi(t)\right) \end{cases} \tag{39}$$

where $x(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]^T$, $y(t) = [y_1(t) \quad y_2(t) \quad y_3(t) \quad y_4(t)]^T$, $z(t) \equiv y_1(t)$,

$h_1(z(t)) = \dfrac{1+\sin(y_1(t))}{2}$, $h_2(z(t)) = \dfrac{1+\cos(y_1(t))}{2}$, $h_3(z(t)) = \dfrac{\cos^2(y_1(t))-1}{2}$,

$h_4(z(t)) = 1 - \left(h_1(z(t)) + h_2(z(t)) + h_3(z(t))\right)$ and the nominal system's matrices:

$$A_1 = \begin{bmatrix} -10 & 0.5 & -1 & 0 \\ 0.5 & -3 & 0 & -1 \\ 1 & 1 & -4 & -1 \\ 0.5 & -1 & 0 & -6 \end{bmatrix},\ A_2 = \begin{bmatrix} -3 & -1 & -1 & 0 \\ 1 & -3 & -1.5 & 0.5 \\ 1.5 & 0 & -3.5 & 0 \\ -0.5 & 0.5 & 0 & -8 \end{bmatrix},\ A_3 = \begin{bmatrix} -5 & 0 & 1 & 0 \\ 0.2 & -4 & 0 & 0 \\ 1.5 & 0.5 & -1 & -1 \\ 0.55 & -0.5 & 0 & -4 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -15 & 0.5 & 1 & 0 \\ 0 & -5 & -2.5 & 0 \\ 0.5 & 0 & -1.5 & 0 \\ -1 & 0.4 & 0 & -7 \end{bmatrix},\ B_1 = \begin{bmatrix} 2.5 \\ 1 \\ 2.8 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 1.5 \\ 0.5 \\ -5.5 \\ 0 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.5 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ B_4 = \begin{bmatrix} -0.5 \\ 0.15 \\ -5.5 \\ 0.5 \end{bmatrix},\ C_1 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ -1 & -0.5 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -5.5 & 0.5 & 0 & 0 \\ 0.5 & -1 & 0 & 0 \end{bmatrix},\quad C_3 = \begin{bmatrix} -2.5 & 1.5 & 0 & 0 \\ 0 & -1.5 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0.5 & -2 & 0 & 0 \end{bmatrix},\quad C_4 = \begin{bmatrix} 1 & 0.8 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ -5.5 & 0.6 & 0 & 0 \\ 2.5 & 2 & 0 & 0 \end{bmatrix},\quad D_1 = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.1 \\ -0.2 \\ 0 \\ 0 \end{bmatrix},\ D_3 = \begin{bmatrix} 0.15 \\ 0.2 \\ 0.25 \\ 0 \end{bmatrix},\ D_4 = \begin{bmatrix} 0.35 \\ -0.4 \\ 0.15 \\ 0 \end{bmatrix},\ F_1 = \begin{bmatrix} 0 \\ -0.25 \\ 0 \\ 0 \end{bmatrix},\ F_2 = \begin{bmatrix} -0.25 \\ 0.25 \\ 0 \\ 0 \end{bmatrix},\ F_3 = \begin{bmatrix} 0.2 \\ -0.5 \\ 0.2 \\ 0 \end{bmatrix},\ F_4 = \begin{bmatrix} -0.5 \\ 0.2 \\ 0.6 \\ 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} -0.3 \\ 0.3 \\ 0 \\ 0 \end{bmatrix},\ G_2 = \begin{bmatrix} 0.35 \\ 0.2 \\ 0.1 \\ 0 \end{bmatrix},\ G_3 = \begin{bmatrix} -0.2 \\ 0.35 \\ 0 \\ 0.2 \end{bmatrix},\ G_4 = \begin{bmatrix} 0.3 \\ 0 \\ 0.45 \\ 0.1 \end{bmatrix},$$

as well as the uncertain matrices $\Delta A_i(t) = H_i^a f_a(t) N_i^a$, $\Delta B_i(t) = H_i^b f_b(t) N_i^b$, $\Delta C_i(t) = H_i^c f_c(t) N_i^c$ and $\Delta D_i(t) = H_i^d f_d(t) N_i^d$ with:

$$H_1^a = \begin{bmatrix} 0 \\ 0.1 \\ 0.25 \\ 0.1 \end{bmatrix},\ H_2^a = \begin{bmatrix} 0 \\ -0.1 \\ 0.15 \\ 0.1 \end{bmatrix},\ H_3^a = \begin{bmatrix} 0 \\ 0.15 \\ 0.05 \\ 0.1 \end{bmatrix},\ H_4^a = \begin{bmatrix} 0 \\ -0.2 \\ 0.25 \\ 0.3 \end{bmatrix},\ H_1^b = \begin{bmatrix} 0 \\ -0.2 \\ 0 \\ 0 \end{bmatrix},\ H_2^b = \begin{bmatrix} 0 \\ -0.1 \\ 0 \\ 0 \end{bmatrix},\ H_3^b = \begin{bmatrix} 0 \\ 0.25 \\ 0 \\ 0 \end{bmatrix},$$

$$H_4^b = \begin{bmatrix} 0 \\ 0.2 \\ 0 \\ 0.1 \end{bmatrix},\ H_1^c = \begin{bmatrix} 0.1 \\ -0.1 \\ 0 \\ 0.1 \end{bmatrix},\ H_2^c = \begin{bmatrix} -0.1 \\ 0.1 \\ 0 \\ 0.1 \end{bmatrix},\ H_3^c = \begin{bmatrix} 0 \\ 0.2 \\ 0 \\ 0 \end{bmatrix},\ H_4^c = \begin{bmatrix} -0.2 \\ 0.15 \\ 0 \\ 0 \end{bmatrix},\ H_1^d = \begin{bmatrix} 0.5 \\ -0.5 \\ 0 \\ 0.2 \end{bmatrix},\ H_2^d = \begin{bmatrix} -0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix},$$

$$H_3^d = \begin{bmatrix} -0.5 \\ 0 \\ 0.25 \\ 0.1 \end{bmatrix},\quad H_4^d = \begin{bmatrix} -0.25 \\ 0.2 \\ 0.3 \\ 0 \end{bmatrix},\quad N_1^a = \begin{bmatrix} 0.1 & 0.1 & -0.1 & 0.2 \end{bmatrix},\quad N_2^a = \begin{bmatrix} -0.1 & 0.1 & 0.1 & -0.2 \end{bmatrix},$$

$N_3^a = \begin{bmatrix} -0.1 & -0.1 & 0.1 & 0 \end{bmatrix}$, $N_1^a = \begin{bmatrix} 0 & -0.2 & 0.3 & -0.4 \end{bmatrix}$, $N_1^b = 0.1$, $N_2^b = -0.175$, $N_3^b = 0.15$, $N_4^b = -0.25$,

$N_1^c = [0.1 \quad 0.1 \quad 0.1 \quad 0.15]$, $N_2^c = [-0.1 \quad -0.1 \quad 0.1 \quad -0.2]$, $N_3^c = [0.2 \quad -0.1 \quad 0 \quad -0.15]$, $N_4^c = [-0.2 \quad -0.15 \quad 0.15 \quad -0.25]$, $N_1^d = -0.15$, $N_2^d = 0.05$, $N_3^d = -0.25$, $N_4^d = 0.15$.

The goal is to design a robust H_∞ $H\infty$ controller that guarantee the attenuation of the external disturbances $\varphi(t)$ affecting the system output and state vector. The following SOFC gain matrices, guaranteeing a minimized H_∞ attenuation level $\lambda = 0.8533$, are obtained using the Matlab LMI Toolbox through theorem 2 with the prescribed lower bounds values $\phi_1 = -1$, $\phi_2 = -0.8$, $\phi_3 = -2$, $\phi_4 = -1.5$.

$$W_1^5 = \begin{bmatrix} 1.3685 & 0.0901 & 0.0221 & -0.0121 \\ 0.0901 & 1.1756 & -0.0078 & -0.0488 \\ 0.0221 & -0.0078 & 1.4464 & -0.0746 \\ -0.0121 & -0.0488 & -0.0746 & 1.4055 \end{bmatrix}, \; W_2^5 = \begin{bmatrix} 1.2426 & 0.0375 & -0.0137 & 0.0422 \\ 0.0375 & 1.4033 & -0.0236 & 0.0112 \\ -0.0137 & -0.0236 & 1.1421 & 0.0506 \\ 0.0422 & 0.0112 & 0.0506 & 1.3595 \end{bmatrix},$$

$$W_3^5 = \begin{bmatrix} 1.1476 & 0.1207 & -0.0695 & -0.0071 \\ 0.1207 & 1.2730 & 0.0722 & -0.1092 \\ -0.0695 & 0.0722 & 1.4095 & 0.0409 \\ -0.0071 & -0.1092 & 0.0409 & 1.1276 \end{bmatrix}, \; W_4^5 = \begin{bmatrix} 1.3887 & -0.0662 & -0.0462 & -0.1025 \\ -0.0662 & 1.4318 & -0.0098 & 0.0479 \\ -0.0462 & -0.0098 & 1.2250 & -0.0237 \\ -0.1025 & 0.0479 & -0.0237 & 1.3264 \end{bmatrix},$$

$L_1 = [-0.0842 \quad 0.218 \quad 0.0701 \quad 0.0352]$, $L_2 = [-0.1448 \quad 0.116 \quad 0.1203 \quad -0.0019]$, $L_3 = [0.2542 \quad -0.2131 \quad 0.0919 \quad -0.2372]$ and $L_4 = [-0.0086 \quad 0.0368 \quad 0.006 \quad -0.0153]$.

For simulation realization, the disturbance signal $\varphi(t)$ have been chosen as a white noise with unit variance characterized by a bounded energy added to a step (amplitude 0.5 for $t \in [5s, 10s]$, 0 elsewhere). Moreover, the uncertain variables have been set as $f_a(t) = 0.25\sin(0.5t)$, $f_b(t) = 0.3\cos(2t)$, $f_c(t) = 0.2\dfrac{\sin(\pi t)}{\pi t}$, $f_d(t) = 0.15\cos^2(2t)$. Fig 2, 3 and 4 show respectively the system's state and output responses, the control signal and the disturbance evolutions for the initial conditions $x(0) = [0.5 \quad 0.3 \quad 0.5 \quad -0.5]^T$ and $y(0) = [-0.5 \quad 0.5 \quad 0.5 \quad -0.5]^T$. The synthesized robust

SOFC is correctly stabilizing the system and attenuates the external disturbances effect with a minimized H_∞ attenuation level $\lambda = 0.8533$.

6.3. Example 3:

Let us consider the crane system depicted in figure 5 with the parameters and variables given in table 1. From the well-known Lagrange formalism, the motion equations of the crane system are given by:

$$\begin{cases} a\ddot{r}(t)+c\ddot{\theta}(t)\cos\theta(t)-c\dot{\theta}^2(t)\sin\theta(t)+k_s r(t)+k_d\dot{r}(t)=u(t)+\varphi_1(t) \\ b\ddot{\theta}(t)+c\ddot{r}(t)\cos\theta(t)+cg\sin\theta(t)=\varphi_2(t) \end{cases} \quad (40)$$

with $a = m + M$, $b = mL^2 + I_{yy}$, $c = mL$ and where $\varphi_1(t)$ and $\varphi_2(t)$ are external disturbance, i.e. respectively an external force vector on the cart and an external torque on the pendulum and the parameters given table 1.

By considering the inertia matrix $M(\theta)=\begin{bmatrix} a & c\cos\theta \\ c\cos\theta & b \end{bmatrix}$ and its inverse $M^{-1}(\theta)=\frac{1}{ab-c^2\cos^2\theta}\begin{bmatrix} b & -c\cos\theta \\ -c\cos\theta & a \end{bmatrix}$, (40) can be rewritten as:

$$\begin{cases} \ddot{r}=\dfrac{1}{ab-c^2\cos^2\theta}\left(-bk_s r+bc\dot{\theta}^2\sin\theta+c^2 g\cos\theta\sin\theta-k_d b\dot{r}+bu+b\varphi_1-c\varphi_2\cos\theta\right) \\ \ddot{\theta}=\dfrac{1}{ab-c^2\cos^2\theta}\left(ck_s r\cos\theta-acg\sin\theta-c^2\dot{\theta}^2\cos\theta\sin\theta+ck_d\dot{r}\cos\theta-cu\cos\theta-c\varphi_1\cos\theta+a\varphi_2\right) \end{cases} \quad (41)$$

Note that the robot's velocities $\dot{r}(t)$ and $\dot{\theta}(t)$ aren't considered available for measurement. Therefore, from the dynamical model (41) and to cope with the goal of designing a robust non-PDC SOFC which doesn't require unmeasurable premise variables (see remark 1), the nonlinear terms $bc\dot{\theta}^2(t)\sin\theta(t)$ and $-c^2\dot{\theta}^2(t)\cos\theta(t)\sin\theta(t)$ will be cast as uncertainties. Moreover, the motion of the system being physically restricted, the angular velocity can be bounded such that $|\dot{\theta}(t)| < \alpha$ and so one can denote the nonlinear term $\delta(\dot{\theta}(t)) = \frac{\dot{\theta}^2(t)}{\alpha^2}$, normalized such that $\delta^2(\dot{\theta}(t)) \le 1$. Then, let $x(t) = \begin{bmatrix} r(t) & \theta(t) & \dot{r}(t) & \dot{\theta}(t) \end{bmatrix}^T$ be the state vector of the crane, $y(t) = q = \begin{bmatrix} r(t) & \theta(t) \end{bmatrix}^T$ the output vector, $\varphi(t) = \begin{bmatrix} \varphi_1(t) & \varphi_2(t) \end{bmatrix}^T$ the vector of external disturbances. From (41) an uncertain state space model is given by:

$$\begin{cases} \dot{x}(t) = \left(A(\theta(t)) + \Delta A\left(\theta(t), \delta\left(\dot{\theta}(t)\right)\right)\right)x(t) + B(\theta(t))u(t) + F(\theta(t))\varphi(t) \\ y(t) = Cx(t) \end{cases} \tag{42}$$

with $A(\theta) = \eta_3(\theta)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_s b & c^2 g\eta_1(\theta)\eta_2(\theta) & -k_d b & 0 \\ k_s c\eta_1(\theta) & -acg\eta_2(\theta) & k_d c\eta_1(\theta) & 0 \end{bmatrix}$,

$$\Delta A\left(\theta, \delta(\dot{\theta})\right) = \eta_3(\theta)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & bc\alpha^2\delta(\dot{\theta})\eta_2(\theta) & 0 & 0 \\ 0 & -c^2\alpha^2\delta(\dot{\theta})\eta_1(\theta)\eta_2(\theta) & 0 & 0 \end{bmatrix} = H_a(\theta)\Delta_a(t)N_a,\ \Delta_a(t) = \delta(\dot{\theta}),\ N_a = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T,$$

$$H_a(\theta) = \eta_3(\theta)\begin{bmatrix} 0 \\ 0 \\ bc\alpha^2\eta_2(\theta) \\ -c^2\alpha^2\eta_1(\theta)\eta_2(\theta) \end{bmatrix},\quad B(\theta) = \eta_3(\theta)\begin{bmatrix} 0 \\ 0 \\ b \\ -c\eta_1(\theta) \end{bmatrix},\quad F(\theta) = \eta_3(\theta)\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b & -c\eta_1(\theta) \\ -c\eta_1(\theta) & a \end{bmatrix},$$

$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and where, for $\theta \in [-\theta_0, \theta_0]$, $\eta_1(\theta) = \cos\theta \in [\beta, 1]$ with $\beta = \cos\theta_0$, $\eta_2(\theta) = \frac{\sin\theta}{\theta} \in [\rho, 1]$ with $\rho = \frac{\sin\theta_0}{\theta_0}$, and $\eta_3(\theta) = \frac{1}{ab - c^2\cos^2\theta} \in \left[\frac{1}{ab-c^2}, \frac{1}{ab-c^2\beta^2}\right]$ are bounded nonlinearities.

Therefore, one may apply the sector nonlinearity approach (Tanaka & Wang, 2001) such that:

$$\eta_1(\theta) = \underbrace{\frac{\eta_1(\theta) - \beta}{1-\beta}}_{w_{11}(\theta(t))}(1) + \underbrace{\frac{1-\eta_1(\theta)}{1-\beta}}_{w_{12}(\theta(t))}(\beta) \tag{43}$$

$$\eta_2(\theta) = \underbrace{\frac{\eta_2(\theta) - \rho}{1-\rho}}_{w_{21}(\theta(t))}(1) + \underbrace{\frac{1-\eta_2(\theta)}{1-\rho}}_{w_{22}(\theta(t))}(\rho) \tag{44}$$

$$\eta_3(\theta) = \underbrace{\frac{(ab-c^2\beta^2)\left((ab-c^2)\eta_3(\theta)-1\right)}{c^2(\beta^2-1)}}_{w_{31}(\theta(t))}\left(\frac{1}{ab-c^2\beta^2}\right) + \underbrace{\frac{(ab-c^2)\left(1-(ab-c^2\beta^2)\eta_2(\theta)\right)}{c^2(\beta^2-1)}}_{w_{32}(\theta_1(t))}\left(\frac{1}{ab-c^2}\right) \tag{45}$$

leading to define the 8 following membership functions as $h_1(\theta) = w_{11}(\theta)w_{21}(\theta)w_{31}(\theta)$, $h_2(\theta) = w_{11}(\theta)w_{21}(\theta)w_{32}(\theta)$, $h_3(\theta) = w_{11}(\theta)w_{22}(\theta)w_{31}(\theta)$, $h_4(\theta) = w_{11}(\theta)w_{22}(\theta)w_{32}(\theta)$, $h_5(\theta) = w_{12}(\theta)w_{21}(\theta)w_{31}(\theta)$, $h_6(\theta) = w_{12}(\theta)w_{21}(\theta)w_{32}(\theta)$, $h_7(\theta) = w_{12}(\theta)w_{22}(\theta)w_{31}(\theta)$, $h_8(\theta) = w_{12}(\theta)w_{22}(\theta)w_{32}(\theta)$ and the relevant 8 rules uncertain T-S model of the crane given by:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{8} h_i(\theta(t))\left((A_i + \Delta A_i(t))x(t) + B_i u(t) + F_i\varphi(t)\right) \\ y(t) = Cx(t) \end{cases} \tag{46}$$

with $A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s b}{ab-c^2\beta^2} & \frac{c^2 g}{ab-c^2\beta^2} & -\frac{k_d b}{ab-c^2\beta^2} & 0 \\ \frac{k_s c}{ab-c^2\beta^2} & -\frac{acg}{ab-c^2\beta^2} & \frac{k_d c}{ab-c^2\beta^2} & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s b}{ab-c^2} & \frac{c^2 g}{ab-c^2} & -\frac{k_d b}{ab-c^{22}} & 0 \\ \frac{k_s c}{ab-c^2} & -\frac{acg}{ab-c^2} & \frac{k_d c}{ab-c^2} & 0 \end{bmatrix}$,

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s b}{ab-c^2\beta^2} & \frac{c^2 g\rho}{ab-c^2\beta^2} & -\frac{k_d b}{ab-c^2\beta^2} & 0 \\ \frac{k_s c}{ab-c^2\beta^2} & -\frac{acg\rho}{ab-c^2\beta^2} & \frac{k_d c}{ab-c^2\beta^2} & 0 \end{bmatrix}, \ A_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s b}{ab-c^2} & \frac{c^2 g\rho}{ab-c^2} & -\frac{k_d b}{ab-c^2} & 0 \\ \frac{k_s c}{ab-c^2} & -\frac{acg\rho}{ab-c^2} & \frac{k_d c}{ab-c^2} & 0 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s b}{ab-c^2\beta^2} & \frac{c^2 g\beta}{ab-c^2\beta^2} & -\frac{k_d b}{ab-c^2\beta^2} & 0 \\ \frac{k_s c\beta}{ab-c^2\beta^2} & -\frac{acg}{ab-c^2\beta^2} & \frac{k_d c\beta}{ab-c^2\beta^2} & 0 \end{bmatrix}, \ A_6 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s b}{ab-c^2} & \frac{c^2 g\beta}{ab-c^2} & -\frac{k_d b}{ab-c^2} & 0 \\ \frac{k_s c\beta}{ab-c^2} & -\frac{acg}{ab-c^2} & \frac{k_d c\beta}{ab-c^2} & 0 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s b}{ab-c^2\beta^2} & \frac{c^2 g\beta\rho}{ab-c^2\beta^2} & -\frac{k_d b}{ab-c^2\beta^2} & 0 \\ \frac{k_s c\beta}{ab-c^2\beta^2} & -\frac{acg\rho}{ab-c^2\beta^2} & \frac{k_d c\beta}{ab-c^2\beta^2} & 0 \end{bmatrix}, \ A_8 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s b}{ab-c^2} & \frac{c^2 g\beta\rho}{ab-c^2} & -\frac{k_d b}{ab-c^2} & 0 \\ \frac{k_s c\beta}{ab-c^2} & -\frac{acg\rho}{ab-c^2} & \frac{k_d c\beta}{ab-c^2} & 0 \end{bmatrix},$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{b}{ab-c^2\beta^2} \\ -\frac{c}{ab-c^2\beta^2} \end{bmatrix}, \quad B_2 = B_4 = \begin{bmatrix} 0 \\ 0 \\ \frac{b}{ab-c^2} \\ -\frac{c}{ab-c^2} \end{bmatrix}, \quad B_5 = B_7 = \begin{bmatrix} 0 \\ 0 \\ \frac{b}{ab-c^2\beta^2} \\ -\frac{c\beta}{ab-c^2\beta^2} \end{bmatrix}, \quad B_6 = B_8 = \begin{bmatrix} 0 \\ 0 \\ \frac{b}{ab-c^2} \\ -\frac{c\beta}{ab-c^2} \end{bmatrix},$$

$$F_1 = F_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{b}{ab-c^2\beta^2} & -\frac{c}{ab-c^2\beta^2} \\ -\frac{c}{ab-c^2\beta^2} & \frac{a}{ab-c^2\beta^2} \end{bmatrix},\ F_2 = F_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{b}{ab-c^2} & -\frac{c}{ab-c^2} \\ -\frac{c}{ab-c^2} & \frac{a}{ab-c^2} \end{bmatrix},$$

$$F_5 = F_7 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{b}{ab-c^2\beta^2} & -\frac{c\beta}{ab-c^2\beta^2} \\ -\frac{c\beta}{ab-c^2\beta^2} & \frac{a}{ab-c^2\beta^2} \end{bmatrix},\ F_6 = F_8 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{b}{ab-c^2} & -\frac{c\beta}{ab-c^2} \\ -\frac{c\beta}{ab-c^2} & \frac{a}{ab-c^2} \end{bmatrix},\ H_{a1} = \begin{bmatrix} 0 \\ 0 \\ \frac{bc\alpha^2}{ab-c^2\beta^2} \\ -\frac{c^2\alpha^2}{ab-c^2\beta^2} \end{bmatrix},$$

$$H_{a2} = \begin{bmatrix} 0 \\ 0 \\ \frac{bc\alpha^2}{ab-c^2} \\ -\frac{c^2\alpha^2}{ab-c^2} \end{bmatrix},\ H_{a3} = \begin{bmatrix} 0 \\ 0 \\ \frac{bc\alpha^2\rho}{ab-c^2\beta^2} \\ -\frac{c^2\alpha^2\rho}{ab-c^2\beta^2} \end{bmatrix},\ H_{a4} = \begin{bmatrix} 0 \\ 0 \\ \frac{bc\alpha^2\rho}{ab-c^2} \\ -\frac{c^2\alpha^2\rho}{ab-c^2} \end{bmatrix},\ H_{a5} = \begin{bmatrix} 0 \\ 0 \\ \frac{bc\alpha^2}{ab-c^2\beta^2} \\ -\frac{c^2\alpha^2\beta}{ab-c^2\beta^2} \end{bmatrix},$$

$$H_{a6} = \begin{bmatrix} 0 \\ 0 \\ \frac{bc\alpha^2}{ab-c^2} \\ -\frac{c^2\alpha^2\beta}{ab-c^2} \end{bmatrix},\ H_{a7} = \begin{bmatrix} 0 \\ 0 \\ \frac{bc\alpha^2\rho}{ab-c^2\beta^2} \\ -\frac{c^2\alpha^2\beta\rho}{ab-c^2\beta^2} \end{bmatrix},\ H_{a7} = \begin{bmatrix} 0 \\ 0 \\ \frac{bc\alpha^2\rho}{ab-c^2} \\ -\frac{c^2\alpha^2\beta\rho}{ab-c^2} \end{bmatrix}.$$

For simulation purpose, the maximal angular velocity has been set as $\alpha = 4\pi\, rad \cdot s^{-1}$ and the maximal angular position as $\theta_0 = \frac{\pi}{2}$. A convenient non-PDC SOFC (6) has been designed through theorem 2 and the Matlab LMI toolbox with the bounds of the membership functions sets as $\phi_i = -10$. The result is given by the following gain matrices for a minimal attenuation level $\lambda = 0.8341$:

$$W_1^5 = \begin{bmatrix} 0.9667 & 0 \\ 0 & 0.9588 \end{bmatrix},\ W_2^5 = \begin{bmatrix} 0.9668 & 0 \\ 0 & 0.9588 \end{bmatrix},\ W_3^5 = \begin{bmatrix} 0.9663 & 0 \\ 0 & 0.9588 \end{bmatrix},\ W_4^5 = \begin{bmatrix} 0.9675 & 0 \\ 0 & 0.9588 \end{bmatrix},$$

$W_5^5 = \begin{bmatrix} 0.964 & 0 \\ 0 & 0.9588 \end{bmatrix}$, $W_6^5 = \begin{bmatrix} 0.9639 & 0 \\ 0 & 0.9588 \end{bmatrix}$, $W_7^5 = \begin{bmatrix} 0.9646 & 0 \\ 0 & 0.9588 \end{bmatrix}$, $W_8^5 = \begin{bmatrix} 0.9638 & 0 \\ 0 & 0.9588 \end{bmatrix}$,

$L_1 = \begin{bmatrix} 0.2202 & \text{-}0.0009 \end{bmatrix}$, $L_2 = \begin{bmatrix} 0.2555 & \text{-}0.0021 \end{bmatrix}$, $L_3 = \begin{bmatrix} 0.2245 & \text{-}0.0001 \end{bmatrix}$, $L_4 = \begin{bmatrix} 0.2202 & \text{-}0.0007 \end{bmatrix}$,

$L_5 = \begin{bmatrix} 0.1238 & 0.0022 \end{bmatrix}$, $L_6 = \begin{bmatrix} 0.0906 & 0.0033 \end{bmatrix}$, $L_7 = \begin{bmatrix} 0.1793 & 0.002 \end{bmatrix}$ and $L_8 = \begin{bmatrix} 0.1612 & 0.0011 \end{bmatrix}$.

For simulation purpose, the external disturbances $\varphi_1(t)$ and $\varphi_2(t)$ has been respectively set as white noises with unit variance characterized by bounded energy added to steps such that $\varphi_1 \approx 10$ for $t \in [1s, 2s]$ (0 elsewhere), and $\varphi_2 \approx -10$ for $t \in [3s, 3.5s]$ (0 elsewhere). Simulations where performed with and without external disturbance to highlight the efficiency of the H_∞ attenuation. Fig 6 and 7 show respectively the system's state, the control signal and the disturbance evolutions for the initial conditions $x(0) = \begin{bmatrix} 1 & \frac{\pi}{4} & 0 & 0 \end{bmatrix}^T$. It can be conclude that the synthesized robust non-PDC SOFC is correctly controlling the crane system in spite of the presence of external disturbances.

7. Conclusion

In this paper, the problem of robust static output feedback stabilization for continuous time uncertain and disturbed Takagi-Sugeno models has been considered. A non-PDC static output feedback control law has been proposed and its design has been involved through a fuzzy Lyapunov approach. Thanks to the descriptor redundancy, crossing terms have been avoided in the closed-loop dynamic formulation and so LMI conditions have been obtained without any assumptions on the output equation of the considered TS model. Then, a H_∞ criterion has been employed to derive conditions which ensure a minimal attenuation level of external disturbances. It has been shown that the proposed SOFC based design lead to low conservatism results regarding to previous works. Moreover, it has been emphasis that, thank to the uncertain TS modeling, such approach is suitable for the design of a robust non-PDC static output feedback controller without the need of unmeasurable variables estimations. Finally, two academic examples has been considered to illustrate the efficiency the proposed fuzzy Lyapunov based SOFC design.

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Figures and tables :

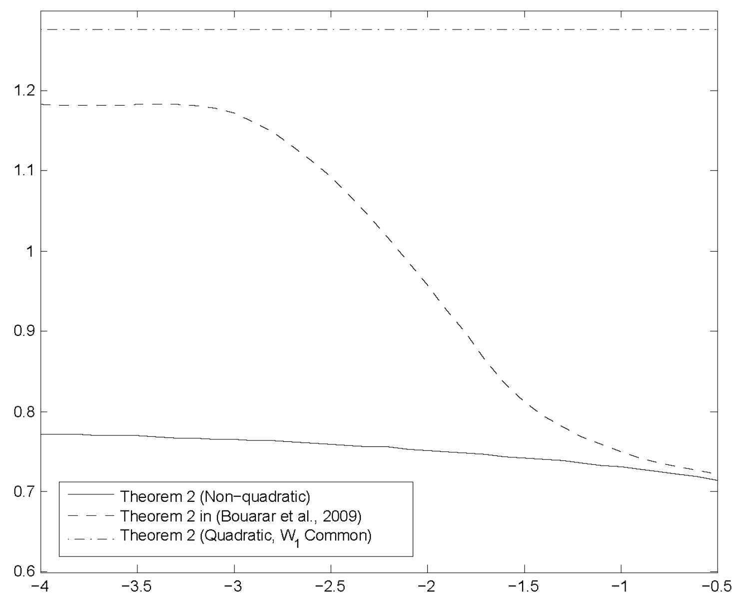

Fig. 1: Attenuation level λ for several values of ϕ_1 (example 1).

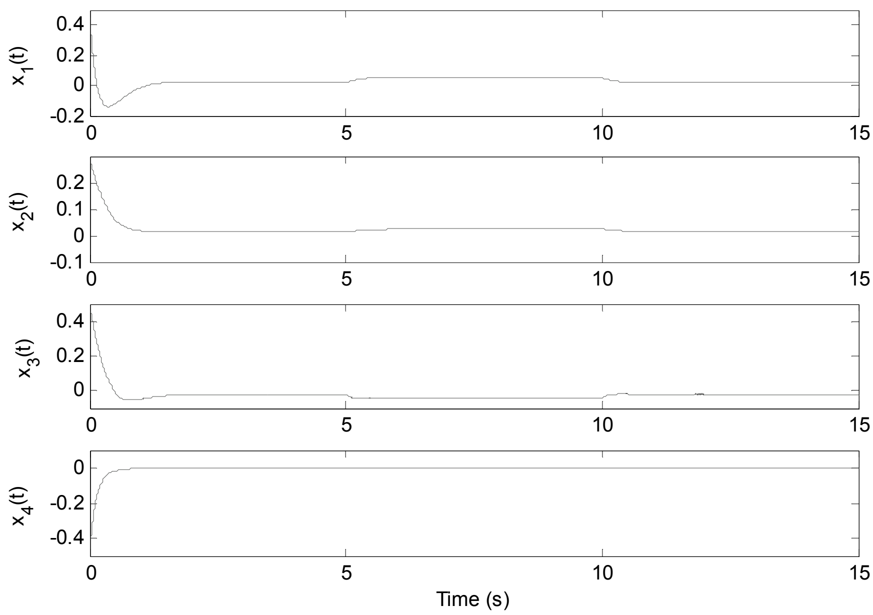

Fig. 2: Time responses of the T-S system's states with external disturbances (example 2)

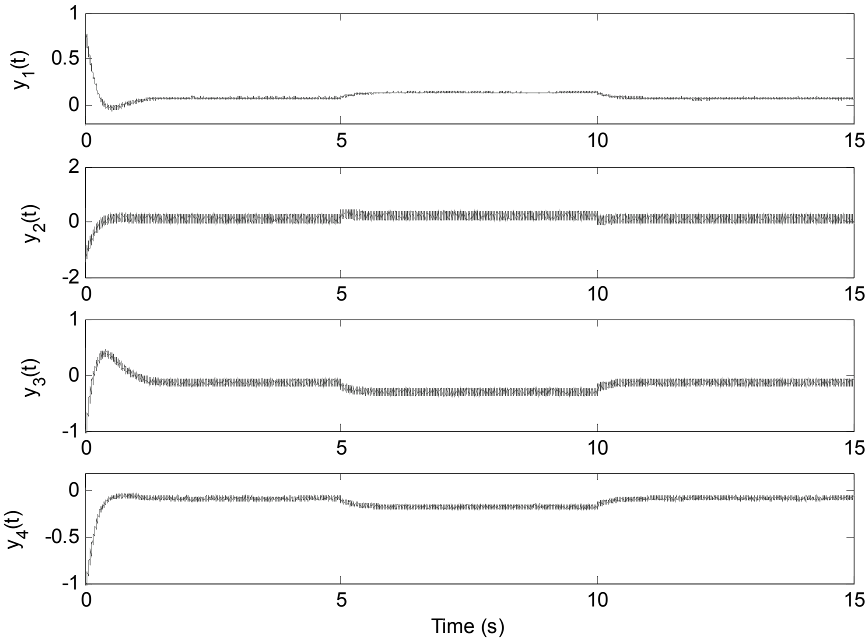

Fig. 3: Evolution of the outputs of the T-S systems with external disturbances (example 2)

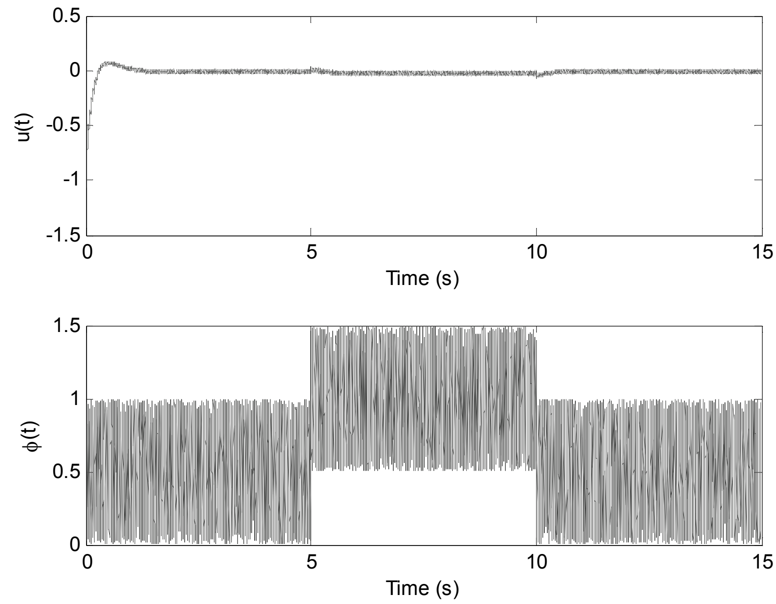

Fig. 4: Evolution of the control signal and external disturbances (example 2).

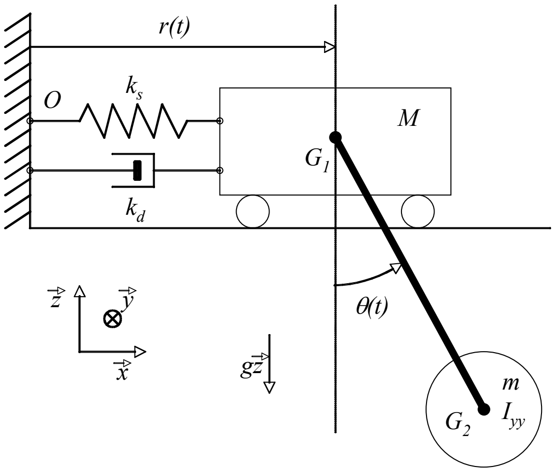

Fig.5. Crane system (example3).

Table 1. Model parameters of the crane system (example 3).

Parameters	Designation	Value
m	Pendulum mass	$10\,(kg)$
M	Cart mass	$3\,(kg)$
k_s	Spring stiffness	10
k_d	Damping coefficient	10
L	Length of the pendulum	$0.5\,(m)$
I_{yy}	Inertia of the pendulum around the $\vec{y}$ axis	$0.3\left(kg/m^2\right)$
g	Gravity acceleration	$9.81\left(m/s^2\right)$
$r(t)$	Position of the cart along the $\vec{x}$ axis	$-(m)$
$\theta(t)$	Pendulum angular position	$-(rad)$
$u(t)$	Input signal (Force on the cart within $\vec{x}$)	$-(Nm)$

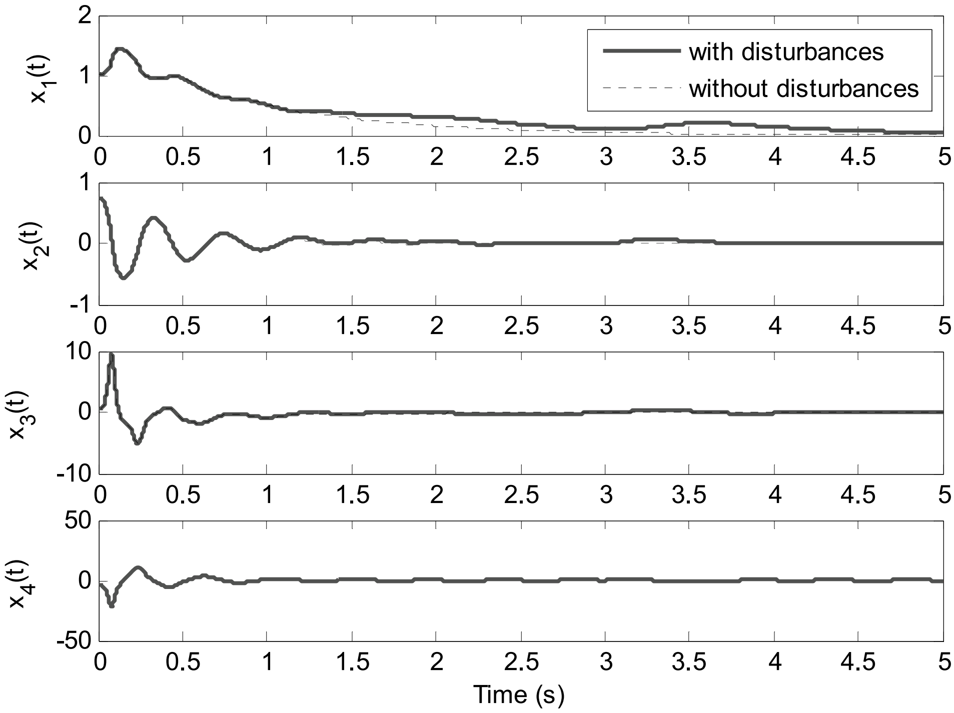

Fig. 6. Evolution of the states of the crane system (example 3).

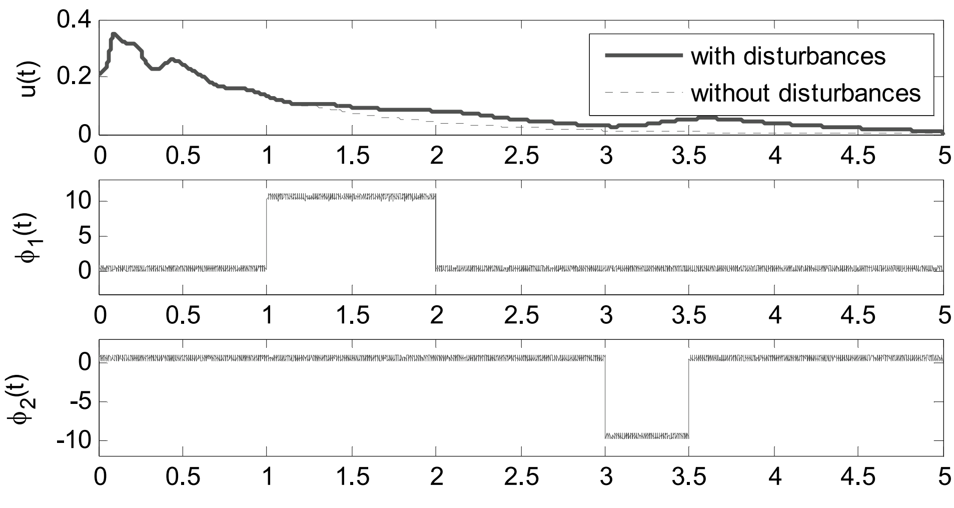

Fig. 7. Evolution of the control signal and external disturbances (example 3).