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**Abstract:**

This work concerns the tracking problem of uncertain Takagi Sugeno (T-S) continuous fuzzy model with external disturbances. The objective is to get a model reference based output feedback tracking control law. The control scheme is based on a PDC structure, a fuzzy observer and a $H_\infty$ performance to attenuate the external disturbances. The stability of the whole closed-loop model is investigated using the well-known quadratic Lyapunov function. The key point of the proposed approaches is to achieve conditions under a LMI (Linear Matrix Inequalities) formulation in the case of an uncertain and disturbed T-S fuzzy model. This formulation facilitates obtaining solutions through interior point optimization methods for some nonlinear output tracking control problems. Finally, a simulation is provide on the well-known inverted pendulum testbed to show the efficiency of the proposed approach.

**Keywords**: Tracking control, Output feedback, Fuzzy Takagi-Sugeno models, linear matrix inequality (LMI), quadratic stability, H-infinity criterion.

# I. Introduction

Design of robust tracking control for uncertain non linear systems has attracted great attention in the past few years. Among nonlinear control theory, the Takagi-Sugeno (T-S) fuzzy model-based approach has nowadays become popular since it showed its efficiency to control complex nonlinear systems and has been used for many applications, see e.g. [7] [9]. Indeed, Takagi and Sugeno have proposed a fuzzy model to describe nonlinear models [26] as a collection of linear time invariant models blended together with nonlinear functions. A control law, called “Parallel Distributed Compensation” (PDC), can be synthesized as a collection of feedback gains that are connected using the same nonlinear functions [36].

Stability and stabilization analyses, for several kind of T-S fuzzy model, have been strongly investigated through Lyapunov direct method see [17] [24] [25] [27] and references therein. The key point of the proposed approaches is to achieve conditions under LMI’s (Linear Matrix Inequalities) formulation. This formulation allows obtaining solutions through interior point optimization methods [3]. A lot of works, involving various specifications, are now available for state space feedback: robustness with bounded uncertainties [6] [15] [23] [24] [28], time delay models with or without uncertainties [4] [37], using pole placement constraints for each linear models [14], including performance specifications $H_2$, $H_\infty$ [19] [20] [23] [29], and more recently using the circle criterion and its graphical interpretation in the frequency domain [2]. Complementary to these works and with the growing interest on engineering applications of T-S models based stabilization, some studies have been done regarding to output stabilization. These can be considered through two approaches. The first one uses a fuzzy observer and is interesting when the premise variables are measured [11] [21] [38]. The second one involves dynamic state feedback [1] [10] [15] [16] [23] [27].

Despite numerous works available, none of them seem able to define a LMI formulation for the problem of robust trajectory tracking for T-S uncertain and/or disturbed models, with $H_\infty$ performance and in output feedback. Usually, the obtained conditions are only expressed in terms of bilinear matrix inequality (BMI) [22] [33]. Moreover, despite abundant literature on stability conditions of T-S fuzzy models, few authors have dealt with the tracking problem recently. Among this literature, some works are concerned with state feedback and $H_\infty$ performances [27] [30] [33]. Let us quote that these works correspond to straightforward extensions of previous results. Nevertheless, when dealing with T-S models with external disturbances [34] the results are not more LMI. For the general case of output tracking, the existing approaches are based on variable structure control techniques [39] or on a switching controller using a reference model [18]. The only results available with PDC structure are given under a BMI form and two steps algorithm based on two LMI problems is generally used [22] [34]. Moreover, [32] has developed a similar result with parametric uncertainties, i.e, BMI conditions solved in two steps. Nevertheless, the developed conditions proposed in [32] are obtained in spite of the confusion of the state reconstruction error and the tracking error leading to unsuitable conditions for tracking control based on an output feedback [5]. Let us quote that the solution (when it exists) using BMI algorithm is strongly depending on the initial conditions and therefore no guarantee of convergence is ensured. This can lead to conservative solutions and then to less practical applicability of the proposed approaches. This lack of results, understood as the deficiency of LMI formulation, may lead to the aim of this paper.

This paper is concerned with uncertain T-S continuous fuzzy model with external disturbances. The goal is to obtain a model reference based robust output feedback tracking control law. This one includes a PDC structure with a fuzzy observer and external disturbances attenuation based on an $H_\infty$ criterion. The stability of the whole closed-loop model is investigated using the well-known quadratic Lyapunov function. The main contribution of the paper is the proposition of a LMI formulation to derive the proposed robust output feedback control law.

The paper is presented as follows: Section II presents the T-S fuzzy models with uncertainties. The observer and the fuzzy control design using a reference model is developed in section III. The stability conditions, for the whole closed loop system, derived in LMI formulation are developed in section IV. Simulation results, showing the tracking performance of the well-known testbed of the inverted pendulum, with a fuzzy observer are given in section V to show the applicability of the proposed approach.

## II. T-S fuzzy models

Takagi–Sugeno fuzzy models allow describing nonlinear dynamical models by a set of Linear Time Invariant (LTI) models interconnected by nonlinear functions. Each rule associates a LTI model as a concluding part to a weight function obtained from the premises [26]. In this paper, we focused on the class of uncertain and disturbed T-S fuzzy models [27]. The bounded uncertainties and external disturbances are added, in a classical way, to each nominal LTI models [28]. Thus, the $ith$ rule can be expressed as:

**If** $z_1\left(x(t)\right)$ is $F_{i1}$ and … and $z_p\left(x(t)\right)$ is $F_{ip}$

**Then** $$\begin{cases} \dot{x}(t) = \left[A_i + \Delta A_i(t)\right]x(t) + \left[B_i + \Delta B_i(t)\right]u(t) + \varphi(t) \\ y(t) = \left[C_i + \Delta C_i(t)\right]x(t) \end{cases} \tag{1}$$

where $F_{ij}$ are the fuzzy set and $r$ is the number of **If–Then** rules, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $\varphi(t) \in \mathbb{R}^n$ is a bounded external disturbance and $z(t) = \left[z_1\left(x(t)\right) \quad \cdots \quad z_p\left(x(t)\right)\right]$ is the premises vector being state dependent. The Lebesgue measurable uncertainties are defined as $\Delta A_i(t) = H_i \Delta a(t) E_{ai}$, $\Delta B_i(t) = H_i \Delta b(t) E_{bi}$, where matrices $H_i$, $E_{ai}$ and $E_{bi}$ are constant and the uncertainties $\Delta a(t)$, $\Delta b(t)$ satisfy the classical bounded conditions [40]: $\Delta a^T(t)\Delta a(t) \le I$, $\Delta b^T(t)\Delta b(t) \le I$.

Given a pair of $\left(x(t), u(t)\right)$, the fuzzy system is inferred as follows.

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) \left[(A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))u(t)\right] + \varphi(t) \\ y(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) C_i x(t) \end{cases} \tag{2}$$

Where $w_i(z(t)) = \prod_{j=1}^{p} F_{ij}(z_j(t))$ with $w_i(z(t)) \geq 0$ and $\sum_{i=1}^{r} w_i(z(t)) > 0$, $i = 1,\ldots,r$. $F_{ij}(z_i(t))$ is the degree of membership of $z_i(t)$ in $F_{ij}$, and:

$$h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}, \text{ for } i = 1,\ldots,r \qquad (3)$$

Therefore the $h_i(z(t))$, $i = 1,\ldots,r$ hold a convex sum property:

$$\sum_{i=1}^{r} h_i(z(t)) = 1,\ h_i(z(t)) \geq 0,\ i = 1,\ldots,r \qquad (4)$$

At last, recall that there exists a systematic way to obtain (2) from a nonlinear model called the sector nonlinearity approach [27]. This one allows the T-S model matching exactly the nonlinear one on a compact set of the state space.

Two types of uncertainties may occur in the modeling of uncertain nonlinear systems. The first one, called “structural uncertainty” is referred to parametric uncertainties that are due to formalized unknown nonlinearities. The second type, known as “unstructured uncertainty” is often due to non-formalized modeling errors and external disturbances. Let us quote that, taking into account these uncertainties in the control design can be understood as more practical applicability. Indeed, with the growing complexity of nonlinear systems, it is often necessary to make approximations in the dynamical modeling process. Therefore, the main objective is now to provide stability conditions, in terms of LMI, that ensure the tracking performance for uncertain T-S models.

## III. CONTROLLER SYNTHESIS

In order to derive an output control law an additional observer is added. This one is based on the nominal model without uncertainties (2) and has the usual form [21] [38]:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)\left[A_i\hat{x}(t) + B_i u(t) + L_i\left(y(t) - \hat{y}(t)\right)\right] \\ \hat{y}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)C_i\hat{x}(t) \end{cases}, \text{ for } i = 1,\dots,r \tag{5}$$

Where $\hat{x}(t) \in \mathbb{R}^n$ is the estimated state and $L_i$ is the observer gain for the $ith$ LTI model. At last, note also that we are in the special case where the premises are supposed measurable, i.e. $z(t)$ instead of $\hat{z}(t)$ in the general case. The former allows in various cases a separation principle [38]. The latter case remains in a more complicated design [11].

To specify the desired trajectory, consider the following reference model [34]:

$$\dot{x}_r(t) = A_r x_r(t) + r(t) \tag{6}$$

where $x_r(t)$ is the reference state, $A_r$ is a specified asymptotically stable matrix, and $r(t)$ is a bounded reference input. The attenuation of external disturbances is guaranteed considering the $H_\infty$ performance related to the tracking error $x_r(t) - x(t)$ as follows [8] [13] [34]:

$$\int_0^{tf}\left(\left[x_r(t) - x(t)\right]^T Q \left[x_r(t) - x(t)\right]\right) dt \leq \eta^2 \int_0^{tf}\left(r(t)^T r(t) + \varphi(t)^T \varphi(t)\right) dt \tag{7}$$

where $t_f$ denotes the final time, $Q$ is a positive definite weighting matrix, and $\eta$ is a specified attenuation level. At last, the control law is based on the classical structure of a PDC law [36] sharing the same nonlinear functions as the T-S model:

$$u(t) = -\sum_{i=1}^{r} h_i(z(t))K_i[x_r(t) - \hat{x}(t)] \tag{8}$$

where $K_i$ are gain matrices with appropriate dimension. Let us consider the estimation error $e_o(t) = x(t) - \hat{x}(t)$, the tracking error $e_p(t) = x(t) - x_r(t)$ and the state reference $x_r(t)$. The state vector for the global closed loop is $\tilde{x}(t) = \begin{bmatrix} e_o & e_p & x_r \end{bmatrix}^T$. Then, combining the control law (8), the system (2) and the observer (5), one obtains, after some easy manipulations, the following closed-loop model:

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\tilde{A}_{ij}\ \tilde{x}(t) + \tilde{S}\tilde{\phi}(t) \tag{9}$$

with

$$\tilde{A}_{ij} = \begin{bmatrix} A_i - L_iC_j - \Delta B_iK_j & \Delta A_i + \Delta B_iK_j & \Delta A_i \\ -B_iK_j - \Delta B_iK_j & A_i + B_iK_j + \Delta A_i + \Delta B_iK_j & A_i - A_r + \Delta A_i \\ 0 & 0 & A_r \end{bmatrix},\ \tilde{S} = \begin{bmatrix} I & 0 \\ I & -I \\ 0 & I \end{bmatrix},\ \tilde{\phi}(t) = \begin{bmatrix} \varphi(t) \\ r(t) \end{bmatrix}$$

Note that with the state vector $\tilde{x}(t)$, (7) can be written with $\tilde{Q} = diag\begin{bmatrix} 0 & Q & 0 \end{bmatrix}$ and the disturbances $r(t)^T r(t) + \varphi(t)^T \varphi(t) = \tilde{\phi}(t)^T \tilde{\phi}(t)$:

$$\int_0^{tf} \tilde{x}^T(t)\ \tilde{Q}\ \tilde{x}(t)\ dt \le \eta^2 \int_0^{tf} \tilde{\phi}^T(t)\tilde{\phi}(t)\ dt \tag{10}$$

The objective is now to compute the gains $K_i$ and $L_i$ from $\tilde{A}_{ij}$ described in (9) to ensure the asymptotic stability of the closed-loop model (9) guaranteeing the $H_\infty$ tracking performance (10) for all $\tilde{\phi}(t)$. A straightforward result is summarized in the following theorem.

**Theorem 1:**

For $t > 0$ and $h_i\left(z(t)\right)h_j\left(z(t)\right) \neq 0$, with $\tilde{A}_{ij}$, $\tilde{S}$ defined in (9), if there exist a matrix $\tilde{P} = \tilde{P}^T > 0$, and a positive constant $\eta$ such that the following matrix inequalities are satisfied, $i, j \in \{1,\dots,r\}$ :

$$\begin{cases} \Upsilon_{ii} < 0 \\ \dfrac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} \leq 0 \quad i \neq j \end{cases} \tag{11}$$

with: $\Upsilon_{ij} = \begin{bmatrix} \tilde{A}_{ij}^T\tilde{P} + \tilde{P}\tilde{A}_{ij} + \tilde{Q} & \tilde{P}\tilde{S} \\ \tilde{S}^T\tilde{P} & -\eta^2 I \end{bmatrix}$. Then the asymptotic stability of the closed loop fuzzy system (9) is ensured and the $H_\infty$ tracking control performance (10) is guaranteed with an attenuation level $\eta$.

**Proof:**

Consider the following candidate Lyapunov function.

$$V(\tilde{x},t) = \tilde{x}^T(t)\tilde{P}\tilde{x}(t) \text{ with } \tilde{P} = \tilde{P}^T > 0 \tag{12}$$

The stability of the closed loop model (9) is satisfied under the $H_\infty$ performance (10) with the attenuation level $\eta$ if [7]:

$$\dot{V}(\tilde{x},t) + \tilde{x}^T(t)\tilde{Q}\tilde{x}(t) - \eta^2\tilde{\phi}^T(t)\tilde{\phi}(t) \leq 0 \tag{13}$$

The condition (13) leads to:

$$\tilde{x}^T(t)\left[\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j\left(\tilde{A}_{ij}^T\tilde{P} + \tilde{P}\tilde{A}_{ij} + \tilde{Q}\right)\right]\tilde{x}(t) + \phi^T(t)\tilde{S}^T\tilde{P}\tilde{x}(t) + \tilde{x}^T(t)\tilde{P}\tilde{S}\tilde{\phi}(t) - \eta^2\tilde{\phi}^T(t)\tilde{\phi}(t) \leq 0 \tag{14}$$

or equivalently:

$$\begin{bmatrix} \tilde{x}(t) \\ \tilde{\phi}(t) \end{bmatrix}^T \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \begin{bmatrix} \tilde{A}_{ij}{}^T \tilde{P} + \tilde{P}\tilde{A}_{ij} + \tilde{Q} & \tilde{P}\tilde{S} \\ \tilde{S}^T \tilde{P} & -\eta^2 I \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{\phi}(t) \end{bmatrix} \leq 0 \qquad (15)$$

which, considering the work of [35], is satisfied if conditions (11) hold.

■

The goal is now to obtain a tractable LMI problem that allows searching the gain matrices (both for the control $K_i$ and the observer $L_i$) and to prove the closed loop stability (finding $\tilde{P} > 0$) ensuring the prescribed attenuation level ($\eta$).

## III. LMI FORMULATION OF THE STABILITY CONDITIONS

The goal is now to propose LMI conditions for T-S models tracking control. The following lemmas are needed to put the further provided conditions into LMI.

**Lemma 1** [40]:

For real matrices $X$, $Y$ and $S = S^T > 0$ with appropriate dimensions and a positive constant $\gamma$, the following inequalities hold:

$$X^T Y + Y^T X \leq \gamma X^T X + \gamma^{-1} Y^T Y \qquad (16)$$

and $X^T Y + Y^T X \leq X^T S^{-1} X + Y^T S Y$ (17)

**Lemma 2**:

For real matrices $A$, $B$, $W$, $Y$, $Z$ and a regular matrix $Q$ with appropriate dimensions one has:

$$\begin{bmatrix} Y + B^T Q^{-1} B & W^T \\ W & Z + AQA^T \end{bmatrix} < 0 \quad \Rightarrow \quad \begin{bmatrix} Y & W^T + B^T A^T \\ W + AB & Z \end{bmatrix} < 0 \tag{18}$$

**Proof of lemma 2:**

For real matrices $A$, $B$, $W$, $Y$, $Z$ and a regular matrix $Q$ with appropriate dimensions, the inequality: $\begin{bmatrix} Y & W^T + B^T A^T \\ W + AB & Z \end{bmatrix} < 0$ can be rewritten as:

$$\begin{bmatrix} Y & W^T \\ W & Z \end{bmatrix} + \begin{bmatrix} 0 & B^T A^T \\ AB & 0 \end{bmatrix} < 0$$

From the inequality (17), it exists a matrix $Q > 0$ such that:

$$\begin{bmatrix} 0 \\ A \end{bmatrix} \begin{bmatrix} B & 0 \end{bmatrix} + \begin{bmatrix} B^T \\ 0 \end{bmatrix} \begin{bmatrix} 0 & A^T \end{bmatrix} \le \begin{bmatrix} 0 \\ A \end{bmatrix} Q \begin{bmatrix} 0 & A^T \end{bmatrix} + \begin{bmatrix} B^T \\ 0 \end{bmatrix} Q^{-1} \begin{bmatrix} B & 0 \end{bmatrix} \tag{19}$$

that leads to (18) and ends the proof.

■

**Lemma 3 [11]**:

Let a matrix $\Omega < 0$, a matrix $X$ with appropriate dimension such that $X^T \Omega X \le 0$, and a scalar $\alpha$, the following inequality holds:

$$X^T \Omega X \le -\alpha (X^T + X) - \alpha^2 \Omega^{-1} \tag{20}$$

**Proof of lemma 3:**

$\Omega$ is a negative definite matrix, then if $X^T \Omega X \le 0$, hence:

$\exists \alpha \in \mathbb{R}$ such that: $(X + \alpha\Omega^{-1})^T \Omega (X + \alpha\Omega^{-1}) \leq 0$

i.e. $X^T \Omega X + \alpha(X^T + X) + \alpha^2 \Omega^{-1} \leq 0$

■

As usual, $(*)$ will indicate a transpose quantity in a symmetric matrix. The main result is given in the following theorem.

**Theorem 2:**

For all $t > 0$ and $h_i\left(z(t)\right)h_j\left(z(t)\right)$, if there exist matrices $P_1 = P_1^T > 0$, $P_3 = P_3^T > 0$, $N = N^T > 0$, $Y_i$, $Z_i$, positive constants $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$, $\mu_5$, $\mu_6$, $\mu_7$, $\mu_8$ and $\eta$, such that the following LMI conditions are satisfied $i, j \in \{1, \ldots, r\}$ :

$$\begin{cases} \Upsilon_{ii} < 0 \\ \dfrac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} \leq 0 \quad i \neq j \end{cases} \tag{21}$$

with $\Upsilon_{ij} = \left[\begin{array}{c|c} \Gamma_{ij} & (*) \\ \hline \left[\begin{array}{cc|c} -B_iY_j & N & 0_{1\times 8} \\ \hline \multicolumn{2}{c|}{0_{8\times 2}} & 0_{8\times 8} \end{array}\right] & \Psi_{ij} \end{array}\right]$ and:

$$\Gamma_{ij}=\begin{bmatrix} -2\alpha N & 0 & (*) & (*) & (*) & 0 & 0 & 0 \\ 0 & -2\alpha N & 0 & 0 & 0 & 0 & 0 & (*) \\ E_{bi}Y_j & 0 & -\mu_1^{-1}I & 0 & 0 & 0 & 0 & 0 \\ E_{bi}Y_j & 0 & 0 & -\mu_5^{-1}I & 0 & 0 & 0 & 0 \\ \alpha I & 0 & 0 & 0 & P_1A_i - Z_iC_j + A_i^TP_1 - C_j^TZ_i^T & (*) & (*) & P_1 \\ 0 & 0 & 0 & 0 & H_i^TP_1 & -(\mu_1^{-1}+\mu_2^{-1}+\mu_3^{-1})^{-1}I & 0 & 0 \\ 0 & 0 & 0 & 0 & H_i^TP_1 & 0 & -\mu_4 I & 0 \\ 0 & \alpha I & 0 & 0 & P_1 & 0 & 0 & -\eta^2 I \end{bmatrix}$$

$$\Psi_{ij}=\begin{bmatrix} \Psi_{ij}(1,1) & (*) & (*) & (*) & (*) & (*) & (*) & 0 & (*) \\ N & -Q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_{bi}Y_j & 0 & -\mu_2^{-1}I & 0 & 0 & 0 & 0 & 0 & 0 \\ E_{bi}Y_j & 0 & 0 & -\mu_7^{-1}I & 0 & 0 & 0 & 0 & 0 \\ E_{ai}N & 0 & 0 & 0 & -\mu_3^{-1}I & 0 & 0 & 0 & 0 \\ E_{ai}N & 0 & 0 & 0 & 0 & -\mu_6^{-1}I & 0 & 0 & 0 \\ A_i^T - A_r^T & 0 & 0 & 0 & 0 & 0 & A_r^TP_3 + P_3A_r + \mu_4 E_{ai}^TE_{ai} & (*) & (*) \\ 0 & 0 & 0 & 0 & 0 & 0 & E_{ai} & -\mu_8^{-1} & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & P_3 & 0 & -\eta^2 I \end{bmatrix}$$

$$\Psi_{ij}(1,1) = A_iN + B_iY_j + N^TA_i^T + Y_j^TB_i^T + (\mu_5^{-1}+\mu_6^{-1}+\mu_7^{-1}+\mu_8^{-1})H_iH_i^T$$

then the asymptotic stability of the closed loop fuzzy system (9) is ensured and the $H_\infty$ tracking control performance (10) is guaranteed with an attenuation level $\eta$ . Moreover, if a solution exists, the gains $K_i$ and $L_i$ are obtained using: $K_i = Y_iN^{-1}$ and $L_i = P_1^{-1}Z_i$ .

**Proof:**

For a convenient design, let us assume that $\tilde{P} = diag[P_1 \quad P_2 \quad P_2]$. (15) can be rewritten as::

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\left(\tilde{\Pi}_{ij} + \Delta\tilde{\Pi}_{ij}\right) \le 0 \tag{22}$$

with

$$\tilde{\Pi}_{ij} = \begin{bmatrix} P_1(A_i - L_iC_j) + (A_i - L_iC_j)^T P_1 & (*) & 0 & (*) & 0 \\ -P_2B_i & P_2(A_i + B_iK_j) + (A_i + B_iK_j)^T P_2 + Q & (*) & (*) & (*) \\ 0 & (A_i^T - A_r^T)P_2 & A_r^T P_3 + P_3A_r & 0 & (*) \\ P_1 & P_2 & 0 & -\eta^2 I & 0 \\ 0 & -P_2 & P_3 & 0 & -\eta^2 I \end{bmatrix}$$

and

$$\Delta\tilde{\Pi}_{ij} = \begin{bmatrix} -P_1\Delta B_iK_j - K_j^T\Delta B_i^T P_1 & (*) & (*) & 0 & 0 \\ -P_2\Delta B_iK_j + K_j^T\Delta B_i^T P_1 + \Delta A_i^T P_1 & P_2(\Delta A_i + \Delta B_iK_j) + (\Delta A_i + \Delta B_iK_j)^T P_2 + Q & (*) & 0 & 0 \\ \Delta A_i^T P_1 & \Delta A_i^T P_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then using the uncertainties structure defined in (2) and the well-known property given in lemma 2, $\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\Delta\tilde{\Pi}_{ij}$ can be bounded as follows:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\Delta\tilde{\Pi}_{ij} \le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\,diag\begin{bmatrix} d_{1ij} & d_{2ij} & d_{3i} & 0 & 0 \end{bmatrix} \tag{23}$$

with

$$d_{1ij} = (\mu_1 + \mu_5)K_j^T E_{bi}^T E_{bi}K_j + (\mu_1^{-1} + \mu_2^{-1} + \mu_3^{-1} + \mu_4^{-1})P_1H_iH_i^T P_1$$
$$d_{2ij} = (\mu_2 + \mu_7)K_j^T E_{bi}^T E_{bi}K_j + (\mu_5^{-1} + \mu_6^{-1} + \mu_7^{-1} + \mu_8^{-1})P_2H_iH_i^T P_2 + (\mu_3 + \mu_6)E_{ai}^T E_{ai}$$
$$d_{3i} = (\mu_4 + \mu_8)E_{ai}^T E_{ai}$$

Then, the inequality (22) holds if:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j \begin{bmatrix} \Theta_{ij}(1,1) & (*) & 0 & (*) & 0 \\ -P_2B_iK_j & \Theta_{ij}(2,2) & (*) & (*) & (*) \\ 0 & (A_i^T - A_r^T)P_2 & A_r^T P_3 + P_3A_r + d_{3i} & 0 & (*) \\ P_1 & P_2 & 0 & -\eta^2 I & 0 \\ 0 & -P_2 & P_3 & 0 & -\eta^2 I \end{bmatrix} \le 0 \tag{24}$$

$\Theta_{ij}(1,1) = P_1(A_i - L_iC_j) + (A_i - L_iC_j)^T P_1 + d_{1ij}$ ,

$\Theta_{ij}\left(2,2\right)=P_2(A_i+B_iK_j)+(A_i+B_iK_j)^T P_2+Q+d_{2ij}$

In order to rearrange the matrices involved in (24) a congruence with the full-rank matrix $\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$ is made. Thus (24) is equivalent to:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j\begin{bmatrix} \Theta_{ij}\left(1,1\right) & (*) & (*) & 0 & 0 \\ P_1 & -\eta^2 I & (*) & 0 & 0 \\ -P_2B_iK_j & P_2 & \Theta_{ij}\left(2,2\right) & (*) & (*) \\ 0 & 0 & (A_i^T-A_r^T)P_2 & A_r^TP_3+P_3A_r+d_3 & (*) \\ 0 & 0 & -P_2 & P_3 & -\eta^2 I \end{bmatrix}\le 0 \tag{25}$$

Then, we proceed a bijective change of variables followed by a pre-post multiply of the inequality (25) by $diag\left[N \quad N \quad N \quad I \quad I\right]$ with $N=P_2^{-1}$, $Y_i=K_iN$ and $Z_i=P_1L_i$, one obtains:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}h_ih_j\left[\begin{array}{c|ccc} \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}\Omega_{1ij}\begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}+\begin{bmatrix} (\mu_1+\mu_5)Y_j^TE_{bi}^TE_{bi}Y_j & 0 \\ 0 & 0 \end{bmatrix} & & (*) & \\ \hline \begin{matrix} -B_iY_j & X_2 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \Omega_{2ij} \\ A_i^T-A_r^T \\ -I \end{matrix} & \begin{matrix} (*) \\ A_r^TP_3+P_3A_r+d_3 \\ P_3 \end{matrix} & \begin{matrix} (*) \\ (*) \\ -\eta^2 I \end{matrix} \end{array}\right]\le 0$$

(26)

with

$$\Omega_{1ij}=\begin{bmatrix} P_1A_i-Z_iC_j+A_i^TP_1-C_j^TZ_i^T+(\mu_1^{-1}+\mu_2^{-1}+\mu_3^{-1}+\mu_4^{-1})P_1H_iH_i^TP_1 & P_1 \\ P_1 & -\eta^2 I \end{bmatrix} \tag{27}$$

$$\begin{aligned}\Omega_{2ij}=\ & NA_i+B_iY_j+A_i^TN+Y_j^TB_i^T+NQN+(\mu_2+\mu_7)Y_j^TE_{bi}^TE_{bi}Y_j \\ & +(\mu_5^{-1}+\mu_6^{-1}+\mu_7^{-1}+\mu_8^{-1})H_iH_i^T+(\mu_3+\mu_6)NE_{ai}^TE_{ai}N\end{aligned} \tag{28}$$

Looking to expressions (26), (27) and (28) shows that the major point to allow an LMI formulation is the product $\begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \Omega_{1ij} \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$. Now, applying lemma 3 to the first diagonal block of (26), it yields:

$$\begin{aligned}
&\begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \Omega_{1ij} \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} + \begin{bmatrix} (\mu_1 + \mu_5) Y_j^{\;T} E_{bi}^{\;T} E_{bi} Y_j & 0 \\ 0 & 0 \end{bmatrix} \\
&\qquad \le -2\alpha \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} - \alpha^2 \Omega_{1ij}^{-1} + \begin{bmatrix} (\mu_1 + \mu_5) Y_j^{\;T} E_{bi}^{\;T} E_{bi} Y_j & 0 \\ 0 & 0 \end{bmatrix} \\
&\qquad = \begin{bmatrix} -2\alpha N + (\mu_1 + \mu_5) Y_j^{\;T} E_{bi}^{\;T} E_{bi} Y_j & 0 \\ 0 & -2\alpha N \end{bmatrix} - \alpha^2 \Omega_{1ij}^{-1}
\end{aligned} \tag{29}$$

Then, applying the Schur complement, (29) becomes:

$$\begin{bmatrix} -2\alpha N + (\mu_1 + \mu_5) Y_j^{\;T} E_{bi}^{\;T} E_{bi} Y_j & 0 & (*) & 0 \\ 0 & -2\alpha N & 0 & (*) \\ \alpha I & 0 & \Xi_{ij}(3,3) & (*) \\ 0 & \alpha I & P_1 & -\eta^2 I \end{bmatrix} \le 0 \tag{30}$$

with $\Xi_{ij}(3,3) = P_1 A_i - Z_i C_j + A_i^T P_1 - C_j^T Z_i^T + (\mu_1^{\;-1} + \mu_2^{\;-1} + \mu_3^{\;-1} + \mu_4^{\;-1}) P_1 H_i H_i^{\;T} P_1$

Substituting (30) in (26), we obtain the following inequality:

$$\left[\begin{array}{cccc|ccc} \mathrm{H}_{ij}(1,1) & 0 & \alpha I & 0 & (*) & 0 & 0 \\ 0 & -2\alpha N & 0 & \alpha I & (*) & 0 & 0 \\ \alpha I & 0 & \mathrm{H}_{ij}(3,3) & P_1 & 0 & 0 & 0 \\ 0 & \alpha I & P_1 & -\eta^2 I & 0 & 0 & 0 \\ \hline -B_i Y_j & N_2 & 0 & 0 & \mathrm{H}_{ij}(5,5) & (*) & (*) \\ 0 & 0 & 0 & 0 & (A_i^{\;T} - A_r^{\;T}) & \mathrm{H}_{ij}(6,6) & (*) \\ 0 & 0 & 0 & 0 & -I & P_3 & -\eta^2 I \end{array}\right] < 0 \tag{31}$$

With $\mathrm{H}_{ij}(1,1) = -2\alpha N + (\mu_1 + \mu_5) Y_j^T E_{bi}^T E_{bi} Y_j$

$$\mathrm{H}_{ij}(3,3) = P_1 A_i - Z_i C_j + A_i^T P_1 - C_j^T Z_i^T + \left(\mu_1^{-1} + \mu_2^{-1} + \mu_3^{-1} + \mu_4^{-1}\right) P_1 H_i H_i^T P_1$$

$$\mathrm{H}_{ij}(5,5) = N A_i + B_i Y_j + A_i^T N + Y_j^T B_i^T + NQN + (\mu_2 + \mu_7) Y_j^T E_{bi}^T E_{bi} Y_j$$
$$+ \left(\mu_5^{-1} + \mu_6^{-1} + \mu_7^{-1} + \mu_8^{-1}\right) H_i H_i^T + (\mu_3 + \mu_6) N E_{ai}^T E_{ai} N$$

$$\mathrm{H}_{ij}(6,6) = A_r^T P_3 + P_3 A_r + (\mu_4 + \mu_8) E_{ai}^T E_{ai}$$

Applying the Schur's complement on the diagonal blocks $\mathrm{H}_{ij}(1,1)$, $\mathrm{H}_{ij}(3,3)$, and $\mathrm{H}_{ij}(6,6)$, the conditions of theorem 2 hold. ■

*Remark:*

The proposed approach provides quasi LMI conditions where only the 4 scalars $\alpha$, $\mu_1$, $\mu_2$ and $\mu_3$ are required to obtain exact LMI conditions. Note that, in the previous literature, several BMI solutions are available [22] [33] that can be solved by iterative algorithms using two general eigenvalue problems (GEVP). Thus, these ones are strongly dependent on the initialization and on the different variables set in each problem (number of epochs, feasibility radius, …) for the GEVP formulation. Providing LMI conditions allows preventing this problem.

# IV. EXAMPLE AND SIMULATION

*System modeling:*

To illustrate the proposed approach, consider the angular position tracking control of an inverted pendulum on a cart (figure 1). The system's dynamical equations are expressed as:

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = \dfrac{(m+M)g\sin(x_1) - mlx_2^2\sin(x_1)\cos(x_1) - \cos(x_1)u}{l\left(\frac{1}{3}m + \frac{4}{3}M + m\sin^2(x_1)\right)}
\end{cases}
\tag{32}
$$

where $x_1(t)$ and $x_2(t)$ are respectively the angular position and velocity of the pendulum, $u(t)$ is the force applied to the cart, $m = 0.1\,kg$ and $M = 1\,kg$ are respectively the masses of the pendulum and the cart, $2l = 1\,m$ is the length of the pendulum and $g = 9,8\ ms^{-2}$.

Note that $m\sin^2(x_1(t))$ is small regarding to $\frac{1}{3}m + \frac{4}{3}M$. Then, it will be assumed to be neglected. Therefore, the system (32) becomes:

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = \dfrac{(m+M)g\sin(x_1) - \cos(x_1)u}{\frac{l}{3}(4M+m)} + \dfrac{-mlx_2^2\sin(x_1)\cos(x_1)}{\frac{l}{3}(4M+m)}
\end{cases}
\tag{33}
$$

Using the well-known sector nonlinearity approach [27], the goal is now to derive a T-S model from (33). Indeed, the above model is constituted by three nonlinearities to be splitted: $\eta_1(x_1(t)) = \sin(x_1(t))$, $\eta_2(x_1(t)) = \cos(x_1(t))$ and $\eta_3(x_2(t)) = x_2^2(t)$.

Let us consider that the velocity signal $x_2(t)$ is not available from measurements. Then, the nonlinear function $x_2^2(t)$ is removed from the certain part of the T-S model. We assume that this nonlinear function is bounded as $\eta_3(x_2(t)) \in [0, \bar{\eta}_3]$. Then, we write it as an uncertainty and (33) can be written as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = d(x_1,u) + \Delta d(x,u) \end{cases} \tag{34}$$

with $d(x_1,u) = \dfrac{(m+M)g\eta_1(x_1) - \eta_2(x_1)u}{\frac{l}{3}(4M+m)}$, $\Delta d(x_1,x_2,u) = \dfrac{-ml\eta_1(x_1)\eta_2(x_1)}{\frac{l}{3}(4M+m)}\bar{\eta}_3 f(x_2)$

and $f(x_2) = \dfrac{\eta_3(x_2)}{\bar{\eta}_3}$.

The T-S membership functions are obtained following the same procedure as presented in [12]. Let $x_1 \in [-\theta_0, \theta_0]$, then the nonlinear functions can be written as follows:

$$\eta_1(x_1(t)) = \omega_1^1(x_1(t))\, x_1(t) + \omega_1^2(x_1(t)) \frac{\sin(\theta_0)}{\theta_0} x_1(t),$$

$$\eta_2(x_1(t)) = \omega_2^1(x_1(t)) + \omega_2^2(x_1(t))\cos(\theta_0),$$

Where, for $i = 1,2$ and $j = 1,2$ we have $0 \le \omega_i^j \le 1$ and

$$\omega_1^1(x_1(t)) = \frac{\theta_0 \sin(x_1(t)) - x_1(t)\sin(\theta_0)}{x_1(t)(\theta_0 - \sin(\theta_0))},$$

$$\omega_1^2(x_1(t)) = 1 - \omega_1^1(x_1(t)) = \frac{\theta_0 x_1(t) - \theta_0 \sin(x_1(t))}{x_1(t)(\theta_0 - \sin(\theta_0))}, \quad \omega_2^1(x_1(t)) = \frac{\cos(x_1(t)) - \cos(\theta_0)}{1 - \cos(\theta_0)},$$

$$\omega_2^2(x_1(t)) = 1 - \omega_2^1(x_1(t)) = \frac{1 - \cos(x_1(t))}{1 - \cos(\theta_0)}.$$

A way to reduce the computational complexity of the LMI conditions is to minimize the number of rules used to model the system [31]. With the previous non linear splitting, the obtained fuzzy model should have 4 rules. Nevertheless, it is still possible to reduce this fuzzy model noticing that $\omega_1^1$ and $\omega_2^1$, respectively $\omega_1^2$ and $\omega_2^2$, are very closed [12]. Thus, for $i = 1, 2$ we assume that $\omega_i = \omega_i^1 = \omega_i^2$ and a 2 rules fuzzy model representing the nonlinear uncertain model (34) can be proposed as:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \omega_i(t)\left[\left(A_i + \Delta A_i(x(t))\right)x(t) + B_i u(t)\right] \\ y(t) = Cx(t) \end{cases} \tag{35}$$

With $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$, $A_1 = \begin{bmatrix} 0 & 1 \\ \dfrac{(M+m)g}{\frac{l}{3}(4M+m)} & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ \dfrac{(M+m)g}{\frac{l}{3}(4M+m)} \dfrac{\sin(\theta_0)}{\theta_0} & 0 \end{bmatrix}$,

$B_1 = \begin{bmatrix} 0 \\ -\dfrac{1}{\frac{l}{3}(4M+m)} \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ -\dfrac{\cos(\theta_0)}{\frac{l}{3}(4M+m)} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$,

$\Delta A_1(x(t)) = \begin{bmatrix} 0 & 0 \\ \dfrac{-ml}{\frac{l}{3}(4M+m)} \bar{\eta}_3 f(x_2(t)) & 0 \end{bmatrix}$ and

$\Delta A_2(x(t)) = \begin{bmatrix} 0 & 0 \\ \dfrac{-ml}{\frac{l}{3}(4M+m)} \dfrac{\sin(\theta_0)\cos(\theta_0)}{\theta_0} \bar{\eta}_3 f(x_2(t)) & 0 \end{bmatrix}$.

Considering the uncertainties structure used to obtain theorem 1 and 2, we can write $\Delta A_1(t) = H_1 F_1(t) E_{a1}$ and $\Delta A_2(t) = H_2 F_2(t) E_{a2}$ with $H_1 = \begin{bmatrix} 0 \\ \dfrac{-ml}{\frac{l}{3}(4M+m)} \end{bmatrix}$, $E_{a1} = [\bar{\eta}_3 \quad 0]$,

$$H_2 = \begin{bmatrix} 0 \\ \dfrac{-ml}{\frac{l}{3}(4M+m)} \dfrac{\sin(\theta_0)\cos(\theta_0)}{\theta_0} \end{bmatrix}, \; E_{a2} = [\bar{\eta}_3 \quad 0].$$

*Simulation results:*

The simulation was performed with a maximum angular velocity set at $0.1 rad/s^2$, then $\bar{\eta}_3 = 0.01$. Note that when $x_1 = \pm\pi/2$, the system presents a singularity and so it is locally uncontrollable. To overcome this problem, the modeling space has been reduced to $x_1(t) \in [-\theta_0, \theta_0]$ with $\theta_0 = 22\pi/45$.

After trials, the presented simulations are performed with the following tuning:

- The reference model was arbitrary chosen with $A_r = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ Hurwitz to set a desired dynamics to follow.

- The dynamics of the closed loop system was fixed by choosing $Q = 10^{-6} \begin{bmatrix} 2.7 & 0 \\ 0 & 2 \end{bmatrix}$.

- The value $\alpha = 100$, $\mu_1 = 100$, $\mu_2 = 100$, $\mu_3 = 0.01$ was arbitrary chosen (note that these value will be balanced by the computed value of $\mu_4$, $\mu_5$, $\mu_6$, $\mu_7$, $\mu_8$).

- The solution $P_1$, $N$, $Y_i$ and $Z_i$ are computed (if feasible) by solving the set of LMI conditions (21) given in theorem 2 with classical LMI toolbox.

- Finally, the gains $K_i$ and $L_i$ are obtained from the bijective change of variables $K_i = Y_i N^{-1}$ and $L_i = P_1^{-1} Z_i$.

Therefore, for the proposed example of the inverted pendulum on a cart, the solution of theorem 2 is obtained using the Matlab LMI toolbox and is given by the gains $K_1 = [294.3233 \quad 126.2873]$, $K_2 = [294.3233 \quad 126.2873]$, $L_1 = [30.0 \quad 332.8878]^T$, $L_2 = [30.0 \quad 330.1325]^T$, the scalars $\mu_4 = 2.7013$, $\mu_5 = 2.7978\ .10^{-9}$, $\mu_6 = 2.7978\ .10^{-9}$, $\mu_7 = 2.7978\ .10^{-9}$, $\mu_8 = 2.7978\ .10^{-9}$ and the matrices $P_1 = 10^{-4}\begin{bmatrix} 94 & -2 \\ -2 & 10 \end{bmatrix}$, $N = 10^7.\begin{bmatrix} 0.1921 & -0.4056 \\ -0.4056 & 1.2086 \end{bmatrix}$, $P_3 = 10^{-4}\begin{bmatrix} 14 & 8 \\ 8 & 7 \end{bmatrix}$.

Figures 2a and 2c show the tracking trajectory position and velocity with respectively initial system states $x(0) = [0.2 \quad 0]^T$ and observed state $\hat{x}(0) = [0 \quad 0.2]^T$ for $r(t) = [0 \quad 2.46\sin(t)]^T$. Note that the system is subject to the external disturbances $\varphi(t) = [0.03\sin(t) \quad 0]^T$ that are set in simulation with amplitude about $10\%$ of the tracking trajectory to test the efficiency of its attenuation. Figures 2b and 2d illustrate the controller and observer efficiency in the transient state. The input signal and the position tracking quadratic error are represented figure 3.

To show the effectiveness of the disturbance attenuation by the $H_\infty$ criterion, a high external disturbance is applied to the inverted pendulum $\varphi(t) = [0.15\sin(t) \quad 0]^T$ that is about $50\%$ of the tracking trajectory. The obtained results are depicted figures 4. Figure

4a and 4b show the tracking trajectory position and velocity with the same initial states as the previous simulation. Note that despite the huge disturbance amplitude, the system does not have an unstable behavior. Even if the system position seems to follow the reference position, in this simulation the tracking velocity performances are lost showing the limits of such control law synthesis. In this case, the input signal and the position tracking quadratic error are presented figure 5.

## V. CONCLUSION

In this paper, a fuzzy tracking control has been designed for an uncertain nonlinear dynamic system with external disturbances using a T-S fuzzy model and a state observer design. A control scheme based on an augmented structure with a guaranteed $H_\infty$ performance and model reference tracking is proposed. The main result of the paper is the quasi-LMI formulation that can be applied for tracking control design of uncertain and disturbed T-S fuzzy model. This can be considered as improvement of previous theoretical study on T-S fuzzy model based output tracking control design and constitute a starting point to further applicative study on complex industrial plants. At last, a design example has illustrated the efficiency of the proposed approach on the well-known testbed of an inverted pendulum.

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LIST OF TABLES AND FIGURES

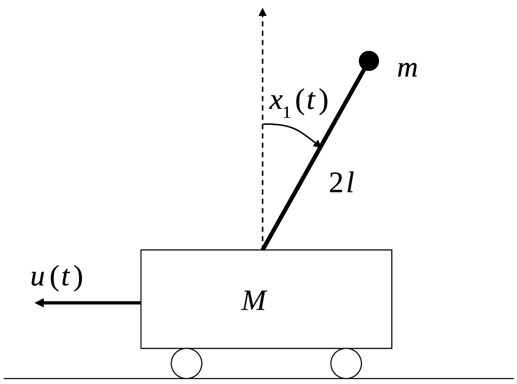


Fig. 1. Inverted pendulum on a car

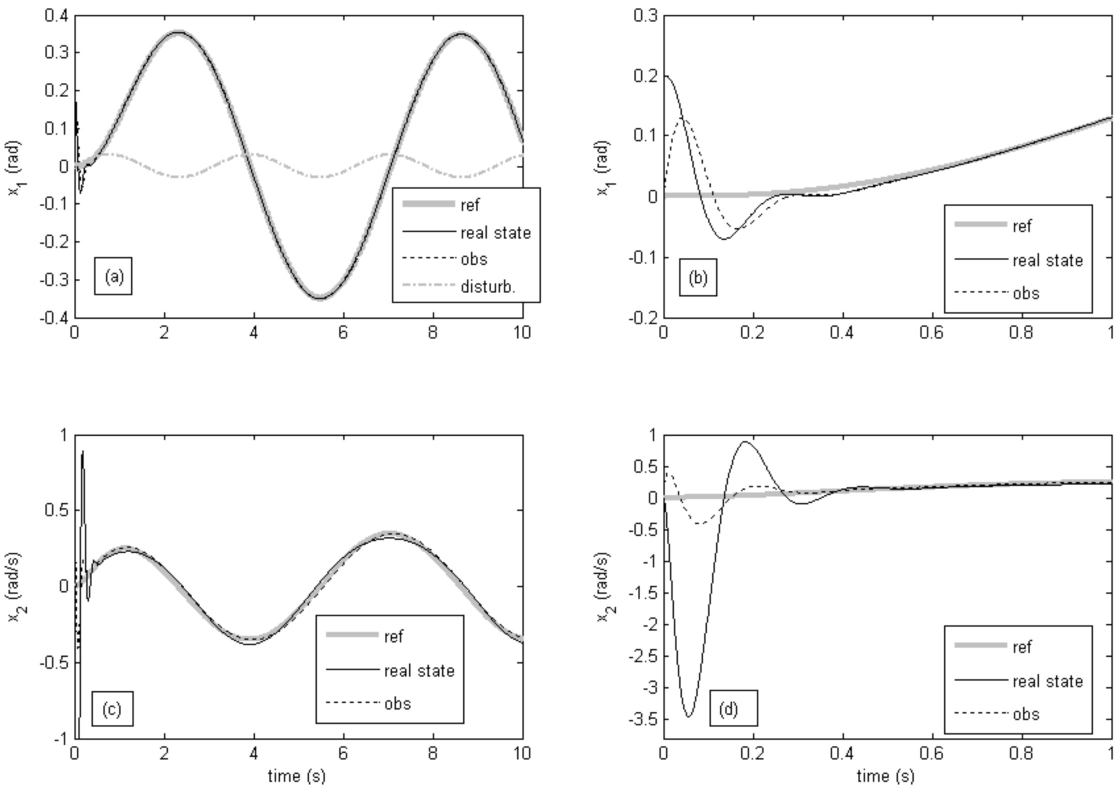


Fig. 2. Tracking performances: (a) Position, (b) Zoom in the transient state of the position, (c) Velocity, (d) Zoom in the transient state of the velocity

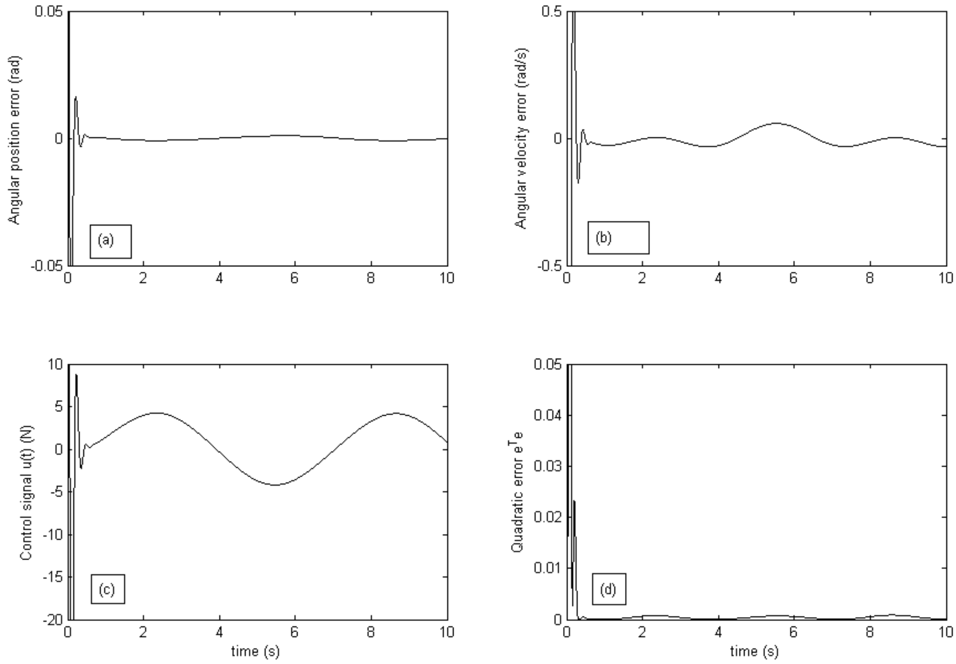


Fig. 3: (a) Position estimation error, (b) Velocity estimation error

(c) Control signal, (d) Quadratic error tracking.

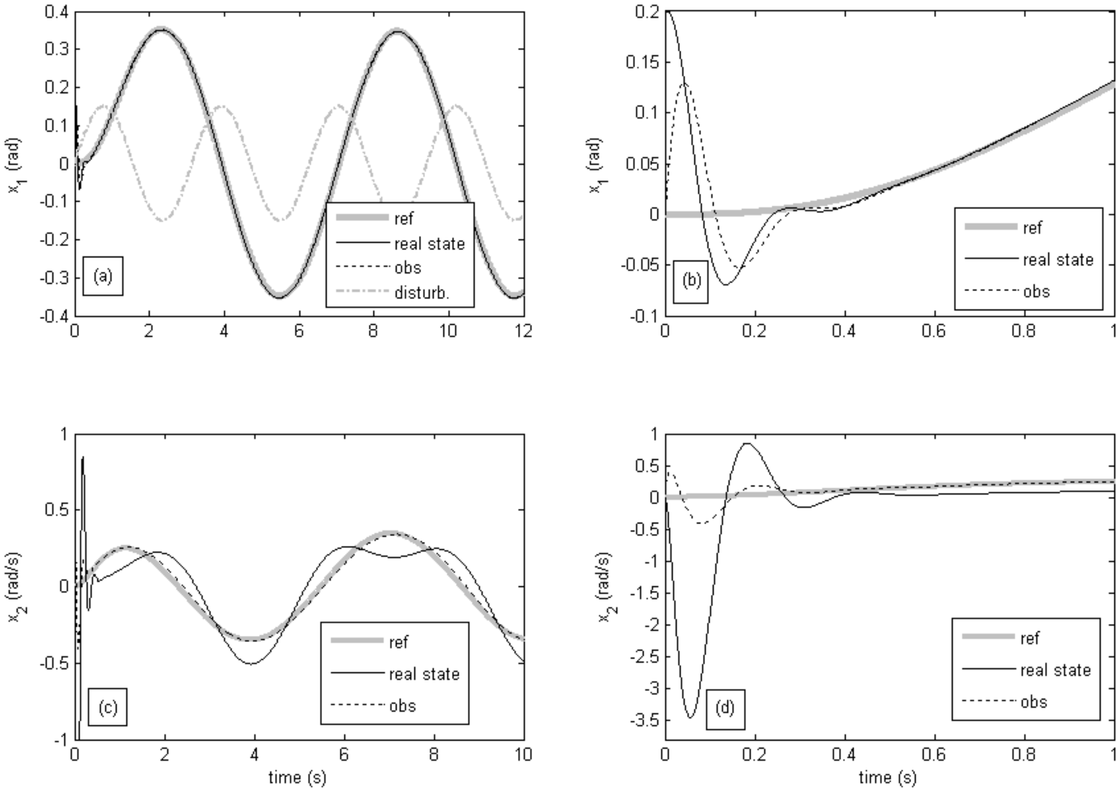


Fig. 4. Tracking performances with high disturbances: (a) Position, (b) Zoom in the transient state of the position, (c) Velocity, (d) Zoom in the transient state of the velocity

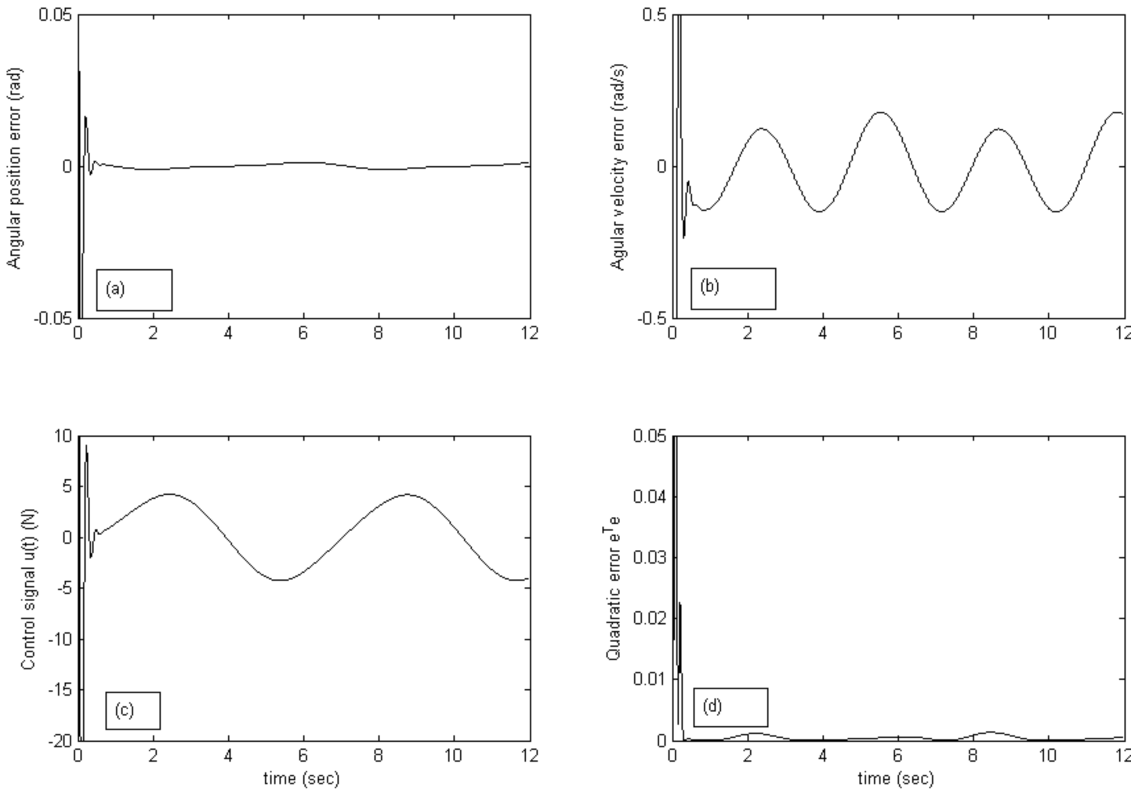


Fig. 5: Simulation with high disturbances: (a) Position estimation error (b) Velocity estimation error (c) Control signal, (d) Quadratic error tracking.