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► **To cite this version:**

G rard Henry Edmond Duchamp, Vincel Hoang Ngoc Minh, Christophe Tollu, B i Chi n, Nguyen
Hoang Nghia. Combinatorics of ϕ -deformed stuffle Hopf algebras. 2013. hal-00793118v6

HAL Id: hal-00793118

<https://hal.science/hal-00793118v6>

Preprint submitted on 23 Feb 2014 (v6), last revised 1 Mar 2014 (v7)

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Combinatorics of deformed shuffle Hopf algebras¹

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Abstract

In order to extend Sch utzenberger's factorization to general perturbations, the combinatorial aspects of the Hopf algebra of a deformed shuffle product is developed systematically in a parallel way with those of the shuffle product, with an emphasis on the Lie elements as studied by Ree. In particular, we will give an effective construction of pair of bases in duality.

Contents

1	Introduction	2
2	First steps	2
3	General results on summability and duality	4
3.1	Total algebras and duality	4
3.1.1	Series and infinite sums	4
3.1.2	Summable families in Hom spaces.	5
3.1.3	Substitutions	7
3.2	Theorem of Cartier-Quillen-Milnor-Moore (analytic form)	10
3.2.1	General properties of bialgebras	10
3.3	Counterexamples and discussion	15
3.3.1	Counterexamples	15
3.3.2	The theorem from the point of view of summability	16
4	Application to the ϕ-deformed shuffle.	18
4.1	General results for the ϕ -deformed shuffle.	18
5	Conclusion	24

¹Version du 23-02-2014 09:56

1 Introduction

Many algebras of functions [11] and many special sums [13, 14] are governed by shuffle products, their perturbations (adding a “superposition term” [12]) or deformations [24].

In order to better understand the mechanisms of these products, we wish here to examine, with full generality the products which are defined by a recursion of the type [7]

$$au \star bv = a(u \star bv) + b(au \star v) + \phi(a, b)u \star v, \quad (1)$$

the empty word being the neutral of this new product.

We then provide some classical combinatorial applications. In most cases, the law ϕ is dual² and under some growth conditions the obtained algebra is an enveloping algebra.

In the second section, is a version of the Cartier-Quillen-Milnor and Moore³ without any use of the Poincaré-Birkhoff-Witt construction. We are obliged to restate the CQMM theorem without supposing any basis because we aim at “varying the scalars” in forthcoming papers (germs of functions, arithmetic functions, etc.) and, in order to do this at ease, we must cope safely with cases where torsion (non-zero annihilators) may appear (and then, one cannot have any basis). See (counter) examples in the section.

Acknowledgements. — The authors wish to thank Darij Grinberg for having thoroughly read the manuscript for having thoroughly read the manuscript, provided a limiting counterexample and participated to fruitful interactions. The authors also would like to acknowledge the support of the “Combinatoire algébrique” Univ. Paris 13, Sorbonne Paris Cité BQR grant.

2 First steps

Let X be a totally ordered alphabet⁴. The free monoid and the set of Lyndon words, over X , are denoted respectively by X^* and $\mathcal{Lyn}X$. The neutral element of X^* , *i.e.* the empty word is denoted by 1_{X^*} . Let $\mathbb{Q}\langle X \rangle$ be equipped with the concatenation and the shuffle product which is defined on the words by

$$\begin{aligned} \forall w \in X^*, \quad w \sqcup 1_{X^*} &= 1_{X^*} \sqcup w = w, \\ \forall x, y \in X, \forall u, v \in X^*, \quad xu \sqcup yv &= x(u \sqcup yv) + y(xu \sqcup v), \end{aligned} \quad (2)$$

or by their dual co-products, $\Delta = \Delta_{\text{conc}}$ and $\Delta = \Delta_{\sqcup}$, defined, for any $w \in X^*$ by,

$$\Delta_{\text{conc}}(w) = \sum_{w=uv} u \otimes v$$

²That is to say comes by dualization of a comultiplication.

³CQMM in the sequel

⁴In the sequel, the order between the words will be understood as the lexicographic total ordering $<$. For example, with $a < b$, one has $ab < b$.

$$\Delta_{\sqcup}(w) = \sum_{I+J=[1..|w|]} w[I] \otimes w[J] \quad (3)$$

One gets two Hopf algebras

$$\begin{aligned} \mathcal{H}_{\sqcup} &= (\mathbb{Q}\langle X \rangle, \text{conc}, 1_{X^*}, \Delta_{\sqcup}, \epsilon, a_{\bullet}) \text{ and} \\ \mathcal{H}_{\sqcup}^{\vee} &= (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*}, \Delta_{\text{conc}}, \epsilon, a_{\sqcup}) \end{aligned} \quad (4)$$

mutually dual with respect to the pairing given by

$$(\forall u, v \in X^*)(\langle u | v \rangle = \delta_{u,v}) . \quad (5)$$

The antipodes and the co-units are the same and given by, for $x_{i_1}, \dots, x_{i_r} \in X$ and $P \in \mathbb{Q}\langle X \rangle$,

$$\begin{aligned} \epsilon(P) &= \langle P | 1_{X^*} \rangle, \\ a_{\sqcup}(w) &= a_{\bullet}(w) = (-1)^r x_{i_r} \dots x_{i_1}, . \end{aligned} \quad (6)$$

By the CQMM theorem, the connected, graded positively, cocommutative Hopf algebra \mathcal{H}_{\sqcup} is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements which here is $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$. Hence any basis of the free algebra $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle^5$ can be completed, by the PBW construction, as a linear basis $\{b_w\}_{w \in X^*}$ of $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle) = \mathbb{Q}\langle X \rangle$ (see below (9) for an example of such a construction) and, when the basis is finely homogeneous, so is $\{b_w\}_{w \in X^*}$ and one can construct, by duality, a basis $\{\check{b}_w\}_{w \in X^*}$ of \mathcal{H}_{\sqcup} (viewed as a \mathbb{Q} -module) such that :

$$\forall u, v \in X^*, \quad \langle \check{b}_u | b_v \rangle = \delta_{u,v} . \quad (7)$$

For $w = l_1^{i_1} \dots l_k^{i_k}$ with $l_1, \dots, l_k \in \mathcal{L}ynX$, $l_1 > \dots > l_k$

$$\check{b}_w = \frac{\check{b}_{l_1}^{\sqcup i_1} \sqcup \dots \sqcup \check{b}_{l_k}^{\sqcup i_k}}{i_1! \dots i_k!} . \quad (8)$$

(see [6, 18, 19]). For example, Chen, Fox and Lyndon [10] constructed the PBW-Lyndon basis $\{P_w\}_{w \in X^*}$ for $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle)$ as follows

$$\begin{aligned} P_x &= x && \text{for } x \in X, \\ P_l &= [P_s, P_r] && \text{for } l \in \mathcal{L}ynX, \text{ with standard factorization } l = (s, r), \\ P_w &= P_{l_1}^{i_1} \dots P_{l_k}^{i_k} && \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k, l_1, \dots, l_k \in \mathcal{L}ynX. \end{aligned} \quad (9)$$

Schützenberger and his school constructed the linear basis $\{S_w\}_{w \in X^*}$ for $\mathcal{A} = (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$ by duality (w.r.t. eq.5) and obtained the transcendence basis of \mathcal{A} , $\{S_l\}_{l \in \mathcal{L}ynX}$ as follows⁶

$$S_l = xS_u, \quad \text{for } l = xu \in \mathcal{L}ynX, \quad (10)$$

$$S_w = \frac{S_{l_1}^{\sqcup i_1} \sqcup \dots \sqcup S_{l_k}^{\sqcup i_k}}{i_1! \dots i_k!} \quad \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k. \quad (11)$$

⁵The basis can be reindexed by Lyndon words and then one uses the canonical factorization of the words.

⁶Therefore \mathcal{A} is a polynomial algebra $\mathcal{A} \simeq \mathbb{Q}[\mathcal{L}ynX]$.

After that, Mélançon and Reutenauer [23] proved that⁷, for any $w \in X^*$,

$$P_w = w + \sum_{v > w, \underline{v} = \underline{w}} c_v v \quad \text{and} \quad S_w = w + \sum_{v < w, \underline{v} = \underline{w}} d_v v. \quad (12)$$

On other words, the elements of the bases $\{S_w\}_{w \in X^*}$ and $\{P_w\}_{w \in X^*}$ are upper and lower triangular respectively and are multihomogeneous.

Moreover, thanks to the duality of the bases $\{P_w\}_{w \in X^k}$ and $\{S_w\}_{w \in X^k}$, if \mathcal{D}_X denotes the diagonal series over X one has

$$\mathcal{D}_X = \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \text{Lyn} X} \exp(S_l \otimes P_l). \quad (13)$$

In fact as stated in [23], this factorization holds in the framework of enveloping algebras and it will be shown in detail how to handle this framework even in the absence of any basis. It is CQMM with an analytic point of view.

3 General results on summability and duality

Let $Y = \{y_i\}_{i \in I}$ be a totally ordered alphabet. The free monoid and the set of Lyndon words, over Y , are denoted respectively by Y^* and $\text{Lyn} Y$. The neutral of Y^* (and then of $A(Y)$) is denoted by 1_{Y^*} .

3.1 Total algebras and duality

3.1.1 Series and infinite sums

We here recall the results used to handle infinite sums in the sequel. The underlying topology is that of the pointwise convergence (the target being undowed with the discrete topology). This section may therefore be skipped by the reader which is familiar with these matters.

In the sequel, we will need to construct spaces of functions on different monoids (mainly direct products of free monoids). We set, once for all the general construction of the corresponding convolution algebra.

Let A be a unitary commutative ring and M a monoid. Let us denote A^M the set⁸ of all (graphs of) mappings $M \rightarrow A$. This set is endowed with its classi-

⁷ Recall that the duality preserves the (multi)homogeneous degrees and interchanges the triangularity of polynomials [23]. For that, one can construct the triangular matrices M and N whose entries are the coefficients of the multihomogeneous triangular polynomials, $\{P_w\}_{w \in X^k}$ and $\{S_w\}_{w \in X^k}$ in the basis $\{w\}_{w \in X^*}$, respectively :

$$M_{u,v} = \langle P_u | v \rangle \quad \text{and} \quad N_{u,v} = \langle S_u | v \rangle.$$

The triangular matrices M and N are unipotent and satisfy the identity $N = ({}^t M)^{-1}$. In Eq. 12, the underlined \underline{u} stand for their multidegree i.e.

$$\underline{u} = (|u|_x)_{x \in X}$$

⁸In general Y^X is the set of all (total) mappings $X \rightarrow Y$ [2] Ch 2.5.2.

cal structure of module. In order to extend the product defined in $A[M]$ (the algebra of the monoid M), it is essential that, in the sums

$$f * g = \sum_{m \in M} \left(\sum_{uv=m} f(u)g(v) \right) m \quad (14)$$

the inner sums $\sum_{uv=m} f(u)g(v)$ make sense. For that, we suppose that the monoid M fulfills condition ‘‘D’’ (i.e. M is of finite decomposition type [3] Ch III.10). Formally, we say that M satisfies condition ‘‘D’’ iff, for all $m \in M$, the set

$$\{(u, v) \in M \times M \mid uv = m\} \quad (15)$$

is finite. In this case eq.14 endows A^M with the structure of an AAU⁹. This algebra is traditionally called the total algebra of M (see [3] Ch III.10) and has very much to do with the algebra of series¹⁰. Here, it will be denoted, with an unambiguous abuse of denotation, by $A\langle\langle M \rangle\rangle$.

The pairing

$$A\langle\langle M \rangle\rangle \otimes A[M] \longrightarrow A \quad (16)$$

defined by¹¹

$$\langle f \mid g \rangle := \sum_{m \in M} f(m)g(m) \quad (17)$$

allows to consider the total algebra as the dual of the module $A[M]$ i.e., through this pairing

$$A\langle\langle M \rangle\rangle \simeq (A[M])^* .$$

One says that a family $(f_i)_{i \in I}$ of $A\langle\langle M \rangle\rangle$ is summable [1] iff, for every $m \in M$, the mapping $i \mapsto \langle f_i \mid m \rangle$ is finitely supported. In this case, the sum $\sum_{i \in I} f_i$ is exactly the mapping $m \mapsto \sum_{i \in I} \langle f_i \mid m \rangle$ so that, one has by definition

$$\left\langle \sum_{i \in I} f_i \mid m \right\rangle = \sum_{i \in I} \langle f_i \mid m \rangle . \quad (18)$$

Finally, let us remark that the set $M_1 \otimes M_2 = \{u \otimes v\}_{(u,v) \in M_1 \times M_2}$ is a (monoidal) basis of $A[M_1] \otimes A[M_2]$ and $M_1 \otimes M_2$ is a monoid (in the product algebra $A[M_1] \otimes A[M_2]$) isomorphic to the direct product $M_1 \times M_2$.

3.1.2 Summable families in Hom spaces.

In fact, $A\langle\langle M \rangle\rangle \simeq (A[M])^* = \text{Hom}(A[M], A)$ and the notion of summability developed above can be seen as a particular case of that of a family of endomorphisms $f_i \in \text{Hom}(V, W)$ for which $\text{Hom}(V, W)$ appears as a complete space.

⁹Associative Algebra with Unit.

¹⁰Actually, the algebra of commutative (resp. noncommutative) series on an alphabet X is the total algebra of the free commutative (resp. free) monoid on X

¹¹Here $A[M]$ is identified with the submodule of finitely supported functions $M \rightarrow A$.

It is indeed the pointwise convergence for the discrete topology. We will not expand that topic here.

The definition is similar of that of a summable family of series [1], viewed as a family of linear forms.

Definition 1. *i) A family $(f_i)_{i \in I}$ of elements in $\text{Hom}(V, W)$ is said to be summable iff for all $x \in V$, the map $i \mapsto f_i(x)$ has finite support. As a quantized criterium it reads*

$$(\forall x \in V)(\exists F \subset I, F \text{ finite})(\forall i \notin F)(f_i(x) = 0) \quad (19)$$

ii) If the family $(f_i)_{i \in I} \in \text{Hom}(V, W)^I$ fulfils the condition 19 above its sum is given by

$$\left(\sum_{i \in I} f_i\right)(x) = \sum_{i \in I} f_i(x) \quad (20)$$

It is an easy exercise to show that the mapping $V \rightarrow W$ defined by the equation 20 is in fact in $\text{Hom}(V, W)$. Remark that, as the limiting process is defined by linear conditions, if a family $(f_i)_{i \in I}$ is summable, so is

$$(a_i f_i)_{i \in I} \quad (21)$$

for an arbitrary family of coefficients $(a_i)_{i \in I} \in A^I$.

This tool will be used in section (3.2) to give an analytic presentation of the theorem of Cartier-Quillen-Milnor-Moore in the case when $V = W = \mathcal{B}$ is a bialgebra.

The most interesting feature of this operation is the interchange of sums. Let us state it formally as a proposition the proof of which is left to the reader.

Proposition 1. *Let $(f_i)_{i \in I}$ be a family of elements in $\text{Hom}(V, W)$ and $(I_j)_{j \in J}$ be a partition of I ([2] ch II §4 n° 7 Def. 6), then, the following statements are equivalent*

i) $(f_i)_{i \in I}$ is summable

ii) for all $j \in J$, $(f_i)_{i \in I_j}$ is summable and the family $(\sum_{i \in I_j} f_i)_{j \in J}$ is summable. In these conditions, one has

$$\sum_{i \in I} f_i = \sum_{j \in J} \left(\sum_{i \in I_j} f_i\right) \quad (22)$$

We derive at once from this the following practical criterium for double sums.

Proposition 2. *Let $(f_{\alpha, \beta})_{(\alpha, \beta) \in X \times Y}$ be a doubly indexed summable family in $\text{Hom}(V, W)$, then, for fixed α (resp. β) the “row-families” $(f_{\alpha, \beta})_{\beta \in Y}$ (resp. the “column-families” $(f_{\alpha, \beta})_{\alpha \in X}$) are summable and their sums are summable. Moreover*

$$\sum_{(\alpha, \beta) \in X \times Y} f_{\alpha, \beta} = \sum_{\alpha \in X} \sum_{\beta \in Y} f_{\alpha, \beta} = \sum_{\beta \in Y} \sum_{\alpha \in X} f_{\alpha, \beta} . \quad (23)$$

3.1.3 Substitutions

Let \mathcal{A} be an AAU and $f \in \mathcal{A}$. For every polynomial $P \in A\langle X \rangle (= A[X^*])$, one can compute $P(f)$ by

$$P(f) = \sum_{n \geq 0} \langle P | X^n \rangle f^n . \quad (24)$$

One checks at once that $P \mapsto P(f)$ is a morphism¹² of AAU's between $A\langle X \rangle$ and \mathcal{A} . Moreover, this morphism is compatible with the substitutions as one checks easily that, for $Q \in A[X]$

$$P(Q)(f) = P(Q(f)) \quad (25)$$

(it suffices to check that $P \mapsto P(Q)(f)$ and $P \mapsto P(Q(f))$ are two morphisms which coincide on $P = X$).

In order to substitute within series, one needs some limiting process. The framework of $\mathcal{A} = \text{Hom}(V, W)$ and summable families will be here sufficient (see paragraph 3.1.2). We suppose that $(V, \delta_V, \epsilon_V)$ is a co-AAU and that $(W, \mu_W, 1_W)$ is an AAU. Then $(\text{Hom}(V, W), *, e)$ is an AAU (with $e = 1_W \circ \epsilon_V$). A series $S \in A[[X]]$ and $f \in \text{Hom}(V, W)$ being given, we say that $f \in \text{Dom}(S)$ iff the family $(\langle S | X^n \rangle f^{*n})_{n \geq 0}$ is summable¹³. We have the following properties

Proposition 3. *If $f \in \text{Dom}(S) \cap \text{Dom}(T)$ and $\alpha \in A$, one has*

$$(\alpha S)(f) = \alpha S(f) ; (S + T)(f) = S(f) + T(f) \quad (26)$$

and

$$(TS)(f) = T(f) * S(f) . \quad (27)$$

If $((f)^{*n})_{n \geq 0}$ is summable and $S(0) = 0$ then

$$f \in \text{Dom}(S) \cap \text{Dom}(T(S)) ; S(f) \in \text{Dom}(T) \quad (28)$$

and

$$T(S)(f) = T(S(f)) \quad (29)$$

Proof. Let us first prove eq.27 . As $f \in \text{Dom}(S) \cap \text{Dom}(T)$, the families $(\langle S | X^n \rangle f^{*n})_{n \geq 0}$ and $(\langle T | X^m \rangle f^{*m})_{m \geq 0}$ are summable, then so is

$$\left(\langle T | X^m \rangle f^{*m} * \langle S | X^n \rangle f^{*n} \right)_{n, m \geq 0} \quad (30)$$

as, for every $x \in V$, $\delta(x) = \sum_{i=1}^N x_i^{(1)} \otimes x_i^{(2)}$ and for every $i \in I$,

$$\text{supp}_{w.r.t. m}(\langle T | X^m \rangle f^{*m}(x_i^{(1)})) ; \text{supp}_{w.r.t. n}(\langle S | X^n \rangle f^{*n}(x_i^{(2)}))$$

¹²In case \mathcal{A} is a geometric space, this morphism is called "evaluation at f " and corresponds to a Dirac measure.

¹³Where f^{*n} denotes straightforwardly the n -th power of f w.r.t. the convolution product.

are finite. Then outside of the cartesian product of the (finite) union of these supports, the product

$$(\langle T | X^m \rangle f^{*m} * \langle S | X^n \rangle f^{*n})(x) = \mu_W((\langle T | X^m \rangle f^{*m} \otimes \langle S | X^n \rangle f^{*n})(\delta(x))) \quad (31)$$

is zero. Hence the summability.

Now

$$\begin{aligned} T(f) * S(f) &= \sum_{m=0}^{\infty} (\langle T | X^m \rangle f^{*m}) * \sum_{n=0}^{\infty} (\langle S | X^n \rangle f^{*n}) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\langle T | X^m \rangle \langle S | X^n \rangle f^{*(n+m)}) \\ &= \sum_{s=0}^{\infty} \left(\sum_{n+m=s}^{\infty} \langle T | X^m \rangle \langle S | X^n \rangle \right) f^{*s} \\ &= \sum_{s=0}^{\infty} (\langle TS | X^s \rangle) f^{*s} = (TS)(f) \end{aligned} \quad (32)$$

We now prove the statements (28) and (29). If $((f)^{*n})_{n \geq 0}$ is summable then f belongs to all domains (i.e. is universally substitutable) by virtue of eq.21 . For all $x \in V$, there exists $N_x \in \mathbb{N}$ such that

$$n > N_x \implies (f)^{*n}(x) = 0 .$$

Now, for S such that $S(0) = 0$, one has $S = \sum_{n=1}^{\infty} \langle S | X^n \rangle X^n$ and then $S^k = \sum_{n=k}^{\infty} \langle S^k | X^n \rangle X^n$. Now, in view of eq.27 , one has

$$S(f)^{*n}(x) = S^n(f)(x) = \sum_{m=n}^{\infty} \langle S^n | X^m \rangle (f)^{*m}(x) \quad (33)$$

which is zero for $n > N_x$. Hence the summability of $(S(f)^{*n})_{n \geq 0}$ which implies that $S(f) \in \text{Dom}(T)$. The family $(\langle T | X^n \rangle \langle S^n | X^m \rangle (f)^{*m})_{(n,m) \in \mathbb{N}^2}$ is summable because, if $x \in V$ and if n or m is greater than N_x then

$$\langle T | X^n \rangle \langle S^n | X^m \rangle (f)^{*m}(x) = 0 \quad (34)$$

thus $T(S(f))$ is the sum

$$\begin{aligned}
T(S(f)) &= \sum_{n=0}^{\infty} \langle T | X^n \rangle S(f)^{*n} \sum_{n=0}^{\infty} \langle T | X^n \rangle \sum_{m=n}^{\infty} \langle S^n | X^m \rangle (f)^{*m} = \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle T | X^n \rangle \langle S^n | X^m \rangle (f)^{*m} = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \langle T | X^n \rangle \langle S^n | X^m \rangle \right) (f)^{*m} = \\
&= \sum_{m=0}^{\infty} \langle T(S) | X^m \rangle (f)^{*m} = T(S)(f)
\end{aligned} \tag{35}$$

□

In the free case (i.e. $V = W$ are the bialgebra $(A\langle X \rangle, \mathbf{conc}, 1_{X^*}, \Delta_{\sqcup}, \epsilon)$), one has a very useful representation of the convolution algebra $\text{Hom}(V, W)$ through images of the diagonal series. This representation will provide us with the key lemma (2). Let

$$\mathcal{D}_X = \sum_{w \in X^*} w \otimes w.$$

be the diagonal series attached to X .

Proposition 4. *Let A be a commutative unitary ring and X an alphabet. Then*

i) For every $f \in \text{End}(A\langle X \rangle)$, the family $(u \otimes f(u))_{u \in X^}$ is summable in $A\langle\langle X^* \otimes X^* \rangle\rangle$.*

ii) The representation

$$f \mapsto \rho(f) = \sum_{u \in X^*} u \otimes f(u) \tag{36}$$

*is faithful from $(\text{End}(A\langle X \rangle), *)$ to $(A\langle\langle X^* \otimes X^* \rangle\rangle, \sqcup \otimes \mathbf{conc})$. In particular, for $f \in \text{End}(A\langle X \rangle)$ and $P \in A[X]$, one has*

$$\rho(P(f)) = P(\rho(f)) \tag{37}$$

iii) If $f(1_{X^}) = 0$ and $S \in A[[X]]$ is a series, then $(\rho(f)^n)_{n \geq 0}$ is summable in $(A\langle\langle X^* \otimes X^* \rangle\rangle, \sqcup \otimes \mathbf{conc})$ and*

$$\rho(S(f)) = S(\rho(f)) \tag{38}$$

Proof. (of Prop.(4)) Let us compute

$$\begin{aligned}
\rho(f)(\sqcup \otimes \mathbf{conc})\rho(g) &= \sum_{u, v \in X^*} (u \otimes f(u)(\sqcup \otimes \mathbf{conc})(v \otimes g(v))) \\
&= \sum_{u, v \in X^*} (u \sqcup v \otimes (\mathbf{conc}(f(u) \otimes g(v))))
\end{aligned}$$

$$\begin{aligned}
&= \sum_{u,v \in X^*} \sum_{w \in X^*} (\langle u \sqcup v \mid w \rangle w \otimes \text{conc}(f(u)g(v))) \\
&= \sum_{w \in X^*} w \otimes \left(\sum_{u,v \in X^*} (\langle u \sqcup v \mid w \rangle \text{conc}(f(u)g(v))) \right) \\
&= \sum_{w \in X^*} w \otimes \left(\sum_{u,v \in X^*} (\langle u \otimes v \mid \Delta(w) \rangle \text{conc}(f(u)g(v))) \right) \\
&= \sum_{w \in X^*} w \otimes (\text{conc}(f \otimes g) \Delta(w)) = \sum_{w \in X^*} w \otimes (f * g(w)) \tag{39}
\end{aligned}$$

□

3.2 Theorem of Cartier-Quillen-Milnor-Moore (analytic form)

3.2.1 General properties of bialgebras

From now on, we suppose that A be a unitary commutative \mathbb{Q} -algebra (i.e. $\mathbb{Q} \subset A$).

The aim of Cartier-Quillen-Milnor-Moore theorem is to provide necessary and sufficient conditions for \mathcal{B} to be an enveloping algebra, we will discuss this condition in detail in the sequel.

Let $(\mathcal{B}, \mu, e_{\mathcal{B}}, \Delta, \epsilon)$ be a (general) A -bialgebra. One can always consider the Lie algebra of primitive elements $\text{Prim}(\mathcal{B})$ and build the map

$$j_{\mathcal{B}} : \mathcal{U}(\text{Prim}(\mathcal{B})) \rightarrow \mathcal{B} .$$

Then, $\mathcal{A} = j_{\mathcal{B}}(\mathcal{U}(\text{Prim}(\mathcal{B})))$ is the subalgebra generated by the primitive elements.

$$\begin{array}{ccccc}
\text{Prim}(\mathcal{B}) & \xrightarrow{i_{\mathcal{A},P}} & \mathcal{A} & \xrightarrow{i_{\mathcal{B},\mathcal{A}}} & \mathcal{B} \\
\downarrow i_{\mathcal{U},P} & \nearrow i_{\mathcal{U},\mathcal{A}} & & \nearrow j_{\mathcal{B}} & \\
\mathcal{U}(\text{Prim}(\mathcal{B})) & & & &
\end{array}$$

Figure 1: The sub-algebra \mathcal{A} generated by primitive elements.

The mapping $i_{\mathcal{B},\mathcal{A}}$ is into but $i_{\mathcal{B},\mathcal{A}} \otimes i_{\mathcal{B},\mathcal{A}}$ may not be so. This is the case for $\mathcal{B} = (\mathbb{Q}[\epsilon][x], \cdot, 1_{\mathbb{Q}[\epsilon][x]}, \Delta, c)$ where $(\mathbb{Q}[\epsilon][x], \cdot, 1_{\mathcal{B}})$ is the usual polynomial algebra with coefficients in the algebra of dual numbers $\mathbb{Q}[\epsilon]$ (with $\epsilon^2 = 0$) and

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \epsilon x \otimes x, \quad c(x) = 0$$

(see details and proofs below, in sec. 3.3).

In general, one has (only) $\Delta_{\mathcal{B}}(\mathcal{A}) \subset \text{Im}(i_{\mathcal{B},\mathcal{A}} \otimes i_{\mathcal{B},\mathcal{A}})$, this can be simply seen

from the following combinatorial argument.

For any list of primitive elements $L = [g_1, g_2, \dots, g_n]$ and $I = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, n\}$, put $L[I] = g_{i_1} g_{i_2} \dots g_{i_k}$, the product of the sublist. One has

$$\Delta(g_1 g_2 \dots g_n) = \Delta(L[\{1, 2, \dots, n\}]) = \sum_{I+J=\{1,2,\dots,n\}} L[I] \otimes L[J]. \quad (40)$$

From (40) one gets also that $j_{\mathcal{B}}$ is a morphism of bialgebras. If for any reason, there exists a lifting of

$$\Delta_{\mathcal{B}} \circ i_{\mathcal{B}, \mathcal{A}} \quad (41)$$

as a comultiplication of \mathcal{A} , then $j_{\mathcal{B}}$ is into (see the statement and the proof below). Formula ?? proves that we have the following maps (save the – hypothetical – dotted one).

$$\begin{array}{ccc} A\langle G \rangle & \xrightarrow{s_G} & \mathcal{A} \\ \Delta_{\perp} \downarrow & & \vdots \Delta_{\mathcal{A}} \\ A\langle G \rangle \otimes A\langle G \rangle & \xrightarrow{s_G \otimes s_G} & \mathcal{A} \otimes \mathcal{A} \end{array}$$

Figure 2: The unique lifting $\Delta_{\mathcal{A}}$ (when it exists).

Where $G \in \text{Prim}(\mathcal{B})$ is any generating set of the AAU \mathcal{A} . We emphasize the fact that, in the diagram above, G must be understood set-theoretically (i.e. with no relation between the elements¹⁴).

In fact, one has the following proposition

Proposition 5. *Let \mathcal{B} be a bialgebra over a (commutative) \mathbb{Q} -algebra A , the notations being those of figures 1 and 2, then the following statements are equivalent*

- i) *For a generating set $G \subset \text{Prim}(\mathcal{B})$, $\ker(s_G) \subset \ker(s_G \otimes s_G) \circ \Delta_{\perp}$.*
- ii) *For any generating set $G \subset \text{Prim}(\mathcal{B})$, $\ker(s_G) \subset \ker(s_G \otimes s_G) \circ \Delta_{\perp}$.*
- iii) *$j_{\mathcal{B}}$ is into.*

Proof. i) \implies iii) In order to prove this, we need to construct the arrows σ, τ which are a decomposition of a section of $j_{\mathcal{B}}$. Let us remark that, when $\text{Prim}(\mathcal{B})$ is free as an A -module, the proof of this fact is a consequence of the PBW theorem¹⁵. But, here, we will construct the section in the general case using projectors which are now classical for the free case but which still can be computed analytically [23] as they lie in $\mathbb{Q}[[X]]$ and still converge in \mathcal{A} .

¹⁴We will see, below and in paragraph 3.3 how it is crucial to consider that $[\lambda x]$ and $\lambda[x]$ are not necessarily equal, when $\lambda x \in G$ (for clarity, $[y] \in A\langle G \rangle$ is the image of $y \in G$).

¹⁵See [4] Ch2 §1 n^o 6 th 1 for a field of characteristic zero and §1 Ex. 10 for the free case (over a ring A with $\mathbb{Q} \subset A$).

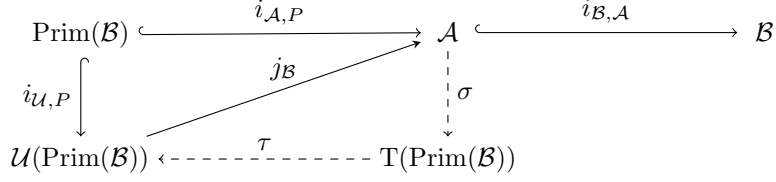


Figure 3: The sub-bialgebra \mathcal{A} generated by primitive elements.

(Injectivity of $j_{\mathcal{B}}$, construction of the section $\tau \circ \sigma$). —

Let \mathcal{A} be the subalgebra of \mathcal{B} generated by $\text{Prim}(\mathcal{B})$, one has $\text{Im}(j_{\mathcal{B}}) = \mathcal{A}$.

Remark that all series $\sum_{n \geq 0} a_n (I_+)^{*n}$ are summable on \mathcal{A} (not in general on \mathcal{B} for example in case \mathcal{B} contains non-trivial group-like elements).

We define

$$c = \log_*(I) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (I_+)^{*n} \quad (42)$$

and remark that, in view of Prop. (4), in the case when $\mathcal{B} = A\langle X \rangle$ one has $\mathcal{A} = \mathcal{B}$ and, with $S(X) = \log(1 + X)$

$$\begin{aligned}
\sum_{w \in X^*} w \otimes \pi_{1,\mathcal{A}}(w) &= \rho(\log(I)) = \rho(S(I^+)) = S(\rho(I^+)) = \\
S\left(\sum_{\substack{w \in X^* \\ w \neq 1_{X^*}}} w \otimes w\right) &= S(\mathcal{D}_X - 1_{X^*} \otimes 1_{X^*}) = \log(\mathcal{D}_X) \quad (43)
\end{aligned}$$

We first prove that $\pi_{1,\mathcal{A}}$ is a projector $\mathcal{A} \rightarrow \text{Prim}(\mathcal{B})$. The key point is that $\Delta_{\mathcal{A}}$ (the restriction of the comultiplication to \mathcal{A}) is a morphism of bialgebras¹⁶ $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$. We first prove that $\Delta_{\mathcal{A}}$ “commutes” with the convolution. This is a consequence of the following property

Lemma 1. *i) Let $f_i \in \text{End}(\mathcal{B}_i)$, be such that $\varphi f_1 = f_2 \varphi$.*

$$\begin{array}{ccc}
\mathcal{B}_1 & \xrightarrow{\varphi} & \mathcal{B}_2 \\
f_1 \downarrow & & \downarrow f_2 \\
\mathcal{B}_1 & \xrightarrow{\varphi} & \mathcal{B}_2
\end{array}$$

Figure 4: Intertwining with a morphism of bialgebras (the functions of f_i below will be computed with the respective convolution products).

i) Then, if $P \in A[X]$, one has

$$\varphi P(f_1) = P(f_2) \varphi . \quad (44)$$

¹⁶In fact it is the case for any cocommutative bialgebra, be it generated by its primitive elements or not.

ii) If the series $\sum_{n \geq 0} (I_i^+)^{*n}$, $i = 1, 2$ are summable and, if $f_1(1) = 0$ (which implies $f_2(1) = 0$) and $S \in A[[X]]$, the families $(\langle S | X^n \rangle f_i^{*n})_{n \in \mathbb{N}}$ are summable, we denote by $S(f_i)$ their sums (note that this definition is coherent with the previous ones when S is a polynomial).
One has, for the convolution product,

$$\varphi S(f_1) = S(f_2) \varphi . \quad (45)$$

Proof. The only delicate part is (ii). First, one remarks that, if φ is a morphism of bialgebras, one has

$$(\varphi \otimes \varphi) \circ \Delta_1^+ = \Delta_2^+ \circ \varphi \quad (46)$$

then, the image by φ of an element of order less than N (i.e. such that $\Delta_1^{+(N)}(x) = 0$) is of order less than N . Let now S be a univariate series $S = \sum_{k=0}^{\infty} a_k X^k$. For every element x of order less than N and $f \in \text{End}(\mathcal{B})$, one has

$$\begin{aligned} S(f)(x) &= \sum_{k=0}^{\infty} a_k f^{*k}(x) = \sum_{k=0}^{\infty} a_k \mu^{(k-1)} f^{\otimes k} \Delta^{(k-1)}(x) \\ &= \sum_{k=0}^{\infty} a_k \mu^{(k-1)} (f^{\otimes k}) \circ (I_+^{\otimes k}) \Delta^{(k-1)}(x) \\ &= \sum_{k=0}^N a_k \mu^{(k-1)} (f^{\otimes k}) \Delta_+^{(k-1)}(x) . \end{aligned} \quad (47)$$

This proves, in view of (i) that $\varphi \circ S(f_1) = S(f_2) \circ \varphi$.

Thanks to Lemma 1, we can now prove that π_1 is a projector $\mathcal{B} \rightarrow \text{Prim}(\mathcal{B})$.

In case \mathcal{B} is cocommutative, the comultiplication Δ is a morphism of bialgebras, so one has

$$\Delta \circ \log_*(I) = \log_*(I \otimes I) \circ \Delta \quad (48)$$

But

$$\begin{aligned} \log_*(I \otimes I) &= \log_*((I \otimes e) * (e \otimes I)) \\ &= \log_*(I \otimes e) + \log_*(e \otimes I) \\ &= \log_*(I) \otimes e + e \otimes \log_*(I) \end{aligned} \quad (49)$$

Then

$$\Delta(\log_*(I)) = (\log_*(I) \otimes e + e \otimes \log_*(I)) \circ \Delta \quad (50)$$

which implies that $\log_*(I)(\mathcal{B}) \subset \text{Prim}(\mathcal{B})$. To finish the proof that π_1 is a projector onto $\text{Prim}(\mathcal{B})$, it suffices to remark that, for $x \in \text{Prim}(\mathcal{B})$ and $n \geq 2$, $(\text{Id}^+)^{*n}(x) = 0$ then

$$\log_*(I)(x) = \text{Id}^+(x) = x . \quad (51)$$

□

Now, we consider

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n \geq 0} \frac{1}{n!} \pi_{1,\mathcal{A}}^{*n}, \quad (52)$$

where $\pi_{1,[\mathcal{A}]} = \log_*(I_{\mathcal{A}})$.

Let us prove that the summands form a resolution of unity.

First, one defines $\mathcal{A}_{[n]}$ as the linear span of the powers $\{P^n\}_{P \in \text{Prim}(\mathcal{B})}$ or, equivalently, of the symmetrized products

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} P_{\sigma(1)} P_{\sigma(2)} \cdots P_{\sigma(n)}. \quad (53)$$

It is obvious that $\text{Im}(\pi_{1,\mathcal{A}})^{*n} \subset \mathcal{A}_{[n]}$. We remark that

$$\pi_{1,\mathcal{A}}^{*n} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1,\mathcal{A}}^{\otimes n} \Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1,\mathcal{A}}^{\otimes n} I_+^{\otimes n} \Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1,\mathcal{A}}^{\otimes n} \Delta_+^{(n-1)} \quad (54)$$

as $\pi_{1,\mathcal{A}} I_+ = \pi_{1,\mathcal{A}}$. Now, let $P \in \text{Prim}(\mathcal{A})$. We compute $\pi_{1,\mathcal{A}}^{*n}(P^m)$. Indeed, if $m < n$, one has

$$\pi_{1,\mathcal{A}}^{*n}(P^m) = \mu_{\mathcal{B}}^{n-1} \Delta_+^{n-1}(P^m) = 0. \quad (55)$$

If $n = m$, one has, from (40)

$$\Delta_+^{n-1}(P^n) = n! P^{\otimes n} \quad (56)$$

and hence $\pi_{1,\mathcal{A}}^{*n}$ is the identity on $\mathcal{A}_{[n]}$. If $m > n$, the nullity of $\pi_{1,\mathcal{A}}^{*n}(P^m)$ is a consequence of the following lemma.

Lemma 2. *Let \mathcal{B} be a bialgebra and P a primitive element of \mathcal{B} . Then*

i) The series $\log_(I)$ is summable on each power P^m*

ii) $\log_(I)(P^m) = 0$ for $m > 2$*

Proof. i) As $\Delta_+^{*N}(P^m) = 0$ for $N > m$, one has $I_+^{*N}(P^m) = 0$ for these values.

ii) Let a be a letter, the morphism of AAU $\varphi_P : A[a] \rightarrow \mathcal{B}$, defined by

$$\varphi_P(a) = P \quad (57)$$

is, in fact, a morphism of bialgebras. One checks easily that $\pi_{1,[A[a]]}(a^m) = 0$

$$\begin{array}{ccc} A[a] & \xrightarrow{\varphi_P} & \mathcal{B} \\ I_{A[a]}^+ \downarrow & & \downarrow I_{\mathcal{B}}^+ \\ A[a] & \xrightarrow{\varphi_P} & \mathcal{B} \end{array}$$

Figure 5: Intertwining with one primitive element.

for $m > 2$ which is a consequence of the general equality (see eq.43)

$$\sum_{w \in X^*} (w \otimes \pi_1(w)) = \log\left(\sum_{w \in X^*} w \otimes w\right) \quad (58)$$

because, for $Y = \{a\}$ (and then $A\langle X \rangle = A[a]$) one has

$$\begin{aligned} \log\left(\sum_{w \in X^*} w \otimes w\right) &= \log\left(\sum_{n \geq 0} a^n \otimes a^n\right) = \\ \log\left(\sum_{n \geq 0} \frac{1}{n!} (a \otimes a)^{(\sqcup \otimes \text{conc})n}\right) &= \log(\exp(a \otimes a)) = a \otimes a \end{aligned} \quad (59)$$

this proves that $\pi_{1,\mathcal{A}}^{*n}(\mathcal{A}_{[m]}) = 0$ for $m \neq n$ and hence the summands of the sum

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n \geq 0} \frac{1}{n!} \pi_{1,\mathcal{A}}^{*n}. \quad (60)$$

are pairwise orthogonal projectors with $\text{Im}(\pi_{1,\mathcal{A}}^{*n}) = \mathcal{A}_{[n]}$ and then

$$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_{[n]}. \quad (61)$$

This decomposition enables to construct σ by

$$\sigma(P^n) = \frac{1}{n!} \Delta_+^{(n-1)}(P^n) \in T_n(\text{Prim}(\mathcal{B})) \quad (62)$$

for $n \geq 1$ and, one sets $\sigma(1_{\mathcal{B}}) = 1_{T(\text{Prim}(\mathcal{B}))}$.

It is easy to check that $j_{\mathcal{B}} \circ \tau \circ \sigma = \text{Id}_{\mathcal{A}}$ as \mathcal{A} is (linearly) generated by the powers $(P^m)_{P \in \text{Prim}(\mathcal{B}), m \geq 0}$. \square

End of the proof of proposition 5. —

iii) \implies ii) If $j_{\mathcal{B}}$ is into, then $i_{\mathcal{U},\mathcal{A}}$ is one-to-one and one gets a comultiplication

$$\Delta_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$$

such that, for any list of primitive elements $L = [g_1, g_2, \dots, g_n]$ (the denotations are the same as previously)

$$\Delta_{\mathcal{A}}(g_1 g_2 \cdots g_n) = \Delta(L[\{1, 2, \dots, n\}]) = \sum_{I+J=\{1,2,\dots,n\}} L[I] \otimes_{\mathcal{A}} L[J] \quad (63)$$

but, this time, the tensor product $\otimes_{\mathcal{A}}$ is understood as being in $\mathcal{A} \otimes \mathcal{A}$. This guarantees that the diagram Fig. 2 commutes for any G .

ii) \implies i) Obvious. \square

3.3 Counterexamples and discussion

3.3.1 Counterexamples

It has been said that, with $\mathcal{B} = (\mathbb{Q}[\epsilon][x], \cdot, 1_{\mathbb{Q}[\epsilon][x]}, \Delta, c)$ (notations as above), $j_{\mathcal{B}}$ is not into, let us show this statement.

The q -infiltration coproduct [8] Δ_q is defined on the free algebra $K\langle X \rangle$ (K is a unitary ring), by its values on the letters

$$\Delta_q(x) = x \otimes 1 + 1 \otimes x + q(x \otimes x) \quad (64)$$

where $q \in K$. One can show easily that, for a word $w \in X^*$,

$$\Delta_q(w) = \sum_{I \cup J = [1..|w|]} q^{|I \cap J|} w[I] \otimes w[J] \quad (65)$$

with, as above (for $I = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, n\}$ and $w = a_1 a_2 \dots a_n$), $w[I] = a_{i_1} a_{i_2} \dots a_{i_k}$.

Then, with $K = \mathbb{Q}[\epsilon]$, $q = \epsilon$, $X = x$, one has (as a direct application of Eq. 65)

$$\Delta_\epsilon(x^n) = \sum_{k=0}^n \binom{n}{k} x^k \otimes x^{n-k} + \epsilon \sum_{k=1}^n k \binom{n}{k} x^k \otimes x^{n-k+1}. \quad (66)$$

This proves that, here, the space of primitive elements is a submodule of Kx and solving $\Delta_\epsilon(\lambda x) = (\lambda x) \otimes 1 + 1 \otimes (\lambda x)$, one finds $\lambda = \lambda_1 \epsilon$. Together with $\epsilon x \in \text{Prim}(\mathcal{B})$ this proves that $\text{Prim}(\mathcal{B})$ is of \mathbb{Q} -dimension one (in fact equal to $\mathbb{Q} \cdot (\epsilon x)$). Now, the consideration of the morphism of Lie algebras $\text{Prim}(\mathcal{B}) \rightarrow K[x]/(\epsilon K[x])$ which sends ϵx to x proves that, in $\mathcal{U}(\text{Prim}(\mathcal{B}))$, we have $(\epsilon x)(\epsilon x) \neq 0$ and $j_{\mathcal{B}}$ cannot be into.

For a graded counterexample¹⁷, one can see that, with $K = \mathbb{Q}[\epsilon]$, $X = x, y, z$, $\mathcal{B} = K\langle X \rangle$ and

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \epsilon(y \otimes z), \quad \Delta(y) = y \otimes 1 + 1 \otimes y, \quad \Delta(z) = z \otimes 1 + 1 \otimes z \quad (67)$$

the same phenomenon occurs (for the gradation, one takes

$$\deg(y) = \deg(z) = 1, \quad \deg(x) = 2).$$

3.3.2 The theorem from the point of view of summability

From now on, the morphism $j_{\mathcal{B}}$ is supposed into.

The bialgebra \mathcal{B} being supposed cocommutative, we discuss the equivalent conditions under which we are in the presence of an enveloping algebra i.e.

$$\mathcal{B} \cong_{A\text{-bialg}} \mathcal{U}(\text{Prim}(\mathcal{B})) \quad (68)$$

from the point of view of the convergence of the series $\log_*(I)$ ¹⁸. These conditions are known as the theorem of Cartier-Quillen-Milnor-Moore (CQMM).

Theorem 1. [4] *Let \mathcal{B} be a A -cocommutative bialgebra (A is a \mathbb{Q} -AAU) and \mathcal{A} , as above, the subalgebra generated by $\text{Prim}(\mathcal{B})$. Then, the following conditions are equivalent :*

¹⁷This example is due to Darij Grinberg.

¹⁸In a A -bialgebra, one can always consider the series of endomorphisms

$$\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (I^+)^{*n}. \quad (69)$$

The family $(\frac{(-1)^{n-1}}{n} (I^+)^{*n})_{n \geq 0}$ is summable iff $((I^+)^{*n})_{n \geq 0}$ is (use eq.21).

i) \mathcal{B} admits an increasing filtration

$$\mathcal{B}_0 = A.1_{\mathcal{B}} \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \cdots$$

compatible with the structures of algebra (i.e. for all $p, q \in \mathbb{N}$, one has $\mathcal{B}_p \mathcal{B}_q \subset \mathcal{B}_{p+q}$) and coalgebra :

$$\forall n \in \mathbb{N}, \quad \Delta(\mathcal{B}_n) \subset \sum_{p+q=n} \mathcal{B}_p \otimes \mathcal{B}_q.$$

ii) $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$ is summable in $\text{End}(\mathcal{B})$.

iii) $\mathcal{B} = \mathcal{A}$.

Proof. We prove

$$(ii) \implies (iii) \implies (i) \implies (ii) \quad (70)$$

(ii) \implies (iii). —

The image of $j_{\mathcal{B}}$ is the subalgebra generated by the primitive elements. Let us prove that, when $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$ is summable, one has $\text{Im}(j_{\mathcal{B}}) = \mathcal{B}$. The series $\log(1 + X)$ is without constant term so, in virtue of (29) and the summability of $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$, one has

$$\exp(\log(e + \text{Id}^+)) = \exp(\log(1 + X))(\text{Id}^+) = 1_{\text{End}(\mathcal{B})} + \text{Id}^+ = e + \text{Id}^+ = I \quad (71)$$

Set $\pi_1 = \log(e + \text{Id}^+)$.

To end this part, let us compute, for $x \in \mathcal{B}$

$$x = \exp(\pi_1)(x) = \left(\sum_{n \geq 0} \frac{1}{n!} \pi_1^{*n} \right)(x) = \left(\sum_{n=0}^N \frac{1}{n!} \mu^{(n-1)} \pi_1^{\otimes n} \right) \Delta^{(n-1)}(x) \quad (72)$$

where N is the first order for which $\Delta^{+(n-1)}(x) = 0$ (as $\pi_1 \circ \text{Id}^+ = \pi_1$). This proves that \mathcal{B} is generated by its primitive elements.

The implications (iii) \implies (i) and (i) \implies (ii) are obvious. □

Remark 1. i) The equivalence (i) \iff (iii) is the classical CQMM theorem (see [4]). The equivalence with (ii) could be called the ‘‘Convolutional CQMM theorem’’. The combinatorial aspects of this last one will be the subject of a forthcoming paper.

ii) When $\text{Prim}(\mathcal{B})$ is free, we have $\mathcal{B} \cong_{k\text{-bialg}} \mathcal{U}(\text{Prim}(\mathcal{B}))$ and \mathcal{B} is an enveloping algebra.

iii) The (counter) example is the following with $A = k[x]$ (k is a field of characteristic zero). Let Y be an alphabet and $A(Y)$ be the usual free algebra (the space of non-commutative polynomials over Y) and ϵ , the ‘‘constant term’’ linear form. Let conc be the concatenation and Δ the dual law of the shuffle product (cf supra).

Then the bialgebra $(A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta, \epsilon)$ is a Hopf algebra (it is the enveloping algebra of the Lie polynomials). Let $A_+(Y) = \ker(\epsilon)$ and, for $N \geq 2$ $J_N = x^N \cdot A_+(Y)$ then, J_N is a Hopf ideal and $\text{Prim}(A\langle Y \rangle / (J_N))$ is never free (no basis).

4 Application to the ϕ -deformed shuffle.

4.1 General results for the ϕ -deformed shuffle.

Let $Y = \{y_i\}_{i \in I}$ be still a totally ordered alphabet and $A\langle Y \rangle$ be equipped with the ϕ -deformed stuffle defined by [7]

- i) for any $w \in Y^*$, $1_{Y^*} \boxplus_\phi w = w \boxplus_\phi 1_{Y^*} = w$,
- ii) for any $y_i, y_j \in Y$ and $u, v \in Y^*$,

$$y_i u \boxplus_\phi y_j v = y_j (y_i u \boxplus_\phi v) + y_i (u \boxplus_\phi y_j v) + \phi(y_i, y_j) u \boxplus_\phi v, \quad (73)$$

where ϕ is an arbitrary mapping

$$\phi : Y \times Y \longrightarrow AY.$$

Definition 2. Let

$$\phi : Y \times Y \longrightarrow AY$$

be defined by its structure constants

$$(y_i, y_j) \longmapsto \phi(y_i, y_j) = \sum_{k \in I} \gamma_{i,j}^k y_k.$$

Proposition 6. The recursion (73) defines a unique mapping

$$\boxplus_\phi : Y^* \times Y^* \longrightarrow A\langle Y \rangle.$$

Proof. Let us denote $(Y^* \times Y^*)_{\leq n}$ the set of words $(u, v) \in Y^* \times Y^*$ such that $|u| + |v| \leq n$. We construct a sequence of mappings

$$\boxplus_{\phi \leq n} : (Y^* \times Y^*)_{\leq n} \longrightarrow A\langle Y \rangle.$$

which satisfy the recursion of eq.73. For $n = 0$, we have only a pre-image and $\boxplus_{\phi \leq 0}(1_{Y^*}) = 1_{Y^*} \otimes 1_{Y^*}$. Suppose $\boxplus_{\phi \leq n}$ already constructed and let $(u, v) \in (Y^* \times Y^*)_{\leq n+1} \setminus (Y^* \times Y^*)_{\leq n}$, i.e. $|u| + |v| = n + 1$. One has three cases : $u = 1_{Y^*}$, $v = 1_{Y^*}$ and $(u, v) \in Y^+ \times Y^+$. For the first two, one uses the initialisation of the recursion, thus

$$\boxplus_{\phi \leq n+1}(w, 1_{Y^*}) = \boxplus_{\phi \leq n+1}(1_{Y^*}, w) = w.$$

For the last case, write $u = y_i u'$, $v = y_j v'$ and use, to get

$$\mathbf{u}_{\phi_{\leq n+1}}(y_i u', y_j v') = y_i \mathbf{u}_{\phi_{\leq n}}(u', y_j v') + y_j \mathbf{u}_{\phi_{\leq n}}(y_i u', v') + y_{i+j} \mathbf{u}_{\phi_{\leq n}}(u', v')$$

this proves the existence of the sequence $(\mathbf{u}_{\phi_{\leq n}})_{n \geq 0}$. Every $\mathbf{u}_{\phi_{\leq n+1}}$ extends the preceding so there is a mapping

$$\mathbf{u}_{\phi} : Y^* \times Y^* \longrightarrow A\langle Y \rangle.$$

which extends all the $\mathbf{u}_{\phi_{\leq n+1}}$ (the graph of which is the union of the graphs of the $\mathbf{u}_{\phi_{\leq n}}$). This proves the existence. For unicity, just remark that, if there were two mappings \mathbf{u}_{ϕ} , \mathbf{u}'_{ϕ} , the fact that they must fulfil the recursion (73) implies that $\mathbf{u}_{\phi} = \mathbf{u}'_{\phi}$. \square

We still denote by ϕ and \mathbf{u}_{ϕ} the linear extension of ϕ and \mathbf{u}_{ϕ} to $AY \otimes AY$ and $A\langle Y \rangle \otimes A\langle Y \rangle$ respectively.

Then \mathbf{u}_{ϕ} is a law of algebra (with 1_{Y^*} as unit) on $A\langle Y \rangle$.

Lemma 3. *Let Δ be the morphism $A\langle Y \rangle \rightarrow A\langle\langle Y^* \otimes Y^* \rangle\rangle$ defined on the letters by*

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{n,m \in I} \gamma_{n,m}^s y_n \otimes y_m \quad (74)$$

Then

i) for all $w \in Y^+$ we have

$$\Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v \in Y^+} \langle \Delta(w) \mid u \otimes v \rangle u \otimes v \quad (75)$$

ii) for all $u, v, w \in Y^*$, one has

$$\langle u \mathbf{u}_{\phi} v \mid w \rangle = \langle u \otimes v \mid \Delta(w) \rangle^{\otimes 2} \quad (76)$$

Proof. i) By recurrence on $|w|$. If $w = y_s$ is of length one, it is obvious from the definition. If $w = y_s w'$, we have, from the fact that Δ is a morphism

$$\begin{aligned} \Delta(w) &= \left(y_s \otimes 1 + 1 \otimes w + \sum_{i,j \in I} \gamma_{i,j}^s y_i \otimes y_j \right) \\ &\quad \left(w' \otimes 1 + 1 \otimes w' + \sum_{u,v \in Y^+} \langle u \otimes v \mid \Delta(w') \rangle \right) \end{aligned} \quad (77)$$

the development of which proves that $\Delta(w)$ is of the desired form.

ii) Let $S(u, v) := \sum_{w \in Y^*} \langle u \otimes v \mid \Delta(w) \rangle w$. It is easy to check (and left to the reader) that, for all $u \in Y^*$, $S(u, 1) = S(1, u) = u$. Let us now prove that, for all $y_i, y_j \in Y$ and $u, v \in Y^*$

$$S(y_i u, y_j v) = y_i S(u, y_j v) + y_j S(y_i u, v) + \phi(y_i, y_j) S(u, v) \quad (78)$$

Indeed, noticing that $\Delta(1) = 1 \otimes 1$, one has

$$\begin{aligned}
S(y_i u, y_j v) &= \sum_{w \in Y^*} \langle y_i u \otimes y_j v \mid \Delta(w) \rangle w = \sum_{w \in Y^+} \langle y_i u \otimes y_j v \mid \Delta(w) \rangle w \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \Delta(y_s w') \rangle y_s w' \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \left(y_s \otimes 1 + 1 \otimes y_s + \sum_{n, m \in I} \gamma_{n, m}^s y_n \otimes y_m \right) \Delta(w') \rangle y_s w' \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid (y_s \otimes 1) \Delta(w') \rangle y_s w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid (1 \otimes y_s) \Delta(w') \rangle y_s w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \left(\sum_{n, m \in I} \gamma_{n, m}^s y_n \otimes y_m \right) \Delta(w') \rangle y_s w' \\
&= \sum_{w' \in Y^*} \langle u \otimes y_j v \mid \Delta(w') \rangle y_i w' + \sum_{w' \in Y^*} \langle y_i u \otimes v \mid \Delta(w') \rangle y_j w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle u \otimes v \mid \gamma_{i, j}^s \Delta(w') \rangle y_s w' \\
&= y_i \sum_{w' \in Y^*} \langle u \otimes y_j v \mid \Delta(w') \rangle w' + y_j \sum_{w' \in Y^*} \langle y_i u \otimes v \mid \Delta(w') \rangle w' \\
&\quad + \sum_{y_s \in Y} \gamma_{i, j}^s y_s \sum_{w' \in Y^*} \langle u \otimes v \mid \Delta(w') \rangle w' \\
&= y_i S(u, y_j v) + y_j S(y_i u, v) + \phi(y_i, y_j) S(u, v)
\end{aligned}$$

then the computation of S shows that, for all $u, v \in Y^*$, $S(u, v) = u \blacktriangleright_{\phi} v$ as S is bilinear, so $S = \blacktriangleright_{\phi}$. \square

Theorem 2. *i) The law $\blacktriangleright_{\phi}$ is commutative if and only if the extension*

$$\phi : AY \otimes AY \longrightarrow AY$$

is so.

ii) The law $\blacktriangleright_{\phi}$ is associative if and only if the extension

$$\phi : AY \otimes AY \longrightarrow AY$$

is so.

iii) Let $\gamma_{x, y}^z := \langle \phi(x, y) \mid z \rangle$ be the structure constants of ϕ (w.r.t. the basis Y), then $\blacktriangleright_{\phi}$ is dualizable if and only if $(\gamma_{x, y}^z)_{x, y, z \in X}$ has the following

decomposition property¹⁹

$$(\forall z \in X)(\#\{(x, y) \in X^2 | \gamma_{x,y}^z \neq 0\} < +\infty). \quad (79)$$

Proof. (i) First, let us suppose that ϕ be commutative and consider T , the twist, i.e. the operator in $A\langle\langle Y^* \otimes Y^* \rangle\rangle$ defined by

$$\langle T(S) | u \otimes v \rangle = \langle S | v \otimes u \rangle \quad (80)$$

it is an easy check to prove that T is a morphism of algebras. If ϕ is commutative, then so is the following diagram.

$$\begin{array}{ccc} Y & \xrightarrow{\Delta_{\boxtimes_\phi}} & A\langle\langle Y^* \otimes Y^* \rangle\rangle \\ & \searrow \Delta_{\boxtimes_\phi} & \downarrow T \\ & & A\langle\langle Y^* \otimes Y^* \rangle\rangle \end{array}$$

and, then, the two morphisms Δ_{\boxtimes_ϕ} and $T \circ \Delta_{\boxtimes_\phi}$ coincide on the generators Y of the algebra $A\langle Y \rangle$ and hence over $A\langle Y \rangle$ itself. Now for all $u, v, w \in Y^*$, one has

$$\begin{aligned} \langle v \boxtimes_\phi u | w \rangle &= \langle v \otimes u | \Delta_{\boxtimes_\phi}(w) \rangle = \langle u \otimes v | T \circ \Delta_{\boxtimes_\phi}(w) \rangle = \\ \langle u \otimes v | \Delta_{\boxtimes_\phi}(w) \rangle &= \langle u \boxtimes_\phi v | w \rangle \end{aligned} \quad (81)$$

which proves that $v \boxtimes_\phi u = u \boxtimes_\phi v$. Conversely, if \boxtimes_ϕ is commutative, one has, for $i, j \in I$

$$\phi(y_j, y_i) = y_j \boxtimes_\phi y_i - (y_j \sqcup y_i) = y_i \boxtimes_\phi y_j - (y_i \sqcup y_j) = \phi(y_i, y_j) \quad (82)$$

(ii) Likewise, if ϕ is associative, let us define the operators

$$\overline{\Delta_{\boxtimes_\phi} \otimes I} : A\langle\langle Y^* \otimes Y^* \rangle\rangle \rightarrow A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle \quad (83)$$

by

$$\langle \overline{\Delta_{\boxtimes_\phi} \otimes I}(S) | u \otimes v \otimes w \rangle = \langle S | (u \boxtimes_\phi v) \otimes w \rangle \quad (84)$$

and, similarly,

$$\overline{I \otimes \Delta_{\boxtimes_\phi}} : A\langle\langle Y^* \otimes Y^* \rangle\rangle \rightarrow A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle \quad (85)$$

by

$$\langle \overline{I \otimes \Delta_{\boxtimes_\phi}}(S) | u \otimes v \otimes w \rangle = \langle S | u \otimes (v \boxtimes_\phi w) \rangle \quad (86)$$

it is easy to check by direct calculation that they are well defined morphisms and that the following diagram

¹⁹One can prove that, in case Y is a semigroup, the associated ϕ fulfils eq.79 iff Y fulfils “condition D” of Bourbaki (see [3])

$$\begin{array}{ccc}
Y & \xrightarrow{\Delta_{\mathbf{u}_\phi}} & A\langle\langle Y^* \otimes Y^* \rangle\rangle \\
\Delta_{\mathbf{u}_\phi} \downarrow & & \downarrow \overline{I \otimes \Delta_{\mathbf{u}_\phi}} \\
A\langle\langle Y^* \otimes Y^* \rangle\rangle & \xrightarrow{\overline{\Delta_{\mathbf{u}_\phi} \otimes I}} & A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle
\end{array}$$

is commutative. This proves that the two composite morphisms

$$\overline{\Delta_{\mathbf{u}_\phi} \otimes I} \circ \Delta_{\mathbf{u}_\phi}$$

and

$$\overline{I \otimes \Delta_{\mathbf{u}_\phi}} \circ \Delta_{\mathbf{u}_\phi}$$

coincide on Y and then on $A\langle Y \rangle$. Now, for $u, v, w, t \in Y^*$, one has

$$\begin{aligned}
\langle (u \mathbf{u}_\phi v) \mathbf{u}_\phi w \mid t \rangle &= \langle (u \mathbf{u}_\phi v) \otimes w \mid \Delta_{\mathbf{u}_\phi}(t) \rangle = \langle u \otimes v \otimes w \mid (\overline{\Delta_{\mathbf{u}_\phi} \otimes I}) \Delta_{\mathbf{u}_\phi}(t) \rangle = \\
&= \langle u \otimes v \otimes w \mid (\overline{I \otimes \Delta_{\mathbf{u}_\phi}}) \Delta_{\mathbf{u}_\phi}(t) \rangle = \langle u \otimes (v \mathbf{u}_\phi w) \mid \Delta_{\mathbf{u}_\phi}(t) \rangle = \langle u \mathbf{u}_\phi (v \mathbf{u}_\phi w) \mid t \rangle
\end{aligned}$$

which proves the associativity of the law \mathbf{u}_ϕ . Conversely, if \mathbf{u}_ϕ is associative, the direct expansion of the right hand side of

$$0 = (y_i \mathbf{u}_\phi y_j) \mathbf{u}_\phi y_k - y_i \mathbf{u}_\phi (y_j \mathbf{u}_\phi y_k) \quad (87)$$

proves the associativity of ϕ .

iii) We suppose that $(\gamma_{x,y}^z)_{x,y,z \in X}$ satisfies eq.79 . In this case $\Delta_{\mathbf{u}_\phi}$ takes its values in $A\langle Y \rangle \otimes A\langle Y \rangle$ so its dual, the law \mathbf{u}_ϕ is dualizable. Conversely, if $Im(\Delta_{\mathbf{u}_\phi}) \subset A\langle Y \rangle \otimes A\langle Y \rangle$, one has, for every $s \in I$

$$\sum_{n,m \in I} \gamma_{n,m}^s y_n \otimes y_m = \Delta(y_s) - (y_s \otimes 1 + 1 \otimes y_s) \in A\langle Y \rangle \otimes A\langle Y \rangle$$

which proves the claim. \square

From now on, we suppose that $\phi : AY \otimes AY \longrightarrow AY$ is an associative and commutative law (of algebra) on AY .

Theorem 3. *Let A be a \mathbb{Q} -algebra. Then if ϕ is dualizable ²⁰, let $\Delta_{\mathbf{u}_\phi} : A\langle Y \rangle \longrightarrow A\langle Y \rangle \otimes A\langle Y \rangle$ denote its dual comultiplication, then*

a) $\mathcal{B}_\phi = (A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\mathbf{u}_\phi}, \varepsilon)$ is a bialgebra.

²⁰For the pairing defined by

$$(\forall x, y \in Y)(\langle x \mid y \rangle = \delta_{x,y}) .$$

b) If A is a \mathbb{Q} -algebra then, the following conditions are equivalent

i) \mathcal{B}_ϕ is an enveloping bialgebra

ii) the algebra AX admits an increasing filtration $\left((AY)_n\right)_{n \in \mathbb{N}}$

$$(AY)_0 = \{0\} \subset (AY)_1 \subset \cdots \subset (AY)_n \subset (AY)_{n+1} \subset \cdots$$

compatible with both the multiplication and the comultiplication $\Delta_{\mathbf{u}\phi}$ i.e.

$$\begin{aligned} (AY)_p (AY)_q &\subset (AY)_{p+q} \\ \Delta_{\mathbf{u}\phi}((AY)_n) &\subset \sum_{p+q=n} (AY)_p \otimes (AY)_q . \end{aligned}$$

iii) \mathcal{B}_ϕ is isomorphic to $(A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\mathbf{u}}, \epsilon)$ as a bialgebra.

iv) I^+ is \star -nilpotent.

Proof. We only prove the following implication (the other ones are easy)

iv) \implies iii) Let us set $y'_s = \pi_1(y_s)$, then using a rearrangement of the star-log of the diagonal series, we have

$$y_s = \sum_{k \geq 1} \frac{1}{k!} \sum_{s'_1 + \cdots + s'_k = s} \pi_1(y_{s'_1}) \cdots \pi_1(y_{s'_k}) \quad (88)$$

This proves that the multiplicative morphism given by $\Phi(y_s) = y'_s$ is an isomorphism. But this morphism is such that $\Delta_{\mathbf{u}\phi} \circ \Phi = (\Phi \otimes \Phi) \circ \Delta_{\mathbf{u}}$ which proves the claim. \square

Remark 2. i) Theorem 3 a) holds for general (dualizable, coassociative) ϕ be it commutative or not.

ii) It can happen that there is no antipode (and then, I^+ cannot be \star -nilpotent) as the following example shows.

Let $Y = \{y_0, y_1\}$ and $\phi(y_i, y_j) = y_{(i+j \bmod 2)}$, then

$$\begin{aligned} \Delta(y_0) &= y_0 \otimes 1 + 1 \otimes y_0 + y_0 \otimes y_0 + y_1 \otimes y_1 \\ \Delta(y_1) &= y_1 \otimes 1 + 1 \otimes y_1 + y_0 \otimes y_1 + y_1 \otimes y_0 \end{aligned} \quad (89)$$

then, from eqns 89, one derives that $1 + y_0 + y_1$ is group-like. As this element has no inverse in $K\langle Y \rangle$. Thus, the bialgebra \mathcal{B}_ϕ cannot be a Hopf algebra.

iii) When I^+ is nilpotent, the antipode exists and is computed by

$$a_{\mathbf{u}\phi} = (I)^{\star-1} = (e + I^+)^{\star-1} = \sum_{n \geq 0} (-1)^n (I^+)^{\star n} \quad (90)$$

(see section (3.2)).

iv) In QFT, the antipode of a vector $h \in \mathcal{B}$ is computed by

$$S(1) = 1, \quad S(h) = -h + \sum_{(1)(2)} S(h_{(1)})h_{(2)} \quad (91)$$

and by using the fact that S is an antimorphism. This formula is used in contexts where I^+ is \star -nilpotent (although the concerned bialgebras are often not cocommutative). Here, one can prove this recursion from eq.90 .

5 Conclusion

We have depicted the framework which is common to different kinds of shuffles. For all these, provided that I_+ be $*$ -nilpotent, the bialgebra

$$(A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\sqcup_\phi}, \varepsilon)$$

is isomorphic to

$$(A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\sqcup}, \varepsilon)$$

and the straightening algorithm is simply the morphism which sends each $y_s \in Y$ to $\pi_1(y_s) = \log(I)(y_s)$ (this bialgebra is then a Hopf algebra). In other cases, such as the infiltration given by

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + y_s \otimes y_s$$

group-like elements without inverse may appear (and therefore no Hopf structure can be hoped).

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