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# Combinatorics of $\phi$ -deformed shuffle Hopf algebras<sup>1</sup>

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## Abstract

In order to extend the Sch utzenberger's factorization to general perturbations, the combinatorial aspects of the Hopf algebra of the  $\phi$ -deformed shuffle product is developed systematically in a parallel way with those of the shuffle product. and in emphasizing the Lie elements as studied by Ree. In particular, we will give an effective construction of pair of bases in duality.

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# 1 Introduction

## 1.1 Motivations

Many algebras of functions [8] and many special sums [10, 11] are ruled out by shuffle products, their perturbations (adding a “superposition term” [9]) or deformations [19].

In order to better understand the mechanisms of this products, we wish here to examine, with full generality the products which are defined by a recursion of the type

$$au \star bv = a(u \star bv) + b(au \star v) + \phi(a, b)u \star v, \quad (1)$$

the empty word being the neutral of this new product.

We give a lot of classical combinatorial applications (as shuffle, stuffles and Hurwitz polzetas), **TODO références**. In most cases, the law  $\phi$  is dual<sup>2</sup> and under some growth conditions the obtained algebra is an enveloping algebra.

In the second section, is a version of the CQMM without PBW. We are obliged to redo the CQMM theorem without supposing any basis because we aim at “varying the scalars” in forthcoming papers (germs of functions, arithmetic functions, etc.) and, in order to do this at ease, we must cope safely with cases where torsion may appear (and then, one cannot have any basis). See (counter) examples in the section.

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## 1.2 First steps

Let  $X$  be an totally ordered alphabet<sup>3</sup>. The free monoid and the set of Lyndon words, over  $X$ , are denoted respectively by  $X^*$  and  $\mathcal{Lyn}X$ . The neutral element of  $X^*$ , *i.e.* the empty word is denoted by  $1_{X^*}$ . Let  $\mathbb{Q}\langle X \rangle$  be equipped by the concatenation and the shuffle which is defined by

$$\begin{aligned} \forall w \in X^*, \quad w \sqcup 1_{X^*} &= 1_{X^*} \sqcup w = w, \\ \forall x, y \in X, \forall u, v \in X^*, \quad xu \sqcup yv &= x(u \sqcup yv) + y(xu \sqcup v), \end{aligned} \quad (2)$$

or by their dual co-products,  $\Delta = \Delta_{\text{conc}}$  and  $\Delta = \Delta_{\sqcup}$ , defined by, for any  $w \in X^*$  by,

$$\Delta_{\text{conc}}(w) = \sum_{w=uv} u \otimes v$$

---

<sup>2</sup>That is to say comes by dualization of a comultiplication.

<sup>3</sup>In the sequel, the order between the words will be understood as the lexicographic by length total ordering  $\prec_{lex}$ . Two words are first compared w.r.t. their length and, in case of equality, w.r.t. the usual lexicographic ordering. For example, with  $a < b$ , one has  $b \prec_{lex} ab$  whereas  $ab \prec_{lex} b$ .

$$\Delta_{\sqcup}(w) = \sum_{I+J=[1..|w|]} w[I] \otimes w[J] \quad (3)$$

One gets two Hopf algebras

$$\begin{aligned} \mathcal{H}_{\sqcup} &= (\mathbb{Q}\langle X \rangle, \text{conc}, 1_{X^*}, \Delta_{\sqcup}, \epsilon, a_{\bullet}) \text{ and} \\ \mathcal{H}_{\sqcup}^{\vee} &= (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*}, \Delta_{\text{conc}}, \epsilon, a_{\sqcup}) \end{aligned} \quad (4)$$

mutually dual with respect to the pairing given by

$$(\forall u, v \in X^*) (\langle u | v \rangle = \delta_{u,v}) . \quad (5)$$

and with, for any  $x_{i_1}, \dots, x_{i_r} \in X$  and  $P \in \mathbb{Q}\langle X \rangle$ ,

$$\begin{aligned} \epsilon(P) &= \langle P | 1_{X^*} \rangle, \\ a_{\sqcup}(w) &= a_{\bullet}(w) = (-1)^r x_{i_r} \dots x_{i_1}, . \end{aligned} \quad (6)$$

By the theorem of Cartier-Quillen-Milnor and Moore (CQMM in the sequel), the connected, graded positively, co-commutative Hopf algebra  $\mathcal{H}_{\sqcup}$  is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements which here is  $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$ . Hence, from any basis of the free algebra  $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$  one can<sup>4</sup> complete, by the Poincaré-Birkhoff-Witt theorem, a linear basis  $\{b_w\}_{w \in X^*}$  for  $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle) = \mathbb{Q}\langle X \rangle$  (see below (9) for an example of such a construction), and, when the basis is finely homogeneous, one can construct, by duality, a basis  $\{\check{b}_w\}_{w \in X^*}$  of  $\mathcal{H}_{\sqcup}$  (viewed as a  $\mathbb{Q}$ -module) such that :

$$\forall u, v \in X^*, \quad \langle \check{b}_u | b_v \rangle = \delta_{u,v} . \quad (7)$$

For  $w = l_1^{i_1} \dots l_k^{i_k}$  with  $l_1, \dots, l_k \in \mathcal{L}ynX$ ,  $l_1 > \dots > l_k$

$$\check{b}_w = \frac{\check{b}_{l_1}^{\sqcup i_1} \sqcup \dots \sqcup \check{b}_{l_k}^{\sqcup i_k}}{i_1! \dots i_k!} . \quad (8)$$

For example, Chen, Fox and Lyndon [7] constructed the PBW-Lyndon basis  $\{P_w\}_{w \in X^*}$  for  $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle)$  as follows

$$\begin{aligned} P_x &= x && \text{for } x \in X, \\ P_l &= [P_s, P_r] && \text{for } l \in \mathcal{L}ynX, \text{ standard factorization of } l = (s, r), \\ P_w &= P_{l_1}^{i_1} \dots P_{l_k}^{i_k} && \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k, l_1, \dots, l_k \in \mathcal{L}ynX. \end{aligned} \quad (9)$$

Schützenberger and his school constructed, the linear basis  $\{S_w\}_{w \in X^*}$  for  $\mathcal{A} = (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$  by duality (w.r.t. eq.5) and obtained the transcendence basis of  $\mathcal{A}$   $\{S_l\}_{l \in \mathcal{L}ynX}$  as follows<sup>5</sup>

$$S_l = x S_u, \quad \text{for } l = xu \in \mathcal{L}ynX, \quad (10)$$

$$S_w = \frac{S_{l_1}^{\sqcup i_1} \sqcup \dots \sqcup S_{l_k}^{\sqcup i_k}}{i_1! \dots i_k!} \quad \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k. \quad (11)$$

<sup>4</sup>The basis can be reindexed by Lyndon words and then one uses the canonical factorization of the words.

<sup>5</sup>Therefore  $\mathcal{A}$  is a polynomial algebra  $\mathcal{A} \simeq \mathbb{Q}[\mathcal{L}ynX]$ .

After that, Mélançon and Reutenauer [18] proved that<sup>6</sup>, for any  $w \in X^*$ ,

$$P_w = w + \sum_{v > w, |v|=|w|} c_v v \quad \text{and} \quad S_w = w + \sum_{v < w, |v|=|w|} c_v v. \quad (12)$$

On other words, the elements of the bases  $\{S_w\}_{w \in X^*}$  and  $\{P_w\}_{w \in X^*}$  are upper and lower triangular respectively and are multihomogeneous.

Moreover, thanks to the duality of the bases  $\{P_w\}_{w \in X^k}$  and  $\{S_w\}_{w \in X^k}$ , if  $\mathcal{D}_X$  denotes the diagonal series over  $X$  one has

$$\mathcal{D}_X = \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{Lyn} X}^{\searrow} \exp(S_l \otimes P_l). \quad (13)$$

In fact as stated in [18], this factorization holds in the framework of enveloping algebras and it will be shown in detail how to handle this framework even in the absence of any basis (it is indeed what could be called CQMM analytic form).

## 2 General results on summability and duality

Let  $Y = \{y_i\}_{i \in I}$  be a totally ordered alphabet. The free monoid and the set of Lyndon words, over  $Y$ , are denoted respectively by  $Y^*$  and  $\mathcal{Lyn} Y$ . The neutral of  $Y^*$  (and then of  $A\langle Y \rangle$ ) is denoted by  $1_{Y^*}$ .

### 2.1 Total algebras and duality

#### 2.1.1 Series and infinite sums

In the sequel, we will need to construct spaces of functions on different monoids (mainly direct products of free monoids). We set, once for all the general construction of the corresponding convolution algebra.

Let  $A$  be a unitary commutative ring and  $M$  a monoid. Let us denote  $A^M$  the set<sup>7</sup> of all (graphs of) mappings  $M \rightarrow A$ . This set is endowed with its classical structure of module. In order to extend the product defined in  $A[M]$  (the algebra of the monoid  $M$ ), it is essential that, in the sums

$$f * g(m) = \sum_{m \in M} \sum_{uv=m} f(u)g(v) \quad (14)$$

---

<sup>6</sup> Recall that the duality preserves the (multi)homogeneous degrees and interchanges the triangularity of polynomials [18]. For that, one can construct the triangular matrices  $M$  and  $N$  admitting as entries the coefficients of the multihomogeneous triangular polynomials,  $\{P_w\}_{w \in X^k}$  and  $\{S_w\}_{w \in X^k}$  in the basis  $\{w\}_{w \in X^*}$  respectively :

$$M_{u,v} = \langle P_u \mid v \rangle \quad \text{and} \quad N_{u,v} = \langle S_u \mid v \rangle.$$

The triangular matrices  $M$  and  $N$  are unipotent and satisfy the identity  $N = ({}^t M)^{-1}$ .

<sup>7</sup>In general  $Y^X$  is the set of all mappings  $X \rightarrow Y$  [2] Ch 2.5.2.

the inner sum  $\sum_{uv=m} f(u)g(v)$  make sense. For that, we suppose that the monoid  $M$  fulfills condition “D” (be of finite decomposition type [3] Ch III.10). Formally, we say that  $M$  satisfies condition “D” iff, for all  $m \in M$ , the set

$$\{(u, v) \in M \times M \mid uv = m\} \quad (15)$$

is finite. In this case eq.14 endows  $A^M$  with the structure of a AAU<sup>8</sup>. This algebra is traditionally called the total algebra of  $M$  (see [3] Ch III.10) and has very much to do with the series<sup>9</sup>. It will be, here (with a slight abuse of denotation which does not cause ambiguity) denoted  $A\langle\langle M \rangle\rangle$ .

The pairing

$$A\langle\langle M \rangle\rangle \otimes A[M] \longrightarrow A \quad (16)$$

defined by<sup>10</sup>

$$\langle f \mid g \rangle := \sum_{m \in M} f(m)g(m) \quad (17)$$

allows to see every element of the total algebra as a linear form on the module  $A[M]$ . One can check easily that, through this pairing, one has

$$A\langle\langle M \rangle\rangle \simeq (A[M])^* .$$

One says that a family  $(f_i)_{i \in I}$  of  $A\langle\langle M \rangle\rangle$  is summable [1] iff, for every  $m \in M$ , the mapping  $i \mapsto \langle f_i \mid m \rangle$  is finitely supported. In this case, the sum  $\sum_{i \in I} f_i$  is exactly the mapping  $m \mapsto \sum_{i \in I} \langle f_i \mid m \rangle$  so that, one has by definition

$$\langle \sum_{i \in I} f_i \mid m \rangle = \sum_{i \in I} \langle f_i \mid m \rangle . \quad (18)$$

To end with, let us remark that the set  $M_1 \otimes M_2 = \{u \otimes v\}_{(u,v) \in M_1 \times M_2}$  is a (monoidal) basis of  $A[M_1] \otimes A[M_2]$  and  $M_1 \otimes M_2$  is a monoid (in the product algebra  $A[M_1] \otimes A[M_2]$ ) isomorphic to the direct product  $M_1 \times M_2$ .

### 2.1.2 Summable families in Hom spaces.

In fact,  $A\langle\langle M \rangle\rangle \simeq (A[M])^* = \text{Hom}(A[M], A)$  and the notion of summability developed above can be seen as a particular case of that of a family of endomorphisms  $f_i \in \text{Hom}(V, W)$  for which  $\text{Hom}(V, W)$  appears as a complete space. It is indeed the pointwise convergence for the discrete topology. We will not detail these considerations here.

The definition is similar of that of a summable family of series [1], viewed as a family of linear forms.

---

<sup>8</sup>Associative Algebra with Unit.

<sup>9</sup>In fact, the algebra of commutative (resp. noncommutative) series on an alphabet  $X$  is the total algebra of the free commutative (resp.  $X^*$ ) monoid on  $X$

<sup>10</sup>Here  $A[M]$  is identified with the submodule of finitely supported functions  $M \rightarrow A$ .

**Definition 1.** *i) A family  $(f_i)_{i \in I}$  of elements in  $\text{Hom}(V, W)$  is said to be summable iff for all  $x \in V$ , the map  $i \mapsto f_i(x)$  has finite support. As a quantized criterium it reads*

$$(\forall x \in V)(\exists F \subset_{\text{finite}} I)(\forall i \notin F)(f_i(x) = 0) \quad (19)$$

*ii) If the family  $(f_i)_{i \in I} \in \text{Hom}(V, W)^I$  fulfils the condition 19 above its sum is given by*

$$\left(\sum_{i \in I} f_i\right)(x) = \sum_{i \in I} f_i(x) \quad (20)$$

It is an easy exercise to show that the mapping  $V \rightarrow W$  defined by the equation 20 is in fact in  $\text{Hom}(V, W)$ . Remark that, as the limiting process is defined by linear conditions, if a family  $(f_i)_{i \in I}$  is summable, so is

$$(a_i f_i)_{i \in I} \quad (21)$$

for an arbitrary family of coefficients  $(a_i)_{i \in I} \in A^I$ .

This tool will be used in section (2.2) to give an analytic presentation of the theorem of Cartier-Quillen-Milnor-Moore in the case when  $V = W = \mathcal{B}$  is a bialgebra.

The most interesting feature of this operation is the interversion of sums. Let us state it formally as a proposition the proof of which is left to the reader.

**Proposition 1.** *Let  $(f_i)_{i \in I}$  be a family of elements in  $\text{Hom}(V, W)$  and  $(I_j)_{j \in J}$  be a partition of  $I$  ([2] ch II §4 n° 7 Def. 6), then TFAE*

*i)  $(f_i)_{i \in I}$  is summable*

*ii) for all  $j \in J$ ,  $(f_i)_{i \in I_j}$  is summable and the family  $(\sum_{i \in I_j} f_i)_{j \in J}$  is summable.*

*In these conditions, one has*

$$\sum_{i \in I} f_i = \sum_{j \in J} \left(\sum_{i \in I_j} f_i\right) \quad (22)$$

We derive at once from this the following practical criterium for double sums.

**Proposition 2.** *Let  $(f_{\alpha, \beta})_{(\alpha, \beta) \in A \times B}$  be a doubly indexed summable family in  $\text{Hom}(V, W)$ , then, for fixed  $\alpha$  (resp.  $\beta$ ) the “row-families”  $(f_{\alpha, \beta})_{\beta \in B}$  (resp. the “column-families”  $(f_{\alpha, \beta})_{\alpha \in A}$ ) are summable and their sums are summable. Moreover*

$$\sum_{(\alpha, \beta) \in A \times B} f_{\alpha, \beta} = \sum_{\alpha \in A} \sum_{\beta \in B} f_{\alpha, \beta} = \sum_{\beta \in B} \sum_{\alpha \in A} f_{\alpha, \beta} . \quad (23)$$

### 2.1.3 Substitutions

Let  $\mathcal{A}$  be a AAU and  $f \in \mathcal{A}$ . For every polynomial  $P \in A\langle X \rangle = A[X]$ , one can compute  $P(f)$  by

$$P(f) = \sum_{n \geq 0} \langle P \mid X^n \rangle f^n \quad (24)$$

one checks at once that  $P \mapsto P(f)$  is a morphism<sup>11</sup> of AAU between  $A[X]$  and  $\mathcal{A}$ . Moreover, this morphism is compatible with the substitutions as one checks easily that, for  $Q \in A[X]$

$$P(Q)(f) = P(Q(f)) \quad (25)$$

(it suffices to check that  $P \mapsto P(Q)(f)$  and  $P \mapsto P(Q(f))$  are two morphisms which coincide at  $P = X$ ).

In order to substitute within series, one needs some limiting process. The framework of  $\mathcal{A} = \text{Hom}(V, W)$  and summable families will be here sufficient (see paragraph 2.1.2). We suppose that  $(V, \delta_V, \epsilon_V)$  is a co-AAU and that  $(W, \mu_W, 1_W)$  is a AAU. Then  $(\text{Hom}(V, W), *, e)$  is a AAU (with  $e = 1_W \circ \epsilon_V$ ). A series  $S \in A[[X]]$  and  $f \in \text{Hom}(V, W)$  being given, we say that  $f \in \text{Dom}(S)$  iff the family  $(\langle S | X^n \rangle f^{*n})_{n \geq 0}$  is summable. We have the following properties

**Proposition 3.** *If  $f \in \text{Dom}(S) \cap \text{Dom}(T)$  and  $\alpha \in A$ , one has*

$$(\alpha S)(f) = \alpha S(f) ; (S + T)(f) = S(f) + T(f) \quad (26)$$

and

$$(TS)(f) = T(f) * S(f) . \quad (27)$$

If  $((f)^{*n})_{n \geq 0}$  is summable and  $S(0) = 0$  then

$$f \in \text{Dom}(S) \cap \text{Dom}(T(S)) ; S(f) \in \text{Dom}(T) \quad (28)$$

and

$$T(S)(f) = T(S(f)) \quad (29)$$

*Proof.* Let us first prove eq.27 . As  $f \in \text{Dom}(S) \cap \text{Dom}(T)$ , the families  $(\langle S | X^n \rangle f^{*n})_{n \geq 0}$  and  $(\langle T | X^m \rangle f^{*m})_{n \geq 0}$  are summable, then so is

$$\left( \langle T | X^m \rangle f^{*m} * \langle S | X^n \rangle f^{*n} \right)_{n, m \geq 0} \quad (30)$$

as, for every  $x \in V$ ,  $\delta(x) = \sum_{i=1}^N x_i^{(1)} \otimes x_i^{(2)}$  and for every  $i \in I$ ,

$$\text{supp}_{w.r.t. m}(\langle T | X^m \rangle f^{*m}(x_i^{(1)})) ; \text{supp}_{w.r.t. n}(\langle S | X^n \rangle f^{*n}(x_i^{(2)}))$$

are finite. Then outside of the cartesian product of the (finite) union of these supports, the product

$$(\langle T | X^m \rangle f^{*m} * \langle S | X^n \rangle f^{*n})(x) = \mu_W((\langle T | X^m \rangle f^{*m} \otimes \langle S | X^n \rangle f^{*n})(\delta(x))) \quad (31)$$

is zero. Hence the summability.

Now

$$T(f) * S(f) = \sum_{m=0}^{\infty} (\langle T | X^m \rangle f^{*m}) * \sum_{n=0}^{\infty} (\langle S | X^n \rangle f^{*n}) =$$

---

<sup>11</sup>In case  $\mathcal{A}$  is a geometric space, this morphism is called "evaluation at  $f$ " and corresponds to a Dirac measure.



$$\begin{aligned}
& \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\langle T | X^m \rangle \langle S | X^n \rangle f^{*n+m}) = \\
& \sum_{s=0}^{\infty} \left( \sum_{n+m=s}^{\infty} \langle T | X^m \rangle \langle S | X^n \rangle \right) f^{*s} = \\
& \sum_{s=0}^{\infty} (\langle TS | X^s \rangle) f^{*s} = (TS)(f) \tag{32}
\end{aligned}$$

We now prove the statements (28) and (29). If  $((f)^{*n})_{n \geq 0}$  is summable then  $f$  belongs to all domains (i.e. is universally substituable) by virtue of eq.21 . For all  $x \in V$ , it exists  $N_x \in \mathbb{N}$  such that

$$n > N_x \implies (f)^{*n}(x) = 0 .$$

Now, for  $S$  such that  $S(0) = 0$ , one has  $S = \sum_{n=1}^{\infty} \langle S | X^n \rangle X^n$  and then  $S^k = \sum_{n=k}^{\infty} \langle S^k | X^n \rangle X^n$ . Now, in view of eq.27 , one has

$$S(f)^{*n}(x) = S^n(f)(x) = \sum_{m=n}^{\infty} \langle S^n | X^m \rangle (f)^{*m}(x) \tag{33}$$

which is zero for  $n > N_x$ . Hence the summability of  $(S(f)^{*n})_{n \geq 0}$  which implies that  $S(f) \in \text{Dom}(T)$ . The family  $(\langle T | X^n \rangle \langle S^n | X^m \rangle (f)^{*m})_{(n,m) \in \mathbb{N}^2}$  is summable because, if  $x \in V$  and if  $n$  or  $m$  is greater than  $N_x$  then

$$\langle T | X^n \rangle \langle S^n | X^m \rangle (f)^{*m}(x) = 0 \tag{34}$$

thus  $T(S(f))$  is the sum

$$\begin{aligned}
T(S(f)) &= \sum_{n=0}^{\infty} \langle T | X^n \rangle S(f)^{*n} = \sum_{n=0}^{\infty} \langle T | X^n \rangle \sum_{m=n}^{\infty} \langle S^n | X^m \rangle (f)^{*m} = \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle T | X^n \rangle \langle S^n | X^m \rangle (f)^{*m} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} \langle T | X^n \rangle \langle S^n | X^m \rangle \right) (f)^{*m} = \\
&= \sum_{m=0}^{\infty} \langle T(S) | X^m \rangle (f)^{*m} = T(S)(f)
\end{aligned} \tag{35}$$

□

In the free case (i.e.  $V = W$  are the bialgebra  $(A\langle X \rangle, \mathbf{conc}, 1_{X^*}, \Delta_{\sqcup}, \epsilon)$ ), one has a very useful representation of the convolution algebra  $\text{Hom}(V, W)$  through images of the diagonal series. This representation will provide us the key lemma (2). Let

$$\mathcal{D}_X = \sum_{w \in X^*} w \otimes w.$$

be the diagonal series attached to  $X$ .

**Proposition 4.** *Let  $A$  be a commutative unitary ring and  $X$  an alphabet. Then*

i) *For every  $f \in \text{End}(A\langle X \rangle)$ , the family  $(u \otimes f(u))_{u \in X^*}$  is summable in  $A\langle\langle X^* \otimes X^* \rangle\rangle$ .*

ii) *The representation*

$$f \mapsto \rho(f) = \sum_{u \in X^*} u \otimes f(u) \tag{36}$$

*is faithful from  $(\text{End}(A\langle X \rangle), *)$  to  $(A\langle\langle X^* \otimes X^* \rangle\rangle, \sqcup \otimes \mathbf{conc})$ . In particular, for  $f \in \text{End}(A\langle X \rangle)$  and  $P \in A[X]$ , one has*

$$\rho(P(f)) = P(\rho(f)) \tag{37}$$

iii) *If  $f(1_{X^*}) = 0$  and  $S \in A[[X]]$  is a series, then  $(\rho(f)^n)_{n \geq 0}$  is summable in  $(A\langle\langle X^* \otimes X^* \rangle\rangle, \sqcup \otimes \mathbf{conc})$  and*

$$\rho(S(f)) = S(\rho(f)) \tag{38}$$

*Proof.* (of Prop.(4)) Let us compute

$$\begin{aligned}
\rho(f)(\sqcup \otimes \mathbf{conc})\rho(g) &= \sum_{u, v \in X^*} (u \otimes f(u)(\sqcup \otimes \mathbf{conc})(v \otimes g(v))) = \\
&= \sum_{u, v \in X^*} (u \sqcup v \otimes (\mathbf{conc}(f(u) \otimes g(v)))) =
\end{aligned}$$

$$\begin{aligned}
& \sum_{u,v \in X^*} \sum_{w \in X^*} (\langle u \sqcup v \mid w \rangle w \otimes \text{conc}(f(u)g(v))) = \\
& \sum_{w \in X^*} w \otimes \left( \sum_{u,v \in X^*} (\langle u \sqcup v \mid w \rangle \text{conc}(f(u)g(v))) \right) = \\
& \sum_{w \in X^*} w \otimes \left( \sum_{u,v \in X^*} (\langle u \otimes v \mid \Delta(w) \rangle \text{conc}(f(u)g(v))) \right) = \\
& \sum_{w \in X^*} w \otimes (\text{conc}(f \otimes g) \Delta(w)) = \sum_{w \in X^*} w \otimes (f * g(w)) \tag{39}
\end{aligned}$$

□

## 2.2 Theorem of Cartier-Quillen-Milnor-Moore (analytic form)

### 2.2.1 General properties of bialgebras

**From now on, we suppose that  $A$  is a unitary commutative  $\mathbb{Q}$ -algebra (i.e.  $\mathbb{Q} \subset A$ ).**

The aim of Cartier-Quillen-Milnor-Moore theorem is to provide necessary and sufficient conditions for  $\mathcal{B}$  to be an enveloping algebra, we will discuss this condition in detail in the sequel.

Let  $(\mathcal{B}, \mu, e_{\mathcal{B}}, \Delta, \epsilon)$  be a (general)  $A$ -bialgebra. One can always consider the Lie algebra of primitive elements  $\text{Prim}(\mathcal{B})$  and build the map

$$j_{\mathcal{B}} : \mathcal{U}(\text{Prim}(\mathcal{B})) \rightarrow \mathcal{B} .$$

Then,  $\mathcal{A} = j_{\mathcal{B}}(\mathcal{U}(\text{Prim}(\mathcal{B})))$  is the subalgebra generated by the primitive elements.

$$\begin{array}{ccccc}
\text{Prim}(\mathcal{B}) & \xrightarrow{i_{\mathcal{A},P}} & \mathcal{A} & \xrightarrow{i_{\mathcal{B},A}} & \mathcal{B} \\
\downarrow i_{\mathcal{U},P} & \nearrow i_{\mathcal{U},A} & & \nearrow j_{\mathcal{B}} & \\
\mathcal{U}(\mathcal{B}) & & & & 
\end{array}$$

Figure 1: The sub-algebra  $\mathcal{A}$  generated by primitive elements.

The mapping is into  $i_{\mathcal{B},A}$  is into but  $i_{\mathcal{B},A} \otimes i_{\mathcal{B},A}$  may not be so. This is the case for  $\mathcal{B} = (\mathbb{Q}[\epsilon][x], \cdot, 1_{\mathbb{Q}[\epsilon][x]}, \Delta, c)$  where  $(\mathbb{Q}[\epsilon][x], \cdot, 1_{\mathcal{B}})$  is the usual polynomial algebra with coefficients in the dual numbers  $(\mathbb{Q}[\epsilon], \epsilon^2 = 0)$  and

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \epsilon x \otimes x, \quad c(x) = 0$$

(see details and proofs below, in sec. 2.3).

In general, one has (only)  $\Delta_{\mathcal{B}}(\mathcal{A}) \subset \text{Im}(i_{\mathcal{B},A} \otimes i_{\mathcal{B},A})$ , this can be simply seen

from the following combinatorial argument.

For any list of primitive elements  $L = [g_1, g_2, \dots, g_n]$  and  $I = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, n\}$ , put  $L[I] = g_{i_1} g_{i_2} \dots g_{i_k}$ , the product of the sublist. One has

$$\Delta(g_1 g_2 \dots g_n) = \Delta(L[\{1, 2, \dots, n\}]) = \sum_{I+J=\{1, 2, \dots, n\}} L[I] \otimes L[J] \quad (40)$$

where, for  $I = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, n\}$ ,

$$L[I] = g_{i_1} g_{i_2} \dots g_{i_k}. \quad (41)$$

From (40) one gets also that  $j_{\mathcal{B}}$  is a morphism of bialgebras. If for any reason, there exists a lifting of

$$\Delta_{\mathcal{B}} \circ i_{\mathcal{B}, \mathcal{A}} \quad (42)$$

as a comultiplication of  $\mathcal{A}$ , then  $j_{\mathcal{B}}$  is into (see the statement and the proof below). Formula 41 proves that we have the following maps (save the - hypothetic - dotted one).

$$\begin{array}{ccc} A\langle G \rangle & \xrightarrow{s_G} & \mathcal{A} \\ \Delta_{\sqcup} \downarrow & & \downarrow \Delta_{\mathcal{A}} \\ A\langle G \rangle \otimes A\langle G \rangle & \xrightarrow{s_G \otimes s_G} & \mathcal{A} \otimes \mathcal{A} \end{array}$$

Figure 2: The unique lifting  $\Delta_{\mathcal{A}}$  (when it exists).

Where  $G$  is any generating set of the AAU  $\mathcal{A}$ . We emphasize the fact that, in the diagram above,  $G$  must be understood set-theoretically (i.e. with no relation between the elements<sup>12</sup>).

In fact, one has the following proposition

**Proposition 5.** *Let  $\mathcal{B}$  be a bialgebra over a (commutative)  $\mathbb{Q}$ -algebra  $A$ , the notations being those of figures 1 and 2, then TFAE*

- i) *For a generating set  $G \subset \text{Prim}(\mathcal{B})$ ,  $\ker(s_G) \subset \ker(s_G \otimes s_G) \circ \Delta_{\sqcup}$ .*
- ii) *For any generating set  $G \subset \text{Prim}(\mathcal{B})$ ,  $\ker(s_G) \subset \ker(s_G \otimes s_G) \circ \Delta_{\sqcup}$ .*
- iii)  *$j_{\mathcal{B}}$  is into.*

*Proof.* i)  $\implies$  iii) In order to prove this, we need to construct the arrows  $\sigma, \tau$  which are a decomposition of a section of  $j_{\mathcal{B}}$ . Let us remark that, when  $\text{Prim}(\mathcal{B})$  is free as a  $A$ -module, the proof of this fact is a consequence of the PBW theorem<sup>13</sup>. But, here, we will construct the section in the general case using projectors which are now classical for the free case but which still can be computed analytically [18] as they lie in  $\mathbb{Q}[[X]]$  and still converge in  $\mathcal{A}$ .

<sup>12</sup>We will see, below and in paragraph 2.3 how it is crucial to consider that  $[\lambda x] \neq \lambda[x]$ , when  $\lambda x \in G$  (for clarity,  $[y] \in A\langle G \rangle$  is the image of  $y \in G$ ).

<sup>13</sup>See [4] Ch2 §1 n° 6 th 1 for a field of characteristic zero and §1 Ex. 10 for the free case (over a ring  $A$  with  $\mathbb{Q} \subset A$ ).

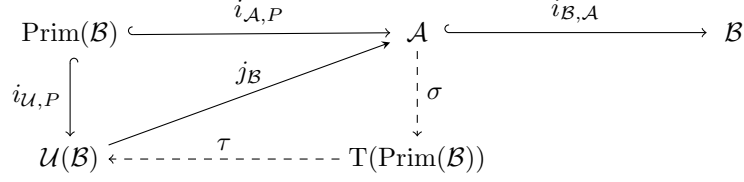


Figure 3: The sub-bialgebra  $\mathcal{A}$  generated by primitive elements.

*Proof.* (Injectivity of  $j_{\mathcal{B}}$ , construction of the section  $\tau \circ \sigma$ ). —

Let  $\mathcal{A}$  be the subalgebra of  $\mathcal{B}$  generated by  $\text{Prim}(\mathcal{B})$ , one has  $\text{Im}(j_{\mathcal{B}}) = \mathcal{A}$ .

Remark that all series  $\sum_{n \geq 0} a_n (I_+)^{*n}$  are summable on  $\mathcal{A}$  (not in general on  $\mathcal{B}$  for example in case  $\mathcal{B}$  contains non-trivial group-like elements).

We define

$$c = \log_*(I) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (I_+)^{*n} \quad (43)$$

and remark that, in view of Prop. (4), in the case when  $\mathcal{B} = A\langle X \rangle$  one has  $\mathcal{A} = \mathcal{B}$  and, with  $S(X) = \log(1 + X)$

$$\begin{aligned}
\sum_{w \in X^*} w \otimes \pi_1(w) &= \rho(\log(I)) = \rho(S(I^+)) = S(\rho(I^+)) = \\
S\left(\sum_{\substack{w \in X^* \\ w \neq 1_{X^*}}} w \otimes w\right) &= S(\mathcal{D}_X - 1_{X^*} \otimes 1_{X^*}) = \log(\mathcal{D}_X) \quad (44)
\end{aligned}$$

We first prove that  $\pi_{1,\mathcal{A}}$  is a projector  $\mathcal{A} \rightarrow \text{Prim}(\mathcal{B})$ . The key point is that  $\Delta_{\mathcal{A}}$  (the restriction of the comultiplication to  $\mathcal{A}$ ) is a morphism of bialgebras<sup>14</sup>  $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ . We begin by to proving that  $\Delta_{\mathcal{A}}$  “commutes” with the convolution. This is a consequence of the following property

**Lemma 1.** *i) Let  $f_i \in \text{End}(\mathcal{B}_i)$ , be such that  $\varphi f_1 = f_2 \varphi$ .*

$$\begin{array}{ccc}
\mathcal{B}_1 & \xrightarrow{\varphi} & \mathcal{B}_2 \\
f_1 \downarrow & & \downarrow f_2 \\
\mathcal{B}_1 & \xrightarrow{\varphi} & \mathcal{B}_2
\end{array}$$

Figure 4: Intertwining with a morphism of bialgebras (the functions of  $f_i$  below will be computed with the respective convolution products).

*i) Then, if  $P \in A[X]$ , one has*

$$\varphi P(f_1) = P(f_2) \varphi . \quad (45)$$

<sup>14</sup>In fact it is the case for any cocommutative bialgebra, be it generated by its primitive elements or not.

ii) If the series  $\sum_{n \geq 0} (I_{(i)}^+)^{*n}$ ,  $i = 1, 2$  are summable, if  $f_1(1) = 0$  (which implies  $f_2(1) = 0$ ) and  $S \in A[[X]]$ , then the families  $(\langle S | X^n \rangle f_i^{*n})_{n \in \mathbb{N}}$  are summable, we denote  $S(f_i)$  their sums (this definition is coherent with the preceding when  $S$  is a polynomial).

One has, for the convolution product,

$$\varphi S(f_1) = S(f_2) \varphi . \quad (46)$$

*Proof.* The only delicate part is (ii). First, one remarks that, if  $\varphi$  is a morphism of bialgebras, one has

$$(\varphi \otimes \varphi) \circ \Delta_1^+ = \Delta_2^+ \circ \varphi \quad (47)$$

then, the image by  $\varphi$  of an element of order less than  $N$  (i.e. such that  $\Delta_1^{+(N)}(x) = 0$ ) is of order less than  $N$ . Let now  $S$  be an univariate series  $S = \sum_{k=0}^{\infty} a_k X^k$ . For every element  $x$  of order less than  $N$  and  $f \in \text{End}(\mathcal{B})$  such that, one has

$$\begin{aligned} S(f)(x) &= \sum_{k=0}^{\infty} a_k f^{*k}(x) = \sum_{k=0}^{\infty} a_k \mu^{(k-1)} f^{\otimes k} \Delta^{(k-1)}(x) \\ &= \sum_{k=0}^{\infty} a_k \mu^{(k-1)} (f^{\otimes k}) \circ (I_+^{\otimes k}) \Delta^{(k-1)}(x) \\ &= \sum_{k=0}^N a_k \mu^{(k-1)} (f^{\otimes k}) \Delta_+^{(k-1)}(x) . \end{aligned} \quad (48)$$

This proves, in view of (i) that  $\varphi \circ S(f_1) = S(f_2) \circ \varphi$ .  $\square$

We reprove now that  $\pi_1$  is a projector [18]  $\mathcal{B} \rightarrow \text{Prim}(\mathcal{B})$  by means of the following lemma.

In case  $\mathcal{B}$  is cocommutative, the comultiplication  $\Delta$  is a morphism of bialgebras, so one has

$$\Delta \circ \log_*(I) = \log_*(I \otimes I) \circ \Delta \quad (49)$$

But

$$\begin{aligned} \log_*(I \otimes I) &= \log_*((I \otimes e) * (e \otimes I)) \\ &= \log_*(I \otimes e) + \log_*(e \otimes I) \\ &= \log_*(I) \otimes e + e \otimes \log_*(I) \end{aligned} \quad (50)$$

Then

$$\Delta(\log_*(I)) = (\log_*(I) \otimes e + e \otimes \log_*(I)) \circ \Delta \quad (51)$$

which implies that  $\log_*(I)(\mathcal{B}) \subset \text{Prim}(\mathcal{B})$ . To finish to prove that  $\pi_1$  is a projector onto  $\text{Prim}(\mathcal{B})$ , one has just to remark that, for  $x \in \text{Prim}(\mathcal{B})$  and  $n \geq 2$   $(\text{Id}^+)^{*n}(x) = 0$  then

$$\log_*(I)(x) = \text{Id}^+(x) = x . \quad (52)$$

$\square$

Now, we consider

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n \geq 0} \frac{1}{n!} \pi_{1, [\mathcal{A}]}^{*n}. \quad (53)$$

where  $\pi_{1, [\mathcal{A}]} = \log_*(I_{\mathcal{A}})$ .

Let us prove that the summands form an resolution of unity.

First, one defines  $\mathcal{A}_{[n]}$  as the linear span of the powers  $\{P^n\}_{P \in \text{Prim}(\mathcal{B})}$  or, equivalently of the symmetrized products

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} P_{\sigma(1)} P_{\sigma(2)} \cdots P_{\sigma(n)}. \quad (54)$$

It is obvious that  $\text{Im}(\pi_{1, [\mathcal{A}]}^{*n}) \subset \mathcal{A}_{[n]}$ . We remark that

$$\pi_{1, [\mathcal{A}]}^{*n} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1, [\mathcal{A}]}^{\otimes n} \Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1, [\mathcal{A}]}^{\otimes n} I_+^{\otimes n} \Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1, [\mathcal{A}]}^{\otimes n} \Delta_+^{(n-1)} \quad (55)$$

as  $\pi_{1, [\mathcal{A}]} I_+ = \pi_{1, [\mathcal{A}]}$ . Now, let  $P \in \text{Prim}(\mathcal{A})$ . We compute  $\pi_{1, [\mathcal{A}]}^{*n}(P^m)$ . Indeed, if  $m < n$ , one has

$$\pi_{1, [\mathcal{A}]}^{*n}(P^m) = \mu_{\mathcal{B}}^{n-1} \Delta_+^{n-1}(P^m) = 0. \quad (56)$$

If  $n = m$ , one has, from (40)

$$\Delta_+^{n-1}(P^n) = n! P^{\otimes n} \quad (57)$$

and hence  $\pi_{1, [\mathcal{A}]}^{*n}$  is the identity on  $\mathcal{A}_{[n]}$ . If  $m > n$ , the nullity of  $\pi_{1, [\mathcal{A}]}^{*n}(P^m)$  is a consequence of the following lemma.

**Lemma 2.** *Let  $\mathcal{B}$  be a bialgebra and  $P$  a primitive element of  $\mathcal{B}$ . Then*

- i) *The series  $\log_*(I)$  is summable on each power  $P^m$*
- ii)  *$\log_*(I)(P^m) = 0$  for  $m > 2$*

*Proof.* i) As  $\Delta_+^{*N}(P^m) = 0$  for  $N > m$ , one has  $I_+^{*N}(P^m) = 0$  for these values.

ii) Let  $a$  be a letter, the morphism of AAU  $\varphi_P : A[a] \rightarrow \mathcal{B}$ , defined by

$$\varphi_P(a) = P \quad (58)$$

is, in fact a morphism of bialgebras and one checks easily that One has just to check that  $\pi_{1, [A[a]]}(a^m) = 0$  for  $m > 2$  which is a consequence of the general equality (see eq.44 )

$$\sum_{w \in X^*} (w \otimes \pi_1(w)) = \log\left(\sum_{w \in X^*} w \otimes w\right) \quad (59)$$

because, for  $Y = \{a\}$  (and then  $A\langle X \rangle = A[a]$ ) one has

$$\begin{array}{ccc}
A[a] & \xrightarrow{\varphi_P} & \mathcal{B} \\
I_{A[a]}^+ \downarrow & & \downarrow I_{\mathcal{B}}^+ \\
A[a] & \xrightarrow{\varphi_P} & \mathcal{B}
\end{array}$$

Figure 5: Intertwining with one primitive element.

$$\begin{aligned}
& \log\left(\sum_{w \in X^*} w \otimes w\right) = \log\left(\sum_{n \geq 0} a^n \otimes a^n\right) = \\
& \log\left(\sum_{n \geq 0} \frac{1}{n!} (a \otimes a)^{(\sqcup \otimes \text{conc})^n}\right) = \log(\exp(a \otimes a)) = a \otimes a \quad (60)
\end{aligned}$$

this proves that  $\pi_{1,[\mathcal{A}]}^{*n}(\mathcal{A}_{[m]}) = 0$  for  $m \neq n$  and hence the summands of the sum

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n \geq 0} \frac{1}{n!} \pi_{1,[\mathcal{A}]}^{*n} \cdot \quad (61)$$

are pairwise orthogonal projectors with  $\text{Im}(\pi_{1,[\mathcal{A}]}^{*n}) = \mathcal{A}_{[n]}$  and then

$$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_{[n]} \cdot \quad (62)$$

This decomposition permits to construct  $\sigma$  by

$$\sigma(P^n) = \frac{1}{n!} \Delta_+^{(n-1)}(P^n) \in T_n(\text{Prim}(\mathcal{B})) \quad (63)$$

for  $n \geq 1$  and, one sets  $\sigma(1_{\mathcal{B}}) = 1_{T(\text{Prim}(\mathcal{B}))}$ .

It is easy to check that  $j_{\mathcal{B}} \circ \tau \circ \sigma = \text{Id}_{\mathcal{A}}$  as  $\mathcal{A}$  is (linearly) generated by the powers  $(P^m)_{P \in \text{Prim}(\mathcal{B}), m \geq 0}$ .  $\square$

**End of the proof of proposition 5.** —

*iii)  $\implies$  ii)* If  $j_{\mathcal{B}}$  is into, then  $i_{\mathcal{U},\mathcal{A}}$  is one-to-one and one gets a comultiplication

$$\Delta_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$$

such that, for any list of primitive elements  $L = [g_1, g_2, \dots, g_n]$  (the denotations are the same as previously)

$$\Delta_{\mathcal{A}}(g_1 g_2 \cdots g_n) = \Delta(L[\{1, 2, \dots, n\}]) = \sum_{I+J=\{1,2,\dots,n\}} L[I] \otimes_{\mathcal{A}} L[J] \quad (64)$$

but, this time, the tensor product  $\otimes_{\mathcal{A}}$  is understood as being in  $\mathcal{A} \otimes \mathcal{A}$ . This guarantees that the diagram Fig. 2 commutes for any  $G$ .

*ii)  $\implies$  i)* Obvious.  $\square$



## 2.3 Counterexamples and discussion

### 2.3.1 Counterexamples

It has been said that, with  $\mathcal{B} = (\mathbb{Q}[\epsilon][x], \cdot, 1_{\mathbb{Q}[\epsilon][x]}, \Delta, c)$  (notations as above),  $j_{\mathcal{B}}$  is not into, let us show this statement.

The  $q$ -infiltration coproduct [5]  $\Delta_q$  is defined on the free algebra  $K\langle X \rangle$  ( $K$  is a unitary ring), by its values on the letters

$$\Delta(x) = x \otimes 1 + 1 \otimes x + q(x \otimes x) \quad (65)$$

where  $q \in K$ . One can show easily that, for a word  $w \in X^*$ ,

$$\Delta_q(w) = \sum_{I \cup J = [1..|w|]} q^{|I \cap J|} w[I] \otimes w[J] \quad (66)$$

with, as above (for  $I = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, n\}$  and  $w = a_1 a_2 \dots a_n$ ),  $w[I] = a_{i_1} a_{i_2} \dots a_{i_k}$ .

Then, with  $K = \mathbb{Q}[\epsilon]$ ,  $q = \epsilon$ ,  $X = x$ , one has (as a direct application of Eq. 66)

$$\Delta(x^n) = \sum_{k=0}^n \binom{n}{k} x^k \otimes x^{n-k} + \epsilon \sum_{k=1}^n \binom{n}{k} k x^k \otimes x^{n-k+1}. \quad (67)$$

This proves that, here, the space of primitive elements is a submodule of  $Kx$  and solving  $\Delta(\lambda x) = (\lambda x) \otimes 1 + 1 \otimes (\lambda x)$ , one finds  $\lambda = \epsilon \lambda_1$ . Together with  $\epsilon x \in \text{Prim}(\mathcal{B})$  this proves that  $\text{Prim}(\mathcal{B}) = \mathbb{Q}(\epsilon x)$ . Now, the consideration of the morphism of Lie algebras  $\text{Prim}(\mathcal{B}) \rightarrow K[x]/(\epsilon K[x])$  which sends  $\epsilon x$  to  $x$  proves that, in  $\mathcal{U}(\text{Prim}(\mathcal{B}))$ , we have  $(\epsilon x)(\epsilon x) \neq 0$  and  $j_{\mathcal{B}}$  cannot be into.

For a graded counterexample, one can see that, with  $K = \mathbb{Q}[\epsilon]$ ,  $X = x, y, z$ ,  $\mathcal{B} = K\langle X \rangle$  and

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \epsilon(y \otimes z), \quad \Delta(y) = y \otimes 1 + 1 \otimes y, \quad \Delta(z) = z \otimes 1 + 1 \otimes z \quad (68)$$

the same phenomenon occurs (for the gradation, one takes  $\deg(y) = \deg(z) = 1$ ,  $\deg(x) = 2$ ).

### 2.3.2 The theorem from the point of view of summability

**From now on, the morphism  $j_{\mathcal{B}}$  is supposed into.**

The bialgebra  $\mathcal{B}$  being supposed cocommutative, we discuss the equivalent conditions under which we are in the presence of an enveloping algebra i.e.

$$\mathcal{B} \cong_{A\text{-bialg}} \mathcal{U}(\text{Prim}(\mathcal{B})) \quad (69)$$

from the point of view of the convergence of the series  $\log_*(I)^{15}$ . These conditions are known as the theorem of Cartier-Quillen-Milnor-Moore (CQMM).

<sup>15</sup>In a  $A$ -bialgebra, one can always consider the series of endomorphisms

$$\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (I^+)^{*n}. \quad (70)$$

The family  $(\frac{(-1)^{n-1}}{n} (I^+)^{*n})_{n \geq 0}$  is summable iff  $((I^+)^{*n})_{n \geq 0}$  is (use eq.21).

**Theorem 1.** [4] Let  $\mathcal{B}$  be a  $A$ -cocommutative bialgebra ( $A$  is a  $\mathbb{Q}$ -AAU) and  $\mathcal{A}$ , as above, the subalgebra generated by  $\text{Prim}(\mathcal{B})$ . Then, the following conditions are equivalent :

i)  $\mathcal{B}$  admits an increasing filtration

$$\mathcal{B}_0 = A.1_{\mathcal{B}} \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \cdots$$

compatible with the structures of algebra (i.e. for all  $p, q \in \mathbb{N}$ , one has  $\mathcal{B}_p \mathcal{B}_q \subset \mathcal{B}_{p+q}$ ) and coalgebra :

$$\forall n \in \mathbb{N}, \quad \Delta(\mathcal{B}_n) \subset \sum_{p+q=n} \mathcal{B}_p \otimes \mathcal{B}_q.$$

ii)  $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$  is summable in  $\text{End}(\mathcal{B})$ .

iii)  $\mathcal{B} = \mathcal{A}$ .

*Proof.* We prove

$$(ii) \implies (iii) \implies (i) \implies (ii) \quad (71)$$

(ii)  $\implies$  (iii). —

The image of  $j_{\mathcal{B}}$  is the subalgebra generated by the primitive elements. Let us prove that, when  $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$  is summable, one has  $\text{Im}(j_{\mathcal{B}}) = \mathcal{B}$ . The series  $\log(1 + X)$  is without constant term so, in virtue of (29) and the summability of  $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$ , one has

$$\exp(\log(e + \text{Id}^+)) = \exp(\log(1 + X))(\text{Id}^+) = 1_{\text{End}(\mathcal{B})} + \text{Id}^+ = e + \text{Id}^+ = I \quad (72)$$

Set  $\pi_1 = \log(e + \text{Id}^+)$ .

To end this part, let us compute, for  $x \in \mathcal{B}$

$$x = \exp(\pi_1)(x) = \left( \sum_{n \geq 0} \frac{1}{n!} \pi_1^{*n} \right)(x) = \left( \sum_{n=0}^N \frac{1}{n!} \mu^{(n-1)} \pi_1^{\otimes n} \right) \Delta^{(n-1)}(x) \quad (73)$$

where  $N$  is the first order for which  $\Delta^{+(n-1)}(x) = 0$  (as  $\pi_1 \circ \text{Id}^+ = \pi_1$ ). This proves that  $\mathcal{B}$  is generated by its primitive elements.

(iii)  $\implies$  (i). —

□

**Remark 1.** i) The equivalence (i)  $\iff$  (iii) is the classical CQMM theorem (see [4]). The equivalence with (ii) could be called the “Convolutional CQMM theorem”. The combinatorial aspects of this last one will be the subject of a forthcoming paper.

ii) When  $\text{Prim}(\mathcal{B})$  is free, we have  $\mathcal{B} \cong_{k\text{-bialg}} \mathcal{U}(\text{Prim}(\mathcal{B}))$  and  $\mathcal{B}$  is an enveloping algebra.

iii) The (counter) example is the following with  $A = k[x]$  ( $k$  is a field of characteristic zero). Let  $Y$  be an alphabet and  $A\langle Y \rangle$  be the usual free algebra (the space of non-commutative polynomials over  $Y$ ) and  $\epsilon$ , the “constant term” linear form. Let  $\text{conc}$  be the concatenation and  $\Delta$  the unshuffling. Then the bialgebra  $(A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta, \epsilon)$  is a Hopf algebra (it is the enveloping algebra of the Lie polynomials). Let  $A_+\langle Y \rangle = \ker(\epsilon)$  and, for  $N \geq 2$   $J_N = x^N \cdot A_+\langle Y \rangle$  then,  $J_N$  is a Hopf ideal and  $\text{Prim}(A\langle Y \rangle / (J_N))$  is never free (no basis).

### 3 Examples of $\phi$ -deformed shuffle.

#### 3.1 Results for the $\phi$ -deformed shuffle.

Let  $Y = \{y_i\}_{i \in I}$  be still a totally ordered alphabet and  $A\langle Y \rangle$  be equipped with the  $\phi$ -deformed shuffle defined by

$$\text{i) for any } w \in Y^*, 1_{Y^*} \bowtie_{\phi} w = w \bowtie_{\phi} 1_{Y^*} = w,$$

$$\text{ii) for any } y_i, y_j \in Y \text{ and } u, v \in Y^*,$$

$$y_i u \bowtie_{\phi} y_j v = y_j (y_i u \bowtie_{\phi} v) + y_i (u \bowtie_{\phi} y_j v) + \phi(y_i, y_j) u \bowtie_{\phi} v, \quad (74)$$

where  $\phi$  is an arbitrary mapping

$$\phi : Y \times Y \longrightarrow AY.$$

**Definition 2.** Let

$$\phi : Y \times Y \longrightarrow AY$$

defined by its structure constants

$$(y_i, y_j) \longmapsto \phi(y_i, y_j) = \sum_{k \in I} \gamma_{i,j}^k y_k.$$

**Proposition 6.** The recursion (74) defines a unique mapping

$$\bowtie_{\phi} : Y^* \times Y^* \longrightarrow A\langle Y \rangle.$$

*Proof.* Let us denote  $(Y^* \times Y^*)_{\leq n}$  the set of words  $(u, v) \in Y^* \times Y^*$  such that  $|u| + |v| \leq n$ . We construct a sequence of mappings

$$\bowtie_{\phi \leq n} : (Y^* \times Y^*)_{\leq n} \longrightarrow A\langle Y \rangle.$$

which satisfy the recursion of eq.74. For  $n = 0$ , we have only a preimage and

$\bowtie_{\phi \leq 0}(1_{Y^*}) = 1_{Y^*} \otimes 1_{Y^*}$ . Suppose  $\bowtie_{\phi \leq n}$  constructed and let

$(u, v) \in (Y^* \times Y^*)_{\leq n+1} \setminus (Y^* \times Y^*)_{\leq n}$ , i.e.  $|u| + |v| = n + 1$ .

One has three cases :  $u = 1_{Y^*}$ ,  $v = 1_{Y^*}$  and  $(u, v) \in Y^+ \times Y^+$ . For the two first, one uses the initialisation of the recursion thus

$$\bowtie_{\phi \leq n+1}(w, 1_{Y^*}) = \bowtie_{\phi \leq n+1}(1_{Y^*}, w) = w$$

for the last case, write  $u = y_i u'$ ,  $v = y_j v'$  and use, to get

$$\mathfrak{L}_{\phi_{\leq n+1}}(y_i u', y_j v') = y_i \mathfrak{L}_{\phi_{\leq n}}(u', y_j v') + y_j \mathfrak{L}_{\phi_{\leq n}}(y_i u', v') + y_{i+j} \mathfrak{L}_{\phi_{\leq n}}(u', v')$$

this proves the existence of the sequence  $(\mathfrak{L}_{\phi_{\leq n}})_{n \geq 0}$ . Every  $\mathfrak{L}_{\phi_{\leq n+1}}$  extends the preceding so there is a mapping

$$\mathfrak{L}_{\phi} : Y^* \times Y^* \longrightarrow A(Y).$$

which extends all the  $\mathfrak{L}_{\phi_{\leq n+1}}$  (the graph of which is the union of the graphs of the  $\mathfrak{L}_{\phi_{\leq n}}$ ). This proves the existence. For unicity, just remark that, if there were two mappings  $\mathfrak{L}_{\phi}$ ,  $\mathfrak{L}'_{\phi}$ , the fact that they must fulfill the recursion (74) implies that  $\mathfrak{L}_{\phi} = \mathfrak{L}'_{\phi}$ .  $\square$

We still denote  $\phi$  and  $\mathfrak{L}_{\phi}$  the linear extension of  $\phi$  and  $\mathfrak{L}_{\phi}$  to  $AY \otimes AY$  and  $A(Y) \otimes A(Y)$  respectively.

Then  $\mathfrak{L}_{\phi}$  is a law of algebra (with  $1_{Y^*}$  as unit) on  $A(Y)$ .

**Lemma 3.** *Let  $\Delta$  be the morphism  $A(Y) \rightarrow A(\langle Y^* \otimes Y^* \rangle)$  defined on the letters by*

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{n,m \in I} \gamma_{n,m}^s y_n \otimes y_m \quad (75)$$

Then

i) for all  $w \in Y^+$  we have

$$\Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v \in Y^+} \langle \Delta(w) \mid u \otimes v \rangle u \otimes v \quad (76)$$

ii) for all  $u, v, w \in Y^*$ , one has

$$\langle u \mathfrak{L}_{\phi} v \mid w \rangle = \langle u \otimes v \mid \Delta(w) \rangle^{\otimes 2} \quad (77)$$

*Proof.* i) By recurrence on  $|w|$ . If  $w = y_s$  is of length one, it is obvious from the definition. If  $w = y_s w'$ , we have, from the fact that  $\Delta$  is a morphism

$$\begin{aligned} \Delta(w) &= \left( y_s \otimes 1 + 1 \otimes w + \sum_{i,j \in I} \gamma_{i,j}^s y_i \otimes y_j \right) \\ &\quad \left( w' \otimes 1 + 1 \otimes w' + \sum_{u,v \in Y^+} \langle u \otimes v \mid \Delta(w') \rangle \right) \end{aligned} \quad (78)$$

the development of which proves that  $\Delta(w)$  is of the desired form.

ii) Let  $S(u, v) := \sum_{w \in Y^*} \langle u \otimes v \mid \Delta(w) \rangle w$ . It is easy to check (and left to the reader) that, for all  $u \in Y^*$ ,  $S(u, 1) = S(1, u) = u$ . Let us now prove that, for all  $y_i, y_j \in Y$  and  $u, v \in Y^*$

$$S(y_i u, y_j v) = y_i S(u, y_j v) + y_j S(y_i u, v) + \phi(y_i, y_j) S(u, v) \quad (79)$$

Indeed, remarking that  $\Delta(1) = 1 \otimes 1$ , one has

$$\begin{aligned}
S(y_i u, y_j v) &= \sum_{w \in Y^*} \langle y_i u \otimes y_j v \mid \Delta(w) \rangle w = \sum_{w \in Y^+} \langle y_i u \otimes y_j v \mid \Delta(w) \rangle w \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \Delta(y_s w') \rangle y_s w' \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \left( y_s \otimes 1 + 1 \otimes y_s + \sum_{n, m \in I} \gamma_{n, m}^s y_n \otimes y_m \right) \Delta(w') \rangle y_s w' \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid (y_s \otimes 1) \Delta(w') \rangle y_s w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid (1 \otimes y_s) \Delta(w') \rangle y_s w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \left( \sum_{n, m \in I} \gamma_{n, m}^s y_n \otimes y_m \right) \Delta(w') \rangle y_s w' \\
&= \sum_{w' \in Y^*} \langle u \otimes y_j v \mid \Delta(w') \rangle y_i w' + \sum_{w' \in Y^*} \langle y_i u \otimes v \mid \Delta(w') \rangle y_j w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle u \otimes v \mid \gamma_{i, j}^s \Delta(w') \rangle y_s w' \\
&= y_i \sum_{w' \in Y^*} \langle u \otimes y_j v \mid \Delta(w') \rangle w' + y_j \sum_{w' \in Y^*} \langle y_i u \otimes v \mid \Delta(w') \rangle w' \\
&\quad + \sum_{y_s \in Y} \gamma_{i, j}^s y_s \sum_{w' \in Y^*} \langle u \otimes v \mid \Delta(w') \rangle w' \\
&= y_i S(u, y_j v) + y_j S(y_i u, v) + \phi(y_i, y_j) S(u, v)
\end{aligned}$$

then the computation of  $S$  shows that, for all  $u, v \in Y^*$ ,  $S(u, v) = u \blacktriangleright_{\phi} v$  as  $S$  is bilinear, one has  $S = \blacktriangleright_{\phi}$ .  $\square$

**Theorem 2.** *i) The law  $\blacktriangleright_{\phi}$  is commutative if and only if the extension*

$$\phi : AY \otimes AY \longrightarrow AY$$

*is so.*

*ii) The law  $\blacktriangleright_{\phi}$  is associative if and only if the extension*

$$\phi : AY \otimes AY \longrightarrow AY$$

*is so.*

*iii) Let  $\gamma_{x, y}^z := \langle \phi(x, y) \mid z \rangle$  be the structure constants of  $\phi$  (w.r.t. the basis  $Y$ ), then  $\blacktriangleright_{\phi}$  is dualizable if and only if  $(\gamma_{x, y}^z)_{x, y, z \in X}$  is of finite decomposition*

~~type~~ has the following decomposition property<sup>16</sup> ~~in its superscript in the following sense~~

$$(\forall z \in X)(\#\{(x, y) \in X^2 | \gamma_{x,y}^z \neq 0\} < +\infty). \quad (80)$$

*Proof.* (i) First, let us suppose that  $\phi$  be commutative and consider  $T$ , the twist, i.e. the operator in  $A\langle\langle Y^* \otimes Y^* \rangle\rangle$  defined by

$$\langle T(S) | u \otimes v \rangle = \langle S | v \otimes u \rangle \quad (81)$$

it is left to the reader to prove that  $T$  is a morphism of algebras. If  $\phi$  is commutative, then so is the following diagram.

$$\begin{array}{ccc} Y & \xrightarrow{\Delta_{\boxtimes\phi}} & A\langle\langle Y^* \otimes Y^* \rangle\rangle \\ & \searrow_{\Delta_{\boxtimes\phi}} & \downarrow T \\ & & A\langle\langle Y^* \otimes Y^* \rangle\rangle \end{array}$$

and, then, the two morphisms  $\Delta_{\boxtimes\phi}$  and  $T \circ \Delta_{\boxtimes\phi}$  coincide on the generators  $Y$  of the algebra  $A\langle Y \rangle$  and hence over  $A\langle Y \rangle$  itself. Now for all  $u, v, w \in Y^*$ , one has

$$\begin{aligned} \langle v \boxtimes_{\phi} u | w \rangle &= \langle v \otimes u | \Delta_{\boxtimes\phi}(w) \rangle = \langle u \otimes v | T \circ \Delta_{\boxtimes\phi}(w) \rangle = \\ \langle u \otimes v | \Delta_{\boxtimes\phi}(w) \rangle &= \langle u \boxtimes_{\phi} v | w \rangle \end{aligned} \quad (82)$$

which proves that  $v \boxtimes_{\phi} u = u \boxtimes_{\phi} v$ . Conversely, if  $\boxtimes_{\phi}$  is commutative, one has, for  $i, j \in I$

$$\phi(y_j, y_i) = y_j \boxtimes_{\phi} y_i - (y_j \sqcup y_i) = y_i \boxtimes_{\phi} y_j - (y_i \sqcup y_j) = \phi(y_i, y_j) \quad (83)$$

(ii) Likewise, if  $\phi$  is associative, let us define the operators

$$\overline{\Delta_{\boxtimes\phi} \otimes I} : A\langle\langle Y^* \otimes Y^* \rangle\rangle \rightarrow A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle \quad (84)$$

by

$$\langle \overline{\Delta_{\boxtimes\phi} \otimes I}(S) | u \otimes v \otimes w \rangle = \langle S | (u \boxtimes_{\phi} v) \otimes w \rangle \quad (85)$$

and, similarly,

$$\overline{I \otimes \Delta_{\boxtimes\phi}} : A\langle\langle Y^* \otimes Y^* \rangle\rangle \rightarrow A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle \quad (86)$$

by

$$\langle \overline{I \otimes \Delta_{\boxtimes\phi}}(S) | u \otimes v \otimes w \rangle = \langle S | u \otimes (v \boxtimes_{\phi} w) \rangle \quad (87)$$

it is easy to check by direct calculation that they are well defined morphisms and that the following diagram

---

<sup>16</sup>One can prove that, in case  $Y$  is a semigroup, the associated  $\phi$  is fulfills eq.80 iff  $Y$  fulfills “condition D” of Bourbaki (see [3])

$$\begin{array}{ccc}
Y & \xrightarrow{\Delta_{\mathfrak{L}\phi}} & A\langle\langle Y^* \otimes Y^* \rangle\rangle \\
\Delta_{\mathfrak{L}\phi} \downarrow & & \downarrow \overline{I \otimes \Delta_{\mathfrak{L}\phi}} \\
A\langle\langle Y^* \otimes Y^* \rangle\rangle & \xrightarrow{\overline{\Delta_{\mathfrak{L}\phi} \otimes I}} & A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle
\end{array}$$

is commutative. This proves that the two composite morphisms

$$\overline{\Delta_{\mathfrak{L}\phi} \otimes I} \circ \Delta_{\mathfrak{L}\phi}$$

and

$$\overline{I \otimes \Delta_{\mathfrak{L}\phi}} \circ \Delta_{\mathfrak{L}\phi}$$

coincide on  $Y$  and then on  $A\langle Y \rangle$ . Now, for  $u, v, w, t \in Y^*$ , one has

$$\begin{aligned}
\langle (u \mathfrak{L}\phi v) \mathfrak{L}\phi w \mid t \rangle &= \langle (u \mathfrak{L}\phi v) \otimes w \mid \Delta_{\mathfrak{L}\phi}(t) \rangle = \langle u \otimes v \otimes w \mid (\overline{\Delta_{\mathfrak{L}\phi} \otimes I}) \Delta_{\mathfrak{L}\phi}(t) \rangle = \\
&= \langle u \otimes v \otimes w \mid (\overline{I \otimes \Delta_{\mathfrak{L}\phi}}) \Delta_{\mathfrak{L}\phi}(t) \rangle = \langle u \otimes (v \mathfrak{L}\phi w) \mid \Delta_{\mathfrak{L}\phi}(t) \rangle = \langle u \mathfrak{L}\phi (v \mathfrak{L}\phi w) \mid t \rangle
\end{aligned}$$

which proves the associativity of the law  $\mathfrak{L}\phi$ . Conversely, if  $\mathfrak{L}\phi$  is associative, the direct expansion of the right hand side of

$$0 = (y_i \mathfrak{L}\phi y_j) \mathfrak{L}\phi y_k - y_i \mathfrak{L}\phi (y_j \mathfrak{L}\phi y_k) \quad (88)$$

proves the associativity of  $\phi$ .

iii) We suppose that  $(\gamma_{x,y}^z)_{x,y,z \in X}$  is of finite decomposition type in its superscript, in this case  $\Delta_{\mathfrak{L}\phi}$  takes its values in  $A\langle Y \rangle \otimes A\langle Y \rangle$  therefore its dual, the law  $\mathfrak{L}\phi$  is dualizable. Conversely, if  $\text{Im}(\Delta_{\mathfrak{L}\phi}) \subset A\langle Y \rangle \otimes A\langle Y \rangle$ , one has, for every  $s \in I$

$$\sum_{n,m \in I} \gamma_{n,m}^s y_n \otimes y_m = \Delta(y_s) - (y_s \otimes 1 + 1 \otimes y_s) \in A\langle Y \rangle \otimes A\langle Y \rangle$$

which proves the claim.  $\square$

**From now on, we suppose that  $\phi : AY \otimes AY \rightarrow AY$  be an associative and commutative law (of algebra) on  $AY$ .**

**Theorem 3.** *Let  $A$  be a  $\mathbb{Q}$ -algebra. Then if  $\phi$  is dualizable<sup>17</sup>, let  $\Delta_{\mathfrak{L}\phi} : A\langle Y \rangle \rightarrow A\langle Y \rangle \otimes A\langle Y \rangle$  denote its dual comultiplication, then*

a)  $\mathcal{B}_\phi = (A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\mathfrak{L}\phi}, \varepsilon)$  is a bialgebra.

<sup>17</sup>For the pairing defined by

$$\forall x, y \in Y, \quad \langle x \mid y \rangle = \delta_{x,y}$$

- b) If  $A$  is a  $\mathbb{Q}$ -algebra then, the following conditions are equivalent
- i)  $\mathcal{B}_\phi$  is an enveloping bialgebra
  - ii) the algebra  $AX$  admits an increasing filtration  $\left((AY)_n\right)_{n \in \mathbb{N}}$  with  $(AY)_0 = \{0\}$

$$(AY)_0 = \{0\} \subset (AY)_1 \subset \cdots \subset (AY)_n \subset (AY)_{n+1} \subset \cdots$$

compatible with both the multiplication and the comultiplication  $\Delta_{\mathbf{w}_\phi}$  i.e.

$$\begin{aligned} (AY)_p (AY)_q &\subset (AY)_{p+q} \\ \Delta_{\mathbf{w}_\phi}((AY)_n) &\subset \sum_{p+q=n} (AY)_p \otimes (AY)_q. \end{aligned}$$

- iii)  $\mathcal{B}_\phi$  is isomorphic to  $(A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\mathbf{w}}, \epsilon)$  as a bialgebra.
- iv)  $I^+$  is  $\star$ -nilpotent.

*Proof.* (Other - easier - implications to be written)

iv)  $\implies$  iii) Let us set  $y'_s = \pi_1(y_s)$ , then using a rearrangement of the star-log of the diagonal series, we have

$$y_s = \sum_{k \geq 1} \frac{1}{k!} \sum_{s'_1 + \cdots + s'_k = s} \pi_1(y_{s'_1}) \cdots \pi_1(y_{s'_k}) \quad (89)$$

This proves that the multiplicative morphism given by  $\Phi(y_s) = y'_s$  is an isomorphism. But this morphism is such that  $\Delta_{\mathbf{w}_\phi} \circ \Phi = (\Phi \otimes \Phi) \circ \Delta_{\mathbf{w}}$  which proves the claim.  $\square$

**Remark 2.** i) Theorem 3 a) holds for general (dualizable, coassociative)  $\phi$  be it commutative or not.

ii) It can happen that there be no antipode (and then,  $I^+$  cannot be  $\star$ -nilpotent) as shows the following example.

Let  $Y = \{y_0, y_1\}$  and  $\phi(y_i, y_j) = y_{(i+j \bmod 2)}$ , then

$$\begin{aligned} \Delta(y_0) &= y_0 \otimes 1 + 1 \otimes y_0 + y_0 \otimes y_0 + y_1 \otimes y_1 \\ \Delta(y_1) &= y_1 \otimes 1 + 1 \otimes y_1 + y_0 \otimes y_1 + y_1 \otimes y_0 \end{aligned} \quad (90)$$

then, from eqns 90, one derives that  $1 + y_0 + y_1$  is group-like. As this element has no inverse in  $K\langle Y \rangle$ , the bialgebra  $\mathcal{B}_\phi$  cannot be a Hopf algebra.

iii) When  $I^+$  is nilpotent, the antipode exists and is computed by

$$a_{\mathbf{w}_\phi} = (I)^{\star-1} = (e + I^+)^{\star-1} = \sum_{n \geq 0} (-1)^n (I^+)^{\star n} \quad (91)$$

(see section (2.2)).

iv) In QFT, the antipode of a vector  $h \in \mathcal{B}$  is computed by

$$S(1) = 1, \quad S(h) = -h + \sum_{(1)(2)} S(h_{(1)})h_{(2)} \quad (92)$$



and from the fact that  $S$  is an antimorphism. This formula is used in contexts where  $I^+$  is  $\star$ -nilpotent (although the concerned bialgebras are often not cocommutative). here, one can prove this recursion from 91.

## 4 Conclusion

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