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G rard Henry Edmond Duchamp, Vincel Hoang Ngoc Minh, Christophe Tollu, B i Chi n, Nguyen
Hoang Nghia. Combinatorics of ϕ -deformed stuffle Hopf algebras. 2013. hal-00793118v1

HAL Id: hal-00793118

<https://hal.science/hal-00793118v1>

Preprint submitted on 21 Feb 2013 (v1), last revised 1 Mar 2014 (v7)

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Combinatorics of ϕ -deformed stuffle Hopf algebras¹

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Abstract

In order to extend the Sch utzenberger's factorization to general perturbations, the combinatorial aspects of the Hopf algebra of the ϕ -deformed stuffle product is developed systematically in a parallel way with those of the shuffle product and in emphasizing the Lie elements as studied by Ree. In particular, we will give an effective construction of pair of bases in duality.

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¹Version du 21-02-2013 16:08

1 Introduction

Let X be an totally ordered alphabet². The free monoid and the set of Lyndon words, over X , are denoted respectively by X^* and $\mathcal{Lyn}X$. The neutral element of X^* , *i.e.* the empty word is denoted by 1_{X^*} . Let $\mathbb{Q}\langle X \rangle$ be equipped by the concatenation and the shuffle which is defined by

$$\begin{aligned} \forall w \in X^*, \quad w \sqcup 1_{X^*} &= 1_{X^*} \sqcup w = w, \\ \forall x, y \in X, \forall u, v \in X^*, \quad xu \sqcup yv &= x(u \sqcup yv) + y(xu \sqcup v), \end{aligned} \quad (1)$$

or by their dual co-products, $\Delta = \Delta_{\text{conc}}$ and $\Delta = \Delta_{\sqcup}$, defined by, for any $w \in X^*$ by,

$$\begin{aligned} \Delta_{\text{conc}}(w) &= \sum_{w=uv} u \otimes v \\ \Delta_{\sqcup}(w) &= \sum_{I+J=[1..|w|]} w[I] \otimes w[J] \end{aligned} \quad (2)$$

One gets two Hopf algebras

$$\begin{aligned} \mathcal{H}_{\sqcup} &= (\mathbb{Q}\langle X \rangle, \text{conc}, 1_{X^*}, \Delta_{\sqcup}, \epsilon, a_{\bullet}) \text{ and} \\ \mathcal{H}_{\sqcup}^{\vee} &= (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*}, \Delta_{\text{conc}}, \epsilon, a_{\sqcup}) \end{aligned} \quad (3)$$

mutually dual with respect to the pairing given by

$$(\forall u, v \in X^*)(\langle u | v \rangle = \delta_{u,v}) . \quad (4)$$

and with, for any $x_{i_1}, \dots, x_{i_r} \in X$ and $P \in \mathbb{Q}\langle X \rangle$,

$$\begin{aligned} \epsilon(P) &= \langle P | 1_{X^*} \rangle, \\ a_{\sqcup}(w) &= a_{\bullet}(w) = (-1)^r x_{i_r} \dots x_{i_1}, . \end{aligned} \quad (5)$$

By the theorem of Cartier-Quillen-Milnor and Moore (CQMM in the sequel), the connected, graded positively, co-commutative Hopf algebra \mathcal{H}_{\sqcup} is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements which here is $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$. Hence, from any basis of the free algebra $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$ one can³ complete, by the Poincaré-Birkhoff-Witt theorem, a linear basis $\{b_w\}_{w \in X^*}$ for $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle) = \mathbb{Q}\langle X \rangle$ (see below (8) for an example of such a construction), and, when the basis is finely homogeneous, one can construct, by duality, a basis $\{\check{b}_w\}_{w \in X^*}$ of \mathcal{H}_{\sqcup} (viewed as a \mathbb{Q} -module) such that :

$$\forall u, v \in X^*, \quad \langle \check{b}_u | b_v \rangle = \delta_{u,v} . \quad (6)$$

For $w = l_1^{i_1} \dots l_k^{i_k}$ with $l_1, \dots, l_k \in \mathcal{Lyn}X$, $l_1 > \dots > l_k$

$$\check{b}_w = \frac{\check{b}_{l_1}^{\sqcup i_1} \sqcup \dots \sqcup \check{b}_{l_k}^{\sqcup i_k}}{i_1! \dots i_k!} . \quad (7)$$

²In the sequel, the order on the words will be understood as the lexicographic by length total ordering.

³The basis can be reindexed by Lyndon words and then one uses the canonical factorization of the words.

For example, Chen, Fox and Lyndon [5] constructed the PBW-Lyndon basis $\{P_w\}_{w \in X^*}$ for $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle)$ as follows

$$\begin{aligned} P_x &= x && \text{for } x \in X, \\ P_l &= [P_s, P_r] && \text{for } l \in \mathcal{L}ynX, \text{ standard factorization of } l = (s, r), \\ P_w &= P_{l_1}^{i_1} \dots P_{l_k}^{i_k} && \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k, l_1, \dots, l_k \in \mathcal{L}ynX. \end{aligned} \quad (8)$$

Schützenberger and his school constructed, the linear basis $\{S_w\}_{w \in X^*}$ for $\mathcal{A} = (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$ by duality (w.r.t. eq.4) and obtained the transcendence basis of \mathcal{A} $\{S_l\}_{l \in \mathcal{L}ynX}$ as follows⁴

$$S_l = xS_u, \quad \text{for } l = xu \in \mathcal{L}ynX, \quad (9)$$

$$S_w = \frac{S_{l_1}^{\sqcup i_1} \sqcup \dots \sqcup S_{l_k}^{\sqcup i_k}}{i_1! \dots i_k!} \quad \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k. \quad (10)$$

After that, Mélançon and Reutenauer [8] proved that⁵, for any $w \in X^*$,

$$P_w = w + \sum_{v > w, |v|=|w|} c_v v \quad \text{and} \quad S_w = w + \sum_{v < w, |v|=|w|} c_v v. \quad (11)$$

On other words, the elements of the bases $\{S_w\}_{w \in X^*}$ and $\{P_w\}_{w \in X^*}$ are upper and lower triangular respectively and are multihomogeneous.

Moreover, thanks to the duality of the bases $\{P_w\}_{w \in X^*}$ and $\{S_w\}_{w \in X^*}$, if \mathcal{D}_X denotes the diagonal series over X one has

$$\mathcal{D}_X = \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynX}^{\rightrightarrows} \exp(S_l \otimes P_l). \quad (12)$$

ACKNOWLEDGEMENTS. —

Nguyen, Tanasa, Deneufchâtel + ANR : The authors are pleased to acknowledge the hospitality of institutions in Paris, Cracow and New York. Special thanks are due to Catherine Borgen for having created a fertile atmosphere in Exeter (UK) where early parts of this manuscript were prepared. We also hereby acknowledge support from the Polish Ministry of Science and Higher Education under Grant N° 202 107 32/2832 for hospitality and the ANR PhysComb “Projet ANR-BLAN08 N° 2 332204”. Finally, we are grateful to ?? and ?? for their reading of and comments on the manuscript.

DEDICATION. —

Pierre Cartier has not only blazed a successful trail in his own areas of Mathematics but, through his support given generously to others, has stimulated

⁴Therefore \mathcal{A} is a polynomial algebra $\mathcal{A} \simeq \mathbb{Q}[\mathcal{L}ynX]$.

⁵ Recall that the duality preserves the (multi)homogeneous degrees and interchanges the triangularity of polynomials [8]. For that, one can construct the triangular matrices M and N admitting as entries the coefficients of the multihomogeneous triangular polynomials, $\{P_w\}_{w \in X^*}$ and $\{S_w\}_{w \in X^*}$ in the basis $\{w\}_{w \in X^*}$ respectively :

$$M_{u,v} = \langle P_u | v \rangle \quad \text{and} \quad N_{u,v} = \langle S_u | v \rangle.$$

The triangular matrices M and N are unipotent and satisfy the identity $N = ({}^t M)^{-1}$.

new branches to develop. On the occasion of 80th birthday we are happy to acknowledge with gratitude his encouragement to the development of our own field of Combinatorial Physics and its interactions with number theory, and look forward to many future years of professional engagement.

2 General results on summability and duality

Let $Y = \{y_i\}_{i \in I}$ be a totally ordered alphabet. The free monoid and the set of Lyndon words, over Y , are denoted respectively by Y^* and $\mathcal{Lyn}Y$. The neutral of Y^* (and then of $A\langle Y \rangle$) is denoted by 1_{Y^*} .

2.1 Total algebras and duality

2.1.1 Series and infinite sums

In the sequel, we will need to construct spaces of functions on different monoids (mainly direct products of free monoids). We set, once for all the general construction of the corresponding convolution algebra.

Let A be a unitary commutative ring and M a monoid. Let us denote A^M the set⁶ of all (graphs of) mappings $M \rightarrow A$. This set is endowed with its classical structure of module. In order to extend the product defined in $A[M]$ (the algebra of the monoid M), it is essential that, in the sums

$$f * g(m) = \sum_{m \in M} \sum_{uv=m} f(u)g(v) \quad (13)$$

the inner sum $\sum_{uv=m} f(u)g(v)$ make sense. For that, we suppose that the monoid M fulfills condition “D” (be of finite decomposition type [3] Ch III.10). Formally, we say that M satisfies condition “D” iff, for all $m \in M$, the set

$$\{(u, v) \in M \times M \mid uv = m\} \quad (14)$$

is finite. In this case eq.13 endows A^M with the structure of a AAU. This algebra is traditionnaly called the total algebra of M (see [3] Ch III.10) and has very much to do with the series⁷. It will be, here (with a slight abuse of denotation which does not cause ambiguity) denoted $A\langle\langle M \rangle\rangle$.

The pairing

$$A\langle\langle M \rangle\rangle \otimes A[M] \longrightarrow A \quad (15)$$

defined by⁸

$$\langle f \mid g \rangle := \sum_{m \in M} f(m)g(m) \quad (16)$$

⁶In general Y^X is the set of all mappings $X \rightarrow Y$ [2] Ch 2.5.2

⁷In fact, the algebra of commutative (resp. noncommutative) series on an alphabet X is the total algebra of the free commutative (resp. X^*) monoid on X

⁸Here $A[M]$ is identified with the submodule of finitely supported functions $M \rightarrow A$.

allows to see every element of the total algebra as a linear form on the module $A[M]$. One can check easily that, with this pairing, one has $A\langle\langle M \rangle\rangle \simeq (A[M])^*$. One says that a family $(f_i)_{i \in I}$ of $A\langle\langle M \rangle\rangle$ is summable [1] iff, for every $m \in M$, the mapping $i \mapsto \langle f_i | m \rangle$ is finitely supported. In this case, the sum $\sum_{i \in I} f_i$ is exactly the mapping $m \mapsto \sum_{i \in I} \langle f_i | m \rangle$ so that, one has by definition

$$\langle \sum_{i \in I} f_i | m \rangle = \sum_{i \in I} \langle f_i | m \rangle . \quad (17)$$

To end with, let us remark that the set $M_1 \otimes M_2 = \{u \otimes v\}_{(u,v) \in M_1 \times M_2}$ is a (monoidal) basis of $A[M_1] \otimes A[M_2]$ and $M_1 \otimes M_2$ is a monoid (in the product algebra $A[M_1] \otimes A[M_2]$) isomorphic to the direct product $M_1 \times M_2$.

2.1.2 Summable families

In fact, $A\langle\langle M \rangle\rangle \simeq (A[M])^* = \text{Hom}(A[M], A)$ and the notion of summability developed here can be seen as a particular case of that of a family of endomorphisms $f_i \in \text{Hom}(V, W)$ for which $\text{Hom}(V, W)$ appears as a complete space. It is indeed the pointwise convergence for the discrete topology. We will not detail these considerations here.

The definition is similar of that of a summable family of series [1], viewed as a family of linear forms.

Definition 1. *i) A family $(f_i)_{i \in I}$ of elements in $\text{Hom}(V, W)$ is said to be summable iff for all $x \in V$, the map $i \mapsto f_i(x)$ has finite support. As a quantized criterium it reads*

$$(\forall x \in V)(\exists F \subset_{\text{finite}} I)(\forall i \notin F)(f_i(x) = 0) \quad (18)$$

ii) If the family $(f_i)_{i \in I} \in \text{Hom}(V, W)^I$ fulfils the condition 18 above its sum is given by

$$\left(\sum_{i \in I} f_i\right)(x) = \sum_{i \in I} f_i(x) \quad (19)$$

It is an easy exercise to show that the mapping $V \rightarrow W$ defined by the equation 19 is in fact in $\text{Hom}(V, W)$. Remark that, as the limiting process is defined by linear conditions, if a family $(f_i)_{i \in I}$ is summable, so is

$$(a_i f_i)_{i \in I} \quad (20)$$

for a set of coefficients $(a_i)_{i \in I} \in A^I$.

if a family When $V = W = \mathcal{B}$ is a bialgebra, this tool will be used in section (2.2) to give an analytic presentation of the theorem of Cartier-Milnor-Moore.

The most interesting feature of this operation is the interversion of sums. Let us state it formally as a proposition the proof of which is left to the reader.

Proposition 1. *Let $(f_i)_{i \in I}$ be a family of elements in $\text{Hom}(V, W)$ and $(I_j)_{j \in J}$ be a partition of I ([2] ch II §4 n° 7 Def. 6), then TFAE*

i) $(f_i)_{i \in I}$ is summable ii) for all $j \in J$, $(f_i)_{i \in I_j}$ is summable and the family $(\sum_{i \in I_j} f_i)_{j \in J}$ is summable.
In these conditions, one has

$$\sum_{i \in I} f_i = \sum_{j \in J} \left(\sum_{i \in I_j} f_i \right) \quad (21)$$

We derive at once from this the following practical criterium for double sums.

Proposition 2. *Let $(f_{\alpha,\beta})_{(\alpha,\beta) \in A \times B}$ be a doubly indexed summable family in $\text{Hom}(V, W)$, then, for fixed α (resp. β) the “row-families” $(f_{\alpha,\beta})_{\beta \in B}$ (resp. the “column-families” $(f_{\alpha,\beta})_{\alpha \in A}$) are summable and their sums are summable. Moreover*

$$\sum_{(\alpha,\beta) \in A \times B} f_{\alpha,\beta} = \sum_{\alpha \in A} \sum_{\beta \in B} f_{\alpha,\beta} = \sum_{\beta \in B} \sum_{\alpha \in A} f_{\alpha,\beta} . \quad (22)$$

2.1.3 Substitutions

Let \mathcal{A} be a AAU and $f \in \mathcal{A}$. For every polynomial $P \in A\langle X \rangle = A[X]$, one can compute $P(f)$ by

$$P(f) = \sum_{n \geq 0} \langle P \mid X^n \rangle f^n \quad (23)$$

one checks at once that $P \mapsto P(f)$ is a morphism⁹ of AAU between $A[X]$ and \mathcal{A} . Moreover, this morphism is compatible with the substitutions as one checks easily that, for $Q \in A[X]$

$$P(Q)(f) = P(Q(f)) \quad (24)$$

(it suffices to check that $P \mapsto P(Q)(f)$ and $P \mapsto P(Q(f))$ are two morphisms which coincide at $P = X$).

In order to substitute within series, one needs some limiting process. The framework of $\mathcal{A} = \text{Hom}(V, W)$ and summable families will be here sufficient (see paragraph 2.1.2). We suppose that $(V, \delta_V, \epsilon_V)$ is a co-AAU and that $(W, \mu_W, 1_W)$ is a AAU. Then $(\text{Hom}(V, W), *, e)$ is a AAU (with $e = 1_W \circ \epsilon_V$). A series $S \in A[[X]]$ being given, we say that $f \in \text{Dom}(S)$ iff the family $(\langle S \mid X^n \rangle f^{*n})_{n \geq 0}$ is summable. It is easy to check that, if $f \in \text{Dom}(S) \cap \text{Dom}(T)$ and $\alpha \in A$, one has

$$(\alpha S)(f) = \alpha S(f) ; (S + T)(f) = S(f) + T(f) \quad (25)$$

and

$$(TS)(f) = T(f) * S(f) . \quad (26)$$

If $((f^n)_{n \geq 0})$ is summable and $S(0) = 0$ then

$$f \in \text{Dom}(S) \cap \text{Dom}(T(S)) ; S(f) \in \text{Dom}(T) \quad (27)$$

and

$$T(S)(f) = T(S(f)) \quad (28)$$

⁹In case \mathcal{A} is a geometric space, this morphism is called “evaluation at f ” and corresponds to a Dirac measure.

Proof. Let us first prove eq.26 . As $f \in \text{Dom}(S) \cap \text{Dom}(T)$, the families $(\langle S | X^n \rangle f^{*n})_{n \geq 0}$ and $(\langle T | X^m \rangle f^{*m})_{n \geq 0}$ are summable, then so is

$$\left(\langle T | X^m \rangle f^{*m} * \langle S | X^n \rangle f^{*n} \right)_{n,m \geq 0} \quad (29)$$

as, for every $x \in V$, $\delta(x) = \sum_{i=1}^N x_i^{(1)} \otimes x_i^{(2)}$ and for every $i \in I$,

$$\text{supp}_{w.r.t. m}(\langle T | X^m \rangle f^{*m}(x_i^{(1)})) ; \text{supp}_{w.r.t. n}(\langle S | X^n \rangle f^{*n}(x_i^{(2)}))$$

are finite. Then outside of the cartesian product of the (finite) union of these supports, the product

$$(\langle T | X^m \rangle f^{*m} * \langle S | X^n \rangle f^{*n})(x) = \mu_W((\langle T | X^m \rangle f^{*m} \otimes \langle S | X^n \rangle f^{*n})(\delta(x))) \quad (30)$$

is zero. Hence the summability. Now

$$\begin{aligned} T(f) * S(f) &= \sum_{m=0}^{\infty} (\langle T | X^m \rangle f^{*m}) * \sum_{n=0}^{\infty} (\langle S | X^n \rangle f^{*n}) = \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\langle T | X^m \rangle \langle S | X^n \rangle f^{*n+m}) = \\ &= \sum_{s=0}^{\infty} \left(\sum_{n+m=s}^{\infty} \langle T | X^m \rangle \langle S | X^n \rangle \right) f^{*s} = \\ &= \sum_{s=0}^{\infty} (\langle TS | X^s \rangle) f^{*s} = (TS)(f) \end{aligned} \quad (31)$$

We now prove the statements (27) and (28). If $((f)^{*n})_{n \geq 0}$ is summable then f belongs to all domains (i.e. is universally substituable). For all $x \in V$, it exists $N_x \in \mathbb{N}$ such that

$$n > N_x \implies (f)^{*n}(x) = 0 .$$

Now, for S such that $S(0) = 0$, one has $S = \sum_{n=1}^{\infty} \langle S | X^n \rangle X^n$ and then $S^k = \sum_{n=k}^{\infty} \langle S^k | X^n \rangle X^n$. Now, in view of eq.26 , one has

$$S(f)^{*n}(x) = S^n(f)(x) = \sum_{m=n}^{\infty} \langle S^n | X^m \rangle (f)^{*m}(x) \quad (32)$$

which is zero for $n > N_x$. Hence the summability of $(S(f)^{*n})_{n \geq 0}$ which implies that $S(f) \in \text{Dom}(T)$. The family $(\langle T | X^n \rangle \langle S^n | X^m \rangle (f)^{*m})_{(n,m) \in \mathbb{N}^2}$ is summable because, if $x \in V$ and if n or m is greater than N_x then

$$\langle T | X^n \rangle \langle S^n | X^m \rangle (f)^{*m}(x) = 0 \quad (33)$$

thus $T(S(f))$ is the sum

$$T(S(f)) = \sum_{n=0}^{\infty} \langle T | X^n \rangle S(f)^{*n} = \sum_{n=0}^{\infty} \langle T | X^n \rangle \sum_{m=n}^{\infty} \langle S^n | X^m \rangle (f)^{*m} =$$

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle T | X^n \rangle \langle S^n | X^m \rangle (f)^{*m} &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \langle T | X^n \rangle \langle S^n | X^m \rangle \right) (f)^{*m} = \\
\sum_{m=0}^{\infty} \langle T(S) | X^m \rangle (f)^{*m} &= T(S)(f)
\end{aligned} \tag{34}$$

□

2.2 Theorem of Cartier-Quillen-Milnor-Moore (analytic form)

2.2.1 General properties of bialgebras

Let A be a unitary commutative ring with all (non-zero) integers invertible (i.e. $\mathbb{Q} \subset A$) and $(\mathcal{B}, \mu, e_{\mathcal{B}}, \Delta, \epsilon)$ a A -bialgebra.

TODO Id_+, Δ_+ .

In a general bialgebra $(\mathcal{B}, \mu, e_{\mathcal{B}}, \Delta, \epsilon)$, one can always consider the Lie algebra of primitive elements $Prim(\mathcal{B})$ and build the map $j_{\mathcal{B}} : \mathcal{U}(Prim(\mathcal{B})) \rightarrow \mathcal{B}$. Then, $\mathcal{A} = j_{\mathcal{B}}(\mathcal{U}(Prim(\mathcal{B})))$ is the subalgebra generated by the primitive elements. It is not difficult to see that \mathcal{A} is a sub-bialgebra of \mathcal{B} as, for any list of primitive elements $L = [g_1, g_2, \dots, g_n]$, one has

$$\Delta(g_1 g_2 \cdots g_n) = \Delta(L[\{1, 2, \dots, n\}]) = \sum_{I+J=\{1, 2, \dots, n\}} L[I] \otimes L[J] \tag{35}$$

where, for $I = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, n\}$,

$$L[I] = g_{i_1} g_{i_2} \cdots g_{i_k} . \tag{36}$$

From (35) one gets also that $j_{\mathcal{B}}$ is a morphism of bialgebras. In order to prove that it is always into we need to construct the arrows σ, τ which are a decomposition of a section of $j_{\mathcal{B}}$. Let us remark that, when $Prim(\mathcal{B})$ is free as a

$$\begin{array}{ccccc}
Prim(\mathcal{B}) & \xleftarrow{i_{\mathcal{A}, P}} & \mathcal{A} & \xleftarrow{i_{\mathcal{B}, \mathcal{A}}} & \mathcal{B} \\
\downarrow i_{\mathcal{U}, P} & & \nearrow j_{\mathcal{B}} & & \downarrow \sigma \\
\mathcal{U}(\mathcal{B}) & \xleftarrow{\tau} & T^{\otimes} (Prim(\mathcal{B})) & &
\end{array}$$

Figure 1: The sub-bialgebra \mathcal{A} generated by primitive elements.

A -module, the proof of this fact is a consequence of the PBW theorem¹⁰. But, here, we will construct the section in the general case using projectors which are now classical for the free case but which still can be computed analytically as they lie in $\mathbb{Q}[[X]]$ and still converge in \mathcal{A} .

¹⁰See [4] Ch2 §1 n^o 6 th 1 for a field of characteristic zero and §1 Ex. 10 for the free case (over a ring A with $\mathbb{Q} \subset A$).

Proof. (Injectivity of $j_{\mathcal{B}}$, construction of the section $\tau \circ \sigma$). —

Let \mathcal{A} be the subalgebra of \mathcal{B} generated by $\text{Prim}(\mathcal{B})$, it is straightforward to check that $\text{Im}(j_{\mathcal{B}}) = \mathcal{A}$.

Remark that all series $\sum_{n \geq 0} a_n (I_+)^{*n}$ are summable on \mathcal{A} (not in general on \mathcal{B} for example in case of non-trivial group-like elements).

We first prove that

$$\pi_1 = \log_*(I) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (I_+)^{*n} \quad (37)$$

is a projector $\mathcal{A} \rightarrow \text{Prim}(\mathcal{B})$. The key point is that $\Delta_{\mathcal{A}}$ (the restriction of the comultiplication to \mathcal{A}) is a morphism of bialgebras¹¹ $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$. We begin by to proving that $\Delta_{\mathcal{A}}$ “commutes” with the convolution. This is a consequence of the following property

Lemma 1. *i) Let $f_i \in \text{End}(\mathcal{B}_i)$, be such that $\varphi f_1 = f_2 \varphi$.*

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{\varphi} & \mathcal{B}_2 \\ f_1 \downarrow & & \downarrow f_2 \\ \mathcal{B}_1 & \xrightarrow{\varphi} & \mathcal{B}_2 \end{array}$$

Figure 2: Intertwining with a morphism of bialgebras (the functions of f_i below will be computed with the respective convolution products).

i) Then, if $P \in A[X]$, one has

$$\varphi P(f_1) = P(f_2) \varphi . \quad (38)$$

*ii) If the series $\sum_{n \geq 0} (I_{(i)}^+)^{*n}$, $i = 1, 2$ are summable, if $f_1(1) = 0$ and $S \in A[[X]]$, then the families $(\langle S | X^n \rangle f_i^{*n})_{n \in \mathbb{N}}$ are summable, we denote $S(f_i)$ their sums (this definition is coherent with the preceding when S is a polynomial). One has, for the convolution product,*

$$\varphi S(f_1) = S(f_2) \varphi . \quad (39)$$

Proof. The only delicate part is (ii). First, one remarks that, if φ is a morphism of bialgebras, one has

$$(\varphi \otimes \varphi) \circ \Delta_1^+ = \Delta_2^+ \circ \varphi \quad (40)$$

then, the image by φ of an element of order less than N (i.e. such that $\Delta_1^{+(N)}(x) = 0$) is of order less than N . Let now S be an univariate series

¹¹In fact it is the case for any cocommutative bialgebra, be it generated by its primitive elements or not.

$S = \sum_{k=0}^{\infty} a_k X^k$. For every element x of order less than N and $f \in \text{End}(\mathcal{B})$ such that $f(1_{\mathcal{B}}) = 0$, one has

$$\begin{aligned}
S(f)(x) &= \sum_{k=0}^{\infty} a_k f^{*k}(x) = \sum_{k=0}^{\infty} a_k \mu^{(k-1)} f^{\otimes k} \Delta^{(k-1)}(x) \\
&= \sum_{k=0}^{\infty} a_k \mu^{(k-1)} (f^{\otimes k}) \circ (I_+^{\otimes k}) \Delta^{(k-1)}(x) \\
&= \sum_{k=0}^N a_k \mu^{(k-1)} (f^{\otimes k}) \Delta_+^{(k-1)}(x). \tag{41}
\end{aligned}$$

This proves, in view of (i) that $\varphi \circ S(f_1) = S(f_2) \circ \varphi$.

We reprove (**credits Reutenauer, Patras, Solomon**) that π_1 is a projector $\mathcal{B} \rightarrow \text{Prim}(\mathcal{B})$ by means of the following lemma. \square

Proof. The only delicate part is (ii). First, one remarks that, if φ is a morphism of bialgebras, one has

$$(\varphi \otimes \varphi) \circ \Delta_1^+ = \Delta_2^+ \circ \varphi \tag{42}$$

then, the image by φ of an element of order less than N (i.e. such that $\Delta_1^{+(N)}(x) = 0$) is of order less than N . Let now S be an univariate series $S = \sum_{k=0}^{\infty} a_k X^k$. For every element x of order less than N and $f \in \text{End}(\mathcal{B})$ such that, one has

$$\begin{aligned}
S(f)(x) &= \sum_{k=0}^{\infty} a_k f^{*k}(x) = \sum_{k=0}^{\infty} a_k \mu^{(k-1)} f^{\otimes k} \Delta^{(k-1)}(x) \\
&= \sum_{k=0}^{\infty} a_k \mu^{(k-1)} (f^{\otimes k}) \circ (I_+^{\otimes k}) \Delta^{(k-1)}(x) \\
&= \sum_{k=0}^N a_k \mu^{(k-1)} (f^{\otimes k}) \Delta_+^{(k-1)}(x). \tag{43}
\end{aligned}$$

This proves, in view of (i) that $\varphi \circ S(f_1) = S(f_2) \circ \varphi$. \square

Now, in case \mathcal{B} is cocommutative, the comultiplication Δ is a morphism of bialgebras, so one has

$$\Delta \circ \log_*(I) = \log_*(I \otimes I) \circ \Delta \tag{44}$$

But

$$\begin{aligned}
\log_*(I \otimes I) &= \log_*(I \otimes e) * (e \otimes I) \\
&= \log_*(I \otimes e) + \log_*(e \otimes I) \\
&= \log_*(I) \otimes e + e \otimes \log_*(I)
\end{aligned} \tag{45}$$

Then

$$\Delta(\log_*(I)) = (\log_*(I) \otimes e + e \otimes \log_*(I)) \circ \Delta \tag{46}$$

which implies that $\log_*(I)(\mathcal{B}) \subset \text{Prim}(\mathcal{B})$. To finish to prove that π_1 is a projector onto $\text{Prim}(\mathcal{B})$, one has just to remark that, for $x \in \text{Prim}(\mathcal{B})$ and $n \geq 2$ $(\text{Id}^+)^{*n}(x) = 0$ then

$$\log_*(I)(x) = \text{Id}^+(x) = x . \quad (47)$$

□

Now, we consider

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n \geq 0} \frac{1}{n!} \pi_{1, [\mathcal{A}]}^{*n} . \quad (48)$$

where $\pi_{1, [\mathcal{A}]} = \log_*(I_{\mathcal{A}})$.

Let us prove that the summands form an resolution of unity.

First, one defines $\mathcal{A}_{[n]}$ as the linear span of the powers $\{P^n\}_{P \in \text{Prim}(\mathcal{B})}$ or, equivalently of the symmetrized products

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} P_{\sigma(1)} P_{\sigma(2)} \cdots P_{\sigma(n)} . \quad (49)$$

It is obvious that $\text{Im}(\pi_{1, [\mathcal{A}]}^{*n}) \subset \mathcal{A}_{[n]}$. We remark that

$$\pi_{1, [\mathcal{A}]}^{*n} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1, [\mathcal{A}]}^{\otimes n} \Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1, [\mathcal{A}]}^{\otimes n} I_+^{\otimes n} \Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1, [\mathcal{A}]}^{\otimes n} \Delta_+^{(n-1)} \quad (50)$$

as $\pi_{1, [\mathcal{A}]} I_+ = \pi_{1, [\mathcal{A}]}$. Now, let $P \in \text{Prim}(\mathcal{A})$. We compute $\pi_{1, [\mathcal{A}]}^{*n}(P^m)$. Indeed, if $m < n$, one has

$$\pi_{1, [\mathcal{A}]}^{*n}(P^m) = \mu_{\mathcal{B}}^{n-1} \Delta_+^{n-1}(P^m) = 0 . \quad (51)$$

If $n = m$, one has, from (35)

$$\Delta_+^{n-1}(P^n) = n! P^{\otimes n} \quad (52)$$

and hence $\pi_{1, [\mathcal{A}]}^{*n}$ is the identity on $\mathcal{A}_{[n]}$. If $m > n$, the nullity of $\pi_{1, [\mathcal{A}]}^{*n}(P^m)$ is a consequence of the following lemma.

Lemma 2. *Let \mathcal{B} be a bialgebra and P a primitive element of \mathcal{B} . Then*

i) *The series $\log_*(I)$ is summable on each power P^m*

ii) *$\log_*(I)(P^m) = 0$ for $m > 2$*

Proof. i) As $\Delta_+^{*N}(P^m) = 0$ for $N > m$, one has $I_+^{*N}(P^m) = 0$ for these values.

ii) The morphism of AAU $\varphi_P : A[X] \rightarrow \mathcal{B}$, defined by

$$\varphi_P(X) = P \quad (53)$$

is, in fact a morphism of bialgebras and one checks easily that One has just to

$$\begin{array}{ccc}
A[X] & \xrightarrow{\varphi_P} & \mathcal{B} \\
I_{A[X]}^+ \downarrow & & \downarrow I_{\mathcal{B}}^+ \\
A[X] & \xrightarrow{\varphi_P} & \mathcal{B}
\end{array}$$

Figure 3:

check that $\pi_{1,[A[X]]}(X^m) = 0$ for $m > 2$ which is a consequence of the equality

$$\sum_{w \in Y^*} (w \otimes \pi_1(w)) = \log\left(\sum_{w \in Y^*} w \otimes w\right) \quad (54)$$

because, for $Y = X$ one has

$$\begin{aligned}
\log\left(\sum_{w \in Y^*} w \otimes w\right) &= \log\left(\sum_{n \geq 0} X^n \otimes X^n\right) = \\
&= \log\left(\sum_{n \geq 0} \frac{1}{n!} (X \otimes X)^{(\sqcup \otimes \text{conc}) n}\right) = \\
&= \log(\exp(X \otimes X)) = X \otimes X \quad (55)
\end{aligned}$$

this proves that $\pi_{1,[A]}^{*n}(\mathcal{A}_{[m]}) = 0$ for $m \neq n$ and hence the summands of the sum

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n \geq 0} \frac{1}{n!} \pi_{1,[A]}^{*n} \quad (56)$$

are pairwise orthogonal projectors with $\text{Im}(\pi_{1,[A]}^{*n}) = \mathcal{A}_{[n]}$ and then

$$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_{[n]} \quad (57)$$

This decomposition permits to construct σ by

$$\sigma(P^n) = \frac{1}{n!} \Delta_+^{(n-1)}(P^n) \in T_n(\text{Prim}(\mathcal{B})) \quad (58)$$

and $\sigma(1_{\mathcal{B}}) = 1_{T(\text{Prim}(\mathcal{B}))}$. One checks easily that $j_{\mathcal{B}} \circ \tau \circ \sigma = Id_{\mathcal{A}}$. \square

2.2.2 The theorem from the point of view of summability

The bialgebra \mathcal{B} being supposed cocommutative, we discuss the equivalent conditions under which we are in the presence of an enveloping algebra i.e.

$$\mathcal{B} \cong_{A\text{-bialg}} \mathcal{U}(\text{Prim}(\mathcal{B})) \quad (59)$$

from the point of view of the convergence of the series $\log_*(I)^{12}$. These conditions are known as the theorem of Cartier-Quillen-Milnor-Moore (CQMM).

Theorem 1. [4] *Let \mathcal{B} be a A -cocommutative bialgebra (A is a \mathbb{Q} -AAU) and \mathcal{A} , as above, the subalgebra generated by $\text{Prim}(\mathcal{B})$. Then, the following conditions are equivalent :*

i) \mathcal{B} admits an increasing filtration

$$\mathcal{B}_0 = A.1_{\mathcal{B}} \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \cdots$$

compatible with the structures of algebra (i.e. for all $p, q \in \mathbb{N}$, one has $\mathcal{B}_p \mathcal{B}_q \subset \mathcal{B}_{p+q}$) and coalgebra :

$$\forall n \in \mathbb{N}, \quad \Delta(\mathcal{B}_n) \subset \sum_{p+q=n} \mathcal{B}_p \otimes \mathcal{B}_q.$$

ii) $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$ is summable in $\text{End}(\mathcal{B})$.

iii) $\mathcal{B} = \mathcal{A}$.

Proof. We prove

$$(ii) \implies (iii) \implies (i) \implies (ii) \quad (61)$$

(ii) \implies (iii). —

The image of $j_{\mathcal{B}}$ is the subalgebra generated by the primitive elements. Let us prove that, when $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$ is summable, one has $\text{Im}(j_{\mathcal{B}}) = \mathcal{B}$. The series $\log(1 + X)$ is without constant term so, in virtue of (28) and the summability of $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$, one has

$$\exp(\log(e + \text{Id}^+)) = \exp(\log(1 + X))(\text{Id}^+) = 1_{\text{End}(\mathcal{B})} + \text{Id}^+ = e + \text{Id}^+ = I \quad (62)$$

Set $\pi_1 = \log(e + \text{Id}^+)$.

To end this part, let us compute, for $x \in \mathcal{B}$

$$x = \exp(\pi_1)(x) = \left(\sum_{n \geq 0} \frac{1}{n!} \pi_1^{*n} \right)(x) = \left(\sum_{n=0}^N \frac{1}{n!} \mu^{(n-1)} \pi_1^{\otimes n} \right) \Delta^{(n-1)}(x) \quad (63)$$

where N is the first order for which $\Delta^{+(n-1)}(x) = 0$ (as $\pi_1 \circ \text{Id}^+ = \pi_1$). This proves that \mathcal{B} is generated by its primitive elements.

(iii) \implies (i). —

□

¹²In a A -bialgebra, one can always consider the series of endomorphisms

$$\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (I^+)^{*n}. \quad (60)$$

The family $(\frac{(-1)^{n-1}}{n} (I^+)^{*n})_{n \geq 0}$ is summable iff $((I^+)^{*n})_{n \geq 0}$ is (use eq.20).

Remark 1. *i) The equivalence (i) \iff (iii) is the classical CQMM theorem (see [4]). The equivalence with (ii) could be called the “Convolutional CQMM theorem”. The combinatorial aspects of this last one will be the subject of a forthcoming paper [CT, HNM, GHED Nguyen ?]*
ii) When $\text{Prim}(\mathcal{B})$ is free, we have $\mathcal{B} \cong_{k\text{-bialg}} \mathcal{U}(\text{Prim}(\mathcal{B}))$ and \mathcal{B} is an enveloping algebra.

3 Case study : ϕ -deformed stuffle

3.1 Results for the ϕ -deformed stuffle

Let $Y = \{y_i\}_{i \in I}$ be still a totally ordered alphabet and $A\langle Y \rangle$ be equipped with the ϕ -deformed stuffle defined by

- i) for any $w \in Y^*$, $1_{Y^*} \boxplus_{\phi} w = w \boxplus_{\phi} 1_{Y^*} = w$,
- ii) for any $y_i, y_j \in Y$ and $u, v \in Y^*$,

$$y_i u \boxplus_{\phi} y_j v = y_j (y_i u \boxplus_{\phi} v) + y_i (u \boxplus_{\phi} y_j v) + \phi(y_i, y_j) u \boxplus_{\phi} v, \quad (64)$$

where ϕ is an arbitrary mapping

$$\phi : Y \times Y \longrightarrow AY.$$

Definition 2. *Let*

$$\phi : Y \times Y \longrightarrow AY$$

defined by its structure constants

$$(y_i, y_j) \longmapsto \phi(y_i, y_j) = \sum_{k \in I} \gamma_{i,j}^k y_k.$$

Proposition 3. *The recursion (64) defines a unique mapping*

$$\boxplus_{\phi} : Y^* \times Y^* \longrightarrow A\langle Y \rangle.$$

Proof. Let us denote $(Y^* \times Y^*)_{\leq n}$ the set of words $(u, v) \in Y^* \times Y^*$ such that $|u| + |v| \leq n$. We construct a sequence of mappings

$$\boxplus_{\phi \leq n} : (Y^* \times Y^*)_{\leq n} \longrightarrow AY.$$

which satisfy the recursion of eq.64 . For $n = 0$, we have only a preimage and $\boxplus_{\phi \leq 0}(1_{Y^*}) = 1_{Y^*} \otimes 1_{Y^*}$. Suppose $\boxplus_{\phi \leq n}$ constructed and let $(u, v) \in (Y^* \times Y^*)_{\leq n+1} \setminus (Y^* \times Y^*)_{\leq n}$, i.e. $|u| + |v| = n + 1$. One has three cases : $u = 1_{Y^*}$, $v = 1_{Y^*}$ and $(u, v) \in Y^+ \times Y^+$. For the two first, one uses the initialisation of the recursion thus

$$\boxplus_{\phi \leq n+1}(w, 1_{Y^*}) = \boxplus_{\phi \leq n+1}(1_{Y^*}, w) = w$$

for the last case, write $u = y_i u'$, $v = y_j v'$ and use, to get

$$\mathbf{\Xi}_{\phi \leq n+1}(y_i u', y_j v') = y_i \mathbf{\Xi}_{\phi \leq n}(u', y_j v') + y_j \mathbf{\Xi}_{\phi \leq n}(y_i u', v') + y_{i+j} \mathbf{\Xi}_{\phi \leq n}(u', v')$$

this proves the existence of the sequence $(\mathbf{\Xi}_{\phi \leq n})_{n \geq 0}$. Every $\mathbf{\Xi}_{\phi \leq n+1}$ extends the preceding so there is a mapping

$$\mathbf{\Xi}_{\phi} : Y^* \times Y^* \longrightarrow A(Y).$$

which extends all the $\mathbf{\Xi}_{\phi \leq n+1}$ (the graph of which is the union of the graphs of the $\mathbf{\Xi}_{\phi \leq n}$). This proves the existence. For unicity, just remark that, if there were two mappings $\mathbf{\Xi}_{\phi}$, $\mathbf{\Xi}'_{\phi}$, the fact that they must fulfill the recursion (64) implies that $\mathbf{\Xi}_{\phi} = \mathbf{\Xi}'_{\phi}$. \square

We still denote ϕ and $\mathbf{\Xi}_{\phi}$ the linear extension of ϕ and $\mathbf{\Xi}_{\phi}$ to $AY \otimes AY$ and $A(Y) \otimes A(Y)$ respectively.

Then $\mathbf{\Xi}_{\phi}$ is a law of algebra (with 1_{Y^*} as unit) on $A(Y)$.

Lemma 3. *Let Δ be the morphism $A(Y) \rightarrow A(\langle Y^* \otimes Y^* \rangle)$ defined on the letters by*

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{n,m \in I} \gamma_{n,m}^s y_n \otimes y_m \quad (65)$$

Then

i) for all $w \in Y^+$ we have

$$\Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v \in Y^+} \langle \Delta(w) \mid u \otimes v \rangle u \otimes v \quad (66)$$

ii) for all $u, v, w \in Y^*$, one has

$$\langle u \mathbf{\Xi}_{\phi} v \mid w \rangle = \langle u \otimes v \mid \Delta(w) \rangle^{\otimes 2} \quad (67)$$

Proof. i) By recurrence on $|w|$. If $w = y_s$ is of length one, it is obvious from the definition. If $w = y_s w'$, we have, from the fact that Δ is a morphism

$$\begin{aligned} \Delta(w) &= \left(y_s \otimes 1 + 1 \otimes w + \sum_{i,j \in I} \gamma_{i,j}^s y_i \otimes y_j \right) \\ &\quad \left(w' \otimes 1 + 1 \otimes w' + \sum_{u,v \in Y^+} \langle u \otimes v \mid \Delta(w') \rangle \right) \end{aligned} \quad (68)$$

the development of which proves that $\Delta(w)$ is of the desired form.

ii) Let $S(u, v) := \sum_{w \in Y^*} \langle u \otimes v \mid \Delta(w) \rangle w$. It is easy to check (and left to the reader) that, for all $u \in Y^*$, $S(u, 1) = S(1, u) = u$. Let us now prove that, for all $y_i, y_j \in Y$ and $u, v \in Y^*$

$$S(y_i u, y_j v) = y_i S(u, y_j v) + y_j S(y_i u, v) + \phi(y_i, y_j) S(u, v) \quad (69)$$

Indeed, remarking that $\Delta(1) = 1 \otimes 1$, one has

$$\begin{aligned}
S(y_i u, y_j v) &= \sum_{w \in Y^*} \langle y_i u \otimes y_j v \mid \Delta(w) \rangle w = \sum_{w \in Y^+} \langle y_i u \otimes y_j v \mid \Delta(w) \rangle w \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \Delta(y_s w') \rangle y_s w' \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \left(y_s \otimes 1 + 1 \otimes y_s + \sum_{n, m \in I} \gamma_{n, m}^s y_n \otimes y_m \right) \Delta(w') \rangle y_s w' \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid (y_s \otimes 1) \Delta(w') \rangle y_s w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid (1 \otimes y_s) \Delta(w') \rangle y_s w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \left(\sum_{n, m \in I} \gamma_{n, m}^s y_n \otimes y_m \right) \Delta(w') \rangle y_s w' \\
&= \sum_{w' \in Y^*} \langle u \otimes y_j v \mid \Delta(w') \rangle y_i w' + \sum_{w' \in Y^*} \langle y_i u \otimes v \mid \Delta(w') \rangle y_j w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle u \otimes v \mid \gamma_{i, j}^s \Delta(w') \rangle y_s w' \\
&= y_i \sum_{w' \in Y^*} \langle u \otimes y_j v \mid \Delta(w') \rangle w' + y_j \sum_{w' \in Y^*} \langle y_i u \otimes v \mid \Delta(w') \rangle w' \\
&\quad + \sum_{y_s \in Y} \gamma_{i, j}^s y_s \sum_{w' \in Y^*} \langle u \otimes v \mid \Delta(w') \rangle w' \\
&= y_i S(u, y_j v) + y_j S(y_i u, v) + \phi(y_i, y_j) S(u, v)
\end{aligned}$$

then the computation of S shows that, for all $u, v \in Y^*$, $S(u, v) = u \boxplus_{\phi} v$ as S is bilinear, one has $S = \boxplus_{\phi}$. \square

Theorem 2. *i) The law \boxplus_{ϕ} is commutative if and only if the extension*

$$\phi : AY \otimes AY \longrightarrow AY$$

is so.

ii) The law \boxplus_{ϕ} is associative if and only if the extension

$$\phi : AY \otimes AY \longrightarrow AY$$

is so.

iii) Let $\gamma_{x, y}^z := \langle \phi(x, y) \mid z \rangle$ be the structure constants of ϕ (w.r.t. the basis Y), then \boxplus_{ϕ} is dualizable if and only if $(\gamma_{x, y}^z)_{x, y, z \in X}$ is of finite decomposition

type¹³ in its superscript in the following sense

$$(\forall z \in X)(\#\{(x, y) \in X^2 | \gamma_{x,y}^z \neq 0\} < +\infty). \quad (70)$$

Proof. (i) First, let us suppose that ϕ be commutative and consider T , the twist, i.e. the operator in $A\langle\langle Y^* \otimes Y^* \rangle\rangle$ defined by

$$\langle T(S) | u \otimes v \rangle = \langle S | v \otimes u \rangle \quad (71)$$

it is left to the reader to prove that T is a morphism of algebras. If ϕ is commutative, then so is the following diagram.

$$\begin{array}{ccc} Y & \xrightarrow{\Delta_{\boxtimes\phi}} & A\langle\langle Y^* \otimes Y^* \rangle\rangle \\ & \searrow_{\Delta_{\boxtimes\phi}} & \downarrow T \\ & & A\langle\langle Y^* \otimes Y^* \rangle\rangle \end{array}$$

and, then, the two morphisms $\Delta_{\boxtimes\phi}$ and $T \circ \Delta_{\boxtimes\phi}$ coincide on the generators Y of the algebra $A\langle Y \rangle$ and hence over $A\langle Y \rangle$ itself. Now for all $u, v, w \in Y^*$, one has

$$\begin{aligned} \langle v \boxtimes_{\phi} u | w \rangle &= \langle v \otimes u | \Delta_{\boxtimes\phi}(w) \rangle = \langle u \otimes v | T \circ \Delta_{\boxtimes\phi}(w) \rangle = \\ \langle u \otimes v | \Delta_{\boxtimes\phi}(w) \rangle &= \langle u \boxtimes_{\phi} v | w \rangle \end{aligned} \quad (72)$$

which proves that $v \boxtimes_{\phi} u = u \boxtimes_{\phi} v$. Conversely, if \boxtimes_{ϕ} is commutative, one has, for $i, j \in I$

$$\phi(y_j, y_i) = y_j \boxtimes_{\phi} y_i - (y_j \sqcup y_i) = y_i \boxtimes_{\phi} y_j - (y_i \sqcup y_j) = \phi(y_i, y_j) \quad (73)$$

(ii) Likewise, if ϕ is associative, let us define the operators

$$\overline{\Delta_{\boxtimes\phi} \otimes I} : A\langle\langle Y^* \otimes Y^* \rangle\rangle \rightarrow A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle \quad (74)$$

by

$$\langle \overline{\Delta_{\boxtimes\phi} \otimes I}(S) | u \otimes v \otimes w \rangle = \langle S | (u \boxtimes_{\phi} v) \otimes w \rangle \quad (75)$$

and, similarly,

$$\overline{I \otimes \Delta_{\boxtimes\phi}} : A\langle\langle Y^* \otimes Y^* \rangle\rangle \rightarrow A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle \quad (76)$$

by

$$\langle \overline{I \otimes \Delta_{\boxtimes\phi}}(S) | u \otimes v \otimes w \rangle = \langle S | u \otimes (v \boxtimes_{\phi} w) \rangle \quad (77)$$

it is easy to check by direct calculation that they are well defined morphisms and that the following diagram

¹³One can prove that, in case Y is a semigroup, the associated ϕ is fulfills eq.70 iff Y fulfills “condition D” of Bourbaki (see [3])

$$\begin{array}{ccc}
Y & \xrightarrow{\Delta_{\mathbf{u}_\phi}} & A\langle\langle Y^* \otimes Y^* \rangle\rangle \\
\Delta_{\mathbf{u}_\phi} \downarrow & & \downarrow \overline{I \otimes \Delta_{\mathbf{u}_\phi}} \\
A\langle\langle Y^* \otimes Y^* \rangle\rangle & \xrightarrow{\overline{\Delta_{\mathbf{u}_\phi} \otimes I}} & A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle
\end{array}$$

is commutative. This proves that the two composite morphisms

$$\overline{\Delta_{\mathbf{u}_\phi} \otimes I} \circ \Delta_{\mathbf{u}_\phi}$$

and

$$\overline{I \otimes \Delta_{\mathbf{u}_\phi}} \circ \Delta_{\mathbf{u}_\phi}$$

coincide on Y and then on $A\langle Y \rangle$. Now, for $u, v, w, t \in Y^*$, one has

$$\begin{aligned}
\langle (u \mathbf{u}_\phi v) \mathbf{u}_\phi w \mid t \rangle &= \langle (u \mathbf{u}_\phi v) \otimes w \mid \Delta_{\mathbf{u}_\phi}(t) \rangle = \langle u \otimes v \otimes w \mid (\overline{\Delta_{\mathbf{u}_\phi} \otimes I}) \Delta_{\mathbf{u}_\phi}(t) \rangle = \\
&= \langle u \otimes v \otimes w \mid (\overline{I \otimes \Delta_{\mathbf{u}_\phi}}) \Delta_{\mathbf{u}_\phi}(t) \rangle = \langle u \otimes (v \mathbf{u}_\phi w) \mid \Delta_{\mathbf{u}_\phi}(t) \rangle = \langle u \mathbf{u}_\phi (v \mathbf{u}_\phi w) \mid t \rangle
\end{aligned}$$

which proves the associativity of the law \mathbf{u}_ϕ . Conversely, if \mathbf{u}_ϕ is associative, the direct expansion of the right hand side of

$$0 = (y_i \mathbf{u}_\phi y_j) \mathbf{u}_\phi y_k - y_i \mathbf{u}_\phi (y_j \mathbf{u}_\phi y_k) \quad (78)$$

proves the associativity of ϕ .

iii) We suppose that $(\gamma_{x,y}^z)_{x,y,z \in X}$ is of finite decomposition type in its superscript, in this case $\Delta_{\mathbf{u}_\phi}$ takes its values in $A\langle Y \rangle \otimes A\langle Y \rangle$ therefore its dual, the law \mathbf{u}_ϕ is dualizable. Conversely, if $\text{Im}(\Delta_{\mathbf{u}_\phi}) \subset A\langle Y \rangle \otimes A\langle Y \rangle$, one has, for every $s \in I$

$$\sum_{n,m \in I} \gamma_{n,m}^s y_n \otimes y_m = \Delta(y_s) - (y_s \otimes 1 + 1 \otimes y_s) \in A\langle Y \rangle \otimes A\langle Y \rangle$$

which proves the claim. \square

Theorem 3. *Let A be a commutative ring with unit and $\phi : AY \otimes AY \rightarrow AY$ be an associative and commutative law (of algebra) on AY . Then if ϕ is dualizable¹⁴, let $\Delta_{\mathbf{u}_\phi} : A\langle Y \rangle \rightarrow A\langle Y \rangle \otimes A\langle Y \rangle$ denote its dual comultiplication, then*

a) $\mathcal{B}_\phi = (A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\mathbf{u}_\phi}, \varepsilon)$ is a bialgebra.

¹⁴For the pairing defined by

$$\forall x, y \in Y, \quad \langle x \mid y \rangle = \delta_{x,y}$$

b) If A is a field of characteristic 0 then \mathcal{B}_ϕ is an enveloping bialgebra if and only if the algebra AX admits an increasing filtration $\left((AY)_n\right)_{n \in \mathbb{N}}$ with $(AY)_0 = \{0\}$ and compatible with both the multiplication and the comultiplication $\Delta_{\mathfrak{L}_\phi}$ i.e.

$$\begin{aligned} (AY)_p(AY)_q &\subset (AY)_{p+q} \\ \Delta_{\mathfrak{L}_\phi}((AY)_n) &\subset \sum_{p+q=n} (AY)_p \otimes (AY)_q . \end{aligned}$$

Proof. i) All the properties of bialgebra have been checked for

$$\mathcal{B}_\phi = (A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\mathfrak{L}_\phi}, \epsilon)$$

save one : the fact that $\Delta_{\mathfrak{L}_\phi}$ be a morphism for the product. This is a consequence of the fact that, in the general case,

$$\Delta_{\mathfrak{L}_\phi} : A\langle Y \rangle \rightarrow A\langle\langle Y^* \otimes Y^* \rangle\rangle$$

is a morphism of algebras.

ii) Let us suppose first that $\mathcal{B}_\phi = \mathcal{U}G$ is an enveloping algebra. Then, the intersection of the standard increasing filtration with AY i.e.

$$(AY)_n := \text{span}(G^n) \cap AY$$

is compatible with product and coproduct and $(AY)_0 := K.1_{\mathcal{U}G} \cap AY = \{0\}$. Conversely let $\left((AY)_n\right)_{n \in \mathbb{N}}$ be an increasing filtration of AY which fulfils the conditions of the theorem and set

$$(\mathcal{B}_\phi)_n = k.1_{\mathcal{B}_\phi} + \sum_{k \geq 0} \sum_{\substack{p_1 + p_2 + \dots + p_k = n \\ p_i > 0}} \text{span}\left((AY)_{p_1} (AY)_{p_2} \dots (AY)_{p_k}\right) \quad (79)$$

has the properties required for the application of the theorem of Cartier-Milnor-Moore. Hence \mathcal{B}_ϕ is an enveloping algebra. \square

With the co-unit

$$\forall P \in A\langle Y \rangle, \quad \epsilon(P) = \langle P \mid 1_{Y^*} \rangle, \quad (80)$$

and the antipode defined by, for any $w = x_{i_1} \dots x_{i_r} \in Y^*$,

$$a_{\mathfrak{L}_\phi}(y_{i_1} \dots y_{i_r}) = - \sum_{k=1}^{r-1} a_{\mathfrak{L}_\phi}(y_{i_1} \dots y_{i_k}) \mathfrak{L}_\phi y_{i_{k+1}} \dots y_{i_r}, \quad (81)$$

one gets mutually dual Hopf algebras $\mathcal{H}_{\mathfrak{L}_\phi} = (\mathbb{Q}\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\mathfrak{L}_\phi}, \epsilon, a_{\mathfrak{L}_\phi})$ and $\mathcal{H}_{\mathfrak{L}_\phi}^\vee = (\mathbb{Q}\langle Y \rangle, \mathfrak{L}_\phi, 1_{Y^*}, \Delta_{\text{conc}}, \epsilon, a_{\mathfrak{L}_\phi})$.

Lemma 4. Let $\mathcal{P} = \{P \in \mathbb{Q}\langle Y \rangle \mid \Delta_{\mathfrak{L}_\phi} P = P \otimes 1 + 1 \otimes P\}$. Then \mathcal{P} is stable by the Lie bracket and by linear combinations.

Proof. Since $\Delta_{\mathfrak{U}_\phi} y_1 = y_1 \otimes 1 + 1 \otimes y_1$ then $\mathcal{P} \neq \emptyset$. Let $P, Q \in \mathcal{P}$ and $p, q \in \mathbb{Q}$. Firstly, since $\Delta_{\mathfrak{U}_\phi}([P, Q]) = \Delta_{\mathfrak{U}_\phi}(PQ - QP) = \Delta_{\mathfrak{U}_\phi}(PQ) - \Delta_{\mathfrak{U}_\phi}(QP)$, $\Delta_{\mathfrak{U}_\phi}(QP) = (\Delta_{\mathfrak{U}_\phi} Q)(\Delta_{\mathfrak{U}_\phi} P)$ and $\Delta_{\mathfrak{U}_\phi}(PQ) = (\Delta_{\mathfrak{U}_\phi} P)(\Delta_{\mathfrak{U}_\phi} Q)$ then one has $\Delta_{\mathfrak{U}_\phi}([P, Q]) = \Delta_{\mathfrak{U}_\phi}([P, Q]) \otimes 1 + 1 \otimes \Delta_{\mathfrak{U}_\phi}([P, Q])$ meaning that $[P, Q] \in \mathcal{P}$ and \mathcal{P} is stable by the Lie bracket. Secondly, $\Delta_{\mathfrak{U}_\phi}(pP + qQ) = p\Delta_{\mathfrak{U}_\phi} P + q\Delta_{\mathfrak{U}_\phi} Q = p(P \otimes 1 + 1 \otimes P) + q(Q \otimes 1 + 1 \otimes Q) = (pP + qQ) \otimes 1 + 1 \otimes (pP + qQ)$ meaning that $pP + qQ \in \mathcal{P}$ and \mathcal{P} is then stable by linear combinations. \square

Lemma 5. *With the notations of Lemma 4, then $Y \subset \mathcal{P} \iff \mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle = \mathcal{P}$.*

Proof. Since $\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$ is the smallest algebra containing Y and it is stable by the Lie bracket and by linear combinations then, by Lemma 4, $\mathcal{P} \subset \mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$. If $\mathcal{P} = \mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$ then $Y \subset \mathcal{P}$. Conversely, if $Y \subset \mathcal{P} \subset \mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$ then, by definition, $\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle \subset \mathcal{P}$. \square

Proposition 4. *With the notations of Lemma 4, then $\mathcal{P} \subsetneq \mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$.*

Proof. For any $k \geq 2$ the letter y_k is a Lie polynomial but, by (??), one has $\Delta_{\mathfrak{U}_\phi} y_k \neq y_k \otimes 1 + 1 \otimes y_k$. Thus, by Lemma 5, it follows the conclusion. \square

Lemma 6 (Friedrichs criterium). *Let $S \in \mathbb{Q}\langle\langle Y \rangle\rangle$ such that $\langle S | 1_{Y^*} \rangle = 1$. Then, for the co-product $\Delta_{\mathfrak{U}_\phi}$,*

i) S is primitive, i.e. $\Delta_{\mathfrak{U}_\phi} S = S \otimes 1 + 1 \otimes S$, if and only if,

$$\forall u, v \in Y^+, \quad \langle S | u \mathfrak{U}_\phi v \rangle = 0.$$

ii) S is group-like, i.e. $\Delta_{\mathfrak{U}_\phi} S = S \otimes S$, if and only if,

$$\forall u, v \in Y^+, \quad \langle S | u \mathfrak{U}_\phi v \rangle = \langle S | u \rangle \langle S | v \rangle.$$

Proof. The expected equivalence is due respectively to the following facts

$$\begin{aligned} \Delta_{\mathfrak{U}_\phi} S &= S \otimes 1 + 1 \otimes S - \langle S | 1_{Y^*} \otimes 1_{Y^*} \rangle 1 \otimes 1 + \sum_{u, v \in Y^+} \langle S | u \mathfrak{U}_\phi v \rangle u \otimes v, \\ \Delta_{\mathfrak{U}_\phi} S &= \sum_{u, v \in Y^*} \langle S | u \mathfrak{U}_\phi v \rangle u \otimes v \quad \text{and} \quad S \otimes S = \sum_{u, v \in Y^*} \langle S | u \rangle \langle S | v \rangle u \otimes v. \end{aligned}$$

\square

Lemma 7. *Let $S \in \mathbb{Q}\langle\langle Y \rangle\rangle$ such that $\langle S | 1_{Y^*} \rangle = 1$. Then, for the co-product $\Delta_{\mathfrak{U}_\phi}$, S is group-like if and only if¹⁵ $\log S$ is primitive.*

¹⁵For any $S \in \mathbb{Q}\langle Y \rangle \hat{\otimes} \mathbb{Q}\langle Y \rangle$, if $\langle S | 1_{Y^*} \otimes 1_{Y^*} \rangle = 0$ then one defines

$$\log(1 + S) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} S^n \quad \text{and} \quad \exp(S) = \sum_{n \geq 1} \frac{S^n}{n!}.$$

and one has usual formulas $\log(\exp(S)) = S$ and $\exp(\log(1 + S)) = 1 + S$.

Proof. Since Δ_{\boxplus_ϕ} and the maps $T \mapsto T \otimes 1, T \mapsto 1 \otimes T$ are continuous homomorphisms then if $\log S$ is primitive then $\Delta_{\boxplus_\phi}(\log S) = \log S \otimes 1 + 1 \otimes \log S$ (see Lemma 6(90)) and since $\log S \otimes 1$ and $1 \otimes \log S$ commute then we get successively

$$\begin{aligned} \Delta_{\boxplus_\phi} S &= \Delta_{\boxplus_\phi}(\exp(\log S)) \\ &= \exp(\Delta_{\boxplus_\phi}(\log S)) \\ &= \exp(\log S \otimes 1) \exp(1 \otimes \log S) \\ &= (\exp(\log S) \otimes 1)(1 \otimes \exp(\log S)) \\ &= S \otimes S. \end{aligned}$$

This means S is group-like. The reciprocal can be obtained in the same way. \square

Lemma 8. *Let $\{\chi_l\}_{l \in \mathcal{L}ynY}$ be a transcendence basis of $(\mathbb{Q}\langle Y \rangle, \boxplus_\phi)$ and let $\{\chi_w\}_{w \in Y^*}$ be the completed basis of $(\mathbb{Q}\langle Y \rangle, \boxplus_\phi)$ (viewed as a \mathbb{Q} -module) defined by, for any $w = l_1 \dots l_n$ with $l_1, \dots, l_n \in \mathcal{L}ynY$ and $l_1 \geq \dots \geq l_n$,*

$$\chi_w = \chi_{l_1} \boxplus_\phi \dots \boxplus_\phi \chi_{l_n}$$

in duality with $\{\xi_w\}_{w \in Y^*}$:

$$\forall u, v \in Y^*, \quad \langle \chi_u \mid \xi_v \rangle = \delta_{u,v}.$$

Then, for any $w = l_1 \dots l_n$ with $l_1, \dots, l_n \in \mathcal{L}ynY$ and $l_1 \geq \dots \geq l_n$, one has

$$\langle \chi_w \mid 1_{Y^*} \rangle = \langle \xi_w \mid 1_{Y^*} \rangle = 0 \quad \text{and} \quad \langle \chi_w \mid \chi_{l_1} \dots \chi_{l_n} \rangle = \langle \xi_w \mid \chi_{l_1} \dots \chi_{l_n} \rangle = 1.$$

Proof. It is immediate by construction and by duality. \square

Lemma 9. *With the notations of Lemma 8, if $w \notin \mathcal{L}ynY$ and if¹⁶*

$$\forall l \in \mathcal{L}ynY, \quad \langle \chi_{l_1} \boxplus_\phi \dots \boxplus_\phi \chi_{l_n} \mid \xi_l \rangle = 0$$

then the polynomials $\{\xi_l\}_{l \in \mathcal{L}ynY}$ are primitive, for the co-product Δ_{\boxplus_ϕ} .

Proof. We have

$$\begin{aligned} \Delta_{\boxplus_\phi} \xi_l &= \sum_{u \in Y^+} \langle u \boxplus_\phi 1_{Y^*} \mid \xi_l \rangle u \otimes 1 + \sum_{v \in Y^+} \langle 1_{Y^*} \boxplus_\phi v \mid \xi_l \rangle 1 \otimes v \\ &+ \sum_{u, v \in Y^+} \langle u \boxplus_\phi v \mid \xi_l \rangle u \otimes v + \langle 1_{Y^*} \boxplus_\phi 1_{Y^*} \mid \xi_l \rangle 1 \otimes 1 \\ &= \xi_l \otimes 1 + 1 \otimes \xi_l. \end{aligned}$$

Because, after decomposing the words u and v on the transcendence basis $\{\chi_l\}_{l \in \mathcal{L}ynY}$ and by assumption, the third sum is vanishing. The last one is also vanishing since the ξ_l 's are proper (see Lemma 6). \square

Lemma 10. *Let S_1, \dots, S_n be proper formal power series in $\mathbb{Q}\langle\langle Y \rangle\rangle$. Let P_1, \dots, P_m be primitive elements in $\mathbb{Q}\langle Y \rangle$, for the co-product Δ_{\boxplus_ϕ} .*

¹⁶This condition can be viewed as an analogous of the generalized Friedrichs' criterion [8].

i) If $n > m$ then $\langle S_1 \boxtimes_{\phi} \dots \boxtimes_{\phi} S_n \mid P_1 \dots P_m \rangle = 0$.

ii) If $n = m$ then

$$\langle S_1 \boxtimes_{\phi} \dots \boxtimes_{\phi} S_n \mid P_1 \dots P_n \rangle = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n \langle S_i \mid P_{\sigma(i)} \rangle.$$

iii) If $n < m$ then, by considering the language \mathcal{M} over $\mathcal{A} = \{P_1, \dots, P_m\}$

$$\mathcal{M} = \{w \in \mathcal{A}^* \mid w = P_{j_1} \dots P_{j_{|w|}}, j_1 < \dots < j_{|w|}, |w| \geq 1\}$$

and the morphism $\mu : \mathbb{Q}\langle \mathcal{A} \rangle \rightarrow \mathbb{Q}\langle Y \rangle$, one has :

$$\langle S_1 \boxtimes_{\phi} \dots \boxtimes_{\phi} S_n \mid P_1 \dots P_m \rangle = \sum_{w_1 \boxtimes_{\phi} \dots \boxtimes_{\phi} w_m = P_1 \dots P_m} \prod_{i=1}^n \langle S_i \mid \mu(w_i) \rangle.$$

Proof. On the one hand, since the P_i 's are primitive then

$$\Delta_{\boxtimes_{\phi}}^{(n-1)}(P_i) = \sum_{p+q=n-1} 1^{\otimes p} \otimes P_i \otimes 1^{\otimes q}.$$

On the other hand, one has $\Delta_{\boxtimes_{\phi}}^{(n-1)}(P_1 \dots P_m) = \Delta_{\boxtimes_{\phi}}^{(n-1)}(P_1) \dots \Delta_{\boxtimes_{\phi}}^{(n-1)}(P_m)$ and $\langle S_1 \boxtimes_{\phi} \dots \boxtimes_{\phi} S_n \mid P_1 \dots P_m \rangle = \langle S_1 \otimes \dots \otimes S_n \mid \Delta_{\boxtimes_{\phi}}^{(n-1)}(P_1 \dots P_m) \rangle$. Hence,

$$\langle S_1 \boxtimes_{\phi} \dots \boxtimes_{\phi} S_n \mid P_1 \dots P_m \rangle = \langle \bigotimes_{i=1}^n S_i \mid \prod_{i=1}^m \sum_{p+q=n-1} 1^{\otimes p} \otimes P_i \otimes 1^{\otimes q} \rangle.$$

i) For $n > m$, by expanding $\Delta_{\boxtimes_{\phi}}^{(n-1)}(P_1) \dots \Delta_{\boxtimes_{\phi}}^{(n-1)}(P_m)$, one obtains a sum of tensors containing at least one factor equal to 1. For $j = 1, \dots, n$, the formal power series S_j is proper and the result follows immediatly.

ii) For $n = m$, since

$$\prod_{i=1}^n \Delta_{\boxtimes_{\phi}}^{(n-1)}(P_i) = \sum_{\sigma \in \mathfrak{S}_n} \bigotimes_{i=1}^n P_{\sigma(i)} + Q,$$

where Q is sum of tensors containing at least one factor equal to 1 and the S_j 's are proper then $\langle S_1 \otimes \dots \otimes S_n \mid Q \rangle = 0$. Thus, the result follows.

iii) For $n < m$, since, for $j = 1, \dots, n$, the power series S_j is proper then the expected result follows by expanding the product

$$\prod_{i=1}^m \Delta_{\boxtimes_{\phi}}^{(n-1)}(P_i) = \prod_{i=1}^m \sum_{p+q=n-1} 1^{\otimes p} \otimes P_i \otimes 1^{\otimes q}..$$

□

Definition 3. Let π_1 and $\tilde{\pi}_1$ be the mutually adjoint degree-preserving linear endomorphisms of $\mathbb{Q}\langle Y \rangle$ given by, for any $w \in Y^+$,

$$\begin{aligned}\pi_1(w) &= w + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in Y^+} \langle w | u_1 \boxplus_\phi \dots \boxplus_\phi u_k \rangle u_1 \dots u_k, \\ \tilde{\pi}_1(w) &= w + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in Y^+} \langle w | u_1 \dots u_k \rangle u_1 \boxplus_\phi \dots \boxplus_\phi u_k.\end{aligned}$$

In particular, for any $y_k \in Y$, the polynomials $\pi_1(y_k)$ and $\tilde{\pi}_1(y_k)$ are given by

$$\pi_1(y_k) = y_k + \sum_{l \geq 2} \frac{(-1)^{l-1}}{l} \sum_{\substack{j_1, \dots, j_l \geq 1 \\ j_1 + \dots + j_l = k}} y_{j_1} \dots y_{j_l} \quad \text{and} \quad \tilde{\pi}_1(y_k) = y_k.$$

Proposition 5. Let \mathcal{D}_Y be the diagonal series over Y :

$$\mathcal{D}_Y = \sum_{w \in Y^*} w \otimes w.$$

Then

$$i) \log \mathcal{D}_Y = \sum_{w \in Y^+} w \otimes \pi_1(w) = \sum_{w \in Y^+} \tilde{\pi}_1(w) \otimes w.$$

ii) For any $w \in Y^*$, we have

$$\begin{aligned}w &= \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \dots, u_k \in Y^*} \langle w | u_1 \boxplus_\phi \dots \boxplus_\phi u_k \rangle \pi_1(u_1) \dots \pi_1(u_k) \\ &= \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \dots, u_k \in Y^*} \langle w | u_1 \dots u_k \rangle \tilde{\pi}_1(u_1) \boxplus_\phi \dots \boxplus_\phi \tilde{\pi}_1(u_k).\end{aligned}$$

Proof. i) Expanding by different ways the logarithm, it follows the results :

$$\begin{aligned}\log \mathcal{D}_Y &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \left(\sum_{w \in Y^+} w \otimes w \right)^k \\ &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in Y^+} (u_1 \boxplus_\phi \dots \boxplus_\phi u_k) \otimes u_1 \dots u_k \\ &= \sum_{w \in Y^+} w \otimes \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in Y^+} \langle w | u_1 \boxplus_\phi \dots \boxplus_\phi u_k \rangle u_1 \dots u_k. \\ \log \mathcal{D}_Y &= \sum_{w \in Y^+} \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in Y^+} \langle w | u_1 \dots u_k \rangle u_1 \boxplus_\phi \dots \boxplus_\phi u_k \otimes w.\end{aligned}$$

ii) Since $\mathcal{D}_Y = \exp(\log(\mathcal{D}_Y))$ then, by the previous results, one has separately,

$$\begin{aligned}
\mathcal{D}_Y &= \sum_{k \geq 0} \frac{1}{k!} \left(\sum_{w \in Y^+} w \otimes \pi_1(w) \right)^k \\
&= \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \dots, u_k \in Y^+} (u_1 \boxtimes_{\phi} \dots \boxtimes_{\phi} u_k) \otimes (\pi_1(u_1) \dots \pi_1(u_k)) \\
&= \sum_{w \in Y^+} w \otimes \sum_{k \geq 1} \frac{1}{k!} \sum_{u_1, \dots, u_k \in Y^+} \langle w \mid u_1 \boxtimes_{\phi} \dots \boxtimes_{\phi} u_k \rangle \pi_1(u_1) \dots \pi_1(u_k). \\
\mathcal{D}_Y &= \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \dots, u_k \in Y^+} (\tilde{\pi}_1(u_1) \boxtimes_{\phi} \dots \boxtimes_{\phi} \tilde{\pi}_1(u_k)) \otimes (u_1 \dots u_k) \\
&= \sum_{w \in Y^+} \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \dots, u_k \in Y^+} \langle w \mid u_1 \dots u_k \rangle \tilde{\pi}_1(u_1) \boxtimes_{\phi} \dots \boxtimes_{\phi} \tilde{\pi}_1(u_k) \otimes w.
\end{aligned}$$

It follows then the expected result. \square

Lemma 11. *For any $w \in Y^+$, one has $\Delta_{\boxtimes_{\phi}} \pi_1(w) = \pi_1(w) \otimes 1 + 1 \otimes \pi_1(w)$.*

Proof. Let α be the alphabet duplication isomorphism defined by

$$\forall \bar{y} \in \bar{Y}, \quad \bar{y} = \alpha(y).$$

Applying the tensor product of algebra isomorphisms $\alpha \otimes \text{Id}$ to the diagonal series \mathcal{D}_Y , we obtain, by Lemma 6, a group-like element and then applying the logarithm of this element (or equivalently, applying $\alpha \otimes \pi_1$ to \mathcal{D}_Y) we obtain \mathcal{S} which is, by Lemma 7, a primitive element :

$$(\alpha \otimes \text{Id})\mathcal{D}_Y = \sum_{w \in Y^*} \alpha(w) w \quad \text{and} \quad \mathcal{S} = (\alpha \otimes \pi_1)\mathcal{D}_Y = \sum_{w \in Y^*} \alpha(w) \pi_1(w).$$

The two members of the identity $\Delta_{\boxtimes_{\phi}} \mathcal{S} = \mathcal{S} \otimes 1 + 1 \otimes \mathcal{S}$ give respectively

$$\sum_{w \in Y^*} \alpha(w) \Delta_{\boxtimes_{\phi}} \pi_1(w) \quad \text{and} \quad \sum_{w \in Y^*} \alpha(w) \pi_1(w) \otimes 1 + \sum_{w \in Y^*} \alpha(w) 1 \otimes \pi_1(w).$$

Since $\{w\}_{w \in \bar{Y}^*}$ as a basis for $\mathbb{Q}\langle \bar{Y} \rangle$ then identifying the coefficients in the previous expressions, we get $\Delta_{\boxtimes_{\phi}} \pi_1(w) = \pi_1(w) \otimes 1 + 1 \otimes \pi_1(w)$ meaning that $\pi_1(w)$ is primitive. \square

3.2 Pair of bases in duality on ϕ -deformed stuffle algebra

Definition 4. *Let $\{\Pi_l\}_{l \in \mathcal{L}_{yn}Y}$ and $\{\Pi_w\}_{w \in Y^*}$ be the families of respectively $\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$ and $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle)$ obtained as follows*

$$\begin{aligned}
\Pi_{y_k} &= \pi_1(y_k) & \text{for } k \geq 1, \\
\Pi_l &= [\Pi_s, \Pi_r] & \text{for } l \in \mathcal{L}_{yn}X, \text{ standard factorization of } l = (s, r), \\
\Pi_w &= \Pi_{l_1}^{i_1} \dots \Pi_{l_k}^{i_k} & \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k, l_1, \dots, l_k \in \mathcal{L}_{yn}Y.
\end{aligned}$$

Let $\{\Sigma_w\}_{w \in Y^*}$ be the family of the quasi-shuffle algebra (viewed as a \mathbb{Q} -module) obtained by duality with $\{\Pi_w\}_{w \in Y^*}$:

$$\forall u, v \in Y^*, \quad \langle \Sigma_v | \Pi_u \rangle = \delta_{u,v}.$$

Proposition 6. Let $\bar{\mathcal{P}} = \{P \in \mathbb{Q}\langle \bar{Y} \rangle \mid \Delta_{\mathbf{u}\phi} P = P \otimes 1 + 1 \otimes P\}$, where $\bar{Y} = \{\pi_1(y_k)\}_{k \geq 1}$. Then¹⁷ $\text{Lie}_{\mathbb{Q}}\langle \bar{Y} \rangle = \bar{\mathcal{P}}$.

Proof. By Lemma 11 and by Lemma 5, we get then the expected result. \square

Proposition 7. i) For $l \in \mathcal{Lyn}Y$, the polynomial Π_l is upper triangular :

$$\Pi_l = l + \sum_{v > w, (v)=(l)} c_v v.$$

ii) The families $\{\Pi_w\}_{w \in Y^*}$ and $\{\Sigma_w\}_{w \in Y^*}$ are upper and lower triangular respectively¹⁸. On other words, for any $w \in Y^+$, by denoting (w) the degree of w with $(y_k) = \deg(y_k) = k$, one has

$$\Pi_w = w + \sum_{v > w, (v)=(w)} c_v v \quad \text{and} \quad \Sigma_w = w + \sum_{v < w, (v)=(w)} d_v v.$$

Proof. i) Let us prove it by induction on the length of l : the result is immediat for $l \in Y$. The result is suppose verified for any $l \in \mathcal{Lyn}Y \cap Y^k$ and $0 \leq k \leq N$. At $N + 1$, by the standard factorization (l_1, l_2) of l , one has $\Pi_l = [\Pi_{l_1}, \Pi_{l_2}]$ and $l_2 l_1 > l_1 l_2 = l$. By induction hypothesis,

$$\begin{aligned} \Pi_{l_1} &= l_1 + \sum_{v > l_1, (v)=(l_1)} c_v v \quad \text{and} \quad \Pi_{l_2} = l_2 + \sum_{u > l_2, (v)=(l_2)} d_u u, \\ &\Rightarrow \Pi_l = l + \sum_{w > l, (w)=(l)} e_w w, \end{aligned}$$

getting e_w 's from c_v 's and d_u 's. Actually, the Lie bracket gives

$$\begin{aligned} \Pi_l &= [l_1, l_2] + \sum_{u > l_2, (v)=(l_2)} d_u l_1 u + \sum_{\substack{v > l_1, (v)=(l_2) \\ u > l_2, (u)=(l_1)}} c_v d_u v u \\ &\quad - \sum_{v > l_1, (v)=(l_1)} c_v l_2 v - \sum_{\substack{v > l_1, (v)=(l_2) \\ u > l_2, (u)=(l_1)}} c_v d_u u v \\ &= [l_1, l_2] + \sum_{u > l_1 l_2, (v)=(l_1 l_2)} d'_u u + \sum_{vu > l_1 l_2, (vu)=(l_1 l_2)} c_v d_u v u \\ &\quad - \sum_{v > l_2 l_1, (v)=(l_2 l_1)} c'_v v - \sum_{uv > l_2 l_1, (uv)=(l_2 l_1)} c_v d_u u v \end{aligned}$$

¹⁷Any $P \in \mathbb{Q}\langle \bar{Y} \rangle$ is a Lie polynomial if and only if P is primitive, for $\Delta_{\mathbf{u}\phi}$.

¹⁸The duality preserves the homogeneous degree and interchanges the triangularity of polynomials (see Note 5 for the same construction of the triangular matrices of coefficients).

$$\begin{aligned}
&= [l_1, l_2] + \sum_{u>l, (v)=(l)} d'_u u + \sum_{vu>l, (vu)=(l)} c_v d_u v u \\
&\quad - \sum_{v>l_2 l_1 >l, (v)=(l)} c'_v v - \sum_{uv>l_2 l_1 >l, (uv)=(l)} c_v d_u u v.
\end{aligned}$$

Hence, the conclusion follows.

ii) Let $w = l_1 \dots l_k$, with $l_1 \geq \dots \geq l_k$ and $l_1, \dots, l_k \in \mathcal{Lyn}Y$. One has

$$\Pi_{l_i} = l_i + \sum_{v>l_i, (v)=(l_i)} c_{i,v} v \quad \text{and} \quad \Pi_w = l_1 \dots l_k + \sum_{u>w, (v)=(w)} d_u u,$$

where the d_u 's are obtained from the $c_{i,v}$'s. Hence, the family $\{\Pi_w\}_{w \in Y^*}$ is upper triangular and, by duality, the family $\{\Sigma_w\}_{w \in Y^*}$ is lower triangular. \square

Theorem 4. *i) The family $\{\Pi_l\}_{l \in \mathcal{Lyn}Y}$ forms a basis of the free Lie algebra.*

ii) The family $\{\Pi_w\}_{w \in Y^}$ forms a basis of the free associative algebra $\mathbb{Q}\langle Y \rangle$.*

iii) The family $\{\Sigma_w\}_{w \in Y^}$ generate freely the quasi-shuffle algebra.*

iv) The family $\{\Sigma_l\}_{l \in \mathcal{Lyn}Y}$ forms a transcendence basis of $(\mathbb{Q}\langle Y \rangle, \mathbf{u}_\phi)$.

Proof. The family $\{\Pi_l\}_{l \in \mathcal{Lyn}Y}$ of primitive upper triangular polynomials is free. By Lemma 4 and then by a theorem of Viennot [9], we get the first result. The second is a direct consequence of the Poincaré-Birkhoff-Witt theorem. By the Cartier-Quillen-Milnor-Moore theorem, we get the third one and the last one is obtained as consequence of the constructions of $\{\Sigma_l\}_{l \in \mathcal{Lyn}Y}$ and $\{\Sigma_w\}_{w \in Y^*}$. \square

Proposition 8. *Let $\pi_Y : (\mathbb{Q} \oplus \mathbb{Q}\langle X \rangle x_1, \cdot) \rightarrow (\mathbb{Q}\langle Y \rangle, \cdot)$ be the morphism mapping $x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1 \in X^* x_1$ to $y_{s_1} \dots y_{s_r} \in Y^*$ and let π_X be its inverse. Then*

i) The homogeneous polynomials $\{\pi_Y P_{\pi_X l}\}_{l \in \mathcal{Lyn}Y}$ are upper triangular and linearly independent¹⁹ and

$$\pi_Y P_{\pi_X l} = \Pi_l + \sum_{v>l, (v)=(l)} p_v v.$$

ii) For any $w \in Y^$, the following homogeneous polynomial*

$$\pi_Y P_{\pi_X w} = \Pi_w + \sum_{v>w, (v)=(w)} c_v v$$

is of degree (w) and the family $\{\pi_Y P_{\pi_X w}\}_{w \in Y^}$ forms a basis for $\mathbb{Q}\langle Y \rangle$.*

¹⁹For any $l \in \mathcal{Lyn}Y$, $P_{\pi_X l}$ and Π_l are primitive but $\pi_Y P_{\pi_X l}$ is not necessarily primitive.. For example, $P_{\pi_Y y_2} = [x_0, x_1]$ and $\Pi_{y_2} = y_2 - \frac{1}{2} y_1^2$ are primitive but $\pi_Y P_{x_0 x_1} = y_2$ is not.

iii) Let $\{\Theta_w\}_{w \in Y^*}$ be the family of homogeneous polynomials in duality with the family $\{\pi_Y P_{\pi_X w}\}_{w \in Y^*}$:

$$\forall u, v \in Y^*, \quad \langle \pi_Y P_{\pi_X u} \mid \Theta_v \rangle = \delta_{u,v}.$$

Then, the family $\{\Theta_w\}_{w \in Y^*}$ generate freely the quasi-shuffle algebra and, for any $w \in Y^*$, Θ_w is upper triangular of degree (w) :

$$\Theta_w = \Sigma_w + \sum_{v < w, (v)=(w)} d_v v.$$

iv) The family $\{\Theta_l\}_{l \in \mathcal{L}ynY}$ does not form a transcendence basis of $(\mathbb{Q}\langle Y \rangle, \mathfrak{L}, \phi)$.

Proof. i) By (11) (resp. by Proposition 7), for any $l \in \mathcal{L}ynX$ (resp. $\mathcal{L}ynY$), one has $\deg(P_l) = |l|$ (resp. $\deg(\Pi_l) = (l)$) and

$$P_l = l + \sum_{v > l, |v|=|l|} a_v v \quad \left(\text{resp.} \quad \Pi_l = l + \sum_{v > l, (v)=(l)} c_v v \right),$$

Hence, for any $l \in \mathcal{L}ynY$, we have $\pi_X l \in \mathcal{L}ynX$ and

$$\begin{aligned} P_{\pi_X l} &= \pi_X \left[\Pi_l - \sum_{v > l, (v)=(l)} c_v v \right] + \sum_{v > \pi_X l, |v|=|l|} a_v v, \\ \Rightarrow \pi_Y P_{\pi_X l} &= \Pi_l + \sum_{u > l, (u)=(l)} (a'_u - c'_u) u. \end{aligned}$$

Hence, we get the expected results by putting $p_u = a'_u - c'_u$, where the coefficients a'_u 's (resp. c'_u 's) are obtained from a_v 's (resp. c_v 's) by completing some nul coefficients when it is necessary and by using the fact

$$\forall w_1, w_2 \in Y^* x_1, \quad w_1 > w_2 \Rightarrow \pi_Y w_1 > \pi_Y w_2.$$

By Proposition 7, the polynomials $\{\Pi_l\}_{l \in \mathcal{L}ynY}$ are upper triangular (see Note 5) and are linearly independent then the $\{\pi_Y P_{\pi_X l}\}_{l \in \mathcal{L}ynY}$ are also.

ii) As in Proposition 7, let $w = l_1 \dots l_k$, with $l_1 > \dots > l_k, l_1, \dots, l_k \in \mathcal{L}ynY$. Firstly, one has $(\pi_X l_1) \dots (\pi_X l_k) = \pi_X w$ and secondly,

$$P_{\pi_X l_i} = \pi_X l_i + \sum_{v > l_i, |v|=|l_i|} c_{i,v} v \quad \text{and} \quad P_{\pi_X w} = \pi_X w + \sum_{u > w, |u|=|w|} d_u u,$$

where the d_u 's are obtained from the $c_{i,v}$'s. Hence, the family $\{P_{\pi_X w}\}_{w \in Y^*}$ is upper triangular. Using the restriction of π_Y , as being morphism from $(\mathbb{Q} \oplus \mathbb{Q}\langle X \rangle x_1, \dots)$ to $(\mathbb{Q}\langle Y \rangle, \dots)$, we get the degree of the upper triangular homogeneous polynomial $\pi_Y P_{\pi_X w}$ as image of Π_w is (see Proposition 7). The family $\{\pi_Y P_{\pi_X w}\}_{w \in Y^*}$ forms then a basis for the free algebra $\mathbb{Q}\langle Y \rangle$.

- iii) It is a consequence of the Cartier-Quillen-Milnor-Moore theorem.
- iv) If $\{\Theta_l\}_{l \in \mathcal{L}ynY}$ constitutes a transcendence basis of $(\mathbb{Q}\langle Y \rangle, \mathfrak{W}_\phi)$ then, for any $l \in \mathcal{L}ynY$, $\pi_Y P_{\pi_X l}$ is primitive but it is false in general (see Note 19). \square

Now, let us clarify the basis $\{\Sigma_w\}_{w \in Y^*}$ and then the transcendence basis $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ of the quasi-shuffle algebra $(\mathbb{Q}\langle Y \rangle, \mathfrak{W}_\phi)$ as follows

Proposition 9. *We have*

- i) For $w = 1_{Y^*}$, $\Sigma_w = 1$.
- ii) For any $w = l_1^{i_1} \dots l_k^{i_k}$, with $l_1, \dots, l_k \in \mathcal{L}ynY$ and $l_1 > \dots > l_k$,

$$\Sigma_w = \frac{\Sigma_{l_1}^{\mathfrak{W}_\phi^{i_1}} \mathfrak{W}_\phi \dots \mathfrak{W}_\phi \Sigma_{l_k}^{\mathfrak{W}_\phi^{i_k}}}{i_1! \dots i_k!}.$$

- iii) For any $y \in Y$, $\Sigma_y = \tilde{\pi}_1(y)$.

Proof. i) Since $\Pi_{1_{Y^*}} = 1$ then $\Sigma_{1_{Y^*}} = 1$.

- ii) Let $u = u_1 \dots u_n = l_1^{i_1} \dots l_k^{i_k}$, $v = v_1 \dots v_m = h_1^{j_1} \dots h_p^{j_p}$ with $l_1, \dots, l_k, h_1, \dots, h_p, u_1, \dots, u_n, v_1, \dots, v_m \in \mathcal{L}ynY$, $l_1 > \dots > l_k, h_1 > \dots > h_p$, $u_1 \geq \dots \geq u_n, v_1 \geq \dots \geq v_m$ and $i_1 + \dots + i_k = n, j_1 + \dots + j_p = m$. Hence, if $m \geq 2$ (resp. $n \geq 2$) then $v \notin \mathcal{L}ynY$ (resp. $u \notin \mathcal{L}ynY$). Since

$$\langle \Sigma_{u_1} \mathfrak{W}_\phi \dots \mathfrak{W}_\phi \Sigma_{u_n} \mid \prod_{i=1}^n \Pi_{u_i} \rangle = \langle \Sigma_{u_1} \otimes \dots \otimes \Sigma_{u_n} \mid \Delta_{\mathfrak{W}_\phi}^{(n-1)}(\Pi_{v_1} \dots \Pi_{v_m}) \rangle$$

then many cases occur :

- (a) Case $n > m$. By Lemma 10(90), $\langle \Sigma_{u_1} \mathfrak{W}_\phi \dots \mathfrak{W}_\phi \Sigma_{u_n} \mid \Pi_{v_1} \dots \Pi_{v_m} \rangle = 0$.
- (b) Case $n = m$. By Lemma 10(87), one has

$$\begin{aligned} \langle \Sigma_{u_1} \mathfrak{W}_\phi \dots \mathfrak{W}_\phi \Sigma_{u_n} \mid \prod_{i=1}^n \Pi_{v_i} \rangle &= \sum_{\sigma \in \Sigma_n} \prod_{i=1}^n \langle \Sigma_{u_i} \mid \Pi_{v_{\sigma(i)}} \rangle \\ &= \sum_{\sigma \in \Sigma_n} \prod_{i=1}^n \delta_{\Sigma_{u_i}, \Pi_{v_{\sigma(i)}}}. \end{aligned}$$

Thus, if $u \neq v$ then $(u_1, \dots, u_n) \neq (v_1, \dots, v_n)$ then the second member is vanishing else, *i.e.* $u = v$, the second member equals 1 because the factorization by Lyndon words is unique.

(c) Case $n < m$. By Lemma 10(iii), let us consider the following language over the alphabet $\mathcal{A} = \{\Pi_{v_1}, \dots, \Pi_{v_m}\}$:

$$\mathcal{M} = \{w \in \mathcal{A}^* \mid w = \Pi_{v_{j_1}} \dots \Pi_{v_{j_{|w|}}}, j_1 < \dots < j_{|w|}, |w| \geq 1\},$$

and the morphism $\mu : \mathbb{Q}\langle \mathcal{A} \rangle \rightarrow \mathbb{Q}\langle Y \rangle$. We get :

$$\langle \Sigma_{u_1} \mathbf{u} \phi \dots \mathbf{u} \phi \Sigma_{u_n} \mid \prod_{i=1}^n \Pi_{u_i} \rangle = \sum_{\substack{w_1 \mathbf{u} \phi \dots \mathbf{u} \phi w_m = P_1 \dots P_m \\ w_1, \dots, w_m \in \mathcal{M}}} \prod_{i=1}^n \langle \Sigma_{u_i} \mid \mu(w_i) \rangle = 0.$$

Because in the side, on the one hand, there is at least one w_i , $|w_i| \geq 2$, corresponding to $\mu(w_i) = \Pi_{v_{j_1}} \dots \Pi_{v_{j_{|w_i|}}}$ such that $v_{j_1} \geq \dots \geq v_{j_{|w_i|}}$ and on the other hand, $\nu_i := v_{j_1} \dots v_{j_{|w_i|}} \notin \mathcal{L}ynY$ and $u_i \in \mathcal{L}ynY$.

By consequent,

$$\langle \Sigma_u \mid \Pi_v \rangle = \left\langle \frac{\Sigma_{l_1} \mathbf{l} \phi^{i_1} \mathbf{l} \phi \dots \mathbf{l} \phi \Sigma_{l_k} \mathbf{l} \phi^{i_k}}{i_1! \dots i_k!} \mid \Pi_{h_1}^{j_1} \dots \Pi_{h_p}^{j_p} \right\rangle = \delta_{u,v}.$$

iii) For any $l \in Y$, $\Pi_l = \pi_1(l)$, $\Sigma_l = \tilde{\pi}_1(l)$ and $\pi_1, \tilde{\pi}_1$ are mutually adjoint. \square

Proposition 10. *i) For $w \in Y^+$, the polynomial Σ_w is proper and homogeneous of degree (w) , for $\deg(y_i) = i$, and of rational positive coefficients.*

$$ii) \mathcal{D}_Y = \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l \in \mathcal{L}ynY} \exp(\Sigma_l \otimes \Pi_l).$$

iii) *The family $\mathcal{L}ynY$ forms a transcendence basis²⁰ of the quasi-shuffle algebra and the family of proper polynomials of rational positive coefficients defined by, for any $w = l_1^{i_1} \dots l_k^{i_k}$ with $l_1 > \dots > l_k$ and $l_1, \dots, l_k \in \mathcal{L}ynY$,*

$$\chi_w = \frac{l_1 \mathbf{l} \phi^{i_1} \mathbf{l} \phi \dots \mathbf{l} \phi l_k \mathbf{l} \phi^{i_k}}{i_1! \dots i_k!}$$

forms a basis of the quasi-shuffle algebra.

iv) *Let $\{\xi_w\}_{w \in Y^*}$ be the basis of the envelopping algebra $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle)$ obtained by duality with the basis $\{\chi_w\}_{w \in Y^*}$:*

$$\forall u, v \in Y^*, \quad \langle \chi_v \mid \xi_u \rangle = \delta_{u,v}.$$

Then the family $\{\xi_l\}_{l \in \mathcal{L}ynY}$ forms a basis of the free Lie algebra $\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$.

²⁰This result is an analogous of a Radford theorem (see [8]). Thus the bases $\mathcal{L}ynY$ and $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ belong to the class of Radford bases, *i.e.* the class of transcendence bases, of the quasi-shuffle algebra, as well as the bases $\mathcal{L}ynX$ and $\{P_l\}_{l \in \mathcal{L}ynX}$ belong to the class of Radford bases of the shuffle algebra.

Proof. i) The proof can be done by induction on the length of w using the fact that the product $\mathbf{\bowtie}_\phi$ conserve the property, l'homogeneity and rational positivity of the coefficients.

ii) Expressing w in the basis $\{\Sigma_w\}_{w \in Y^*}$ of the quasi-shuffle algebra and then in the basis $\{\Pi_w\}_{w \in Y^*}$ of the envelopping algebra, we obtain successively

$$\begin{aligned}
\mathcal{D}_Y &= \sum_{w \in Y^*} \left(\sum_{u \in Y^*} \langle \Pi_u \mid w \rangle \Sigma_u \right) \otimes w \\
&= \sum_{u \in Y^*} \Sigma_u \otimes \left(\sum_{w \in Y^*} \langle \Pi_u \mid w \rangle w \right) \\
&= \sum_{u \in Y^*} \Sigma_u \otimes \Pi_u \\
&= \sum_{\substack{l_1 > \dots > l_k \\ i_1, \dots, i_k \geq 1}} \frac{\Sigma_{l_1}^{\mathbf{\bowtie}_\phi^{i_1}} \mathbf{\bowtie}_\phi \dots \mathbf{\bowtie}_\phi \Sigma_{l_k}^{\mathbf{\bowtie}_\phi^{i_k}}}{i_1! \dots i_k!} \otimes \Pi_{l_1}^{i_1} \dots \Pi_{l_k}^{i_k} \\
&= \prod_{l \in \mathcal{Lyn}Y} \sum_{i \geq 0} \frac{\Sigma_l^{\mathbf{\bowtie}_\phi^i}}{i!} \otimes \Pi_l^i \\
&= \prod_{l \in \mathcal{Lyn}Y} \exp(\Sigma_l \otimes \Pi_l).
\end{aligned}$$

iii) For $w = l_1^{i_1} \dots l_k^{i_k}$ with $l_1, \dots, l_k \in \mathcal{Lyn}Y$ and $l_1 > \dots > l_k$, by Proposition 7, the proper polynomial of positive coefficients Σ_w is lower triangular :

$$\Sigma_w = \frac{\Sigma_{l_1}^{\mathbf{\bowtie}_\phi^{i_1}} \mathbf{\bowtie}_\phi \dots \mathbf{\bowtie}_\phi \Sigma_{l_k}^{\mathbf{\bowtie}_\phi^{i_k}}}{i_1! \dots i_k!} = w + \sum_{v < w, (v)=(w)} c_v v.$$

In particular, for any $l_j \in \mathcal{Lyn}Y$, Σ_{l_j} is lower triangular :

$$\Sigma_{l_j} = l_j + \sum_{v < l_j, (v)=(l_j)} c_v v.$$

Hence, $\Sigma_w = \chi_w + \chi'_w$, where χ'_w is a proper polynomial of $\mathbb{Q}\langle Y \rangle$ of rational positive coefficients. We deduce then the support of χ_w contains words which are less than w and $\langle \chi_w \mid w \rangle = 1$. Thus, the proper polynomial χ_w of rational positive coefficients is lower triangular :

$$\begin{aligned}
\chi_w &= w + \sum_{v < w, (v)=(w)} c_v v, \\
\Rightarrow \forall l \in \mathcal{Lyn}Y, \chi_l &= l + \sum_{v < l, (v)=(l)} c_v v.
\end{aligned}$$

It follows then expected results.

iv) By duality, for $w \in Y^*$, the proper polynomial ξ_w is upper triangular. In particular, for any $l \in \mathcal{Lyn}Y$, the proper polynomial ξ_l is upper triangular :

$$\xi_l = l + \sum_{v>l, (v)=(l)} d_v v.$$

Hence, the family $\{\xi_l\}_{l \in \mathcal{Lyn}Y}$ is free and its elements verify an analogous of the generalized criterion of Friedrichs :

- for $w \in \mathcal{Lyn}Y$, one has $\langle \chi_w \mid \xi_l \rangle = \delta_{w,l}$,
- for $w = l_1 \dots l_n \notin \mathcal{Lyn}Y$ with $l_1, \dots, l_n \in \mathcal{Lyn}Y$ and $l_1 \geq \dots \geq l_n$, one has $\langle \chi_{l_1 \sqcup \phi \dots \sqcup \phi l_n} \mid \xi_l \rangle = \langle \chi_w \mid \xi_l \rangle = 0$ (since $l \in \mathcal{Lyn}Y$).

Moreover, by Lemma 9, the polynomials ξ_l 's are primitive.. By Lemma 4 and then by a Viennot's theorem [9], we get the expected result. \square

3.3 Particular case : q -deformation of the shuffle product

In the sequel, we focus on the particular case where

$$\forall y_i, y_j \in Y, \quad \phi(y_i, y_j) = qy_{i+j}.$$

We call a standard sequence of Lyndon words to be a sequence

$$S = (\ell_1, \dots, \ell_k), k \geq 1 \tag{82}$$

if for all i , either ℓ_i to be a letter or the standard factorization $\sigma(\ell_i) = (\ell'_i, \ell''_i)$ and $\ell''_i \geq \ell_{i+1}, \dots, \ell_n$. Note that a decreasing sequence of Lyndon words is also a standard sequence. A rise of a sequence S is an index i such that $\ell_i < \ell_{i+1}$. A legal rise of sequence S is a rise of i such that $\ell_{i+1} \geq \ell_{i+2}, \dots, \ell_k$; with the legal rise i , we define

$$\lambda_i(S) = (\ell_1, \dots, \ell_{i-1}, \ell_i \ell_{i+1}, \ell_{i+2}, \dots, \ell_n), \tag{83}$$

$$\rho_i(S) = (\ell_1, \dots, \ell_{i-1}, \ell_{i+1}, \ell_i, \ell_{i+2}, \dots, \ell_n) \tag{84}$$

We denote $S \Rightarrow T$ if $T = \lambda_i(S)$ or $T = \rho_i(S)$ for some legal rise i ; and $S \xrightarrow{*} T$, transitive closure of \Rightarrow .

A derivation tree $\mathcal{T}(S)$ of S to be a labelled rooted tree with the following properties: if S is decreasing, then $\mathcal{T}(S)$ is reduced to its root, labelled S ; if not, $\mathcal{T}(S)$ is the tree with root labelled S , with left and right immediate subtree $\mathcal{T}(S')$ and $\mathcal{T}(S'')$, where $S' = \lambda_i(S)$, $S'' = \rho_i(S)$ for some legal rise i of S ; we define $\Pi(S) = \Pi_{\ell_1} \dots \Pi_{\ell_n}$ (note that, $\Pi(S) \neq \Pi_{\ell_1 \dots \ell_k}$ because ℓ_1, \dots, ℓ_k can be not a decreasing sequence).

Conversely, we call a fall of sequence S is an index i such that $\ell_1, \dots, \ell_i \in Y, \ell_i > \ell_{i+1}$. We define

$$\rho_i^{-1}(S) = (\ell_1, \dots, \ell_{i+1}, \ell_i, \dots, \ell_n).$$

We call a landmark of sequence S is an index i such that $\ell_1, \dots, \ell_{i-1} \in Y, \ell_i \in Y^* \setminus Y$, and we define

$$\lambda_i^{-1}(S) = (\ell_1, \dots, \ell_{i-1}, \ell'_i, \ell''_i, \ell_{i+1}, \dots, \ell_n),$$

where $\sigma(\ell_i) = (\ell'_i, \ell''_i)$. We will denote by $S \Leftarrow T$ if $T = \rho_i^{-1}(S)$ or $T = \lambda_i^{-1}(S)$ for some fall or landmark i ; and $S \Leftarrow^* T$, transitive closure of \Leftarrow . Similarly, we call the conversely derivation tree $\mathcal{T}^{-1}(S)$ with root labelled S , with left and right immediate subtree $\mathcal{T}^{-1}(S')$ and $\mathcal{T}^{-1}(S'')$, where $S' = \rho_i^{-1}(S)$ for some fall i , $S'' = \lambda_i^{-1}(S)$ for some landmark i .

Lemma 12. *For each standard sequence S , $\Pi(S)$ is the sum of all $\Pi(T)$ for T a leaf in a fixed derivation tree of S .*

Proof. The Lemma is a consequence of the definitions (83), (84) of $\lambda_i(S)$ and $\rho_i(S)$, of that of $\mathcal{T}(S)$ and $\Pi(S)$, and of the identity in $\mathbb{Q}[q]\langle X \rangle$:

$$\Pi_{\ell_i} \Pi_{\ell_{i+1}} = [\Pi_{\ell_i}, \Pi_{\ell_{i+1}}] + \Pi_{\ell_{i+1}} \Pi_{\ell_i} = \Pi_{\ell_i \ell_{i+1}} + \Pi_{\ell_{i+1}} \Pi_{\ell_i}.$$

□

Example 1. $\Pi(y_4, y_2, y_1) = \Pi_{y_4 y_2 y_1} + \Pi_{y_2 y_1} \Pi_{y_4} + \Pi_{y_4 y_1 y_2} + \Pi_{y_2} \Pi_{y_4 y_1} + \Pi_{y_1} \Pi_{y_4 y_2} + \Pi_{y_1} \Pi_{y_2} \Pi_{y_4}$, we can see the following diagram (note that $y_4 < y_2 < y_1$)

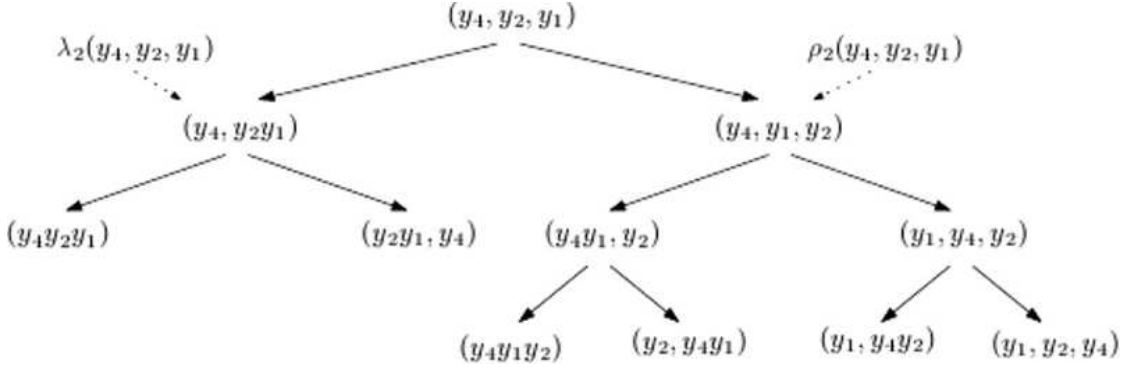


Figure 4: Derivation tree $\mathcal{T}(y_4, y_2, y_1)$

Proposition 11. *i) For any Lyndon word $y_{s_1} \dots y_{s_k}$, we have*

$$\Sigma_{y_{s_1} \dots y_{s_k}} = \sum_{\substack{\{s'_1, \dots, s'_i\} \subset \{s_1, \dots, s_k\} \\ \ell_1 \geq \dots \geq \ell_n \text{ in } \mathcal{L}y_n Y \\ (y_{s_1} \dots y_{s_k}) \Leftarrow^* (y_{s'_1}, \dots, y_{s'_i}, \ell_1, \dots, \ell_n)}} \frac{q^{i-1}}{i!} y_{s'_1 + \dots + s'_i} \Sigma_{\ell_1 \dots \ell_n}. \quad (85)$$

ii) In special case, if $y_{s_1} \leq \dots \leq y_{s_k}$ then

$$\Sigma_{y_{s_1} \dots y_{s_k}} = \sum_{i=1}^k \frac{q^{i-1}}{i!} y_{s_1 + \dots + s_i} \Sigma_{y_{s_{i+1}} \dots y_{s_k}}. \quad (86)$$

Proof. (Proposition 11) At first, we remark that the equality (85) is equivalent to saying that for any word u and any letter y_s ,

$$\langle \Sigma_{y_{s_1} \dots y_{s_k}} | y_s u \rangle = \sum_{\substack{\{s'_1, \dots, s'_i\} \subset \{s_1, \dots, s_k\} \\ \ell_1 \geq \dots \geq \ell_n \text{ in } \mathcal{L}ynY \\ (y_{s_1} \dots y_{s_k}) \stackrel{*}{\neq} (y_{s'_1}, \dots, y_{s'_i}, \ell_1, \dots, \ell_n)}} \frac{q^{i-1}}{i!} \delta_{s'_1 + \dots + s'_i, s} \langle \Sigma_{\ell_1 \dots \ell_n} | u \rangle. \quad (87)$$

Since the duality, we have

$$u = \sum_{w \in Y^*} \langle \Sigma_w | u \rangle \Pi_w,$$

this and (??) we imply

$$y_s u = \sum_w \langle \Sigma_w | u \rangle y_s \Pi_w \quad (88)$$

$$= \sum_w \langle \Sigma_w | u \rangle \left(\sum_{i \geq 1} \frac{q^{i-1}}{i!} \sum_{s'_1 + \dots + s'_i = s} \Pi_{y_{s'_1}} \dots \Pi_{y_{s'_i}} \right) \Pi_w \quad (89)$$

$$= \sum_w \langle \Sigma_w | u \rangle \sum_{i \geq 1} \frac{q^{i-1}}{i!} \sum_{s'_1 + \dots + s'_i = s} \Pi_{y_{s'_1}} \dots \Pi_{y_{s'_i}} \Pi_w. \quad (90)$$

For each w fixed, we write w form factorization of Lyndon words $w = \ell_1 \dots \ell_n, \ell_1 \geq \dots \geq \ell_n$, then we have $S := (y_{s'_1}, \dots, y_{s'_i}, \ell_1, \dots, \ell_n)$ is a standard sequence, so we obtain from Lemma 12

$$\Pi(S) = \Pi(y_{s'_1}, \dots, y_{s'_i}, \ell_1, \dots, \ell_n) = \sum_{S \stackrel{*}{\Rightarrow} T} \alpha_T \Pi(T).$$

Consequently

$$\begin{aligned} \langle \Sigma_{y_1 \dots y_k} | y_s u \rangle &= \sum_w \langle \Sigma_w | u \rangle \sum_{i \geq 1} \frac{q^{i-1}}{i!} \sum_{s'_1 + \dots + s'_i = s} \langle \Sigma_{y_1 \dots y_k} | \Pi_{y_{s'_1}} \dots \Pi_{y_{s'_i}} \Pi_w \rangle \\ &= \sum_{\ell_1 \geq \dots \geq \ell_n \text{ in } \mathcal{L}ynY} \langle \Sigma_{\ell_1 \dots \ell_n} | u \rangle \sum_{i \geq 1} \frac{q^{i-1}}{i!} \sum_{\substack{s'_1 + \dots + s'_i = s \\ (y_{s_1}, \dots, y_{s_i}, \ell_n, \dots, \ell_n) \stackrel{*}{\Rightarrow} T}} \alpha_T \langle \Sigma_{y_1 \dots y_k} | \Pi(T) \rangle. \end{aligned}$$

Note that, the leaves T 's of derivation tree $\mathcal{T}(S)$ are decreasing sequences of Lyndon words with length ≥ 2 except leaves form $T = (\ell)$, where $\ell \in \mathcal{L}ynY$. Therefore $\langle \Sigma_{y_1 \dots y_k} | \Pi(T) \rangle \neq 0$ if $T = (y_{s_1} \dots y_{s_k})$. By maps ρ^{-1} and λ^{-1} , we construct a conversely derivation tree from the standard sequence of one Lyndon word $S =$

$(y_{s_1} \dots y_{s_k})$, we take standard sequences form $(y_{s_1}, \dots, y_{s_i}, \ell_n, \dots, \ell_n), i \geq 1$; at that time, for each S of these sequences, we get unique leaf $T = (y_{s_1} \dots y_{s_k})$ in the derivation tree $\mathcal{T}(S)$, it mean $\alpha_T = 1$. We thus get (87).

In other words, if $y_{s_1} \leq \dots \leq y_{s_k}$ then the standard sequence $(y_{s_1} \dots y_{s_k})$ may only be a leaf of a derivation tree $\mathcal{T}(S)$ after applying map λ_i more times, we imply that $\langle \Sigma_{y_{s_1} \dots y_{s_k}} | \Pi_{y_{s'_1}} \dots \Pi_{y_{s'_i}} \Pi_w \rangle \neq 0$ if and only if $y_{s_1} \dots y_{s_k} = y_{s'_1} \dots y_{s'_i} \ell_1 \dots \ell_n$, then $y_{s_1} = y_{s'_1}, \dots, y_{s'_i} = y_{s_i}$ and $y_{s_{i+1}} \dots y_{s_k} = \ell_1 \dots \ell_n$. Hence

$$\langle \Sigma_{y_{s_1} \dots y_{s_k}} | \Pi_{y_{s'_1}} \dots \Pi_{y_{s'_i}} \Pi_w \rangle = \delta_{s_1 + \dots + s_i, s} \delta_{y_{s_{i+1}} \dots y_{s_k}, w},$$

we thus get

$$\begin{aligned} \langle \Sigma_{y_{s_1} \dots y_{s_k}} | y_{s u} \rangle &= \sum_w \langle \Sigma_w | u \rangle \sum_{i \geq 1} \frac{q^{i-1}}{i!} \sum_{s'_1 + \dots + s'_i = s} \langle \Sigma_{y_{s_1} \dots y_{s_k}} | \Pi_{y_{s'_1}} \dots \Pi_{y_{s'_i}} \Pi_w \rangle \\ &= \frac{q^{i-1}}{i!} \delta_{s_1 + \dots + s_i, s} \langle \Sigma_{y_{s_{i+1}} \dots y_{s_k}} | u \rangle. \end{aligned}$$

□

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