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► **To cite this version:**

Jeremie Guilhot. Cellularity of the lowest two-sided ideal of an affine Hecke algebra. *Advances in Mathematics*, 2014, 255, pp.525-561. hal-00790711v2

HAL Id: hal-00790711

<https://hal.science/hal-00790711v2>

Submitted on 10 Oct 2013

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CELLULARITY OF THE LOWEST TWO-SIDED IDEAL OF AN AFFINE HECKE ALGEBRA

JÉRÉMIE GUILHOT

ABSTRACT. In this paper we show that the lowest two-sided ideal of an affine Hecke algebra is affine cellular for all choices of parameters. We explicitly describe the cellular basis and we show that the basis elements have a nice decomposition when expressed in the Kazhdan-Lusztig basis. In type A we provide a combinatorial description of this decomposition in term of number of paths.

1. INTRODUCTION

The notion of cellular algebra was first introduced by Graham and Lehrer in [6] for finite dimensional algebras. Roughly speaking, a cellular algebra A possesses a distinguished basis (possibly more than one), known as a "cellular basis", that is particularly well-adapted to studying the representation theory of A . Indeed, a cellular basis yields a filtration of A with composition factors isomorphic to the cell modules. Further, the structure constants with respect to the cellular basis define bilinear forms on cell modules and one can show that the quotient of a cell module by the radical of this bilinear form is either 0 or irreducible. In this way, one obtains a complete set of isomorphism classes of simple modules for the given cellular algebra. Examples of cellular algebras include many finite-dimensional Hecke algebras [4].

Recently, Koenig and Xi [10] have generalized this concept to possibly infinite dimensional algebras over a principal ideal domain k by introducing the notion of affine cellular algebras. In a cellular algebra, cell modules are isomorphic to matrix rings with coefficients in k and twisted multiplication with respect to the bilinear form. In the affine case, they still are isomorphic to matrix rings with twisted multiplication but the matrix ring is now defined over a quotient of a polynomial ring over k . As in the finite dimensional case, affine cellular structure for an algebra A yields a parametrisation of the irreducible A -modules. In their paper, Koenig and Xi have shown that extended affine Hecke algebras of type A are affine cellular. In [8], it is shown that affine Hecke algebras of rank 2 are affine cellular for all generic choices of parameters.

In this paper we are concerned with extended affine Weyl groups. Let W_e be an extended affine Weyl group and let W_0 be the associated finite Weyl group. Then it is a well-known fact that the set $c_0 := \{w \in W_e \mid w = xw_0y, \ell(w) = \ell(x) + \ell(w_0) + \ell(y)\}$ is a Kazhdan-Lusztig two-sided cell and that it is the lowest one with respect to a certain order on cells. We can associate to this cell a two-sided ideal \mathcal{M}_0 of the corresponding affine Hecke algebra (with possibly unequal parameters).

The purpose of this paper is twofold:

- (1) Show that the lowest two-sided ideal \mathcal{M}_0 is affine cellular and explicitly construct a cellular basis; see Section 4.

Date: October 10, 2013.

- (2) Study the decomposition of the previously found cellular basis in the Kazhdan-Lusztig basis; see Section 5.

It is in type A that point (2) above is the nicest : we will show in section 6 that the coefficients that appear in the expression of the cellular basis when expressed in the Kazhdan-Lusztig basis can be interpreted as the number of certain kind of paths in a Weyl chamber.

2. AFFINE WEYL GROUPS AND GEOMETRIC REALISATION

In this paper, we fix an euclidean \mathbb{R} -vector space V of dimension $r \geq 1$ and we denote by Φ an *irreducible* root system in V of rank r : the scalar product will be denoted by $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$. The dual of V will be denoted by V^* and $\langle \cdot, \cdot \rangle : V \times V^* \rightarrow \mathbb{R}$ will denote the canonical pairing. If $\alpha \in \Phi$, we denote by $\alpha^\vee \in V^*$ the associated coroot (if $x \in V$, then $\langle x, \alpha^\vee \rangle = 2(x, \alpha)/(\alpha, \alpha)$) and by Φ^\vee the dual root system. We will denote by Q the root lattice. We fix a positive system Φ^+ and a simple system $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi^+$.

2.1. Affine Weyl group. For $\alpha \in \Phi^+$ and $n \in \mathbb{Z}$, we set

$$H_{\alpha, n} = \{x \in V \mid \langle x, \alpha^\vee \rangle = n\} \quad \text{and} \quad \mathcal{F} = \{H_{\alpha, n} \mid \alpha \in \Phi^+ \text{ and } n \in \mathbb{Z}\}.$$

We will say that H is of *direction* α if H is of the form $H_{\alpha, k}$ for some $k \in \mathbb{Z}$ and, for $\alpha \in \Phi^+$, we denote by \mathcal{F}_α the set of hyperplanes of direction α . We will say that H and H' are parallel if they have same directions. For $H = H_{\alpha, k}$ with $\alpha \in \Phi^+$, we set

$$H^+ := \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle > k\} \quad \text{and} \quad H^- := \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle < k\}.$$

If $H \in \mathcal{F}$, we denote by σ_H the orthogonal reflection with respect to H . Then the group W_a generated by $\{\sigma_H \mid H \in \mathcal{F}\}$ is an affine Weyl group of type Φ^\vee . Let $\alpha_0 \in \Phi^+$ be such that α_0^\vee is the highest coroot in Φ^\vee . Then W_a is a Coxeter group generated by $S_a := S_0 \cup \{\sigma_{\alpha_0, 1}\}$ where $S_0 := \{\sigma_{\alpha, 0} \mid \alpha \in \Delta\}$. The Weyl group W_0 of Φ is generated by S_0 . The translation t_α of vector $\alpha \in Q$ indeed lie in W_a as have $\sigma_{\alpha, 1}\sigma_{H_{\alpha, 0}} = t_\alpha$. In fact it is a well-known fact that $W_a \simeq W_0 \rtimes Q$.

To simplify the notation we will write σ_i instead of $\sigma_{\alpha_i, 0}$ for all $1 \leq i \leq n$ and σ_0 instead of $\sigma_{\alpha_0, 1}$. Then W_0 is generated by $S_0 := \{\sigma_1, \dots, \sigma_n\}$ and W_a is generated by $S_a := \{\sigma_0, \dots, \sigma_n\}$.

An *alcove* is a connected component of the set

$$V - \bigcup_{H \in \mathcal{F}} H.$$

Then W_a acts simply transitively on the set of alcoves $\text{Alc}(\mathcal{F})$. Recall also that, if A is an alcove, then its closure \bar{A} is a fundamental domain for the action of W_a on V . We denote by A_0 the fundamental alcove associated to Φ

$$A_0 = \{x \in V \mid 0 < \langle x, \check{\alpha} \rangle < 1 \text{ for all } \alpha \in \Phi^+\}$$

and by \mathcal{C}^+ the fundamental Weyl chamber

$$\mathcal{C}^+ = \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle > 0\}.$$

We also set $\mathcal{C}^- = \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle < 0\} = -\mathcal{C}^+$.

The group W_a acts on the set of faces (the codimension 1 facets) of alcoves. We denote by S the set of W_a -orbits in the set of faces. If $A \in \text{Alc}(\mathcal{F})$, then the faces of A is a set of representatives of S since \bar{A} is a fundamental domain for the action of W_a . If a face f is contained in the orbit $s \in S$, we say that f is of type s . To each $s \in S$ we can associate an involution $A \rightarrow sA$ of $\text{Alc}(\mathcal{F})$: the alcove sA is defined to be $A\sigma_{H_s}$ where H_s is the

unique hyperplane that contains the face of type s of A . Let W be the group generated by all such involutions. Then (W, S) is a Coxeter system and it is isomorphic to the affine Weyl group W_a . We shall regard W as acting on the left of $\text{Alc}(\mathcal{F})$. The action of W_a on the right and the action of W on the left commute.

Let $A \in \text{Alc}(\mathcal{F})$. Then there exists a unique $w \in W$ such that $wA_0 = A$. We will freely identify W with the set of alcoves $\text{Alc}(\mathcal{F})$. We set $S = \{s_0, \dots, s_n\}$ in such a way that $s_i A_0 = A_0 \sigma_i$ for all $0 \leq i \leq n$. We introduce the following notation:

- for $w \in W$, let σ_w be the unique element of W_a such that $wA_0 = A_0 \sigma_w$;
- for $A \in \text{Alc}(\mathcal{F})$, let σ_A be the unique element of W_a such that $A_0 \sigma_A = A$;
- for $\alpha \in Q$ let p_α be the unique element of W such that $p_\alpha A_0 = A_0 t_\alpha$.

Remark 2.1. (1). Let us investigate the relationship between the actions of W_a and W . Let $w \in W$ and let $s_{i_n} \dots s_{i_1}$ be a reduced expression of w . If we denote by H_{i_k} the unique hyperplane separating $s_{i_k} \dots s_{i_n} A_0$ and $s_{i_{k-1}} \dots s_{i_1} A_0$, we see that the set of hyperplanes which separates A_0 and wA_0 is $\{H_{i_1}, \dots, H_{i_n}\}$ and we get

$$wA_0 = s_{i_n} \dots s_{i_1} A_0 = A_0 \sigma_{H_{i_1}} \sigma_{H_{i_2}} \dots \sigma_{H_{i_n}}.$$

But we also have

$$wA_0 = s_{i_n} \dots s_{i_1} A_0 = A_0 \sigma_{i_n} \sigma_{i_{n-1}} \dots \sigma_{i_1}$$

From there, we can show that

$$\{H_{i_1}, \dots, H_{i_n}\} = \{H_{\alpha_{i_1,0}}, H_{\alpha_{i_2,0} \sigma_{i_1}}, H_{\alpha_{i_3,0} \sigma_{i_2} \sigma_{i_1}}, \dots, H_{\alpha_{i_n,0} \sigma_{n-1} \dots \sigma_{i_1}}\}.$$

We refer to [9, Chapter 4.5] for details.

(2). Let $A \in \text{Alc}(\mathcal{F})$. For all $\alpha \in Q$, we have

$$p_\alpha A = p_\alpha A_0 \sigma_A = A_0 t_\alpha \sigma_A = A_0 \sigma_A t_{\alpha \sigma_A} = A t_{\alpha \sigma_A}.$$

Hence $p_\alpha A$ is a translate of A of vector $\alpha \sigma_A$.

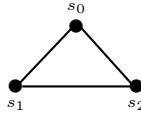
Example 2.2. Let us recall the classic realisation of the root system of type A . Let E be a $n+1$ -dimensional \mathbb{R} -vector space with basis ϵ_i ($1 \leq i \leq n+1$) and let V be the n -dimensional subspace of E define by

$$V = \{v = (v_1, \dots, v_{n+1}) \in V \mid \sum v_i = 0\}.$$

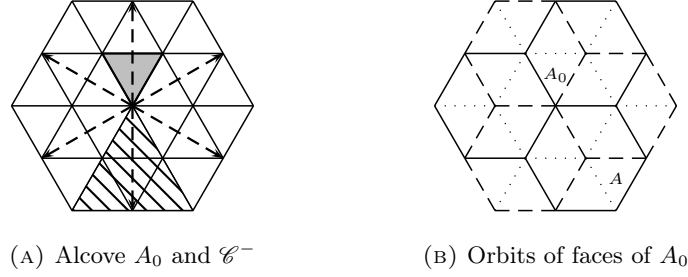
Then the root system of type A can be describe as follows

- Roots: $\epsilon_i - \epsilon_j$, $1 \leq i, j \leq n+1$;
- Simple roots: $\alpha_i := \epsilon_i - \epsilon_{i+1}$ for all $1 \leq i \leq n$;
- Positive roots: $\epsilon_i - \epsilon_j = \sum_{i \leq k < j} \alpha_k$.

When $n = 2$, the affine Weyl group \tilde{A}_2 has the following Dynkin diagram:



We draw the alcoves in this type in Figure 1. In Figure 1.(A), the dashed arrows represent the root system, the gray alcove represents the fundamental alcove A_0 , and the dashed alcoves represent the set $-\mathcal{C}^+$. In Figure 1.(B) we describe the orbits of the faces of A_0 under the action of the group W_a . The dashed faces of A_0 corresponds to s_1 , the plain one to s_2 and the dotted one to s_0 . Then, the unique element of W_a which sends A_0 to A is $s_1 s_3 s_2 s_1$.

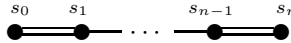
FIGURE 1. Roots, hyperplanes and alcoves in type A_2

2.2. Weight functions and L -weights. Let $L : W \rightarrow \mathbb{N}$ be a weight function on W , that is a function which satisfies $L(ww') = L(w) + L(w')$ whenever $\ell(ww') = \ell(w) + \ell(w')$ where ℓ denotes the usual length function and such that $L(w) = 0$ if and only if w is the identity of W . Recall that this implies the following property: if s, t are conjugate in W then $L(s) = L(t)$. (Note that the length function is indeed a positive weight function.) From now on and until the end of this paper, we will fix a positive weight function L on W .

Let $x, y \in W$. We will write $w = x \bullet y$ if and only if $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. By definition of weight functions, we see that if we have $w = x \bullet y$ then $L(w) = L(x) + L(y)$. We generalise this notation to more than two words in a straightforward fashion.

Let $H \in \mathcal{F}$ and assume that H supports a face of type $t \in S$. Then, following [3], we set $L_H = L(t)$. This is well-defined since, according to [3, Lemma 2.1], if H supports a face of type s and t then s are conjugate in W and $L(s) = L(t) = L_H$. For $\alpha \in \Phi^+$ we set $L_\alpha := \max_{H \in \mathcal{F}_\alpha} L_H$. We then say that H is of maximal weight if $L_H = L_\alpha$ where H is of direction α . The only case where two parallel hyperplanes might have different weights is type C . In all other cases, parallel hyperplanes have same weight and therefore all hyperplanes are of maximal weight.

Convention 2.3. *In the case where W is of type \tilde{C}_r with Dynkin diagram*



by symmetry, we may (and we will) assume that $L(s_0) \geq L(s_n)$. (Recall that \tilde{C}_1 is also of type \tilde{A}_1)

Let $\alpha \in \Phi^+$. We set

$$b_\alpha = \begin{cases} 1 & \text{if } L_{H_{\alpha,0}} = L_{H_{\alpha,1}} \\ 2 & \text{otherwise} \end{cases}$$

We see that $b_\alpha = 2$ only in type C when $L(s_0) > L(s_n)$.

For $\lambda \in V$ we set

$$L_\lambda := \sum_{H \in \mathcal{F}, \lambda \in H} L_H.$$

If we denote by W_λ the subgroup of W which stabilises the set of alcoves which contains λ in their closure, then W_λ is a parabolic subgroup of W , generated by $S_\lambda := S \cap W_\lambda$ and we have $L_\lambda = L(w_\lambda)$ where w_λ is the longest element of W_λ . Note that, by definition, we have $L_{\lambda\sigma} = L_\lambda$ for all $\sigma \in W_a$. We set $\nu_L := \max_{\lambda \in V} L_\lambda$. We then say that $\lambda \in V$ is an L -weight if $L_\lambda = \nu_L$. We will denote by P the set of all L -weights. With the convention above, 0 is always a special point and W_0 is isomorphic to the Weyl group associated to

W and therefore our notation is coherent with the definition of W_0 given before.

Let λ be an L -weight. A quarter with vertex λ is a connected component of the set

$$V - \bigcup_{\lambda \in H} H.$$

Example 2.4. When $L = \ell$ then $L_H = 1$ for all $H \in \mathcal{F}$ and $\nu_\ell = |\Phi^+|$. Then the following are equivalent

- (1) $\lambda \in V$ is an L -weight;
- (2) λ lies in the intersection of $|\Phi^+|$ hyperplanes;
- (3) for all $\alpha \in \Phi^+$, there exists $\alpha_\lambda \in \mathbb{Z}$ such that $\langle \lambda, \alpha \rangle = \alpha_\lambda$ (i.e. $\lambda \in H_{\alpha, \alpha_\lambda}$);
- (4) λ is a weight according to the definition in Bourbaki [2, Proposition 26 and after].

We will denote by P^+ be the set of dominant L -weights, that is the set of L -weights which lies in the closure of the fundamental Weyl chamber \mathcal{C}^+ . Finally let \mathbf{P}^+ be the set of fundamental L -weights, that is the dual basis of

$$\{b_\alpha \alpha^\vee \mid \alpha \in \Delta\} \subset \Phi^\vee.$$

Remark 2.5. Once again, it is clear that the notion of dominant L -weights and fundamental L -weights coincide with the usual one when $L = \ell$. In fact (once again!) the only case when the notion of weights and L -weights do not coincide is when W is of type \tilde{C}_r and $L(s_0) > L(s_n)$.

Example 2.6. The next figure describes the L -weights of \tilde{C}_2 . The thick arrows represent the positive roots Φ^+ and the gray alcoves represents the fundamental alcove. In the case $L(s_0) = L(s_n)$, all the circled points are L -weights. In the case $L(s_0) > L(s_n)$, only the gray points are L -weights.

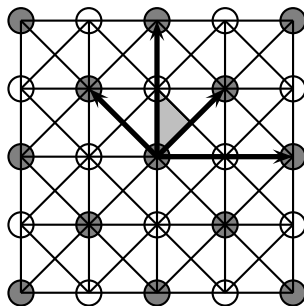


FIGURE 3. L -weights of \tilde{C}_2 .

2.3. Extended affine Weyl groups. The extended affine Weyl group is defined by $W_e = W_0 \times P$; it acts naturally on (the right) of V and on $\text{Alc}(\mathcal{F})$ but the action is no longer faithful. If we denote by Π the stabiliser of A_0 in W_e then we have $W_e = \Pi \times W_a$. Further the group Π is isomorphic to P/Q , hence abelian, and its action on W_a is given by an automorphism of the Dynkin diagram; see Planches I–IX in [2]. We use the following notation. Let $\pi \in \Pi$. Since π permutes the generators S_a of W_a , it induces a permutation of $\{0, \dots, n\}$. We will still denote by π this permutation so that we have $\pi \cdot \sigma_i = \sigma_{\pi(i)}$. We write σ^π for the image of σ under π that is

$$\sigma^\pi = \sigma_{\pi(i_1)} \dots \sigma_{\pi(i_n)} \text{ where } \sigma = \sigma_{i_1} \dots \sigma_{i_n}.$$

We have $W_e = \Pi \ltimes W_a \simeq \Pi \ltimes W$ where the action of Π on W is of course given by $\pi \cdot s_i = s_{\pi(i)}$. We would like to define two commutative and faithful (left and right) actions of W_e on a set of alcoves. To this end, we introduce the set of extended alcoves, denoted by $\text{Alc}_e(\mathcal{F})$, which is the cartesian product $\Pi \times \text{Alc}(\mathcal{F})$. Then for $\pi, \pi' \in \Pi$, $w \in W$ and $\sigma \in W_a$ we set

$$\begin{aligned}\pi' \cdot (\pi, A) &= (\pi' \pi, A) & (\pi, A) \cdot \pi' &= (\pi \pi', A_0 \sigma_A^{\pi'}) \\ w \cdot (\pi, A) &= (\pi, w^\pi A) & (\pi, A) \cdot \sigma &= (\pi, A \sigma)\end{aligned}$$

Proposition 2.7. *The two actions are faithful and they commute.*

Proof. The fact that these two actions are faithful is straightforward since the right action of W_a (respectively the left action of W) on $\text{Alc}(\mathcal{F})$ is faithful. Let us show that they commute. First let $w = s_1 \dots s_n \in W$ and $\sigma = \sigma_1 \dots \sigma_n$ so that $w A_0 = A_0 \sigma$. We have

$$w^\pi A_0 = s_{\pi(1)} \dots s_{\pi(n)} A_0 = A_0 \sigma_{\pi(1)} \dots \sigma_{\pi(n)} = A_0 \sigma^\pi$$

Let $\pi, \pi_1, \pi_2 \in \Pi$, $w \in W$, $\sigma \in W_a$ and $A = A_0 \sigma_A \in \text{Alc}(\mathcal{F})$. We have

$$\begin{aligned}\pi_1 w \cdot [(\pi, A) \cdot \pi_2 \sigma] &= \pi_1 w [(\pi \pi_2, A_0 \sigma_A^{\pi_2} \sigma)] \\ &= (\pi_1 \pi \pi_2, w^{\pi \pi_2} A_0 \sigma_A^{\pi_2} \sigma)\end{aligned}$$

and

$$\begin{aligned}[\pi_1 w \cdot (\pi, A)] \cdot \pi_2 \sigma &= (\pi_1 \pi, w^\pi A) \cdot \pi_2 \sigma \\ &= (\pi_1 \pi, w^\pi A_0 \sigma_A) \cdot \pi_2 \sigma \\ &= (\pi_1 \pi, A_0 \sigma_w^\pi \sigma_A) \cdot \pi_2 \sigma \\ &= (\pi_1 \pi \pi_2, A_0 \sigma_w^{\pi \pi_2} \sigma_A^{\pi_2} \sigma) \\ &= (\pi_1 \pi \pi_2, w^{\pi \pi_2} A_0 \sigma_A^{\pi_2} \sigma)\end{aligned}$$

hence the result. \square

In order to simplify the notation, we will write πA instead of (π, A) . One needs to be careful though, we do have $\pi A_0 = (\pi, A_0) = (1, A_0) \pi = A_0 \pi$ but this only holds for A_0 .

Remark 2.8. Each alcove in $\text{Alc}(\mathcal{F})$ can be identified with its set of vertices $\{f_0^A, \dots, f_n^A\}$ where $f_i^A \in V$ for all $0 \leq i \leq n$. Then the action of W_a on $\text{Alc}(\mathcal{F})$ is given by

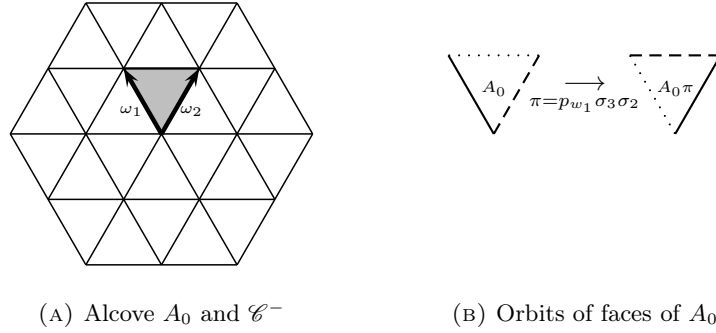
$$A \cdot \sigma = \{f_0^A, \dots, f_n^A\} \cdot \sigma = \{f_0^A \sigma, \dots, f_n^A \sigma\}.$$

The action on the left is simply deduced from the right action by setting $s_i \cdot A_0 = A_0 \sigma_i$. In the extended setting, we identify A_0 with the ordered sequence $(f_0^{A_0}, \dots, f_n^{A_0}) \in V^{n+1}$. Now W_e naturally acts diagonally on V^{n+1} . Note that since $\pi \in \Pi$ stabilises A_0 , it induces a permutation of the vertices of A_0 . By labelling the vertices carefully we have $(f_0^{A_0}, \dots, f_n^{A_0}) \cdot \pi = (f_{\pi(0)}^{A_0}, \dots, f_{\pi(n)}^{A_0})$. The set of extended alcoves can be identified with the orbit of $(f_0^{A_0}, \dots, f_n^{A_0})$ under the action of W_e by setting

$$(\pi, A) = (\pi, A_0 \sigma_A) \longleftrightarrow A_0 \cdot \pi \sigma_A$$

The left action is then given by $\pi w \cdot (\pi' A) = \pi w A_0 \pi' \sigma_A = A_0 \pi \sigma_w \pi' \sigma_A$.

Example 2.9. Let us consider the example of type \tilde{A}_2 as in Example 2.2. In Figure 2, the arrows represent the fundamental weights ω_1 and ω_2 and the gray alcove represents the fundamental alcove A_0 . Consider the element $\pi = t_{\omega_1} \sigma_3 \sigma_2$. Then one can check that π stabilises the alcove A_0 but only globally, not pointwise. Indeed the faces of A_0 are permuted circularly as shown on figure 2.(B).


 FIGURE 2. Action of Π in type \tilde{A}_2

The weight function L can be extended to W_e by setting $L(\pi w) = L(w)$ for all $\pi \in \Pi$ and $w \in W$. The Bruhat order can also be extended by setting $\pi w \leq \pi' w$ if and only if $\pi = \pi'$ and $w \leq w'$ where $\pi, \pi' \in \Pi$ and $w, w' \in W$.

As before, for all $\alpha \in P$, we will denote by p_α the unique element of W_e satisfying $p_\alpha A_0 = A_0 t_\alpha$ (where the equality holds in $\text{Alc}_e(\mathcal{F})$).

2.4. The lowest two-sided cell of an affine Weyl groups. We will write w_0 for the longest element of Weyl group (W_0, S_0) . We define the following subset of W_e :

$$c_0 = \{w \in W_e \mid w = x \bullet w_0 \bullet y, \ x, y \in W_e\}.$$

This set is called the lowest two-sided cell of W_e . The reason for this terminology is that c_0 is a Kazhdan-Lusztig cell (we refer to Section 3 for the definition of cells) and it is the lowest one for a certain partial order. This cell has been extensively studied, see for instance [3, 7, 12, 14, 15, 16].

In this section, we give a nice description of the elements lying in c_0 as in [1, 15]. Recall the definition of b_α in section 2.2. Let \mathbf{B}_0 be the subset of $\text{Alc}_e(\mathcal{F})$ which consist of all the alcoves πA with $A \in \text{Alc}(\mathcal{F})$ satisfying the following property for all $\alpha \in \Delta$:

$$\forall \lambda \in A, 0 < \langle \lambda, \alpha \rangle < b_\alpha.$$

The set \mathbf{B}_0 is very closely related to the set c_0 as shown in the following theorem.

Theorem 2.10. *Let $w \in c_0$. There exist unique $\tau \in P^+$ and $z, z' \in \mathbf{B}_0$ such that $w = z \bullet p_\lambda \bullet w_0 \bullet z'^{-1}$.*

The proof of this theorem can be found in [15, Proposition 4.3]. The result is slightly different as the author consider the non-extended case when $L = \ell$ but all the main idea easily generalize to our case. Another proof of this result in the extended case can be found in [1, Proposition 3.1].

Theorem 2.10 should be understood as follows. The set c_0 can be decomposed into connected components:

$$c_0 = \bigsqcup_{z' \in \mathbf{B}_0} N_{z'} \text{ where } N_{z'} := \{w \in W_e \mid w = x \bullet w_0 \bullet z'^{-1}, \ x \in W_e\}.$$

In turn each connected component can be decomposed in the following way:

$$N_{z'} = \{z p_\tau w_0 z'^{-1} \mid \tau \in P^+, z \in \mathbf{B}_0\}.$$

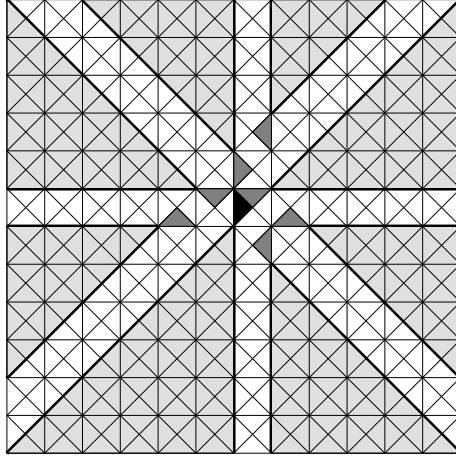
This shows that the set $N_{z'}$ can be covered by translates of \mathbf{B}_0 ; see Example 2.12 below.

Remark 2.11. In the above expression of $N_{z'}$, we do not need to specify that $w = z \bullet p_\lambda \bullet w_0 \bullet z'^{-1}$ as it is a (non-trivial!) consequence of the definition of \mathbf{B}_0 and w_0 .

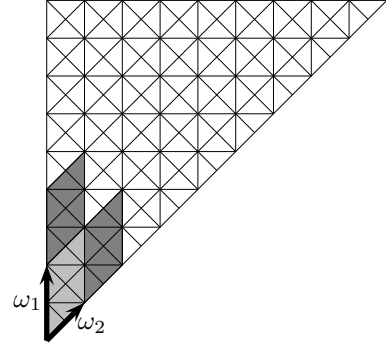
Example 2.12. Let W be of type \tilde{C}_2 and let $a, b, c \in \mathbb{N}^*$ be such that $L(s_0) = a, L(s_1) = b$ and $L(s_2) = c$. By convention, we assume that $a \geq c$. In the following figures, we describe the set \mathbf{B}_0 , the set c_0 as well as each of its connected components. We need to distinguish the case $a > c$ in which $\Pi = \{1\}$ and $a = c$ in which $\Pi \simeq \mathbb{Z}/2\mathbb{Z}$.

In Figure 3.(a) and 3.(c), the black alcove is the fundamental alcove, the dark grey alcoves correspond to the alcoves $z^{-1}A_0$ for $z \in \mathbf{B}_0$ and the light gray areas to the set c_0 . In Figure 3.(b) and 3.(d), we represent the connected component $N_{z'}$ for $z' \in \mathbf{B}_0$. The light grey alcoves correspond $zw_0z'^{-1}A_0$ for all $z \in \mathbf{B}_0$ and the darker ones to $zp_{\omega_1}w_0z'^{-1}A_0$ and $zp_{\omega_2}w_0z'^{-1}A_0$ where ω_1 and ω_2 are the fundamental L -weights.

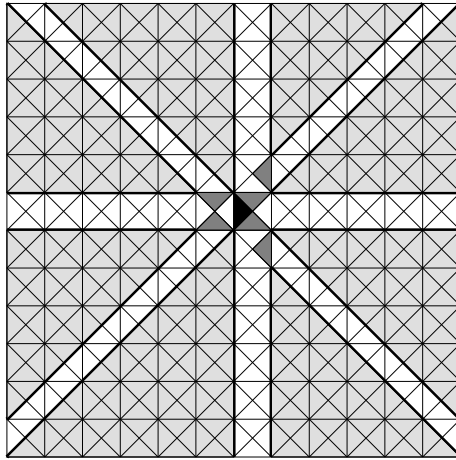
FIGURE 3. Description of c_0



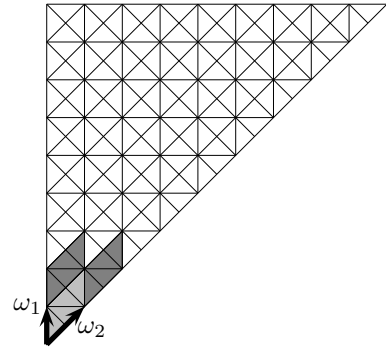
(a) The set c_0 when $a > c$



(b) Connected component $N_{z'}$ of c_0



(c) The set c_0 when $a = c$



(d) Connected component $N_{z'}$ of c_0

3. KAZHDAN-LUSZTIG THEORY

In this section $W_e = \Pi \ltimes W$ denotes an extended affine Weyl group with generating set S . The extended Bruhat order and the extended weight function will be denoted by \leq and L .

3.1. Hecke algebras and Kazhdan-Lusztig basis. Let $\mathcal{A} = \mathbb{C}[q, q^{-1}]$ where q is an indeterminate. Let \mathcal{H} be the Iwahori-Hecke algebra associated to W , with \mathcal{A} -basis $\{T_w | w \in W_e\}$ and multiplication rule given by

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + (q^{L(s)} - q^{-L(s)})T_w, & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for all $s \in S$ and $w \in W_e$.

Remark 3.1. If we denote by \mathcal{H}' the subalgebra of \mathcal{H} generated by T_s with $s \in S$ then \mathcal{H} is isomorphic to the twisted tensor product $\mathbb{Z}[\Pi] \otimes \mathcal{H}'$ by setting $T_{\omega w} \mapsto \omega \otimes T_w$. Here $\mathbb{Z}[\Pi]$ denotes the group algebra of Π over \mathbb{Z} and $\omega \in \Pi$.

We will denote by $\bar{\cdot}$ the ring involution of \mathcal{A} which takes q to q^{-1} . This involution can be extended to a ring involution of \mathcal{H} via the formula

$$\overline{\sum_{w \in W_e} a_w T_w} = \sum_{w \in W_e} \bar{a}_w T_w^{-1} \quad (a_w \in \mathcal{A}).$$

We set

$$\begin{aligned} \mathcal{A}_{<0} &= q^{-1}\mathbb{Z}[q^{-1}] & , & & \mathcal{H}_{<0} &= \bigoplus_{w \in W_e} \mathcal{A}_{<0} T_w \\ \mathcal{A}_{\leq 0} &= \mathbb{Z}[q^{-1}] & \text{and} & & \mathcal{H}_{\leq 0} &= \bigoplus_{w \in W_e} \mathcal{A}_{\leq 0} T_w. \end{aligned}$$

For each $w \in W_e$ there exists a unique element $C_w \in \mathcal{H}$ (see [13, Theorem 5.2]) such that (1) $\overline{C_w} = C_w$ and (2) $C_w \equiv T_w \pmod{\mathcal{H}_{<0}}$. For any $w \in W_e$ we set

$$C_w = T_w + \sum_{y \in W_e} P_{y,w} T_y \quad \text{where } P_{y,w} \in \mathcal{A}_{<0}.$$

The coefficients $P_{y,w}$ are called as the Kazhdan-Lusztig polynomials. It is well known ([13, §5.3]) that $P_{y,w} = 0$ whenever $y \not\leq w$. It follows that $\{C_w | w \in W_e\}$ forms an \mathcal{A} -basis of \mathcal{H} known as the ‘‘Kazhdan-Lusztig basis’’.

Lemma 3.2. *Let $h \in \mathcal{H}$ be such that $\bar{h} = h$ and $h \equiv \sum a_z T_z \pmod{\mathcal{H}_{<0}}$ where $a_z \in \mathbb{Z}$. Then $h = \sum a_z C_z$.*

Proof. We know [13, §5.2.(e)] that if $h' \in \mathcal{H}_{<0}$ satisfies $\bar{h}' = h$ then $h' = 0$. The lemma is an easy consequence of this result setting $h' = h - \sum a_z C_z$. \square

Following Lusztig [13, §3.4], let us now introduce another involution of \mathcal{H} which plays a crucial role in the sequel.

Definition 3.3. There exists a unique involutive antiautomorphism, i.e. an \mathcal{A} -involution, $\flat : \mathcal{H} \rightarrow \mathcal{H}$ which carries T_w to $T_{w^{-1}}$.

Using this map, we obtain right handed version of the multiplication of \mathcal{H} :

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w), \\ T_{ws} + (q^{L(s)} - q^{-L(s)})T_w, & \text{if } \ell(ws) < \ell(w). \end{cases}$$

Further, since \flat sends $\mathcal{H}_{<0}$ to itself it can be shown that [13, §5.6] that $C_w^\flat = C_{w^{-1}}$, from where it follows that

$$P_{y,w} = P_{y^{-1}, w^{-1}}.$$

3.2. Kazhdan-Lusztig cells. For any $x, y \in W_e$, we set

$$C_x C_y = \sum_{z \in W_e} h_{x,y,z} C_z.$$

Note that $\bar{h}_{x,y,z} = h_{x,y,z}$ and using the \flat -involution we have that $h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}$.

We write $z \leftarrow_{\mathcal{L}} y$ if one of the following condition holds:

- (1) $z = \omega y$ with $\omega \in \Pi$
- (2) C_z appears with a non-zero coefficient in the expression of $C_s C_y$ in the Kazhdan-Lusztig basis: that is, if there exists $s \in S$ such that $h_{s,y,z} \neq 0$.

We then denote by $\leq_{\mathcal{L}}$ the transitive closure of this relation: it is known as the Kazhdan-Lusztig left pre-order $\leq_{\mathcal{L}}$ on W_e . We will denote by $\sim_{\mathcal{L}}$ equivalence relation associated to $\leq_{\mathcal{L}}$ and the equivalence classes will be called left cells.

Similarly, multiplying on the right in the defining relation, we can define a pre-order $\leq_{\mathcal{R}}$. Unsurprisingly, The associated equivalence relation will be denoted by $\sim_{\mathcal{R}}$ and the corresponding equivalence classes are called the right cells of W_e . Using the antiautomorphism \flat , we have (see [13, §8])

$$x \leq_{\mathcal{L}} y \iff x^{-1} \leq_{\mathcal{R}} y^{-1}.$$

Finally we will write $x \leq_{\mathcal{LR}} y$ if one can find a sequence $x = x_0, x_1, \dots, x_n = y$ of W_e such that for each $0 \leq i \leq n-1$ we have either $x_i \leftarrow_{\mathcal{L}} x_{i+1}$ or $x_i \leftarrow_{\mathcal{R}} x_{i+1}$. The equivalence relation associated to $\leq_{\mathcal{LR}}$ will be denoted by $\sim_{\mathcal{LR}}$ and the equivalence classes are called the two-sided cells of W_e .

Example 3.4. Let $s \in S$ and $w \in W_e$ be such that $sw > w$. Then it can be shown [13, Theorem 6.6] that

$$C_s C_w = C_{sw} + \sum_{z < w} a_z C_z \text{ where } a_z \in \mathcal{A}.$$

Therefore if $sw > w$ we always have $sw \leq_{\mathcal{L}} w$. A straightforward generalisation of this result shows that

- (1) for all $x, w \in W_e$ such that $xw = x \bullet w$ we have $xw \leq_{\mathcal{L}} w$
- (2) for all $y, w \in W_e$ such that $wy = w \bullet y$ we have $wy \leq_{\mathcal{R}} w$
- (3) for all $x, y, w \in W_e$ such that $xwy = x \bullet w \bullet y$ we have $xwy \leq_{\mathcal{LR}} w$

3.3. Lowest two-sided cell modules. Recall that (W_0, S_0) denotes the Weyl group associated to W , that w_0 is the longest element of W_0 and the definition of c_0 :

$$c_0 := \{w \in W_e \mid w = x \bullet w_0 \bullet y, x, y \in W_e\}.$$

It is a well known result that c_0 is a two-sided cell and that it is the lowest one with respect to the partial order on two-sided cells induced by $\leq_{\mathcal{LR}}$. In other words, for all $y \in W_e$ and all $w \in c_0$ we have

$$(3.4) \quad y \leq_{\mathcal{LR}} w \implies y \in c_0.$$

Remark 3.5. If one knows that c_0 is a two-sided cell, it is fairly easy to see why it has to be the lowest one. Let $z \in W_e$. Let us show that there exists $w \in c_0$ such that $z \geq_{\mathcal{LR}} w$. We construct a sequence $z = z_0, \dots, z_n$ in the following way

- (1) if $sz_i < z_i$ for all $s \in S_0$ then we stop and we set $z_i = w$;
- (2) if there exists $s \in S_0$ such that $sz_i > z_i$ then we set $z_{i+1} = sz_i$.

The process finishes as S_0 generates a finite group. Further, when it finishes, say for z_n , then $sz_n < z_n$ for all $s \in S_0$, that is $z_n = w_0 z$ and it is easy to see that by construction we have $z_n = w_0 \bullet z$. It follows that $z \geq_{\mathcal{L}} z_n \in c_0$ by Example 3.4 and, in particular we have $z \geq_{\mathcal{LR}} z_n$. Now this implies that c_0 is the lowest two-sided cell. Indeed let $z \in W_e$ be such that $z \leq_{\mathcal{LR}} w \in c_0$. Then we can construct $z_n \in c_0$ such that $z_n \leq_{\mathcal{LR}} z \leq_{\mathcal{LR}} w$ which implies $z_n \sim_{\mathcal{LR}} z \sim_{\mathcal{LR}} w$ as both z_n and w lies in c_0 .

Relation 3.4 shows that the \mathcal{A} -module $\mathcal{M}_0 := \langle C_w \mid w \in c_0 \rangle_{\mathcal{A}}$ is a two-sided ideal. By definition of the pre-order $\leq_{\mathcal{L}\mathcal{R}}$, we have for all $h, h' \in \mathcal{H}$ and $w \in c_0$:

$$hC_w h' = \sum_{y \leq_{\mathcal{L}\mathcal{R}} w} C_y$$

which lies in \mathcal{M}_0 .

Recall the definition of \mathbf{B}_0 in Section 2.4. For $y \in \mathbf{B}_0^{-1}$ we set

$$N_y = \{w \in W \mid w = xw_0y, x \in X_0\} \quad \text{and} \quad \mathcal{M}_y = \langle C_w \mid w \in N_y \rangle_{\mathcal{A}}.$$

We will show in the next section that \mathcal{M}_y is a left ideal.

Remark 3.6. Note that the involution $\bar{}$ can be restricted to \mathcal{M}_0 as it sends C_w to $C_{w^{-1}}$ and c_0 is easily seen to be stable under taking the inverse.

3.4. Relative Kazhdan-Lusztig polynomials. We denote by X_0 the set of representative of minimal length of the left cosets of W_0 in W_e . We have

$$X_0 = \{x \in W_e \mid xw_0 = x \bullet w_0\} \quad \text{and} \quad X_0^{-1} = \{x \in W_e \mid w_0x = w_0 \bullet x\}.$$

Theorem 3.7. *Let $y \in X_0^{-1}$. For all $x \in X_0$, there exists a unique family of polynomials $(\mathfrak{p}_{x',x})_{x' \in X_0}$ in $\mathcal{A}_{<0}$ such that $\mathfrak{p}_{x',x} = 0$ whenever $x' \not\leq x$ and*

$$(3.7) \quad \tilde{C}_{xw_0y} = T_x C_{w_0y} + \sum_{x' \in X_0} \mathfrak{p}_{x',x} T_{x'} C_{w_0y}$$

is stable under the $\bar{}$ involution.

Proof. The proof of this theorem is given in [7] (see Lemma 5.5, Lemma 5.6 and Proposition 5.7). It is based on the fact that the \mathcal{A} -submodule

$$\mathcal{M} := \langle T_x C_{w_0y} \mid x \in X_0 \rangle_{\mathcal{A}}$$

is a left ideal of \mathcal{H} for all $y \in X_0^{-1}$. Then the construction of the polynomials $\mathfrak{p}_{x',x}$ only depends on the action of the T_s 's on the elements $T_x C_{w_0y}$. We show here that \mathcal{M} is indeed a left ideal and we describe the action of T_s on $T_x C_{w_0y}$ as we will need it later on.

Fix $T_x C_{w_0y} \in \mathcal{M}$ and $s \in S$. In order to show that \mathcal{M} is a left ideal, it is enough to show that $T_s T_x C_{w_0y} \in \mathcal{M}$. To simplify we will assume that $x \in W$, the case where $x = \pi x'$ with $(\pi, x) \in \Pi \times W$ is similar. According to Deodhar's lemma (see [5, Lemma 2.1.2]), there are three cases to consider

- (1) $sx \in X_0$ and $\ell(sx) > \ell(x)$. Then $T_s T_x C_{w_0y} = T_{sx} C_{w_0y}$.
- (2) $sx \in X_0$ and $\ell(sx) < \ell(x)$. Then $T_s T_x C_{w_0y} = T_{sx} C_{w_0y} + (q^{L(s)} - q^{-L(s)}) T_x C_{w_0y}$.
- (3) $t := x^{-1}sx \in S_0$. Then $\ell(sx) = \ell(x) + 1 = \ell(xt)$. Now, since $tw_0y < w_0y$, we have [13, §5.5, Theorem 6.6.b]

$$T_t C_v = q^{L(t)} C_v.$$

Thus, we see that

$$T_s T_x C_{w_0y} = T_{sx} C_{w_0y} = T_{xt} C_{w_0y} = T_x T_t C_{w_0y} = q^{L(t)} T_x C_{w_0y}$$

In all cases, we do have $T_s T_x C_{w_0y} \in \mathcal{M}$ as required \square

Looking at the proof, we see that the action of T_s on \mathcal{M} do not depend on y . This means that the construction of the polynomials $\mathfrak{p}_{x',x}$ do not depend on y either. As a consequence, there exists a unique family of polynomials $(\mathfrak{p}_{x',x})_{x' \in X_0}$ in $\mathcal{A}_{<0}$ such that $\mathfrak{p}_{x',x} = 0$ whenever $x' \not\leq x$ and such that:

$$(3.7) \quad \tilde{C}_{xw_0y} = T_x C_{w_0y} + \sum_{x' \in X_0} \mathfrak{p}_{x',x} T_{x'} C_{w_0y}$$

is stable under the bar involution for all $y \in X_0$.

Theorem 3.8. *Let $y \in \mathbf{B}_0^{-1}$. We have $\tilde{C}_{xw_0y} = C_{xw_0y}$ for all $x \in X_0$. For all $y \in \mathbf{B}_0^{-1}$ the \mathcal{A} -submodule:*

$$\mathcal{M}_y = \langle C_{xw_0y} \mid x \in X_0 \rangle_{\mathcal{A}}$$

is a left ideal.

Proof. According to the proof of [7, Proposition 5.7] we have

$$\tilde{C}_{xw_0y} \equiv T_{xw_0y} \pmod{\mathcal{H}_{\leq 0}}.$$

We get that $\tilde{C}_{xw_0y} = C_{xw_0y}$ using Lemma 5.3. Next using the previous theorem we see that

$$\langle T_x C_{w_0y} \mid x \in X_0 \rangle_{\mathcal{A}} = \langle \tilde{C}_{xw_0y} \mid x \in X_0 \rangle$$

hence the second part of the theorem. \square

Remark 3.9. Using the \flat -involution, we obtain right handed versions of all the results in this section. For instance, for all $y \in X_0$ and $x \in X_0^{-1}$, there exists a unique family of polynomials $(p_{x',y}^r)_{x' \in X_0, x' < y}$ in $\mathcal{A}_{< 0}$ such that

$$\tilde{C}_{yw_0x} = C_{yw_0x} + \sum_{x' \in X_0^{-1}} p_{x',yw_0}^r C_{yw_0} T_{x'} = C_{yw_0} (T_x + \sum_{x' \in X_0^{-1}} p_{x',yw_0}^r T_{x'})$$

is stable under the bar involution. Also, if $y \in \mathbf{B}_0^{-1}$, then we have $\tilde{C}_{yw_0x} = C_{yw_0x}$ for all $x \in X_0$.

4. ON THE CELLULAR STRUCTURE

In this section we prove the main result of this paper, that the lowest two-sided ideal \mathcal{M}_0 of W_e is affine cellular in the sense of Koenig and C.Xi [10].

4.1. Affine cell ideal. Let k be a principal ideal domain. For a k -algebra A , a k -linear anti-automorphism i of A satisfying $i^2 = id_A$ is called a k -involution on A . A quotient $B = k[t_1, \dots, t_r]/I$ where I is an ideal of $k[t_1, \dots, t_r]$ will be called an affine k -algebra.

For an affine k -algebra B with a k -involution ν , a free k -module V of finite rank and a k -bilinear form $\varphi : V \times V \rightarrow B$, denote by $\mathbb{A}(V, B, \varphi)$ the (possibly non-unital) algebra given as a k -module by $V \otimes_k B \otimes_k V$, on which we impose the multiplication $(v_1 \otimes b_1 \otimes w_1)(v_2 \otimes b_2 \otimes w_2) := v_1 \otimes b_1 \varphi(w_1, v_2) b_2 \otimes w_2$.

Remark 4.1. Let Ψ be the matrix representing the bilinear form ψ with respect to some choice of basis $\{v_1, \dots, v_n\}$ of V . Then the algebra $\mathbb{A}(V, B, \varphi)$ is nothing else than a generalised matrix algebra over B , that is the ordinary matrix algebra with coefficients in B in which the multiplication is twisted by Ψ . Indeed we can identify an element

$$\sum_{1 \leq i, j \leq n} v_i \otimes b_{i,j} \otimes v_j$$

of $\mathbb{A}(V, B, \varphi)$ with a $n \times n$ matrix $M = (b_{i,j})_{1 \leq i, j \leq n}$ with coefficients in B . The multiplication of M_1, M_2 is then defined to be

$$M_1 \cdot M_2 = M_1 \Psi M_2.$$

In this paper, we will use the following definition of affine cell ideal.

Definition 4.2. [10, Proposition 2.3] Let k be a principal ideal domain, A a unitary k -algebra with a k -involution i . A two-sided ideal J in A is an affine cell ideal if and only if there exists an affine k -algebra B with a k -involution ν , a free k -module V of finite rank and a bilinear form $\varphi : V \otimes V \rightarrow B$, and an A - A -bimodule structure on $V \otimes_k B \otimes_k V$, such that $J \cong \mathbb{A}(V, B, \varphi)$ as an algebra and as an A - A -bimodule, and such that under this isomorphism the k -involution i restricted to J corresponds to the k -involution given by $v \otimes b \otimes w \mapsto w \otimes \nu(b) \otimes v$.

An algebra A is then affine cellular if one can find a chain of two-sided ideals $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$, such that each subquotient J_i/J_{i-1} is an affine cell ideal in A/J_{i-1} . Then J_i/J_{i-1} is isomorphic to $\mathbb{A}(V_i, B_i, \varphi_i)$ for some finite-dimensional vector space V_i , a commutative k -algebra B_i and a bilinear form $\varphi_i : V_i \times V_i \rightarrow B_i$. Let (ϕ_{st}^i) be the matrix representing the bilinear form ϕ with respect to some choice of basis of V_i . Then Koenig and Xi obtain a parametrisation of simple modules of an affine cellular algebra by establishing a bijection between isomorphism classes of simple A -modules and the set

$$\{(j, \mathfrak{m}) \mid 1 \leq j \leq n, \mathfrak{m} \in \text{MaxSpec}(B_j) \text{ such that some } \phi_{st}^j \notin \mathfrak{m}\}$$

where $\text{MaxSpec}(B_j)$ denotes the maximal ideal spectrum of B_j .

Remark 4.3. The notion of affine cellularity should really be understood as a generalisation of cellularity for finite dimensional algebra as defined by Graham and Lehrer in [6]. In fact, we obtain the definition of cellularity for finite dimensional algebras by setting $B = k$ in the above definition. We refer to [11] for details.

4.2. The elements \mathbf{P} . We now introduce the \mathbf{P} -elements, which are crucial ingredients for the definition of the cellular basis. Recall the definition of the polynomials $\mathfrak{p}_{x,x'} \in \mathcal{A}$ for $x, x' \in X_0$ in Section 3.4. We (have) set for all $z \in \mathbf{B}_0$ and $y \in \mathbf{B}_0^{-1}$

$$\begin{aligned} N_y &:= \{xw_0y \mid x \in X_0\} & \mathcal{M}_y &:= \langle C_w \mid w \in N_y \rangle_{\mathcal{A}} \\ N_z^{\mathcal{R}} &:= \{zw_0x \mid x \in X_0^{-1}\} & \mathcal{M}_z^{\mathcal{R}} &:= \langle C_w \mid w \in N_z^{\mathcal{R}} \rangle_{\mathcal{A}}. \end{aligned}$$

We also set

$$\mathcal{M}_+ := \mathcal{M}_1 \cap \mathcal{M}_1^{\mathcal{R}} = \langle C_{p_\tau w_0} \mid \tau \in P^+ \rangle_{\mathcal{A}}.$$

Definition 4.4. For $z \in \mathbf{B}_0$ and $\omega \in \mathbf{P}^+$ we set

$$\mathbf{P}(z) = \sum_{x \in X_0} \mathfrak{p}_{x,z} T_x \quad \text{and} \quad \mathbf{P}(\omega) = \sum_{x \in X_0} \mathfrak{p}_{x,p_\omega} T_x$$

and

$$\mathbf{P}_R(z^{-1}) = \sum_{x \in X_0^{-1}} \mathfrak{p}_{x,z^{-1}}^r T_x \quad \text{and} \quad \mathbf{P}_R(-\omega) = \sum_{x \in X_0^{-1}} \mathfrak{p}_{x,p_\omega^{-1}}^r T_x.$$

By definition, we see that for all $y \in \mathbf{B}_0^{-1}$, $z \in \mathbf{B}_0$ and $\omega \in \mathbf{P}^+$ we have

$$\mathbf{P}(z)C_{w_0y} = C_{zw_0y} \quad \text{and} \quad \mathbf{P}(\omega)C_{w_0y} = C_{p_\omega w_0y}$$

and

$$C_{zw_0} \mathbf{P}_R(y) = C_{zw_0y} \quad \text{and} \quad C_{zw_0} \mathbf{P}_R(-\omega) = C_{zw_0 p_\omega^{-1}}.$$

Further, we have $(\mathbf{P}(z))^b = \mathbf{P}_R(z^{-1})$ and $(\mathbf{P}(\omega))^b = \mathbf{P}_R(-\omega)$.

The element w_0 acts on the set of roots. For instance in type A , if we keep the notation of Example 2.2, we have $w_0(\alpha_i) = -\alpha_{n-i+1}$ for all $i = 1, \dots, n$. In type $B, C, D_{2n}, F, E_7, E_8, G$, w_0 simply acts as minus the identity. We refer to the tables in [2] for a precise description of this action. Then the action of $-w_0$ induces

- a permutation of \mathbf{P}^+ which in turn induces an action on P^+ ;
- an automorphism of W_0 which sends the simple reflection with respect to α to the simple reflection associated to $-w_0(\alpha)$.

We will denote both the action on P^+ and the group automorphism by ν . It can easily be shown that the group automorphism ν of W_0 extends to a group automorphism of W_a and W_e which, when non-trivial, is a diagram automorphism. In turn, we obtain an automorphism of \mathcal{H} defined by $\nu(T_w) = T_{\nu(w)}$ for all $w \in W_e$. Now by definition of ν we obtain the following important equalities:

$$\omega w_0 = -w_0 \nu(\omega) \quad \text{and} \quad p_\omega w_0 = w_0 p_{\nu(\omega)}^{-1} \quad \text{for all } \omega \in P^+.$$

Proposition 4.5. *Let $\omega, \omega' \in \mathbf{P}^+$. We have*

- (1) $\mathbf{P}(\omega)C_{w_0} = C_{w_0}\mathbf{P}_R(-\nu(\omega))$;
- (2) $\mathbf{P}(\omega)\mathbf{P}(\omega')C_{w_0} = \mathbf{P}(\omega')\mathbf{P}(\omega)C_{w_0}$.

Proof. We prove (1). For all $\omega \in \mathbf{P}^+$ we have

$$\mathbf{P}(\omega)C_{w_0} = C_{p_\omega w_0} = C_{w_0 p_\nu^{-1}} = C_{w_0}\mathbf{P}_R(-\nu(\omega)).$$

We prove (2). First we show that ν and \flat commutes. Indeed we have for all $w \in W_e$

$$\nu(C_w^\flat) = \nu(C_{w^{-1}}) = C_{\nu(w^{-1})} = C_{(\nu(w))^{-1}} = C_{\nu(w)}^\flat.$$

Next we show that $\nu \circ \flat$ acts as the identity on $\mathbf{P}(\omega)\mathbf{P}(\omega')C_{w_0}$. Since

$$\mathbf{P}(\omega)\mathbf{P}(\omega')C_{w_0} = C_{w_0}\mathbf{P}_R(-\nu(\omega))\mathbf{P}_R(-\nu(\omega'))$$

we see that the element $\mathbf{P}(\omega)\mathbf{P}(\omega')C_{w_0}$ lies in $\mathcal{M}_1 \cap \mathcal{M}_1^{\mathcal{R}} = \mathcal{M}_+$. Hence $\mathbf{P}(\omega)\mathbf{P}(\omega')C_{w_0}$ is an \mathcal{A} -linear combination of elements of the form $C_{p_\tau w_0}$. Now we have

$$\nu(C_{p_\tau w_0})^\flat = \nu(C_{w_0 p_\tau^{-1}}) = \nu(C_{p_{\nu(\tau)} w_0}) = C_{p_\tau w_0}.$$

On the one hand we have

$$\nu(\mathbf{P}(\omega)\mathbf{P}(\omega')C_{w_0})^\flat = \mathbf{P}(\omega)\mathbf{P}(\omega')C_{w_0}$$

and on the other hand

$$\begin{aligned} \nu(\mathbf{P}(\omega)\mathbf{P}(\omega')C_{w_0})^\flat &= \nu(C_{w_0}^\flat \mathbf{P}(\omega')^\flat \mathbf{P}(\omega)^\flat) \\ &= \nu(C_{w_0} \mathbf{P}_R(-\omega') \mathbf{P}_R(-\omega)) \\ &= \nu(\mathbf{P}(\nu(\omega')) \mathbf{P}(\nu(\omega)) C_{w_0}) \\ &= \mathbf{P}(\omega') \mathbf{P}(\omega) C_{w_0} \end{aligned}$$

and the result follows. \square

Definition 4.6. For $\tau = \omega_1 + \omega_2 \dots + \omega_k \in P^+$ (with $\omega_i \in \mathbf{P}^+$ for all i) we set

$$\mathbf{P}(\tau) = \mathbf{P}(\omega_1)\mathbf{P}(\omega_2)\dots\mathbf{P}(\omega_k)$$

and

$$\mathbf{P}_R(-\tau) = \mathbf{P}_R(-\omega_k)\mathbf{P}_R(-\omega_{k-1})\dots\mathbf{P}_R(-\omega_1).$$

Note that since P^+ is an abelian group, we need part (2) of the previous proposition for the elements $\mathbf{P}(\tau)$ to be well-defined.

Remark 4.7. By construction, we have

$$\mathbf{P}(z)C_{w_0 y} = C_{z w_0 y} \text{ and } \mathbf{P}(\omega)C_{w_0 y} = C_{p_\omega w_0 y}$$

for all $y \in \mathbf{B}_0^{-1}$, $z \in \mathbf{B}_0$ and $\omega \in \mathbf{P}^+$. It is important to notice however that we *do not* have $\mathbf{P}(\tau)C_{w_0 y} = C_{p_\tau w_0 y}$. Indeed, we *did not* define $\mathbf{P}(\tau)$ as $\sum_{x \in X_0} \mathfrak{p}_{x, p_\tau} T_x$. The decomposition of $\mathbf{P}(\tau)C_{w_0 y}$ in the Kazhdan-Lusztig basis will be discuss in Section 5.

Lemma 4.8. (1) *The set $\{\mathbf{P}(\tau)C_{w_0} \mid \tau \in P^+\}$ is an \mathcal{A} -basis of \mathcal{M}_+ .*

(2) *The set $\{\mathbf{P}(z)\mathbf{P}(\tau)C_{w_0 y} \mid z \in \mathbf{B}_0, \tau \in P^+\}$ is an \mathcal{A} -basis of \mathcal{M}_y for all $y \in \mathbf{B}_0^{-1}$.*

Proof. (1) Since $\mathbf{P}(\tau)C_{w_0} = C_{w_0}\mathbf{P}_R(-\tau)$ we see that

$$\mathbf{P}(\tau)C_{w_0} \in \mathcal{M}_1 \cap \mathcal{M}_1^{\mathcal{R}} = \mathcal{M}_+.$$

Then the result follows easily by a triangularity property:

$$\mathbf{P}(\tau)C_{w_0} = C_{p_\tau w_0} + \sum_{z < p_\tau w_0} \mathcal{A} C_z.$$

(2) It is clear that $\mathbf{P}(z)\mathbf{P}(\tau)C_{w_0 y} \in \mathcal{M}_y$ since $C_{w_0 y} \in \mathcal{M}_y$ and \mathcal{M}_y is a left ideal. Once again, the result follows easily by a triangularity argument. \square

4.3. Main result. As our principal ideal domain k , we choose \mathcal{A} and we set B to be the monoid algebra $\mathcal{A}[P^+] = \{e^\tau \mid \tau \in P^+\}$. Note that B is isomorphic to the ring of polynomials in $\text{Card}(\mathbf{P}^+)$ indeterminates. The proof is similar to the one in [8]. The only difference here is that we need to introduce an involution on B defined by $\nu e^\tau = e^{\nu(\tau)}$ where ν is the involution introduced at the beginning of Section 4.2. It did not appear in [8] because in type G_2 and B_2 , it is simply the identity.

Let $z, z' \in \mathbf{B}_0$. On the one hand we have

$$C_{w_0 z^{-1}} C_{z'^{-1} w_0} \in C_{w_0 z^{-1}} \mathcal{H} \in \mathcal{M}_1^{\mathcal{B}}$$

and on the other hand

$$C_{w_0 z^{-1}} C_{z'^{-1} w_0} \in \mathcal{H} C_{z'^{-1} w_0} \in \mathcal{M}_1$$

therefore $C_{w_0 z^{-1}} C_{z'^{-1} w_0} \in \mathcal{M}_+$. Thus, by Lemma 4.8, we have

$$C_{w_0 z^{-1}} C_{z'^{-1} w_0} = \sum_{\tau \in P^+} a_\tau^{z, z'} \mathbf{P}(\tau) C_{w_0} \text{ where } a_\tau^{z, z'} \in \mathcal{A}.$$

Let V be the free \mathcal{A} -module of rank $\text{Card}(\mathbf{B}_0)$ on basis $\{v_z, z \in \mathbf{B}_0\}$ and define the \mathcal{A} -bilinear form φ by

$$\begin{aligned} \varphi: V \times V &\longrightarrow B \\ (v_z, v_{z'}) &\longmapsto \sum_{\tau \in P^+} a_\tau^{z, z'} e^\tau. \end{aligned}$$

This defines an algebra $\mathbb{A}(V, B, \varphi) \cong V \otimes_{\mathcal{A}} B \otimes_{\mathcal{A}} V$ with multiplication \mathcal{A} -bilinearly extended from $(v_{z_i} \otimes e^\tau \otimes v_{z_j})(v_{z_k} \otimes e^{\tau'} \otimes v_{z_\ell}) = v_{z_i} \otimes e^\tau \varphi(v_{z_j}, v_{z_k}) e^{\tau'} \otimes v_{z_\ell}$ as in Section 4.1.

We now define a map

$$\Phi: \mathbb{A}(V, B, \varphi) \rightarrow \mathcal{H}$$

by

$$v_z \otimes e^\tau \otimes v_{z'^{-1}} \mapsto \mathbf{P}(z) \mathbf{P}(\tau) C_{w_0} \mathbf{P}_R(z'^{-1})$$

for basis elements $v_z, v_{z'}$ of V and $\tau \in P^+$. We have

$$\mathbf{P}(z) \mathbf{P}(\tau) C_{w_0} \mathbf{P}_R(z'^{-1}) \in \mathcal{H} C_{w_0} \mathcal{H} \subseteq \sum_{z \leq \mathcal{L} \mathcal{R} w_0} \mathcal{A} C_z.$$

Hence, the image of Φ is contained in \mathcal{M}_0 , the two-sided ideal associated to the lowest two-sided cell c_0 .

Theorem 4.9. *The two-sided ideal \mathcal{M}_0 is a an affine cell ideal. More precisely:*

- (1) *The map $\Phi: \mathbb{A}(V, B, \varphi) \rightarrow \mathcal{M}_0$ is an isomorphism of \mathcal{A} -algebras.*
- (2) *Using (1) to define left and right \mathcal{H} -module structures on $\mathbb{A}(V, B, \varphi)$ by letting $h \in \mathcal{H}$ act, for $v, w \in V$ and $b \in B$, as*

$$h(v \otimes b \otimes w) = \Phi^{-1}(h \Phi(v \otimes b \otimes w))$$

and

$$(v \otimes b \otimes w)h = \Phi^{-1}(\Phi(v \otimes b \otimes w)h)$$

respectively, Φ is an isomorphism of \mathcal{H} - \mathcal{H} -bimodules.

- (3) *We have $\Phi(v \otimes b \otimes w)^b = \Phi(w \otimes \nu(b) \otimes v)$ for $v, w \in V$ and $b \in B$ and where ν is the involution defined by $\nu(e^\tau) = e^{\nu(\tau)}$.*

Proof. The map Φ is \mathcal{A} -linear by definition. We have, for basis elements $v_{z_i}, v_{z_j}, v_{z_k}, v_{z_\ell}$ of V and $\tau, \tau' \in P^+$,

$$\begin{aligned}
& \Phi(v_{z_i} \otimes e^\tau \otimes v_{z_j}) \Phi(v_{z_k} \otimes e^{\tau'} \otimes v_{z_\ell}) \\
&= \mathbf{P}(z_i) \mathbf{P}(\tau) C_{w_0} \mathbf{P}_R(z_j^{-1}) \mathbf{P}(z_k) \mathbf{P}(\tau') C_{w_0} \mathbf{P}_R(z_\ell^{-1}) \\
&= \mathbf{P}(z_i) \mathbf{P}(\tau) C_{w_0 z_j^{-1}} \mathbf{P}(z_k) C_{w_0} \mathbf{P}_R(-\tau') \mathbf{P}_R(z_\ell^{-1}) \\
&= \mathbf{P}(z_i) \mathbf{P}(\tau) C_{w_0 z_j^{-1}} C_{z_k w_0} \mathbf{P}_R(-\tau') \mathbf{P}_R(z_\ell^{-1}) \\
&= \mathbf{P}(z_i) \mathbf{P}(\tau) \left(\sum_{\tau \in P^+} a_{\tau}^{z_j, z_k} \mathbf{P}(\tau) \right) C_{w_0} \mathbf{P}_R(-\tau') \mathbf{P}_R(z_\ell) \\
&= \mathbf{P}(z_i) \mathbf{P}(\tau) \left(\sum_{\tau \in P^+} a_{\tau}^{z_j, z_k} \mathbf{P}(\tau) \right) \mathbf{P}(\tau') C_{w_0} \mathbf{P}_R(z_\ell^{-1}) \\
&= \Phi(v_{z_i} \otimes e^\tau \varphi(v_{z_j}, v_{z_k}) e^{\tau'} \otimes v_{z_\ell}).
\end{aligned}$$

So Φ is a morphism of \mathcal{A} -algebras. The fact that Φ is bijective follows easily from the fact that

$$\mathbf{P}(z_i) \mathbf{P}(\tau) C_{w_0} \mathbf{P}_R(z_j^{-1}) = C_{z_i p_\tau w_0 z_j^{-1}} + \sum_{z < z_i p_\tau w_0 z_j^{-1}} \mathcal{A} C_z.$$

This completes the proof of (1). Statement (2) follows directly from the definition and the fact that \mathcal{M}_0 is an \mathcal{H} - \mathcal{H} -bimodule. We prove Claim (3). Let v_{z_i}, v_{z_j} be basis elements of V and $\tau \in P^+$. We have

$$\begin{aligned}
\Phi(v_{z_i} \otimes e^\tau \otimes v_{z_j})^b &= (\mathbf{P}(z_i) \mathbf{P}(\tau) C_{w_0} \mathbf{P}_R(z_j^{-1}))^b \\
&= (\mathbf{P}_R(z_j^{-1}))^b (C_{w_0})^b (\mathbf{P}(\tau))^b (\mathbf{P}(z_i))^b \\
&= \mathbf{P}(z_j) C_{w_0} \mathbf{P}_R(-\tau) \mathbf{P}_R(z_i^{-1}) \\
&= \mathbf{P}(z_j) \mathbf{P}(\nu(\tau)) C_{w_0} \mathbf{P}_R(z_i^{-1}) \\
&= \Phi(v_{z_j} \otimes \nu(e^\tau) \otimes v_{z_i}).
\end{aligned}$$

Statement (3) follows from \mathcal{A} -linearity. □

5. ON THE CELLULAR BASIS

In this section, we study the decomposition of the cellular basis in the Kazhdan-Lusztig basis. We set $\mathbf{B}_0 := \{z_1, \dots, z_n\}$ and for all $1 \leq i, j \leq n$ and $\tau \in P^+$. We will show that

$$(*) \quad \mathbf{P}(\tau) C_{w_0} = \sum_{\tau' \in P^+} m_{\tau'} C_{p_{\tau'} w_0}$$

where the coefficients $m_{\tau'}$ are integers. Note that, if we prove (*) then we can deduce the expression of any element of the cellular basis in the Kazhdan-Lusztig basis using Corollary 5.4 below:

$$\mathbf{P}(z_i) \mathbf{P}(\tau) C_{w_0} \mathbf{P}_R(z_j^{-1}) = \mathbf{P}(z_i) \left(\sum m_{\tau'} C_{p_{\tau'} w_0} \right) \mathbf{P}_R(z_j^{-1}) = \sum m_{\tau'} C_{z_i p_{\tau'} w_0 z_j^{-1}}.$$

Remark 5.1. In this section, it will sometime be more convenient to enumerate the elements of the lowest two-sided cell as $z_i w_0 p_\lambda z_j$ with $\lambda \in P^- = -P^+$ instead of $z_i p_\tau w_0 z_j$ with $\tau \in P^+$. Indeed, these are the same since we have $p_\tau w_0 = w_0 p_{\nu(\tau)}^{-1} = w_0 p_{-\nu(\tau)}$. As much as possible in order to avoid confusion, we will use the letter λ for negative weights and τ for positive weights.

5.1. **Multiplication of the standard and the Kazhdan-Lusztig bases.** For $x, y \in W_e$, we set

$$T_x T_y = \sum_{z \in W_e} f_{x,y,z} T_z \text{ where } f_{x,y,z} \in \mathcal{A}.$$

In this section, we present some results of [7, §2.3] that will be needed later on and we introduce some notation. For $\alpha \in \Phi^+$, we have set $\mathcal{F}_\alpha = \{H_{\alpha,n} \mid n \in \mathbb{Z}\}$. For $x, y \in W$ we set

$$H_{x,y} = \{H \in \mathcal{F} \mid H \in H(A_0, yA_0) \cap H(yA_0, xyA_0)\} \text{ and} \\ I_{x,y} = \{\alpha \in \Phi^+ \mid H_{x,y} \cap \mathcal{F}_\alpha \neq \emptyset\}.$$

Finally for $\alpha \in I_{x,y}$, let

$$c_{x,y}(\alpha) = \max_{H \in H_{x,y} \cap \mathcal{F}_\alpha} L_H \text{ and } c_{x,y} = \sum_{\alpha \in I_{x,y}} c_{x,y}(\alpha).$$

Then we have the following result on the degree of the structure constants $f_{x,y,z}$ with respect to the standard basis [7, Theorem 2.4].

Theorem 5.2. *Let $x, y \in W_e$. We have $\deg(f_{x,y,z}) \leq c_{x,y}$ for all $z \in W_e$.*

Proposition 5.3. *Let $\tau \in P^+$, $y \in X_0^{-1}$, $x \in X_0$ and $v \in W_0$. The following holds*

- (1) $c_{x,vy} \leq L(w_0) - L(v)$ for all $v \in W_0$; if $x \in \mathbf{B}_0$ then $c_{x,vy} < L(w_0) - L(v)$
- (2) $T_x C_{w_0 y} \equiv \sum_{v \in W_0} q^{L(v) - L(w_0)} T_x T_{vy} \pmod{\mathcal{H}_{<0}}$;
- (3) $T_x C_{w_0 y} \in \mathcal{H}_{\leq 0}$.

Proof. Fix $v \in W_0$. Let $u = w_0 v^{-1}$ so that $w_0 = u \cdot v$. Write $w_0 = s_{i_n} \dots s_{i_{m+1}} s_{i_m} \dots s_{i_1}$ where $u = s_{i_n} \dots s_{i_{m+1}}$ and $v = s_{i_m} \dots s_{i_1}$ are reduced expression. Let $H_{i_n}, \dots, H_{i_{m+1}}$ (respectively H_{i_m}, \dots, H_{i_1}) with direction $\alpha_{i_n}, \dots, \alpha_{i_{m+1}}$ (respectively $\alpha_{i_m}, \dots, \alpha_{i_1}$) be the set of hyperplanes separating vyA_0 and w_0yA_0 (respectively yA_0 and vyA_0). Let τ be the unique L -weight lying in the intersection of all the H_{i_k} . Then since $x \in X_0$, we see that $xvyA_0$ lies in the quarter \mathcal{C} with vertex τ which contains vyA_0 .

Let $1 \leq r \leq m$ and let $k \in \mathbb{Z}$ such that $H_{i_r} = H_{\alpha_{i_r}, k}$. We will assume that $k > 0$, the case $k \leq 0$ being similar. We have $vyA_0 \in V_{H_{i_r}}^+$. Now, since τ lies in the closure of $vp_\tau A_0$ and $\tau \in H_{i_r}$, one can see that

$$k < \langle \mu, \check{\alpha}_{i_r} \rangle < k + 1 \text{ for all } \mu \in vyA_0.$$

Moreover, $xvyA_0 \in \mathcal{C}$ implies that

$$k < \langle \mu, \check{\alpha}_{i_r} \rangle \text{ for all } \mu \in xvyA_0.$$

From there, we conclude that all the hyperplanes $H_{\alpha_{i_r}, l}$ with $l \leq k$ do not lie in $H(vyA_0, xvyA_0)$ and that all the hyperplanes $H_{\alpha_{i_r}, l}$ with $l > k$ do not lie in $H(A_0, vyA_0)$. Thus $\alpha_{i_r} \notin I_{x,vy}$ and we have

$$I_{x,vy} \subset \{\alpha_{i_{m+1}}, \dots, \alpha_{i_n}\},$$

which implies

$$c_{x,vy} \leq \sum_{r=m+1}^{r=n} c_{x,vy}(\alpha_{i_r}) = L(w_0) - L(v)$$

as required. In the case where $x \in \mathbf{B}_0$, we see that

- if W is not of type C or if W is of type C and $L(t) = L(t')$ then $\alpha_{i_{m+1}} \notin I_{x,vy}$;
- if W is of type C and $L(t) > L(t')$, then we may have $\alpha_{i_{m+1}} \in I_{x,vy}$ but in this case the hyperplane of direction $\alpha_{i_{m+1}}$ that lies in $H_{x,vy}$ cannot be of maximal weight.

In both cases we see that

$$c_{x,vy} < \sum_{r=m+1}^{r=n} c_{x,vy}(\alpha_i) = L(w_0) - L(v).$$

Next we have

$$\begin{aligned} T_x C_{w_0 y} &= T_x \left(\sum_{z \in W_e} P_{z, w_0 y} T_z \right) \\ &= T_{x w_0 y} + \sum_{z < w_0 y} P_{z, w_0 y} T_x T_z. \end{aligned}$$

Write $z = vy'$ where $y' \in X_0^{-1}$ and $v \in W_0$ (any element of W_e can uniquely be written in this way; [5]). Then, using the recursive formula for Kazhdan-Lusztig polynomials, we see that

$$P_{z, w_0 y} = P_{vy', w_0 y} = q^{L(v) - L(w_0)} P_{w_0 y', w_0 y}.$$

Since the degree of the polynomials appearing in the expression of $T_x T_z$ is at most $c_{x,z} = c_{x,vy'}$ and $c_{x,vy'} \leq L(w_0) - L(v)$ we see that we must have $P_{z, w_0 y} T_x T_z \in \mathcal{H}_{\leq 0}$. More precisely, if $w_0 y' < w_0 y$, then $P_{w_0 y', w_0 y} \in \mathcal{A}_{< 0}$ and $P_{z, w_0 y} T_x T_z \in \mathcal{H}_{< 0}$ in this case. Putting all this together, we get

$$\begin{aligned} T_x C_{w_0 y} &= T_{x w_0 y} + \sum_{z < w_0 y} P_{z, w_0 y} T_x T_z \\ &\equiv T_{x w_0 y} + \sum_{z \in W_0 y} P_{z, w_0 y} T_x T_z \pmod{\mathcal{H}_{< 0}} \\ &\equiv T_{x w_0 y} + \sum_{v \in W_0} P_{vy, w_0 y} T_x T_{vy} \pmod{\mathcal{H}_{< 0}} \\ &\equiv T_{x w_0 y} + \sum_{v \in W_0} q^{L(v) - L(w_0)} T_x T_{vy} \pmod{\mathcal{H}_{< 0}} \end{aligned}$$

as required in (2). Statement (3) is a direct consequence of (1) and the above relation. \square

Corollary 5.4. *For all $z, z' \in \mathbf{B}_0$ and $\tau \in P^+$ we have*

$$\mathbf{P}(z) C_{p_\tau w_0} \mathbf{P}_R(z'^{-1}) = C_{z p_\tau w_0 z'^{-1}}$$

Proof. Let $z \in \mathbf{B}_0$, $\tau \in P^+$ and $y \in X_0^{-1}$. We have

$$\begin{aligned} \mathbf{P}(z) C_{p_\tau w_0 y} &= \sum_{x \in X_0, x \leq z} \mathfrak{p}_{x,z} T_x C_{w_0 p_{\nu(\tau)}^{-1} y} \\ &= T_z C_{w_0 p_{\nu(\tau)}^{-1} y} + \sum_{x < z, x \in X_0} \mathfrak{p}_{x,z} T_x C_{w_0 p_{\nu(\tau)}^{-1} y} \\ &\equiv T_{z p_\tau w_0 y} \pmod{\mathcal{H}_{< 0}} \end{aligned}$$

by the proposition above and since $T_x C_{p_\tau w_0 y} \in \mathcal{H}_{\leq 0}$ and $\mathfrak{p}_{x,z} \in \mathcal{A}_{< 0}$. Further, by definition, we know that $\mathbf{P}(z) C_{p_\tau w_0 y}$ is stable under the $\bar{}$ -involution, hence we get $\mathbf{P}(z) C_{p_\tau w_0 y} = C_{z p_\tau w_0 y}$. In particular we have

$$P(z) C_{p_\tau w_0} = C_{z p_\tau w_0} \quad \text{and} \quad C_{p_\tau w_0} \mathbf{P}_R(z^{-1}) = C_{p_\tau w_0 z^{-1}}$$

the last equality being easily obtained using the $\bar{}$ -involution. Putting all this together, we get

$$\mathbf{P}(z) C_{p_\tau w_0} \mathbf{P}(z'^{-1}) = \mathbf{P}(z) C_{p_\tau w_0 z'^{-1}} = C_{z p_\tau w_0 z'^{-1}}$$

as required. \square

Let $x, w \in W$ and $x = \pi_x s_N \dots s_1$ be a reduced expression of x . We denote $\mathfrak{J}_{x,w}$ the collection of all subsets $I = \{i_1, \dots, i_p\}$ such that $1 \leq i_1 < \dots < i_p \leq N$ and

$$s_{i_t} \dots \hat{s}_{i_{t-1}} \dots \hat{s}_{i_1} \dots s_1 w < \hat{s}_{i_t} \dots \hat{s}_{i_{t-1}} \dots \hat{s}_{i_1} \dots s_1 w.$$

For $I = \{i_1, \dots, i_p\} \in \mathfrak{J}_{x,w}$ we set $x_I = s_N \dots \hat{s}_{i_p} \dots \hat{s}_{i_1} \dots s_1$. Then we have [3, Proof of Proposition 5.1]

$$(5.4) \quad T_x T_y = \sum_{I \in \mathfrak{J}_{x,y}} \xi_{s_{i_1}} \dots \xi_{s_{i_p}} T_{x_I y}.$$

Note that by definition we have $x_I \leq x$ in the Bruhat order, therefore

$$T_x T_y = \sum_{z \leq x} a_z T_{zy} \text{ where } a_z \in \mathcal{A}.$$

We will write

$$T_x T_y \approx T_x T_{y'}$$

if and only if

$$T_x T_y = \sum_{z \leq x} a_z T_{zy} \quad \text{and} \quad T_x T_{y'} = \sum_{z \leq x} a_z T_{zy'}.$$

Let $\alpha \in \Phi^+$ and $\omega \in \mathbf{P}^+$. We define the following integer:

$$m_\alpha(\omega) := \max_{\substack{x \in [1, p_\omega] \\ v \in W_0}} \left| (xvA_0)_\alpha \right|$$

where $[1, p_\omega]$ denotes the Bruhat interval from the identity to p_ω and $(xvA_0)_\alpha$ is the largest integer k in absolute value such that $H_{\alpha,k}$ separates A_0 and xvA_0 . It is a straightforward consequence of this definition that if $H_{\alpha,k} \in H(A_0, xvA_0)$ for some $x \in [1, p_\omega]$ then $|k| \leq m_\alpha(\omega)$. We will then say that $\lambda \in P^-$ is ω -far away from the α -wall if $\lambda_\alpha \leq -m_\alpha$. For $\lambda \in P$ and $\alpha \in \Phi$ we set $\lambda_\alpha = \langle \lambda, \alpha^\vee \rangle$ and $A_\lambda := A_0 t_\lambda$.

Lemma 5.5. *Let $\lambda \in P^-$, $v \in W_0$ and $x \in [1, p_\omega]$. Then if $H_{\alpha,k} \in H(xvA_\lambda, vA_\lambda)$ we have*

$$|k - \lambda_\alpha| \leq m_\alpha(\omega).$$

Proof. Translating the relation $H_{\alpha,k} \in H(xvA_\lambda, vA_\lambda)$ by $-\lambda$ we get that $H_{\alpha,k-\lambda_\alpha} \in H(xvA_0, vA_0)$. If $H_{\alpha,k-\lambda_\alpha} \in H(xvA_0, A_0)$ then by definition of $m_\alpha(\omega)$ we get that $|k - \lambda_\alpha| \leq m_\alpha(\omega)$ as required. If $H_{\alpha,k-\lambda_\alpha} \notin H(xvA_0, A_0)$ then $H_{\alpha,k-\lambda_\alpha}$ must separate A_0 and vA_0 . Therefore $k - \lambda_\alpha = 0$ and the result is obvious. \square

The end of this section is devoted to the study of product of the form $T_{p_\omega} T_{vp_\lambda}$ where $\omega \in \mathbf{P}^+$, $\lambda \in P^-$ and $v \in W_0$.

Proposition 5.6. *Let $\lambda, \lambda' \in P^-$ and $\Phi^+ = \Phi_c^+ \sqcup \Phi_f^+$ be a disjoint union such that*

- (1) λ and λ' are ω -far away from the α -wall for all $\alpha \in \Phi_f^+$,
- (2) $\lambda_\alpha = \lambda'_\alpha$ for all $\alpha \in \Phi_c^+$.

Then $T_{p_\omega} T_{vp_\lambda} \approx T_{p_\omega} T_{vp_{\lambda'}}$ for all $v \in W_0$.

Proof. Let $p_\omega = \pi_\omega s_r \dots s_1$ be a reduced expression and let $I = \{i_1, \dots, i_p\} \in \mathfrak{J}_{p_\omega, vp_\lambda}$. For $1 \leq t \leq p$ we set $x_{i_t} = \hat{s}_{i_t} \dots \hat{s}_{i_{t-1}} \dots \hat{s}_{i_1} \dots s_1$. Then by definition of $\mathfrak{J}_{p_\omega, vp_\lambda}$ we have $s_{i_t} x_{i_t} vp_\lambda < x_{i_t} vp_\lambda$ for all $t \in \{1, \dots, p\}$. We will show that $s_{i_t} x_{i_t} vp_\lambda < x_{i_t} vp_{\lambda'}$ for all $t \in \{1, \dots, p\}$. This is enough to prove the lemma, since this will show, by exchanging the role of λ and λ' that $\mathfrak{J}_{p_\omega, vp_\lambda} = \mathfrak{J}_{p_\omega, vp_{\lambda'}}$ and the result will follow from 5.4.

Let $H_t := H_{\alpha_{i_t}, k_{i_t}}$ with $\alpha_{i_t} \in \Phi^+$ and $k_{i_t} \in \mathbb{Z}$ be the unique hyperplane separating $x_{i_t}vA_\lambda$ and $s_{i_t}x_{i_t}vA_\lambda$. Since $s_{i_t}x_{i_t}vp_\lambda < x_{i_t}vp_\lambda$ we have $H_t \in H(A_0, x_{i_t}vA_\lambda)$. We denote by H'_t the translate of H_t by $\lambda - \lambda'$. We set $k'_{i_t} = k_{i_t} + \lambda'_{\alpha_{i_t}} - \lambda_{\alpha_{i_t}}$ so that

$$H'_t := H_t t(\lambda' - \lambda) = H_{\alpha_{i_t}, k_{i_t} + \lambda'_{\alpha_{i_t}} - \lambda_{\alpha_{i_t}}} = H_{\alpha_{i_t}, k'_{i_t}}.$$

Now, $s_{i_t}x_{i_t}vA_{\lambda'}$ and $x_{i_t}vA_{\lambda'}$ are translates by $\lambda' - \lambda$ of $s_{i_t}x_{i_t}vA_\lambda$ and $x_{i_t}vA_\lambda$ respectively, therefore we have $H'_t \in H(s_{i_t}x_{i_t}vA_{\lambda'}, x_{i_t}vA_{\lambda'})$. We claim that $H'_t \in H(A_0, x_{i_t}vA_{\lambda'})$.

First of wall, assume that $\alpha_{i_t} \in \Phi_f^+$, that is λ and λ' are ω -far away from the wall. Since $s_{i_t}x_{i_t}, x_{i_t} \in [1, p_\omega]$ and $H_t \in H(x_{i_t}vA_\lambda, s_{i_t}x_{i_t}vA_\lambda)$, H_t separates A_λ and one of the alcoves $x_{i_t}vA_\lambda$ or $s_{i_t}x_{i_t}vA_\lambda$, hence by the previous lemma we see that

$$\lambda_{\alpha_{i_t}} - m_{\alpha_{i_t}}(\omega) \leq k_{i_t} \leq \lambda_{\alpha_{i_t}} + m_{\alpha_{i_t}}(\omega) \leq 0.$$

Further we must have $s_{i_t}x_{i_t}vA_\lambda \in H_t^+$ and $x_{i_t}vA_\lambda \in H_t^-$ as A_0 and $s_{i_t}x_{i_t}vA_\lambda$ lie on the same side of H_t which has to be H_t^+ . Adding $\lambda'_{\alpha_{i_t}} - \lambda_{\alpha_{i_t}}$ throughout the previous equality, we get

$$\lambda'_{\alpha_{i_t}} - m_{\alpha_{i_t}}(\omega) \leq k'_{i_t} \leq \lambda'_{\alpha_{i_t}} + m_{\alpha_{i_t}}(\omega).$$

But λ' is ω -far away from the α_{i_t} -wall, therefore $\lambda'_{\alpha_{i_t}} + m_{\alpha_{i_t}}(\omega) \leq 0$ and $k'_{i_t} \leq 0$. We have seen that $s_{i_t}x_{i_t}vA_\lambda \in H_t^+$ and $x_{i_t}vA_\lambda \in H_t^-$. Translating by $\lambda' - \lambda$, we get $s_{i_t}x_{i_t}vA'_\lambda \in H'^+_t$ and $x_{i_t}vA'_\lambda \in H'^-_t$. Then as, $k'_{i_t} \leq 0$, this implies that $H'_t \in H(x_{i_t}vA_{\lambda'}, A_0)$ as required.

Next assume that $\alpha_{i_t} \in \Phi_c^+$. In that case, we have

$$\langle x, \alpha_{i_t}^\vee \rangle = \langle x', \alpha_{i_t}^\vee \rangle \quad \text{for all } x \in x_{i_t}vA_\lambda \text{ and } x' \in x_{i_t}vA'_\lambda$$

and

$$\langle y, \alpha_{i_t}^\vee \rangle = \langle y', \alpha_{i_t}^\vee \rangle \quad \text{for all } y \in s_{i_t}x_{i_t}vA_\lambda \text{ and } y' \in s_{i_t}x_{i_t}vA'_\lambda$$

therefore it is easy to see that $H'_t \in H(x_{i_t}vA_{\lambda'}, A_0)$ in this case since $H_t \in H(x_{i_t}vA_\lambda, A_0)$.

Finally, in both cases, we have shown that $H'_t \in H(x_{i_t}vA_{\lambda'}, A_0)$. We have seen that $H'_t \in H(s_{i_t}x_{i_t}vA_{\lambda'}, x_{i_t}vA_{\lambda'})$. It follows that we have $s_{i_t}x_{i_t}vp_{\lambda'} < x_{i_t}vp_{\lambda'}$ as required to complete the proof of the lemma. \square

Corollary 5.7. *Let $\lambda \in P^-$, $\omega \in \mathbf{P}^+$ and $v \in W_0$. There exists $\lambda' \in P^-$ such that $0 \leq \lambda'_\alpha \leq m_\alpha$ for all $\alpha \in \Phi^+$ and such that $T_{p_\omega}T_{vp_\lambda} \approx T_{p_\omega}T_{vp_{\lambda'}}$.*

5.2. Decomposition of the cellular basis.

Theorem 5.8. *Let $\lambda, \lambda' \in P^-$ and $\Phi^+ = \Phi_c^+ \sqcup \Phi_f^+$ be a disjoint union such that*

- (1) λ and λ' are ω -far away from the α -wall for all $\alpha \in \Phi_f^+$,
- (2) $\lambda_\alpha = \lambda'_\alpha$ for all $\alpha \in \Phi_c^+$.

Then there exists a family $(a_\alpha)_{\alpha \in P}$ of integers such that

$$(1) \quad \mathbf{P}(\omega)C_{w_0p_\lambda} = C_{p_\omega w_0p_\lambda} + \sum_{\alpha \in P, \lambda - \alpha \in P^-} a_\alpha C_{p_\alpha w_0p_\lambda}$$

and

$$(2) \quad \mathbf{P}(\omega)C_{w_0p_{\lambda'}} = C_{p_\omega w_0p_{\lambda'}} + \sum_{\alpha \in P, \lambda' - \alpha \in P^-} a_\alpha C_{p_\alpha w_0p_{\lambda'}}$$

Proof. We start by proving that

$$\mathbf{P}(\omega)C_{w_0p_\lambda} = C_{p_\omega w_0p_\lambda} + \sum_{\alpha \in P, \lambda - \alpha \in P^-} a_\alpha C_{p_\alpha w_0p_\lambda}$$

for some integers a_α . We know that $\overline{\mathbf{P}(\omega)C_{w_0p_\lambda}} = \mathbf{P}(\omega)C_{w_0p_\lambda}$ hence according to Lemma 5.3, in order to prove the result it is enough to show that

$$(*) \quad \mathbf{P}(\omega)C_{\lambda w_0} \equiv T_{p_\alpha w_0 p_\lambda} + \sum_{\alpha \in P, \lambda - \alpha \in P^-} a_\alpha T_{p_\alpha w_0 p_\lambda} \pmod{\mathcal{H}_{<0}}.$$

We have

$$\begin{aligned} \mathbf{P}(\omega)C_{w_0p_\lambda} &= (T_{p_\omega} + \sum_{z \in X_0, z < p_\omega} \mathfrak{p}_{z, p_\omega} T_z) C_{w_0p_\lambda} \\ &= T_{p_\omega} C_{w_0p_\lambda} + \sum_{z \in X_0, z < p_\omega} \mathfrak{p}_{z, p_\omega} T_z C_{w_0p_\lambda}. \end{aligned}$$

By Corollary 5.3, we know that $T_z C_{w_0p_\lambda} \in \mathcal{H}_{\leq 0}$ and therefore, since $\mathfrak{p}_{z, w} \in \mathcal{A}_{<0}$ for all $z \in X_0$, we get that $\mathfrak{p}_{z, w} T_z C_{w_0p_\lambda} \in \mathcal{H}_{<0}$. Thus

$$T_{p_\omega} C_{w_0p_\lambda} + \sum_{z \in X_0, z < p_\omega} \mathfrak{p}_{z, \omega} T_z C_{w_0p_\lambda} \equiv T_{p_\omega} C_{w_0p_\lambda} \pmod{\mathcal{H}_{<0}}.$$

Again by Corollary 5.3 (3) we obtain

$$T_{p_\omega} C_{w_0p_\lambda} \equiv T_{p_\omega w_0 p_\lambda} + \sum_{v \in W_0} q^{L(v) - L(w_0)} T_{p_\omega} T_{v p_\lambda} \pmod{\mathcal{H}_{<0}}.$$

On the one hand, the above expression is an integer combination of elements of the standard basis by Corollary 5.3 (3). On the other hand, the terms appearing in the product $T_{p_\omega} T_{v p_\lambda}$ are of the form $T_{z v p_\lambda}$ for some $z \in [1, p_\omega]$ (see formula 5.4). Thus we have

$$T_{p_\omega} C_{w_0p_\lambda} \equiv T_{p_\omega w_0 p_\lambda} + \sum_{z \in [1, p_\omega]} a_z T_{z v p_\lambda} \pmod{\mathcal{H}_{<0}} \text{ where } a_z \in \mathbb{Z}.$$

From there we see that

$$\mathbf{P}(\omega)C_{\lambda w_0} \equiv T_{p_\omega w_0 p_\lambda} + \sum_{z \in [1, p_\omega]} a_z T_{z v p_\lambda} \pmod{\mathcal{H}_{<0}}.$$

Next we have $\mathbf{P}(\omega)C_{w_0p_\lambda} \in \mathcal{M}_+ = \langle C_{w_0 p_\gamma} \mid \gamma \in P^- \rangle_{\mathcal{A}}$. Thus, the only terms with non-zero coefficients in the above expression have to be of the form $T_{w_0 p_\gamma}$ with $\gamma \in P^-$. Hence, for all $z \leq p_\omega$ there exists $\gamma \in P^-$ such that $z v p_\lambda = w_0 p_\gamma = w_0 p_{\gamma - \lambda} p_\lambda = p_{\lambda - \gamma} w_0 p_\lambda$ and we get the desired expression by setting $\alpha = p_{\lambda - \gamma}$.

Now, the fact that

$$\mathbf{P}(\omega)C_{w_0 p_{\lambda'}} = C_{p_\omega w_0 p_{\lambda'}} + \sum_{\alpha \in P, \lambda - \alpha \in P^-} a_\alpha C_{p_\alpha w_0 p_{\lambda'}}$$

is a direct consequence of Proposition 5.6. Indeed the decomposition of $\mathbf{P}(\omega)C_{w_0 p_{\lambda'}}$ in the Kazhdan-Lusztig basis only depends on the products of the form $T_{z v p_\lambda}$ where $v \in W_0$ and $z \in [1, p_\omega]$. \square

As a direct corollary of this theorem, we see that the multiplication of the form $\mathbf{P}(\omega)C_{w_0 p_\lambda}$ can be determined for all $\lambda \in P^-$ only by calculating the expression in the Kazhdan-Lusztig basis of all the products $\mathbf{P}(\omega)C_{w_0 p_\lambda}$ for all λ satisfying $|\lambda_\alpha| \leq m_\alpha(\omega)$. Then, one can find the expression of a cellular basis element by induction by successively computing products of the form $\mathbf{P}(\omega)C_{p_\lambda w_0}$.

6. AFFINE WEYL GROUP OF TYPE A

In this section we study in more detail the decomposition of the cellular basis in the Kazhdan-Lusztig basis for affine Weyl groups of type A . We will see that the integer coefficients that appear in Theorem 5.8 in that case can be interpreted as the number of certain kind of paths from 0 to λ . We will assume that $W = \langle s_0, \dots, s_n \rangle$ is an affine Weyl group of type \tilde{A}_n and that $W_e = \Pi \ltimes W$ is the extended group where $\Pi = \langle \pi \rangle$ is a cyclic group of order n .

Remark 6.1. In type A we have $\langle \omega, \alpha^\vee \rangle = \pm 1$ for all fundamental weight ω and all $\alpha \in \Phi$ (see [2, Planche 1]), therefore one can easily see that $m_\alpha(\omega) = 1$; see Section 5.1 for the definition of $m_\alpha(\omega)$. Also, we have $\alpha^\vee = \alpha$ for all $\alpha \in \Phi$.

6.1. A refinement of Theorem 5.8. Let $\omega \in \mathbf{P}^+$. We denote by $\mathcal{O}(\omega)$ the orbit of ω under the action of $W_0 = \langle \sigma_0, \dots, \sigma_n \rangle$.

Theorem 6.2. *Let $\lambda \in P^-$ and $\omega \in \mathbf{P}^+$. We have*

$$\mathbf{P}(\omega)C_{w_0 p_\lambda} = \sum_{\rho \in \mathcal{O}(\omega), \lambda - \rho \in P^-} C_{p_\rho w_0 p_\lambda}.$$

Proof. In view of the proof of Theorem 5.8, we know that

$$T_{p_\omega} C_{w_0 p_\lambda} \equiv T_{p_\omega w_0 p_\lambda} + \sum_{v \in W_0} q^{\ell(v) - \ell(w_0)} T_{p_\omega} T_{v p_\lambda} \pmod{\mathcal{H}_{<0}}.$$

By Lemma 5.3, we need to show that

$$(*) \quad \sum_{v \in W_0} q^{\ell(v) - \ell(w_0)} T_{p_\omega} T_{v p_\lambda} \equiv \sum_{\rho \in \mathcal{O}(\omega), \lambda - \rho \in P^-} T_{p_\rho w_0 p_\lambda} \pmod{\mathcal{H}_{<0}}.$$

Let $v \in W_0$ and $u = w_0 v^{-1}$ so that $w_0 = u \bullet v$. Let $w_0 = s_{i_n} \dots s_{i_{m+1}} s_{i_m} \dots s_{i_1}$ be a reduced expression of w_0 such that $u = s_{i_n} \dots s_{i_{m+1}}$ and $v = s_{i_m} \dots s_{i_1}$ (note that these are necessarily reduced expression). We will denote by $H_{i_n}, \dots, H_{i_{m+1}}$ (respectively H_{i_m}, \dots, H_{i_1}) with direction $\alpha_{i_n}, \dots, \alpha_{i_{m+1}}$ (respectively $\alpha_{i_m}, \dots, \alpha_{i_1}$) the set of hyperplanes separating $vp_\lambda A_0$ and $w_0 p_\lambda A_0$ (respectively $p_\lambda A_0$ and $vp_\lambda A_0$).

We have seen in the proof of Proposition 5.3, that the direction of the hyperplanes lying in H_{p_ω, vp_λ} belongs to $\{\alpha_{i_n}, \dots, \alpha_{i_{m+1}}\}$ and are of the form $H_{\alpha_{i_k}, r_{i_k}}$ where $\langle \lambda, \alpha_{i_k} \rangle = \lambda_{i_k} < r_{i_k} \leq 0$. (In particular, if $\lambda_{i_k} = 0$ there are no hyperplane of direction α_{i_k} in H_{p_ω, vp_λ} .)

In type A , since $\langle \omega, \alpha^\vee \rangle = 0$ or 1 for all $\alpha \in \Phi^+$, we see that we must have

$$H_{p_\omega, vp_\lambda} \subset \{H_{\alpha_{i_n}, \lambda_{\alpha_{i_n}} + 1}, \dots, H_{\alpha_{i_{m+1}}, \lambda_{\alpha_{i_{m+1}}} + 1}\}.$$

This set contains $\ell(w_0) - \ell(v)$ elements, thus to get $q^{\ell(v) - \ell(w_0)} T_{p_\omega} T_{v p_\lambda} \not\equiv 0 \pmod{\mathcal{H}_{<0}}$ we need to have equality in the above inclusion.

Remark 6.3. If one can show that there are no hyperplane of direction α_r for $m+1 \leq r \leq n$ separating the alcoves A_0 and $vp_\lambda A_0$ or $vp_\lambda A_0$ and $p_\omega vp_\lambda A_0$ then

$$q^{\ell(v) - \ell(w_0)} T_{p_\omega} T_{v p_\lambda} \equiv 0 \pmod{\mathcal{H}_{<0}}.$$

We will make frequent use of this remark in the sequel.

Let Δ' be the subset of Δ which consists of all simple roots satisfying $\langle \omega, \alpha^\vee \rangle = 0$. Then the subgroup W'_0 of W_0 stabilising p_ω is generated by $S'_0 = \{s \in S \mid sA_0 = A_0 \sigma_\alpha, \alpha \in \Delta'\}$.

To prove the theorem, we will proceed in four steps:

Claim 1. If there exists $s \in S'$ such that $sv > v$ then $q^{\ell(v) - \ell(w_0)} T_{p_\omega} T_{v p_\lambda} \equiv 0 \pmod{\mathcal{H}_{<0}}$.

Claim 2. There is at most one term in the product $T_{p_\omega} T_{vp_\lambda}$ expressed in the standard basis that can have a coefficient of degree $\ell(w_0) - \ell(v)$.

Claim 3. If v is of maximal length in its coset $W'v$ then $p_\omega v = u^{-1} \bullet (p_{\omega\sigma_u} w_0)$.

Claim 4. If $\lambda - \omega\sigma_u \notin P^-$ then $q^{\ell(v) - \ell(w_0)} T_{p_\omega} T_{vp_\lambda} \equiv 0 \pmod{\mathcal{H}_{<0}}$.

Before giving the proofs of those claims, we show that they imply the theorem. By Claim 1 the only terms in (*) that can contribute $\pmod{\mathcal{H}_{<0}}$ are those of the form $q^{\ell(v) - \ell(w_0)} T_{p_\omega} T_{vp_\lambda}$ where $v \in W_0$ is of maximal length in its right cosets $W'v$. Then, by Claim 2, each of these products contribute with at most one term. Fix v of maximal length in $W'v$. Then

$$T_{p_\omega} T_{vp_\lambda} = T_{p_\omega} T_v T_{p_\lambda} = T_{p_\omega v} T_{p_\lambda} = T_{u^{-1} p_{\omega\sigma_u} w_0} T_{p_\lambda} = T_{u^{-1}} T_{p_{\omega\sigma_u} w_0} T_{p_\lambda}.$$

The last equality is true, thanks to Claim 3. The product $T_{p_{\omega\sigma_u} w_0} T_{p_\lambda}$ gives a term $T_{p_{\omega\sigma_u} w_0 p_\lambda}$ with coefficient 1. But we have $T_{p_{\omega\sigma_u} w_0 p_\lambda} = T_{w_0 p_{\lambda - \omega\sigma_u}}$. So either $\lambda - \omega\sigma \in P^-$ in which case $p_{\lambda - \omega\sigma} \in X_0^{-1}$ and we get

$$T_{u^{-1}} T_{w_0 p_{(\lambda - \omega\sigma_u)}} = T_{u^{-1}} T_{w_0} T_{p_{(\lambda - \omega\sigma_u)}} = q^{\ell(u^{-1})} T_{w_0 p_{(\lambda - \omega\sigma_u)}} = q^{\ell(w_0) - \ell(v)} T_{w_0 p_{(\lambda - \omega\sigma_u)}}$$

and

$$q^{\ell(v) - \ell(w_0)} T_{p_\omega} T_{vp_\lambda} \equiv T_{p_{\omega\sigma_u} w_0 p_\lambda} \pmod{\mathcal{H}_{<0}}.$$

Otherwise, $\lambda - \omega\sigma \notin P^-$ and $q^{\ell(v) - \ell(w_0)} T_{p_\omega} T_{vp_\lambda} \equiv 0 \pmod{\mathcal{H}_{<0}}$ by Claim 4.

Finally when v runs through all the maximal length right cosets representative of W' in W we see that u runs through all the minimal length left cosets representative and so $\omega\sigma_u$ runs exactly through the orbit of ω under the action of W_0 . Hence the desired expression.

We now prove the four claims.

Proof of the Claim 1. If $sv > v$ then we also have $svp_\lambda > vp_\lambda$ and the hyperplane H_s which separates the alcoves $vp_\lambda A_0$ and $svp_\lambda A_0$ lies in $\{H_{i_n}, \dots, H_{i_{m+1}}\}$ and is of the form $H_{\alpha_{i_k}, \lambda_{i_k}}$ for some $m+1 \leq k \leq n$. We have

$$sp_\omega vp_\lambda A_0 = svp_\lambda A_0 t_{\omega\sigma_v} = vp_\lambda A_0 \sigma_{H_s} t_{\omega\sigma_v}$$

and

$$p_\omega svp_\lambda A_0 = p_\omega vp_\lambda A_0 \sigma_{H_s} = vp_\lambda A_0 t_{\omega\sigma_v} \sigma_{H_s}.$$

Thus $\sigma_{H_s} t_{\omega\sigma_v} = t_{\omega\sigma_v} \sigma_{H_s}$ which implies that the vector $\omega\sigma_v$ lies in H_s , that is $\omega\sigma_v$ is orthogonal to α_{i_k} . As a consequence, since $p_\omega(vp_\lambda A_0) = vp_\lambda A_0 t_{\omega\sigma_v}$, both alcoves $p_\omega(vp_\lambda A_0)$ and $vp_\lambda A_0 t_{\omega\sigma_v}$ lie on the same side of H_s and there can be no hyperplane of direction α_{i_k} lying in $H(vp_\lambda A_0, p_\omega vp_\lambda A_0)$. Therefore we have

$$H_{p_\omega, vp_\lambda} \subsetneq \{H_{\alpha_{i_n}, \lambda_{\alpha_{i_n}} + 1}, \dots, H_{\alpha_{i_{m+1}}, \lambda_{\alpha_{i_{m+1}}} + 1}\}$$

and the claim follows with Remark 6.3.

Proof of the Claim 2. We use the notation of Section 5.1. We show that there is at most one set $I \in \mathfrak{J}_{p_\omega, vp_\lambda}$ which can give a term with a coefficient of degree $\ell(w_0) - \ell(v)$. Let $p_\omega = \pi t_n \dots t_1$ where $\pi \in \Pi$ and $t_n \dots t_1 \in W$ is a reduced expression. Let j_1 the smallest index such that $t_{j_1} \dots t_1 vp_\lambda < \hat{t}_{j_1} \dots t_1 vp_\lambda$ and let H_{j_1} be the unique hyperplane separating the two corresponding alcoves. Note that $H_{i_1} \in H_{p_\omega, vp_\lambda}$. We have

$$\begin{aligned} T_x T_{vp_\lambda} &= T_{t_k \dots t_{j_1+1}} T_{\hat{t}_{j_1}} T_{\hat{t}_{j_1} \dots t_1 vp_\lambda} \\ &= T_{t_k \dots t_{j_1+1}} T_{t_j \dots t_1 vp_\lambda} + (q - q^{-1}) T_{t_k \dots t_{j_1+1}} T_{\hat{t}_{j_1} \dots t_1 vp_\lambda} \end{aligned}$$

Firstly we see that $H_{t_k \dots t_{j_1+1}, t_i \dots t_1 v p_\lambda} = H_{p_\omega, v p_\tau} - \{H_{j_1}\}$. But $H_{p_\omega, v p_\lambda}$ only contains one hyperplane of direction α_{j_1} hence there are no hyperplane of direction α_{j_1} in $H_{x, v p_\tau} - \{H_{j_1}\}$ and we have

$$I_{t_k \dots t_{j_1+1}, t_i \dots t_1 v p_\lambda} = I_{p_\omega, v p_\lambda} - \{\alpha_{j_1}\}.$$

By Theorem 5.2, the maximal degree appearing in the coefficients in $T_{t_k \dots t_{j_1+1}} T_{t_i \dots t_1 v p_\tau}$ is $\ell(w_0) - \ell(v) - 1$.

Secondly, we can see that the set \mathfrak{H} of hyperplanes separating the alcoves

$$t_n \dots \hat{t}_{j_1} \dots t_1 v p_\lambda A_0 = t_n \dots t_{j_1} \dots t_1 v p_\lambda A_0 \sigma_{H_{j_1}}$$

and

$$\hat{t}_{j_1} \dots t_1 v p_\lambda A_0 = t_{j_1} \dots t_1 v p_\lambda A_0 \sigma_{H_{j_1}}$$

is included in $H_{p_\omega, v p_\lambda} \cdot \sigma_{H_{j_1}} - \{H_{j_1}\}$ and this set contains at most $\ell(w_0) - \ell(v) - 1$ elements. But we have

$$H_{t_k \dots t_{j_1+1}, \hat{t}_{j_1} \dots t_1 v p_\lambda} \subset \mathfrak{H}$$

hence if \mathfrak{H} contains strictly less than $\ell(w_0) - \ell(v) - 1$ we cannot have a term of degree $\ell(w_0) - \ell(v) - 1$ in $T_{t_k \dots t_{i_1+1}} T_{\hat{t}_{i_1} \dots t_1 v p_\lambda}$.

By induction, we construct a set $I = \{j_1, \dots, j_k\}$ with $k \leq \ell(w_0) - \ell(v)$. (For instance j_2 is the smallest integer such that $t_{j_2} \dots \hat{t}_{j_1} \dots t_1 v p_\lambda < \hat{t}_{j_2} \dots \hat{t}_{j_1} \dots t_1 v p_\lambda$.) If we have $k = \ell(w_0) - \ell(v)$, then the term corresponding to I in the product $T_{p_\omega} T_{v p_\lambda}$ has a coefficient of degree $\ell(w_0) - \ell(v)$ otherwise there are none.

Proof of the Claim 3. The equality $p_\omega v = p_\omega u^{-1} w_0 = u^{-1} p_{\omega, \sigma_u} w_0$ is clear. Let $n(\sigma_u^{-1}) := \{\alpha \in \Phi^+ \mid \alpha \sigma_u^{-1} \in \Phi^-\}$. It is a well known fact that $n(\sigma_u^{-1}) = \ell(u)$. We define the following subset of Φ^+ :

$$\begin{aligned} \mathcal{P}_0 &= \{\alpha \in \Phi^+ \mid (\omega \sigma_u, \alpha) = 0\} \\ \mathcal{P}_< &= \{\alpha \in \Phi^+ \mid (\omega \sigma_u, \alpha) < 0\} \\ \mathcal{P}_> &= \{\alpha \in \Phi^+ \mid (\omega \sigma_u, \alpha) > 0\}. \end{aligned}$$

Note that, if $\alpha \in n(\sigma_u)$, then $(\omega \sigma_u, \alpha) = (\omega, \alpha \sigma_u^{-1}) \leq 0$ since $\omega \in \mathbf{P}^+$. On the other hand, if $\alpha \sigma_u \in \Phi^+$ then $(\omega \sigma_u, \alpha) = (\omega, \alpha \sigma_u^{-1}) \geq 0$. Therefore we see that $\mathcal{P}_< \subset n(\sigma_u^{-1})$. Assume that there exists $\alpha \in n(\sigma_u^{-1})$ such that $(\omega \sigma_u, \alpha) = (\omega, \alpha \sigma_u^{-1}) = 0$. Then $\alpha \sigma_u^{-1} \in \Phi'$, the root system generated by the simple system Δ' . Further, since $\alpha \sigma_u^{-1} \in \Phi^-$, we have that $\alpha \sigma_u \in \Phi'^-$, hence we can write

$$\alpha \sigma_u^{-1} = \sum_{\gamma \in \Delta'} a_\gamma \gamma \text{ where } a_\gamma \leq 0$$

But then we obtain that

$$\alpha = \sum_{\gamma \in \Delta'} a_\gamma \gamma \sigma_u \text{ where } a_\gamma \leq 0.$$

Since v is of maximal length in its coset, we can check that u is of minimal length in its right coset with respect to W' , that is $us > u$ for all $s \in S'$. Therefore, $\gamma \sigma_u$ is a positive root for all $\gamma \in \Delta'$ and the last equality is a contradiction with the fact $\alpha \in \Phi^+$. We have proved that $\mathcal{P}_< = n(\sigma_u^{-1})$.

Next, since the length of an element is equal to the number of hyperplanes separating the corresponding alcoves and since $p_{\omega\sigma_u}w_0A_0 = w_0A_0t_{-\omega\sigma_u}$ we can see that

$$\begin{aligned}\ell(p_{\omega\sigma_u}w_0) &= \ell(w_0) + |\mathcal{P}_>| - |\mathcal{P}_<| \\ &= \ell(w_0) + \ell(p_\omega) - 2|\mathcal{P}_<| \\ &= \ell(w_0) + \ell(p_\omega) - 2\ell(u).\end{aligned}$$

Finally we obtain

$$\ell(u^{-1}) + \ell(p_{\omega\sigma_u}w_0) = \ell(u) + \ell(w_0) + \ell(p_\omega) - 2\ell(u) = \ell(p_\omega) + (\ell(w_0) - \ell(u)) = \ell(p_\omega) + \ell(v)$$

The claim is proved.

Proof of the Claim 4. Assume $\lambda - \omega\sigma_u \notin P^-$. As we are in type A we know that for all $\sigma \in W_0$ and all $\alpha \in \Phi$ we have $|(\omega, \alpha\sigma)| \leq \pm 1$. As $\lambda \in P^-$, to have $\lambda - \omega\sigma_u \notin P^-$, we must have $(\lambda, \alpha_i) = 0$ and $(\omega\sigma_u, \alpha_i) = (\omega, \alpha\sigma_u^{-1}) = -1$. In turn this implies that $u^{-1}s_i < u^{-1}$. Thus the alcoves $w_0p_\lambda A_0$ and $u^{-1}w_0p_\lambda A_0 = vp_\lambda A_0$ lies on two different sides of $H_{\alpha_i, 0}$. This implies that $(x, \alpha_i) > 0$ for all $x \in vp_\lambda A_0$. Thus there can't be any hyperplane of direction α_i in H_{p_ω, vp_λ} as A_0 and $vp_\lambda A_0$ both lie between $H_{\alpha_i, 0}$ and $H_{\alpha_i, 1}$. The Claim follows. \square

6.2. Combinatorics of the cellular basis. In this section we study the decomposition in the Kazhdan-Lusztig basis of the element $\mathbf{P}(\tau)C_{w_0}$ for all $\tau \in P^+$. Recall that it is enough to find the expression of any element of the cellular basis in the Kazhdan-Lusztig basis, as noted in the beginning of Section 5.

We denote by $\{\omega_1, \dots, \omega_n\}$ the set of fundamental weights and for each $\omega_i \in \mathbf{P}^+$ we denote by $\mathcal{O}(\omega_i) = \{\omega_i = \omega_i^{(1)}, \dots, \omega_i^{(N)}\}$ its orbit under the action of W_0 . Let $x, y \in P^-$. A path of length N in P^- from x to y is an (ordered) sequence $x = x_0, x_1, \dots, x_N = y$ in P^- such that for all $1 \leq \ell \leq N-1$ we have $x_{\ell+1} = x_\ell - \omega_{i_\ell}^{(j_\ell)}$ and $x_\ell \in P^-$. The element $\omega_{i_\ell}^{(j_\ell)}$ is called the ℓ^{th} -step of the path from x to y . A path is said to be of type $\mathbf{m} = (m_1, \dots, m_N) \in \{1, \dots, n\}^N$ if its ℓ^{th} -step lies in $\mathcal{O}(\omega_{m_\ell})$. Let $\gamma \in P^-$ and $\mathbf{m} \in \{1, \dots, n\}^N$. We define $\mathcal{P}_{\mathbf{m}}(\gamma)$ to be the number of paths of type \mathbf{m} from 0 to γ .

Remark 6.4. Let $\lambda \in P^-$ be such that $\lambda = -\sum a_i \omega_i$ for some $a_i \geq 0$. Let $\mathbf{m} = (m_1, \dots, m_N) \in \{1, \dots, n\}^N$ be any N -tuples which contains exactly a_i i 's for all i 's. Then $\mathcal{P}_{\mathbf{m}}(\lambda) = 1$ as the ℓ^{th} -step needs to be equal to $\omega_{m_\ell}^1 = \omega_{m_\ell}$.

Corollary 6.5. *Let $(a_1, \dots, a_n) \in \mathbb{N}^n$, $\tau = \sum a_i \omega_i \in P^+$ and $N = \sum a_i$. Let $\mathbf{m} = (m_1, \dots, m_N) \in \{1, \dots, n\}^N$ be any N -tuples which contains exactly a_i i 's for all i 's. Then the element $\mathbf{P}(\tau)C_{w_0}$ of the cellular basis has the following expression in the Kazhdan-Lusztig basis :*

$$\mathbf{P}(\tau)C_{w_0} = \sum_{\lambda \in P^-} \mathcal{P}_{\mathbf{m}}(\lambda)C_{w_0 p_\lambda}$$

Proof. Let $\lambda \in P^-$ and let $\mathbf{m}' = (m_1, \dots, m_{N-1})$. Then we have

$$\mathcal{P}_{\mathbf{m}}(\lambda) = \sum_{\lambda' \in P^-, \lambda' - \lambda \in \mathcal{O}(\omega_{m_N})} \mathcal{P}_{\mathbf{m}'}(\lambda').$$

We have $\mathbf{P}(\tau)C_{w_0} = \mathbf{P}(\omega_{m_N}) \dots \mathbf{P}(\omega_{m_1})C_{w_0}$. Let us prove the result by induction. For $N = 1$ the result is true. Indeed we have $\mathbf{P}(\omega_i)C_{w_0} = C_{p_{\omega_i} w_0}$ and there is only one path of type (i) from 0 to ω_i . Next assume that we have

$$\mathbf{P}(\omega_{m_{N-1}}) \dots \mathbf{P}(\omega_1)C_{w_0} = \sum_{\lambda \in P^-} \mathcal{P}_{\mathbf{m}'}(\lambda)C_{w_0 p_\lambda} \text{ where } \mathbf{m}' = (m_{N-1}, \dots, 1).$$

Then

$$\begin{aligned}
\mathbf{P}(\omega_{n_N}) \dots \mathbf{P}(\omega_1) C_{w_0} &= \mathbf{P}(\omega_{n_N}) \sum_{\lambda \in P^-} \mathcal{P}_{\mathbf{m}'}(\lambda) C_{w_0 p_\lambda} \\
&= \sum_{\lambda \in P^-} \mathcal{P}_{\mathbf{m}'}(\lambda) \mathbf{P}(\omega_{n_N}) C_{w_0 p_\lambda} \\
&= \sum_{\lambda \in P^-} \mathcal{P}_{\mathbf{m}'}(\lambda) \left(\sum_{\substack{\gamma \in P^- \\ \lambda - \gamma \in \theta(\omega_{i_n})}} C_{w_0 p_\gamma} \right) \\
&= \sum_{\lambda \in P^-} \sum_{\substack{\gamma \in P^- \\ \lambda - \gamma \in \theta(\omega_{n_N})}} \mathcal{P}_{\mathbf{m}'}(\lambda) C_{w_0 p_\gamma} \\
&= \sum_{\gamma \in P^-} \sum_{\substack{\lambda \in P^- \\ \lambda - \gamma \in \theta(\omega_{n_N})}} \mathcal{P}_{\mathbf{m}'}(\lambda) C_{w_0 p_\gamma} \\
&= \sum_{\gamma \in P^-} \mathcal{P}_{\mathbf{m}}(\gamma) C_{w_0 p_\gamma}
\end{aligned}$$

as required. \square

Example 6.6. In this example, we study the case of the affine Weyl group of type \tilde{A}_2 . The quarter $\mathcal{C}^- = -\mathcal{C}^+$ is represented in figure 4. We denote by ω_1 and ω_2 the fundamental weights. The set of anti-dominant weights, that is $-P^+$, is just the set of points lying in \mathcal{C}^- and at the intersection of 3 hyperplanes. For each antidominant weight $\lambda \in P^-$, the red (respectively green) arrows leaving from λ represents the orbit of ω_1 (respectively ω_2) under the action of W_0 .

To find the expression of the cellular basis element $\mathbf{P}(2\omega_1 + 2\omega_2) C_{w_0}$ one needs to count the number of path of type $(1, 1, 2, 2)$ from 0 to λ for all $\lambda \in P^-$. This number (when non-zero) is indicated in figure 5. Hence we find

$$\mathbf{P}(2\omega_1 + 2\omega_2) C_{w_0} = C_{p_{\omega_1}^2 p_{\omega_2}^2 w_0} + C_{p_{\omega_1}^3 w_0} + C_{p_{\omega_2}^3 w_0} + 4C_{p_{\omega_1} p_{\omega_2} w_0} + 2C_{w_0}.$$

The description of all paths form 0 to $-(\omega_1 + \omega_2)$ of type $(1, 1, 2, 2)$ can be found in Figure 5.

Remark 6.7. The interpretation in term of paths also works for other types, however the situation gets more complicated as we get closer to the wall and one needs to define many different kind of paths depending on how close we are from each wall.

