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Breaking a magnetic zero locus: asymptotic analysis

N. Raymond*

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Abstract

This paper deals with the spectral analysis of the Laplacian in presence of a magnetic field vanishing along a broken line. Denoting by θ the breaking angle, we prove complete asymptotic expansions of all the lowest eigenpairs when θ goes to 0. The investigation deeply uses a coherent states decomposition and a microlocal analysis of the eigenfunctions.

Keywords: Semiclassical analysis, vanishing magnetic field, coherent states.

Classification MSC: 35P15, 81Q10, 81Q20

1 Introduction

This paper is concerned with the effect of breaking a magnetic zero locus in the Schrödinger equation. We investigate the spectrum of self-adjoint realizations on the plane \mathbb{R}^2 or on the half-plane \mathbb{R}_+^2 of the operator $(-ih\nabla + \mathbf{A})^2$ when the magnetic field $\beta = \nabla \times \mathbf{A}$ cancels along a line. The case when the magnetic field non-degenerately vanishes along a smooth line is considered in the initial paper of R. Montgomery [27] where the behavior of the spectrum is investigated in the semiclassical limit $h \rightarrow 0$. In particular, the model operator, in the semiclassical limit, which is considered there is the self-adjoint realization on $L^2(\mathbb{R}^2)$ of:

$$\mathfrak{L} = D_t^2 + (D_s - st)^2,$$

where we use the standard notation $D_x = -i\partial_x$. In this case the magnetic field is given by $\beta(s, t) = s$ so that the zero locus of β is the line $s = 0$. Let us write the following change of gauge:

$$\mathfrak{L}^{\text{Mont}} = e^{-i\frac{s^2 t}{2}} \mathfrak{L} e^{i\frac{s^2 t}{2}} = D_s^2 + \left(D_t + \frac{s^2}{2}\right)^2.$$

The Fourier transform (after changing ξ in $-\xi$) with respect to t gives the direct integral:

$$\mathfrak{L}^{\text{Mont}} = \int^\oplus \mathfrak{L}_\xi^{\text{Mont}} d\xi, \quad \text{where} \quad \mathfrak{L}_\xi^{\text{Mont}} = D_s^2 + \left(-\xi + \frac{s^2}{2}\right)^2.$$

Notation 1.1 *If \mathfrak{A} is an operator, we denote by $\mathfrak{s}(\mathfrak{A})$ its spectrum and by $\mathfrak{s}_{\text{ess}}(\mathfrak{A})$ its essential spectrum.*

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From this representation, we deduce that:

$$\mathfrak{s}(\mathfrak{L}) = \mathfrak{s}_{\text{ess}}(\mathfrak{L}) = [\mu_{\text{Mont}}, +\infty), \quad (1.1)$$

where μ_{Mont} is defined as:

$$\mu_{\text{Mont}} = \inf_{\xi \in \mathbb{R}} \mu_1^{\text{Mont}}(\xi),$$

where $\mu_1^{\text{Mont}}(\xi)$ denotes the first eigenvalue of $\mathfrak{L}_\xi^{\text{Mont}}$. The main properties of $\mu_1^{\text{Mont}}(\xi)$ can be found in [28, 17, 20]).

Waveguides induced by magnetic fields We notice that the reduction, through the Fourier transform, to the one dimensional Montgomery operator is possible due to the translation invariance of the zero line of the magnetic field. This invariance is also the reason for which the spectrum is essential. This situation is reminiscent of the paper of Duclos and Exner [13] (see also [6, 22, 16] and [14, 5, 23, 24, 25, 29] where the same philosophy appears for quantum layers). In particular Duclos and Exner notice that, in the waveguides framework, there is discrete spectrum below the threshold of the essential spectrum if and only if the waveguide is not translation invariant. In the present situation, we can think that the zero locus of the magnetic field “plays the role” of a waveguide. Let us also mention the paper [11] where the problem of quantum transport in magnetic waveguides is analyzed.

Semiclassical motivation In the semiclassical limit the leading operator of Montgomery which appears (after a suitable rescaling) is translation invariant because of the smoothness of the magnetic zero locus. In particular it does not create bound states on its own. It is even proved in [12] that bound states appear under an additional assumption on the normal derivative of the magnetic field along its zero locus. Concerning the semiclassical analysis in presence of smooth vanishing magnetic fields, we refer to [19, 18, 12]. The results of this paper could be used in the semiclassical spectral analysis of vanishing magnetic fields in the case with boundary (which is still an open problem). In the case when the zero locus has a corner, the limiting model is no more translation invariant. Therefore we can hope for bound states and another behavior for the semiclassical problem related to the principal symbol of the magnetic Laplacian and no more to the subprincipal terms as in the smooth case.

Breaking a magnetic zero locus The question that we tackle in this paper and which echoes to the paper of R. Montgomery [27] is the following:

“Can we hear the smoothness of a magnetic zero locus?”.

More precisely we want to analyze the two following model situations (with $h = 1$):

1. making the magnetic zero locus meet a Neumann boundary,
2. breaking the zero locus.

It turns out that the first situation is investigated in [28] and that both of them are analyzed through numerical experiments in [4]. In particular the paper [4] displays new model operators which naturally appears when we break the translation invariance. The spectral analysis of these operators gives rise to numerical simulations which enlighten nice structures of the eigenfunctions of the magnetic Laplacian (see [4, Section 4]). The aim of this paper is to explain such structures by using the strong “breaking limit” $\theta \rightarrow 0$ where θ is the angle which breaks the magnetic zero locus.

Organization of the paper Section 2 aims at introducing the main operators with which this paper is concerned and at stating our main results. In Section 3 we perform a formal series (with respect to $\theta^{1/2}$) analysis to provide accurate estimates of the first eigenvalues of the magnetic Laplacian. In Section 4, by using the formalism of the coherent states, we establish microlocalization properties satisfied by the lowest eigenfunctions of the magnetic Laplacian. In Section 5 we use the microlocal estimates to reduce the analysis to perturbation theory and deduce complete asymptotic expansions of the eigenpairs.

2 Main operators and results

In order to deal with each model situation we use the notation \bullet where $\bullet = \text{Dir}, \text{Neu}, \emptyset$. We denote $\mathbb{R}_\bullet^2 = \mathbb{R}^2$ when $\bullet = \emptyset$ and $\mathbb{R}_\bullet^2 = \mathbb{R}_+^2 = \{(\hat{s}, \hat{t}) \in \mathbb{R}^2 : \hat{t} > 0\}$ when $\bullet = \text{Dir}, \text{Neu}$. Let us fix the breaking parameter $\theta \in (0, \frac{\pi}{2})$. We are concerned by the Friedrichs extension $\hat{\mathcal{L}}_{\tan\theta}^\bullet$ on $L^2(\mathbb{R}_\bullet^2)$ of:

$$D_{\hat{t}}^2 + \left(D_{\hat{s}} + \text{sgn}(\hat{t}) \frac{\hat{t}^2}{2} - \hat{s}\hat{t} \tan\theta \right)^2,$$

where sgn is the sign function. When $\bullet \neq \emptyset$, the boundary $\hat{t} = 0$ carries the Dirichlet (Dir) or the Neumann (Neu) condition.

Remark 2.1 *The corresponding magnetic field is given by $\beta(\hat{s}, \hat{t}) = |\hat{t}| - \hat{s} \tan\theta$ and vanishes along the line $|\hat{t}| = \hat{s} \tan\theta$.*

Notation 2.2 *We let $\varepsilon = \tan\theta$.*

For $(\alpha, \xi) \in \mathbb{R}^2$ and $\varepsilon > 0$, we introduce the unitary transform:

$$V_{\varepsilon, \alpha, \xi} \psi(\hat{s}, \hat{t}) = e^{-i\xi\hat{s}} \psi\left(\hat{s} - \frac{\alpha}{\varepsilon}, \hat{t}\right)$$

and the conjugate operator:

$$\hat{\mathcal{L}}_{\varepsilon, \alpha, \xi}^\bullet = V_{\varepsilon, \alpha, \xi}^{-1} \hat{\mathcal{L}}_\bullet^\bullet V_{\varepsilon, \alpha, \xi}.$$

Its expression is given by:

$$\hat{\mathcal{L}}_{\varepsilon, \alpha, \xi}^\bullet = D_{\hat{t}}^2 + \left(-\xi - \alpha\hat{t} + \text{sgn}(\hat{t}) \frac{\hat{t}^2}{2} + D_{\hat{s}} - \varepsilon\hat{s}\hat{t} \right)^2.$$

We perform the change of variables:

$$\hat{s} = \varepsilon^{-1/2}\sigma, \quad \hat{t} = \tau. \tag{2.1}$$

Main operator The operator $\hat{\mathcal{L}}_{\varepsilon, \alpha, \xi}^\bullet$ is unitarily equivalent to $\mathfrak{M}_{\varepsilon, \alpha, \xi}^\bullet$ whose expression is given by:

$$\mathfrak{M}_{\varepsilon, \alpha, \xi}^\bullet = D_\tau^2 + \left(-\xi - \alpha\tau + \text{sgn}(\tau) \frac{\tau^2}{2} + \varepsilon^{1/2} D_\sigma - \varepsilon^{1/2} \sigma\tau \right)^2. \tag{2.2}$$

Notation 2.3 *Since the spectrum of $\mathfrak{M}_{\varepsilon, \alpha, \xi}^\bullet$ does not depend on (α, ξ) , we just denote by $\lambda_n^\bullet(\varepsilon)$ its n -th Rayleigh quotient.*

Model operators Taking $\varepsilon = 0$ in (2.2), we are led to introduce a family of model operators depending on two parameters.

Notation 2.4 We introduce the notation \mathbb{R}_\bullet . If $\bullet = \text{Dir}, \text{Neu}$, \mathbb{R}_\bullet denotes \mathbb{R}_+ . If $\bullet = \emptyset$, we let $\mathbb{R}_\bullet = \mathbb{R}$.

For $\alpha, \xi \in \mathbb{R}$, let us introduce the following 1D model operator, on $L^2(\mathbb{R}_\bullet)$:

$$\mathcal{M}_{\alpha, \xi}^\bullet = D_\tau^2 + \left(-\xi - \alpha\tau + \text{sgn}(\tau) \frac{\tau^2}{2} \right)^2,$$

with boundary condition at $\tau = 0$ given by \bullet . This operator has compact resolvent and its form domain is independent from (α, ξ) . $(\mathcal{M}_{\alpha, \xi}^\bullet)_{\alpha, \xi \in \mathbb{R}}$ is an analytic family of type (B) (see [21]). Let us denote $\mu_1^\bullet(\alpha, \xi)$ the lowest eigenvalue of $\mathcal{M}_{\alpha, \xi}^\bullet$ and by $u_{\alpha, \xi}^\bullet$ a normalized and positive associated eigenfunction. This eigenvalue is simple and analytically depends on (α, ξ) .

2.1 Known results

Let us gather a few results obtained in [4]. The methods used to prove them are not directly related to the issue of this paper.

The operator $\mathfrak{M}_{\varepsilon, \alpha, \xi}^\bullet$ We can relate the essential spectrum with the bottom of the spectrum of the Montgomery operator.

Proposition 2.5 For all $\varepsilon > 0$, we have:

$$\mathfrak{s}_{\text{ess}}(\mathfrak{M}_{\varepsilon, \alpha, \xi}^\bullet) = \left[(1 + \varepsilon^2)^{1/3} \mu_{\text{Mont}}, +\infty \right).$$

In the case with Dirichlet boundary condition, the spectrum is essential.

Proposition 2.6 For all $\varepsilon > 0$, we have:

$$\mathfrak{s}(\mathfrak{M}_{\varepsilon, \alpha, \xi}^{\text{Dir}}) = \left[(1 + \varepsilon^2)^{1/3} \mu_{\text{Mont}}, +\infty \right).$$

As soon as an eigenvalue lies below the essential spectrum, we can prove an Agmon type estimate (which indicates that the first eigenfunctions live near the axis $\{\tau = 0\}$):

Proposition 2.7 We assume that $\bullet = \emptyset, \text{Neu}$. There exist $C > 0, c > 0$ such that for all $\varepsilon > 0, \alpha, \xi \in \mathbb{R}$ and all eigenpair (λ, ψ) of $\mathfrak{M}_{\varepsilon, \alpha, \xi}^\bullet$ such that $\lambda < (1 + \varepsilon^2)^{1/3} \mu_{\text{Mont}}$, we have:

$$\int_{\mathbb{R}_\bullet^2} e^{2c|\tau| \sqrt{(1+\varepsilon^2)^{1/3} \mu_{\text{Mont}} - \lambda}} |\psi|^2 \, d\sigma \, d\tau \leq C (\mu_{\text{Mont}} - (1 + \varepsilon^2)^{-1/3} \lambda)^{-1} \|\psi\|^2$$

Remark 2.8 It is proved in [28] that for all $\varepsilon > 0$ there is always an eigenvalue below the essential spectrum in the case $\bullet = \text{Neu}$. The present paper gives a new proof of this fact in the limit $\varepsilon \rightarrow 0$. We will also notice that, as soon as ε is small enough, there is at least one eigenvalue below the essential spectrum when $\bullet = \emptyset$.

The operator $\mathcal{M}_{\alpha,\xi}^\bullet$

Proposition 2.9 *We assume that $\bullet = \emptyset, \text{Neu}$. The function $\mathbb{R} \times \mathbb{R} \ni (\alpha, \xi) \mapsto \mu_1^\bullet(\alpha, \xi)$ admits a minimum μ_0^\bullet and $\limsup_{|\alpha|+|\xi| \rightarrow +\infty} \mu_1^\bullet(\alpha, \xi) > \mu_0^\bullet$. Moreover we have $\mu_0^\bullet < \mu_{\text{Mont}}$.*

Remark 2.10 *In order to prove Proposition 2.9 we use the fact that $\mu_{\text{Mont}} \geq 0.5$ for the case $\bullet = \emptyset$. This fact is not proved, but numerically conjectured in [4].*

Conjecture 2.11 *We assume that $\bullet = \emptyset, \text{Neu}$. The function $\mathbb{R} \times \mathbb{R} \ni (\alpha, \xi) \mapsto \mu_1^\bullet(\alpha, \xi)$ admits a unique and non-degenerate minimum μ_0^\bullet at a point denoted by $(\alpha_0^\bullet, \xi_0^\bullet)$.*

Convention 2.12 *In what follows we omit the superscript $\bullet \in \{\text{Neu}, \emptyset\}$.*

2.2 Spirit of the analysis and main result

The main purpose of this paper is to analyze the influence of the breaking parameter on the spectrum. In particular, we would like to make ε tend to 0 to increase the effect of the symmetry breaking and exhibit a spectral structure as it is the case for waveguides with small apertures (see [9, 10]).

Approximating the spectrum with power series We introduce:

$$\mathcal{H}^{\text{Harm}} = \frac{\partial_\xi^2 \mu_1(\alpha_0, \xi_0)}{2} D_\sigma^2 - \frac{\partial_\xi \partial_\alpha \mu_1(\alpha_0, \xi_0)}{2} \sigma D_\sigma - \frac{\partial_\xi \partial_\alpha \mu_1(\alpha_0, \xi_0)}{2} D_\sigma \sigma + \frac{\partial_\alpha^2 \mu_1(\alpha_0, \xi_0)}{2} \sigma^2. \quad (2.3)$$

Using the non-degeneracy given by Conjecture 2.11, we infer that the operator $\mathcal{H}^{\text{Harm}}$ is unitarily equivalent to an harmonic oscillator whose spectrum is given by:

$$\left\{ \nu_n^{\text{Harm}} = \frac{2n-1}{2} (\partial_\xi^2 \mu_1(\alpha_0, \xi_0) \partial_\alpha^2 \mu_1(\alpha_0, \xi_0) - (\partial_\alpha \partial_\xi \mu_1(\alpha_0, \xi_0))^2)^{1/2}, \quad n \geq 1 \right\}.$$

Proposition 2.13 *We assume that Conjecture 2.11 is true. For all $n \geq 1$, there exists a family $(\gamma_j^n)_{j \geq 0}$ such that for all $J \geq 0$ there exist $C > 0, \varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, we have:*

$$\text{dist} \left(\sum_{j=0}^J \gamma_j^n \varepsilon^{j/2}, \mathfrak{s}(\mathfrak{M}_\varepsilon) \right) \leq C \varepsilon^{(J+1)2},$$

with:

$$\gamma_0^n = \mu_0, \quad \gamma_1^n = 0, \quad \gamma_2^n = \nu_n^{\text{Harm}}.$$

The following corollary is a straightforward consequence.

Corollary 2.14 *We assume that Conjecture 2.11 is true. For all $n \geq 1$, there exist $C > 0, \varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\lambda_n(\varepsilon) \leq \mu_0 + C\varepsilon.$$

Remark 2.15 *With Proposition 2.5, we infer that $\lambda_n(\varepsilon)$ is an eigenvalue as soon as ε is small enough.*

Using Propositions 2.9 and 2.7, we infer:

Corollary 2.16 *We assume that Conjecture 2.11 is true. For all $n \geq 1$, there exist $C > 0, \varepsilon_0 > 0, c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all eigenpair $(\lambda_n(\varepsilon), \psi)$ of $\mathfrak{M}_{\varepsilon, \alpha_0, \xi_0}$, we have:*

$$\int_{\Omega} e^{2c|\tau|} |\psi|^2 \, d\sigma \, d\tau \leq C \|\psi\|^2.$$

Heuristics and methods Thanks to Proposition 2.13, we can guess what the expansions of the lowest eigenvalues are when ε goes to 0. With Corollary 2.16, we also know that the lowest eigenfunctions are “bounded” with respect to τ . Nevertheless the behavior of the eigenfunctions with respect to the variable σ is still not understood. We can just guess with the formal power series analysis leading to Proposition 2.13 that the eigenfunctions should be “bounded” with respect σ in a suitable sense. Let us very roughly examine the situation. We consider:

$$\mathfrak{M}_\varepsilon = D_\tau^2 + \left(-\xi_0 - \alpha_0\tau + \operatorname{sgn}(\tau)\frac{\tau^2}{2} + \varepsilon^{1/2}D_\sigma - \varepsilon^{1/2}\sigma\tau \right)^2 = D_\tau^2 + P_\varepsilon^2. \quad (2.4)$$

We cannot perform a Fourier transform with respect to σ (which would reduce the analysis to a family of model operators). Nevertheless, a coherent states decomposition (Fourier decomposition with an Gaussian weight) will permit to play with σ and D_σ as if they were parameters. Let consider our operator when acting on a function localized in the phase space at the point $(\sigma, D_\sigma) = (u, p)$. The operator \mathfrak{M}_ε is approximated by the microlocalized operator $\mathfrak{M}_{\varepsilon,u,p}$:

$$\mathfrak{M}_{\varepsilon,u,p} = D_\tau^2 + \left(-\xi_0 - \alpha_0\tau + \operatorname{sgn}(\tau)\frac{\tau^2}{2} + \varepsilon^{1/2}p - \varepsilon^{1/2}u\tau \right)^2 = \mathcal{M}_{\alpha_0 + u\varepsilon^{1/2}, \xi_0 - p\varepsilon^{1/2}}.$$

Therefore we guess that the lowest eigenvalues should be described by the one of the 1D pseudo-differential operator:

$$\mu_1(\alpha_0 + \varepsilon^{1/2}\sigma, \xi_0 - \varepsilon^{1/2}D_\sigma)$$

and the non-degeneracy of the minimum of μ_1 should involve nice microlocalization properties of the eigenfunctions. We will combine a coherent states decomposition with variational and spectral techniques to make this heuristics rigorous.

Main result In the spirit of the Gårding inequality (see for instance [26] to see the relation with the FBI transform), this leads to a first rough estimate of the lowest eigenvalues.

Proposition 2.17 *We assume that Conjecture 2.11 is true. For all $n \geq 1$, there exist $C > 0, \varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$:*

$$|\lambda_n(\varepsilon) - \mu_0| \leq C\varepsilon.$$

The last step is to provide an accurate estimate of the spectral gap between the eigenvalues. Such an estimate is the most delicate part of the analysis and obliges to investigate the microlocal behavior of the eigenfunctions with respect to the variable σ . In particular we will need to commute \mathfrak{M}_ε with the creation and annihilation operators and to use generalizations of the so-called “IMS” formula (see [8]).

Proposition 2.18 *We assume that Conjecture 2.11 is true. For all $n \geq 1$, there exist $C > 0, \varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\lambda_n(\varepsilon) \geq \mu_0 + \nu_n^{\text{Harm}}\varepsilon - C\varepsilon^{1+1/8}.$$

Propositions 2.13 and 2.18 imply our main theorem:

Theorem 2.19 *We assume that Conjecture 2.11 is true. For all $n \geq 1$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, we have:*

$$\lambda_n(\varepsilon) \underset{\varepsilon \rightarrow 0}{\sim} \sum_{j \geq 0} \gamma_j^n \varepsilon^{j/2}.$$

Remark 2.20 *Theorem 2.19 implies that the lowest eigenvalues become simple when ε is small enough so that we get an approximation at any order of the corresponding eigenfunctions by the formal power series constructed in Section 3 (see (3.6)). These eigenfunctions are microlocalized near (α_0, ξ_0) . In addition, this theorem is confirmed by the numerical simulations of [4].*

The next sections are devoted to the proofs of our main results.

3 Formal series

This section is devoted to the proof of Proposition 2.13.

3.1 Feynman-Hellmann formulas with two parameters

Let us prove the so-called Feynman-Hellmann formulas (which are consequences of the perturbation theory of Kato, see [21]) associated with $\mathcal{M}_{\alpha, \xi}$.

Proposition 3.1 *If (α_1, ξ_1) is a critical point of μ_1 , we have the following formulas:*

$$\begin{aligned} \int_{\mathbb{R}_\bullet} \left(-\xi_1 - \alpha_1 \tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2} \right) \tau u_{\alpha_1, \xi_1} u_{\alpha_1, \xi_1} \, d\tau \\ = \int_{\mathbb{R}_\bullet} \left(-\xi_1 - \alpha_1 \tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2} \right) u_{\alpha_1, \xi_1} u_{\alpha_1, \xi_1} \, d\tau = 0. \end{aligned} \quad (3.1)$$

and at the point (α_1, ξ_1) , we have:

$$\begin{aligned} \partial_\alpha \partial_\xi \mu_1 &= 2 \int_{\mathbb{R}_\bullet} \tau u_{\alpha_1, \xi_1}^2 \, d\tau - 2 \int_{\mathbb{R}_\bullet} \left(-\xi_1 - \alpha_1 \tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2} \right) ((\tau \partial_\xi + \partial_\alpha) u)_{\alpha_1, \xi_1} u_{\alpha_1, \xi_1} \, d\tau, \\ \partial_\alpha^2 \mu_1 &= 2 \int_{\mathbb{R}_\bullet} \tau^2 u_{\alpha_1, \xi_1}^2 \, d\tau - 4 \int_{\mathbb{R}_\bullet} \left(-\xi_1 - \alpha_1 \tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2} \right) \tau (\partial_\alpha u)_{\alpha_1, \xi_1} u_{\alpha_1, \xi_1} \, d\tau, \\ \partial_\xi^2 \mu_1 &= 2 - 4 \int_{\mathbb{R}_\bullet} \left(-\xi_1 - \alpha_1 \tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2} \right) (\partial_\xi u_{\alpha, \xi})_{\alpha_1, \xi_1} u_{\alpha_1, \xi_1} \, d\tau. \end{aligned} \quad (3.2)$$

Proof: We can write:

$$\mathcal{M}_{\alpha, \xi} u_{\alpha, \xi} = \mu_1(\alpha, \xi) u_{\alpha, \xi}.$$

For short, we let: $u = u_{\alpha, \xi}$ and $\mu_1(\alpha, \xi) = \mu_1$. We get:

$$\begin{aligned} (\mathcal{M}_{\alpha, \xi} - \mu_1(\alpha, \xi)) \partial_\alpha u &= (\partial_\alpha \mu_1 - \partial_\alpha \mathcal{M}_{\alpha, \xi}) u, \\ (\mathcal{M}_{\alpha, \xi} - \mu_1(\alpha, \xi)) \partial_\xi u &= (\partial_\xi \mu_1 - \partial_\xi \mathcal{M}_{\alpha, \xi}) u. \end{aligned} \quad (3.3)$$

We have:

$$\partial_\alpha \mathcal{M}_{\alpha, \xi} = -2 \left(-\xi + \frac{\tau^2}{2} - \alpha \tau \right) \tau, \quad \partial_\xi \mathcal{M}_{\alpha, \xi} = -2 \left(-\xi + \frac{\tau^2}{2} - \alpha \tau \right).$$

At a critical point (α_1, ξ_1) of μ_1 , we take the scalar product of the r.h.s. of the equations (3.3) with u_{α_1, ξ_1} and we use the Fredholm condition to get (3.1). We will need the derivatives of second order. Taking again the derivative, we find the equation, at a critical point of μ_1 :

$$\begin{aligned} (\mathcal{M}_{\alpha, \xi} - \mu(\alpha, \xi)) \partial_\alpha \partial_\xi u &= \partial_\alpha \partial_\xi \mu_1 u - \partial_\alpha \partial_\xi \mathcal{M}_{\alpha, \xi} u - \partial_\alpha \mathcal{M}_{\alpha, \xi} \partial_\xi u - \partial_\xi \mathcal{M}_{\alpha, \xi} \partial_\alpha u, \\ (\mathcal{M}_{\alpha, \xi} - \mu_1(\alpha, \xi)) \partial_\alpha^2 u &= \partial_\alpha^2 \mu_1 u - \partial_\alpha^2 \mathcal{M}_{\alpha, \xi} u - 2 \partial_\alpha \mathcal{M}_{\alpha, \xi} \partial_\alpha u, \\ (\mathcal{M}_{\alpha, \xi} - \mu_1(\alpha, \xi)) \partial_\xi^2 u &= \partial_\xi^2 \mu_1 u - \partial_\xi^2 \mathcal{M}_{\alpha, \xi} u - 2 \partial_\xi \mathcal{M}_{\alpha, \xi} \partial_\xi u. \end{aligned} \quad (3.4)$$

A straight forward computation gives:

$$\partial_\alpha \partial_\xi \mathcal{M}_{\alpha, \xi} = 2\tau, \quad \partial_\alpha^2 \mathcal{M}_{\alpha, \xi} = 2\tau^2, \quad \partial_\xi^2 \mathcal{M}_{\alpha, \xi} = 2.$$

We obtain the relations (3.2) after taking the scalar product of the r.h.s. of the equations (3.4) with u_{α_1, ξ_1} and using the Fredholm condition. ■

3.2 Formal series expansions

We recall that:

$$\mathfrak{M}_{\varepsilon, \alpha_0, \xi_0} = D_\tau^2 + \left(-\xi_0 - \alpha_0 \tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2} + \varepsilon^{1/2} D_\sigma - \varepsilon^{1/2} \sigma \tau \right)^2.$$

For short, we let $\mathfrak{M}_{\varepsilon, \alpha_0, \xi_0} = \mathfrak{M}_\varepsilon$. We get:

$$\mathfrak{M}_\varepsilon = \mathcal{L}_0 + \varepsilon^{1/2} \mathcal{L}_1 + \varepsilon \mathcal{L}_2,$$

where:

$$\mathcal{L}_0 = \mathcal{M}_{\alpha_0, \xi_0}, \quad \mathcal{L}_1 = 2 \left(-\xi_0 - \alpha_0 \tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2} \right) (D_\sigma - \sigma \tau), \quad \mathcal{L}_2 = (D_\sigma - \sigma \tau)^2. \quad (3.5)$$

We look for quasi-eigenpairs in the form

$$\psi_\varepsilon \sim \sum_{j \geq 0} \psi_j \varepsilon^{j/2}, \quad \lambda_\varepsilon \sim \sum_{j \geq 0} \gamma_j \varepsilon^{j/2}$$

such that, in the sense of formal series:

$$\mathfrak{M}_\varepsilon \psi_\varepsilon \sim \lambda_\varepsilon \psi_\varepsilon.$$

Let us solve the formal system of PDEs that we get.

Term in ε^0 We must solve the first equation:

$$(\mathcal{L}_0 - \gamma_0) \psi_0 = 0.$$

We choose $\gamma_0 = \mu_1(\alpha_0, \xi_0)$ and $\psi_0(\sigma, \tau) = f_0(\sigma) u_{\alpha_0, \xi_0}(\tau) = f_0(\sigma) u_0(\tau)$.

Term in $\varepsilon^{1/2}$ Then, we solve:

$$(\mathcal{L}_0 - \gamma_0) \psi_1 = (\gamma_1 - \mathcal{L}_1) \psi_0.$$

Writing the Fredholm condition and using Proposition 3.1, it follows that $\gamma_1 = 0$. Moreover, using the proof of Proposition 3.1 (see (3.3)), we can write an explicit expression for ψ_1 :

$$\psi_1(\sigma, \tau) = -D_\sigma f_0 (\partial_\xi u_{\alpha, \xi})_{\alpha_0, \xi_0}(\tau) + \sigma f_0 (\partial_\alpha u)_{\alpha_0, \xi_0}(\tau) + f_1(\sigma) u_0(\tau).$$

Term in ε^1 We now solve the crucial equation :

$$(\mathcal{L}_0 - \gamma_0)\psi_2 = \gamma_2\psi_0 - \mathcal{L}_1\psi_1 - \mathcal{L}_2\psi_0.$$

Let us write the Fredholm condition:

$$\gamma_2 f_0 = \langle \mathcal{L}_1\psi_1, u_0 \rangle_\tau + \langle \mathcal{L}_2\psi_0, u_0 \rangle_\tau,$$

where $\langle \cdot, \cdot \rangle_\tau$ denotes the partial scalar product with respect to τ . This equation takes the form:

$$\gamma_2 f_0 = (AD_\sigma^2 + B_1\sigma D_\sigma + B_2D_\sigma\sigma + C\sigma^2)f_0.$$

Using Proposition 3.1 (see (3.2)), we get:

$$A = \frac{\partial_\xi^2 \mu_1(\alpha_0, \xi_0)}{2}, \quad C = \frac{\partial_\alpha^2 \mu_1(\alpha_0, \xi_0)}{2}.$$

We have:

$$B_1 = - \int_{\mathbb{R}_\bullet} \tau u_0^2 d\tau - \int_{\mathbb{R}_\bullet} (\partial_\alpha V)_{\alpha_0, \xi_0} (\partial_\xi u)_{\alpha_0, \xi_0} u_0 d\tau,$$

$$B_2 = - \int_{\mathbb{R}_\bullet} \tau u_0^2 d\tau - \int_{\mathbb{R}_\bullet} (\partial_\xi V)_{\alpha_0, \xi_0} (\partial_\alpha u)_{\alpha_0, \xi_0} u_0 d\tau,$$

where

$$V_{\alpha, \xi}(\tau) = \left(-\xi - \alpha\tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2} \right)^2.$$

With Proposition 3.1, we infer:

$$B_1 + B_2 = -\partial_\alpha \partial_\xi \mu_1(\alpha_0, \xi_0).$$

Let us now check that $B_1 = B_2$ or equivalently, let us check that:

$$\int_{\mathbb{R}_\bullet} (\partial_\alpha V)_{\alpha_0, \xi_0} (\partial_\xi u)_{\alpha_0, \xi_0} u_0 d\tau = \int_{\mathbb{R}_\bullet} (\partial_\xi V)_{\alpha_0, \xi_0} (\partial_\alpha u)_{\alpha_0, \xi_0} u_0 d\tau.$$

Using (3.3), we infer:

$$\partial_\alpha \mu_1 = \int_{\mathbb{R}_\bullet} \partial_\alpha V_{\alpha, \xi} u_{\alpha, \xi}^2 d\tau, \quad \partial_\xi \mu_1 = \int_{\mathbb{R}_\bullet} \partial_\xi V_{\alpha, \xi} u_{\alpha, \xi}^2 d\tau$$

so that:

$$\partial_\xi \partial_\alpha \mu_1 = \int_{\mathbb{R}_\bullet} \partial_\xi (\partial_\alpha V_{\alpha, \xi} u_{\alpha, \xi}^2) d\tau, \quad \partial_\alpha \partial_\xi \mu_1 = \int_{\mathbb{R}_\bullet} \partial_\alpha (\partial_\xi V_{\alpha, \xi} u_{\alpha, \xi}^2) d\tau.$$

Therefore, we get the equation:

$$\mathcal{H}^{\text{Harm}} f_0 = \gamma_2 f_0,$$

where $\mathcal{H}^{\text{Harm}}$ is defined in (2.3). This leads to chose $\gamma_2 = \nu_n^{\text{Harm}}$ the n -th eigenvalue of $\mathcal{H}^{\text{Harm}}$ and to take for f_0 the corresponding normalized eigenfunction. At this step this determines ψ_0 and permits to solve the equation satisfied by ψ_2 .

Further terms The construction can be continued at any order through a standard induction argument (see for instance [12]) and we construct a family $(\gamma_j^n, \psi_j^n)_{j \geq 0}$ indexed by $n \geq 1$ (determined at the step ε) which solves the formal system. It is also standard that the functions ψ_j have exponential decay with respect to σ and τ (see for instance [3]). We can now prove Proposition 2.13. We have, by construction and using the exponential decay of the ψ_j^n :

$$\left\| \left(\mathfrak{M}_\varepsilon - \sum_{j=0}^J \gamma_j^n \varepsilon^{j/2} \right) \left(\sum_{j=0}^J \psi_j^n \varepsilon^{j/2} \right) \right\| \leq C_J \varepsilon^{(J+1)/2}. \quad (3.6)$$

It remains to apply the spectral theorem and the conclusion follows.

4 Microlocalization and coherent states

This section is devoted to establish microlocal estimates of the eigenfunctions.

4.1 Coherent states

In this section we recall the formalism of coherent states (partially with respect to σ). We refer to the books [15, 7] concerning the Bargmann representation. We let:

$$g_0(\sigma) = \pi^{-1/4} e^{-\sigma^2/2}$$

and the usual creation and annihilation operators:

$$a = \frac{1}{\sqrt{2}}(\sigma + \partial_\sigma), \quad a^* = \frac{1}{\sqrt{2}}(\sigma - \partial_\sigma)$$

which satisfy the commutator identity:

$$[a, a^*] = 1.$$

We notice that:

$$\sigma = \frac{a + a^*}{\sqrt{2}}, \quad \partial_\sigma = \frac{a - a^*}{\sqrt{2}}, \quad aa^* = \frac{1}{2}(D_\sigma^2 + \sigma^2 + 1).$$

We introduce the coherent states:

$$f_{u,p}(\sigma) = e^{ip\sigma} g_0(\sigma - u)$$

and the associated projection:

$$\Pi_{u,p} \psi = \langle \psi, f_{u,p} \rangle f_{u,p} = \psi_{u,p} f_{u,p}$$

which satisfies:

$$\psi = \int \Pi_{u,p} \psi \, du \, dp$$

and the Parseval formula:

$$\|\psi\|^2 = \int_{\mathbb{R}_\bullet} \int |\psi_{u,p}|^2 \, du \, dp \, d\tau.$$

We recall that:

$$af_{u,p} = \frac{u+ip}{\sqrt{2}} f_{u,p}$$

and

$$(a)^m (a^*)^n \psi = \int \left(\frac{u-ip}{\sqrt{2}} \right)^n \left(\frac{u+ip}{\sqrt{2}} \right)^m \Pi_{u,p} \psi \, du \, dp.$$

In particular, we find:

$$\sigma \psi = \int u \Pi_{u,p} \psi \, du \, dp, \quad D_\sigma \psi = \int p \Pi_{u,p} \psi \, du \, dp$$

and:

$$\sigma^2 \psi = \int \left(u^2 - \frac{1}{2} \right) \Pi_{u,p} \psi \, du \, dp, \quad (4.1)$$

$$D_\sigma^2 \psi = \int \left(p^2 - \frac{1}{2} \right) \Pi_{u,p} \psi \, du \, dp, \quad (4.2)$$

$$(\sigma D_\sigma + D_\sigma \sigma) \psi = \int 2up \Pi_{u,p} \psi \, du \, dp. \quad (4.3)$$

We let:

$$V_0(\tau) = -\xi_0 - \alpha_0 \tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2}.$$

Let us recall that (see (3.5)):

$$\mathfrak{M}_\varepsilon = \mathcal{L}_0 + \varepsilon^{1/2} \mathcal{L}_1 + \varepsilon \mathcal{L}_2. \quad (4.4)$$

We can write the Wick ordered operators (see [15, Chapter 2]):

$$\mathcal{L}_1 = 2V_0(\tau) \left(-i \frac{a-a^*}{\sqrt{2}} - \tau \frac{a+a^*}{\sqrt{2}} \right), \quad (4.5)$$

$$\mathcal{L}_2 = \mathcal{L}_{2,\text{reo}} + \left(-\frac{1}{2} - \frac{\tau^2}{2} \right).$$

where:

$$\mathcal{L}_{2,\text{reo}} = -\frac{a^2 - 2aa^* + (a^*)^2}{2} + \frac{\tau^2(a^2 + 2aa^* + (a^*)^2)}{2} + i\tau(a^2 - (a^*)^2).$$

The operator becomes:

$$\mathfrak{M}_\varepsilon \psi = \int \left\{ \mathfrak{M}_{u,p,\varepsilon} + \varepsilon \left(-\frac{\tau^2}{2} - \frac{1}{2} \right) \right\} \Pi_{u,p} \psi \, du \, dp,$$

with:

$$\mathfrak{M}_{u,p,\varepsilon} = D_\tau^2 + \left(-\xi_0 - \alpha_0 \tau + \operatorname{sgn}(\tau) \frac{\tau^2}{2} - \varepsilon^{1/2} u \tau + \varepsilon^{1/2} p \right)^2 = \mathcal{M}_{\alpha_0 + u\varepsilon^{1/2}, \xi_0 - \varepsilon^{1/2} p}.$$

In terms of quadratic form, we have the analog of the so-called ‘‘IMS’’ formula:

$$\mathfrak{Q}_\varepsilon(\psi) = \int \mathfrak{Q}_{u,p,\varepsilon}(\psi_{u,p}) \, du \, dp + \varepsilon \int_{\mathbb{R}_\bullet} \int \left(-\frac{\tau^2}{2} - \frac{1}{2} \right) |\psi_{u,p}|^2 \, du \, dp \, d\tau, \quad (4.6)$$

where $\mathfrak{Q}_{u,p,\varepsilon}$ is the quadratic form associated with $\mathfrak{M}_{u,p,\varepsilon}$.

Proof of Proposition 2.17 Let us prove Proposition 2.17. Let us fix $n \geq 1$ and an eigenpair $(\lambda_n(\varepsilon), \psi)$. With (4.6), we can write:

$$\mathfrak{Q}_\varepsilon(\psi) = \int \mathfrak{Q}_{u,p,\varepsilon}(\psi_{u,p}) \, du \, dp + \varepsilon \int \left(-\frac{\tau^2}{2} - \frac{1}{2} \right) |\psi|^2 \, d\tau \, d\sigma.$$

Thanks to Proposition 2.16, we deduce that:

$$\mathfrak{Q}_\varepsilon(\psi) \geq \int \mathfrak{Q}_{u,p,\varepsilon}(\psi_{u,p}) \, du \, dp - C\varepsilon \|\psi\|^2.$$

so that:

$$\mathfrak{Q}_\varepsilon(\psi) \geq \int \mu_1(\alpha_0 + \varepsilon^{1/2}u, \xi_0 - \varepsilon^{1/2}p) \|\psi_{u,p}\|_\tau^2 \, du \, dp - C\varepsilon \|\psi\|^2$$

and:

$$\lambda_n(\varepsilon) \|\psi\|^2 = \mathfrak{Q}_\varepsilon(\psi) \geq \mu_0 \|\psi\|^2 - C\varepsilon \|\psi\|^2.$$

so that Proposition 2.17 follows.

Space generated by the first eigenfunctions Let us introduce the space generated by the eigenfunctions associated with the lowest eigenvalues. For all $N \geq 1$ and $n \in \{1, \dots, N\}$, we consider a normalized eigenpair $(\lambda_n(\varepsilon), \Psi_{n,\varepsilon})$ such that the family $(\Psi_{n,\varepsilon})_{n=1, \dots, N}$ is orthonormal. We let:

$$\mathfrak{E}_N(\varepsilon) = \text{span}_{1 \leq n \leq N} \Psi_{n,\varepsilon}.$$

Remark 4.1 The estimate given in Proposition 2.16 can be extended to $\psi \in \mathfrak{E}_N(\varepsilon)$.

The next section aims at establishing microlocalization properties of the elements of $\mathfrak{E}_N(\varepsilon)$.

4.2 Microlocalization with respect to σ

We will need the following formula which is the key point in our next estimates.

Lemma 4.2 Let us consider P, A two pseudo-differential operators. We assume that P is symmetric. We have, for ψ in the Schwartz class and in the domain of P^2 :

$$\begin{aligned} \Re \langle P^2 \psi, AA^* \psi \rangle &= \|P(A^* \psi)\|^2 - \|[A^*, P]\psi\|^2 + \Re \langle P\psi, [[P, A], A^*] \psi \rangle \\ &\quad + \Re \left(\langle P\psi, A^*[P, A]\psi \rangle - \overline{\langle P\psi, A[P, A^*]\psi \rangle} \right). \end{aligned} \quad (4.7)$$

Proof: We write, thanks to an integration by parts:

$$\langle P^2 \psi, AA^* \psi \rangle = \langle P\psi, PAA^* \psi \rangle.$$

Then, we introduce the commutators:

$$\langle P\psi, PAA^* \psi \rangle = \langle P\psi, APA^* \psi \rangle + \langle P\psi, [P, A]A^* \psi \rangle$$

and we get:

$$\langle P\psi, PAA^* \psi \rangle = \langle A^* P\psi, PA^* \psi \rangle + \langle P\psi, [P, A]A^* \psi \rangle$$

and:

$$\langle A^*P\psi, PA^*\psi \rangle = \langle [A^*, P]\psi, PA^*\psi \rangle + \langle PA^*\psi, PA^*\psi \rangle.$$

We have:

$$\langle [A^*, P]\psi, PA^*\psi \rangle = \langle [A^*, P]\psi, [P, A^*]\psi \rangle + \langle [A^*, P]\psi, A^*P\psi \rangle.$$

and it remains to notice that:

$$\langle P\psi, [P, A]A^*\psi \rangle = \langle P\psi, [[P, A], A^*]\psi \rangle + \langle P\psi, A^*[P, A]\psi \rangle$$

and the formula follows by taking the real part. \blacksquare

Remark 4.3 *In the following we will use this formula with the magnetic momentum $P = P_\varepsilon$ or $P = D_\tau$ (see (2.4)) and $A = a$ for instance. This formula can be seen as a generalization of the so-called “IMS” formula (see [8, Theorem 3.2]) and of the Agmon identities (cf. [1, 2]).*

We start by proving a lemma providing a localization of the eigenfunctions with respect to τ .

Lemma 4.4 *For all $N \geq 1$, $m \in \mathbb{N}$, $l \geq 0$, $V \in \mathbb{C}_l[X, Y]$, there exist $C > 0, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all eigenpair (λ, ψ) with $\lambda = \lambda_n(\varepsilon)$ and $n \in \{1, \dots, N\}$, we have:*

$$\mathfrak{Q}_\varepsilon(\tau^m V(\sigma, D_\sigma)\psi) \leq C \sum_{j=0}^l \sum_{M \in \mathfrak{M}_j} \|M(\sigma, D_\sigma)\psi\|^2 \quad (4.8)$$

and, in particular:

$$\|\tau^m V(\sigma, D_\sigma)\psi\|^2 \leq C \sum_{j=0}^l \sum_{M \in \mathfrak{M}_j} \|M(\sigma, D_\sigma)\psi\|^2.$$

Proof: We prove this by induction on l . We can notice that this assertion is trivial for $l = 0$ due to the estimates with respect to τ (see Proposition 2.16). Let us consider $m, L \geq 0$ and assume that this inequality is satisfied for $l = 0, \dots, L$ and $k \in \mathbb{N}$. Let us consider $V \in \mathbb{C}_{L+1}[X, Y]$. We may assume that V is a monomial and, thanks to the induction assumption, we may assume that it is in the form $V = \sigma^k D_\sigma^{L+1-k}$ (since the commutators provide lower order polynomials). We use (4.7) with the symmetric operator $A = \tau^m (D_\sigma^{L+1-k} \sigma^k + \sigma^k D_\sigma^{L+1-k})$. We have :

$$\mathfrak{Q}_\varepsilon(A\psi) = \lambda \|A\psi\|^2 + \|[A, P_\varepsilon]\psi\|^2 + \|m\tau^{m-1} (D_\sigma^{L+1-k} \sigma^k + \sigma^k D_\sigma^{L+1-k})\psi\|^2 - \Re \langle P_\varepsilon \psi, [[P_\varepsilon, A], A]\psi \rangle.$$

We notice that $[A, P_\varepsilon]$ is of order less than L and with degree at most $m+1$ with respect to τ so that this term is controlled by the induction assumption. With the same argument, we get the control of $\Re \langle P_\varepsilon \psi, [[P_\varepsilon, A], A]\psi \rangle$. This provides the estimate of the quadratic form:

$$\mathfrak{Q}_\varepsilon(A\psi) \leq \lambda \|A\psi\|^2 + C \sum_{j=0}^L \sum_{M \in \mathfrak{M}_j} \|M(\sigma, D_\sigma)\psi\|^2 + C \sum_{j=0}^{L+1} \sum_{M \in \mathfrak{M}_j} \|m\tau^{m-1} M(\sigma, D_\sigma)\psi\|^2.$$

Since we have $D_\sigma^{L+1-k}\sigma^k + \sigma^k D_\sigma^{L+1-k} = 2\sigma^k D_\sigma^{L+1-k} + \text{lower order terms}$, we infer:

$$\begin{aligned} \mathfrak{Q}_\varepsilon(\tau^m \sigma^k D_\sigma^{L+1-k} \psi) &\leq \lambda \|\tau^m \sigma^k D_\sigma^{L+1-k} \psi\|^2 + C \sum_{j=0}^L \sum_{M \in \mathfrak{M}_j} \|M(\sigma, D_\sigma) \psi\|^2 \\ &\quad + C \sum_{j=0}^{L+1} \sum_{M \in \mathfrak{M}_j} \|m \tau^{m-1} M(\sigma, D_\sigma) \psi\|^2. \end{aligned} \quad (4.9)$$

If $m = 0$, the conclusion easily follows (the term in τ^{m-1} does not appear). If $m \geq 1$, we use that $\tau^m \sigma^k D_\sigma^{L+1-k} \psi$ satisfies a Dirichlet condition on $\tau = 0$ to infer that (see Proposition 2.6):

$$\mathfrak{Q}_\varepsilon(\tau^m \sigma^k D_\sigma^{L+1-k} \psi) \geq (\mu_{\text{Mont}} - C\varepsilon) \|\tau^m \sigma^k D_\sigma^{L+1-k} \psi\|^2.$$

Using that $\mu_{\text{Mont}} > \mu_0$, we deduce:

$$\|\tau^m \sigma^k D_\sigma^{L+1-k} \psi\|^2 \leq C \sum_{j=0}^L \sum_{M \in \mathfrak{M}_j} \|M(\sigma, D_\sigma) \psi\|^2 + C \sum_{j=0}^{L+1} \sum_{M \in \mathfrak{M}_j} \|\tau^{m-1} M(\sigma, D_\sigma) \psi\|^2.$$

We infer that:

$$\mathfrak{Q}_\varepsilon(\tau^m \sigma^k D_\sigma^{L+1-k} \psi) \leq C \sum_{j=0}^L \sum_{M \in \mathfrak{M}_j} \|M(\sigma, D_\sigma) \psi\|^2 + C \sum_{j=0}^{L+1} \sum_{M \in \mathfrak{M}_j} \|\tau^{m-1} M(\sigma, D_\sigma) \psi\|^2.$$

For $m = 1$, we get the conclusion. Then, by induction on m , we find:

$$\mathfrak{Q}_\varepsilon(\tau^m \sigma^k D_\sigma^{L+1-k} \psi) \leq C \sum_{j=0}^{L+1} \sum_{M \in \mathfrak{M}_j} \|M(\sigma, D_\sigma) \psi\|^2.$$

■

Lemma 4.5 *Under Conjecture 2.11, there exist $\eta_0 > 0$ and $c_0 > 0$ such that for all \hat{u}, \hat{p} such that $|\hat{u}| + |\hat{p}| \leq \eta_0$:*

$$\mu_1(\alpha_0 + \hat{u}, \xi_0 - \hat{p}) - \mu_0 \geq c_0(\hat{u}^2 + \hat{p}^2).$$

Then, there exists $c_1 > 0$ such that for all \hat{u}, \hat{p} such that $|\hat{u}| + |\hat{p}| \geq \eta_0$:

$$\mu_1(\alpha_0 + \hat{u}, \xi_0 - \hat{p}) - \mu_0 \geq c_1.$$

Let us now use Lemma 4.5 to get a control of the eigenfunctions with respect to (σ, D_σ) .

Proposition 4.6 *For all $N \geq 1$, $m \geq 0$, there exist $C > 0, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\psi \in \mathfrak{E}_N(\varepsilon)$, we have:*

$$\|\tau^m \sigma \psi\| + \|\tau^m D_\sigma \psi\| \leq C \|\psi\|. \quad (4.10)$$

Proof: We consider an eigenfunction associated with $\lambda = \lambda_n(\varepsilon)$, with $n \in \{1, \dots, N\}$. We have $\mathcal{L}_\varepsilon \psi = \lambda \psi$ so that $\mathfrak{Q}_\varepsilon(\psi) = \lambda \|\psi\|^2$. Due to Lemma 4.4, (4.10) is a consequence of the inequality when $m = 0$.

A bound near the minimum We recall that (4.6) holds. With Corollary 2.16, it follows that:

$$\mathfrak{Q}_\varepsilon(\psi) \geq \int \mathfrak{Q}_{u,p,\varepsilon}(\psi_{u,p}) \, du \, dp - C\varepsilon \|\psi\|^2.$$

Since $\lambda \leq \mu_0 + C\varepsilon$ (see Corollary 2.14), we infer that:

$$\mathfrak{Q}_\varepsilon(\psi) \geq \int \mathfrak{Q}_{u,p,\varepsilon}(\psi_{u,p}) \, du \, dp - \mu_0 \int \|\psi_{u,p}\|_\tau^2 \, du \, dp - C\varepsilon \|\psi\|^2$$

and thus:

$$\int (\mu(\alpha_0 + \varepsilon^{1/2}u, \xi_0 - \varepsilon^{1/2}p) - \mu_0) \|\psi_{u,p}\|_\tau^2 \, du \, dp \leq C\varepsilon \|\psi\|^2. \quad (4.11)$$

We use Lemma 4.5 to deduce from (4.11) that:

$$c_1 \int_{|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \geq \eta_0} \|\psi_{u,p}\|_\tau^2 \, du \, dp + c_0 \varepsilon \int_{|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \leq \eta_0} (u^2 + p^2) \|\psi_{u,p}\|_\tau^2 \, du \, dp \leq C\varepsilon \|\psi\|^2.$$

In particular, we have:

$$\int_{|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \leq \eta_0} (u^2 + p^2) \|\psi_{u,p}\|_\tau^2 \, du \, dp \leq c_0^{-1} C \|\psi\|^2, \quad (4.12)$$

$$\int_{|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \geq \eta_0} \|\psi_{u,p}\|_\tau^2 \, du \, dp \leq c_1^{-1} C \varepsilon \|\psi\|^2. \quad (4.13)$$

The estimate (4.13) is not enough to get the control with respect to $u^2 + p^2$ on the region away from the minimum.

A bound away from the minimum We recall that $(a^*\psi)_{u,p} = \frac{u-ip}{\sqrt{2}}\psi_{u,p}$. We replace ψ by $a^*\psi$ in (4.6) and we are led to:

$$\mathfrak{Q}_\varepsilon(a^*\psi) = \int \mathfrak{Q}_{u,p,\varepsilon}((a^*\psi)_{u,p}) \, du \, dp + \varepsilon \int \left(-\frac{\tau^2}{2} - \frac{1}{2} \right) |(a^*\psi)_{u,p}|_\tau^2 \, du \, dp.$$

We get:

$$\mathfrak{Q}_\varepsilon(a^*\psi) \geq \int \mathfrak{Q}_{u,p,\varepsilon}((a^*\psi)_{u,p}) \, du \, dp - C\varepsilon (\|\tau a^*\psi\|^2 + \|a^*\psi\|^2)$$

and:

$$\mathfrak{Q}_\varepsilon(a^*\psi) \geq \int \mathfrak{Q}_{u,p,\varepsilon}((a^*\psi)_{u,p}) \, du \, dp - C\varepsilon (\|\tau\sigma\psi\|^2 + \|\sigma\psi\|^2 + \|\tau D_\sigma\psi\|^2 + \|D_\sigma\psi\|^2). \quad (4.14)$$

We have to estimate $\mathfrak{Q}_\varepsilon(a^*\psi)$. For that purpose, we use the formula given in (4.7) with $A = a$. Let us estimate the different terms. We have:

$$[P_\varepsilon, a^*] = i\varepsilon^{1/2} - \tau\varepsilon^{1/2}, \quad [P_\varepsilon, a] = i\varepsilon^{1/2} + \tau\varepsilon^{1/2}$$

so that the formula becomes:

$$\Re \langle P_\varepsilon^2 \psi, a a^* \psi \rangle = \|P_\varepsilon(a^*\psi)\|^2 - \|[a^*, P_\varepsilon]\psi\|^2 + \Re \left(\langle P_\varepsilon \psi, a^* [P_\varepsilon, a] \psi \rangle - \overline{\langle P_\varepsilon \psi, a [P_\varepsilon, a^*] \psi \rangle} \right).$$

We infer that:

$$\lambda \|a^*\psi\|^2 \geq \mathfrak{Q}_\varepsilon(a^*\psi) - C\varepsilon \|\tau\psi\|^2 - C\varepsilon^{1/2} \|P_\varepsilon\psi\| (\|D_\sigma\psi\| + \|\sigma\psi\| + \|\tau D_\sigma\psi\| + \|\tau\sigma\psi\|)$$

and:

$$\lambda \|a^* \psi\|^2 \geq \mathfrak{Q}_\varepsilon(a^* \psi) - C\varepsilon^{1/2}(\|\psi\|^2 + \|D_\sigma \psi\|^2 + \|\sigma \psi\|^2 + \|\tau D_\sigma \psi\|^2 + \|\tau \sigma \psi\|^2).$$

With (4.14), it follows that:

$$\lambda \|a^* \psi\|^2 \geq \int \mathfrak{Q}_{u,p,\varepsilon}((a^* \psi)_{u,p}) \, du \, dp - C\varepsilon^{1/2}(\|\psi\|^2 + \|\tau \sigma \psi\|^2 + \|\sigma \psi\|^2 + \|\tau D_\sigma \psi\|^2 + \|D_\sigma \psi\|^2)$$

and:

$$\int (\mathfrak{Q}_{u,p,\varepsilon}((a^* \psi)_{u,p}) - \mu_0) \|\psi_{u,p}\|_\tau^2 \, du \, dp \leq C\varepsilon^{1/2}(\|\psi\|^2 + \|\tau \sigma \psi\|^2 + \|\sigma \psi\|^2 + \|\tau D_\sigma \psi\|^2 + \|D_\sigma \psi\|^2).$$

We infer the following estimates:

$$\begin{aligned} \int_{|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \leq \eta_0} (u^2 + p^2) \|(a^* \psi)_{u,p}\|_\tau^2 \, du \, dp \\ \leq C\varepsilon^{-1/2}(\|\psi\|^2 + \|\tau \sigma \psi\|^2 + \|\sigma \psi\|^2 + \|\tau D_\sigma \psi\|^2 + \|D_\sigma \psi\|^2) \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} \int_{|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \geq \eta_0} \|(a^* \psi)_{u,p}\|_\tau^2 \, du \, dp \\ \leq C\varepsilon^{1/2}(\|\psi\|^2 + \|\tau \sigma \psi\|^2 + \|\sigma \psi\|^2 + \|\tau D_\sigma \psi\|^2 + \|D_\sigma \psi\|^2). \end{aligned} \quad (4.16)$$

Combining (4.16) and (4.12), we deduce that:

$$\int \int (u^2 + p^2) \|\psi_{u,p}\|_\tau^2 \, du \, dp \leq C\|\psi\|^2 + C\varepsilon^{1/2}(\|\tau \sigma \psi\|^2 + \|\sigma \psi\|^2 + \|\tau D_\sigma \psi\|^2 + \|D_\sigma \psi\|^2).$$

It remains to use (4.1), (4.2) and Lemma 4.4 to get (4.10) for $m = 0$. The extension of (4.10) to $\psi \in \mathfrak{E}_N(\varepsilon)$ is then standard. \blacksquare

Let us now prove a higher order microlocalization proposition.

Proposition 4.7 *For all $N \geq 1$, $m \geq 0$, there exist $C > 0, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\psi \in \mathfrak{E}_N(\varepsilon)$, we have:*

$$\|\tau^m \sigma^2 \psi\| + \|\tau^m D_\sigma^2 \psi\| + \|\tau^m \sigma D_\sigma \psi\| + \|\tau^m D_\sigma \sigma \psi\| \leq C\varepsilon^{-1/4} \|\psi\|. \quad (4.17)$$

Proof: We prove (4.17) when ψ is an eigenfunction associated with $\lambda = \lambda_n(\varepsilon)$, with $n \in \{1, \dots, N\}$. By Lemma 4.4 and Proposition 4.6, it is sufficient to deal with the case when $m = 0$. We recall that (4.15) and (4.16) still hold. With Proposition 4.6, we infer:

$$\int_{|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \leq \eta_0} (u^2 + p^2) \|(a^* \psi)_{u,p}\|_\tau^2 \, du \, dp \leq C\varepsilon^{-1/2} \|\psi\|^2 \quad (4.18)$$

and

$$\int_{|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \geq \eta_0} \|(a^* \psi)_{u,p}\|_\tau^2 \, du \, dp \leq C\varepsilon^{1/2} \|\psi\|^2. \quad (4.19)$$

As previously, (4.19) does not provide enough information. We write (4.7) with $A = aa^*$ so that it becomes:

$$\Re \langle P_\varepsilon^2 \psi, (aa^*)^2 \psi \rangle = \|P_\varepsilon(aa^* \psi)\|^2 - \|[aa^*, P_\varepsilon] \psi\|^2 + \Re \langle P_\varepsilon \psi, [[P_\varepsilon, aa^*], aa^*] \psi \rangle. \quad (4.20)$$

A straightforward computation provides:

$$[P_\varepsilon, aa^*] = -i\varepsilon^{1/2}(\sigma + \tau D_\sigma), \quad [[P_\varepsilon, aa^*], aa^*] = \varepsilon^{1/2}(D_\sigma - \tau\sigma).$$

We deduce that:

$$\lambda \|aa^*\psi\|^2 \geq \mathfrak{Q}_\varepsilon(aa^*\psi) - C\varepsilon^{1/2}\|\psi\|^2.$$

Let us now write the coherent states microlocalization formula to estimate $\mathfrak{Q}_\varepsilon(aa^*\psi)$. We have:

$$\mathfrak{Q}_\varepsilon(aa^*\psi) = \langle aa^* \mathcal{L}_\varepsilon aa^* \psi, \psi \rangle.$$

We recall that (4.4) and (4.5) hold. We have the Wick ordering:

$$aa^* \mathcal{L}_0 aa^* = \mathcal{L}_0 aa^* aa^* = \mathcal{L}_0 (a^2 a^*)^2 - \mathcal{L}_0 aa^*.$$

Let us introduce a convenient notation.

Notation 4.8 We denote by $\mathcal{O}_j(\sigma, D_\sigma)$ every polynomial in σ and D_σ with degree less than j .

We can write the Wick ordering:

$$aa^* \mathcal{L}_1 aa^* = \mathcal{L}_1 a^2 (a^*)^2 + V_0 \mathcal{O}_3(D_\sigma, \sigma) + \tau V_0 \mathcal{O}_3(D_\sigma, \sigma)$$

and:

$$aa^* \mathcal{L}_{2, \text{reo}} aa^* = \mathcal{L}_{2, \text{reo}} a^2 (a^*)^2 + \varepsilon \mathcal{O}_4(\sigma, D_\sigma) + \varepsilon \tau^2 \mathcal{O}_4(\sigma, D_\sigma).$$

It follows that:

$$\begin{aligned} aa^* \mathcal{L}_\varepsilon aa^* \psi &= \int \frac{u^2 + p^2}{2} \mathcal{L}_{u,p,\varepsilon} \frac{u^2 + p^2}{2} \Pi_{u,p} \psi \, du \, dp \\ &\quad - \mathcal{L}_0 aa^* \psi + \varepsilon^{1/2} (V_0 \mathcal{O}_3(D_\sigma, \sigma) + \tau V_0 \mathcal{O}_3(D_\sigma, \sigma)) \psi + \varepsilon (\mathcal{O}_4(\sigma, D_\sigma) + \tau^2 \mathcal{O}_4(\sigma, D_\sigma)) \psi \\ &\quad + \varepsilon (\mathcal{O}_2(\sigma, D_\sigma) + \tau^2 \mathcal{O}_2(\sigma, D_\sigma)) \psi. \end{aligned} \quad (4.21)$$

In terms of quadratic form, we get:

$$\begin{aligned} \mathfrak{Q}_\varepsilon(aa^*\psi) &\geq \int \mathfrak{Q}_{u,p,\varepsilon} \left(\frac{u^2 + p^2}{2} \psi_{u,p} \right) \, du \, dp - Q_0(a^*\psi) \\ &\quad - C\varepsilon^{1/2} (\|V_0 \mathcal{O}_1 \psi\| \|\mathcal{O}_2 \psi\| + \|\tau V_0 \mathcal{O}_1 \psi\| \|\mathcal{O}_2 \psi\|) \\ &\quad - C\varepsilon (\|\mathcal{O}_1 \psi\|^2 + \|\tau \mathcal{O}_1 \psi\|^2) - C\varepsilon (\|\mathcal{O}_2 \psi\|^2 + \|\tau \mathcal{O}_2 \psi\|^2). \end{aligned} \quad (4.22)$$

With Proposition 4.6, we have:

$$\begin{aligned} \|V_0 \mathcal{O}_1 \psi\| \|\mathcal{O}_2 \psi\| + \|\tau V_0 \mathcal{O}_1 \psi\| \|\mathcal{O}_2 \psi\| &\leq C(\|\psi\|^2 + \|\mathcal{O}_2 \psi\|^2), \\ \|\mathcal{O}_1 \psi\|^2 + \|\tau \mathcal{O}_1 \psi\|^2 &\leq C\|\psi\|^2, \\ Q_0(a^*\psi) &\leq C\|\psi\|^2. \end{aligned}$$

This reduces to the inequality:

$$\int \mathfrak{Q}_{u,p,\varepsilon} ((u^2 + p^2) \psi_{u,p}) - \mu_0 \|(u^2 + p^2) \psi_{u,p}\|_\tau^2 \, du \, dp \leq C\|\psi\|^2 + C\varepsilon^{1/2} \|\mathcal{O}_2 \psi\|^2$$

which provides:

$$\varepsilon \int_{|\varepsilon^{1/2}u|+|\varepsilon^{1/2}p|\leq\eta_0} (u^2 + p^2)^3 \|\psi_{u,p}\|_\tau^2 du dp \leq C\|\psi\|^2 + C\varepsilon^{1/2}\|\mathcal{O}_2\psi\|^2 \quad (4.23)$$

and

$$\int_{|\varepsilon^{1/2}u|+|\varepsilon^{1/2}p|\geq\eta_0} (u^2 + p^2)^2 \|\psi_{u,p}\|_\tau^2 du dp \leq C\|\psi\|^2 + C\varepsilon^{1/2}\|\mathcal{O}_2\psi\|^2. \quad (4.24)$$

From (4.18) and (4.24), we get:

$$\int (u^2 + p^2)^2 \|\psi_{u,p}\|_\tau^2 du dp \leq C\varepsilon^{-1/2}\|\psi\|^2 + C\|\psi\|^2 + C\varepsilon^{1/2}\|\mathcal{O}_2\psi\|^2. \quad (4.25)$$

We have:

$$\|\mathcal{O}_2\psi\|^2 \leq C \int (u^2 + p^2)^2 \|\psi_{u,p}\|_\tau^2 du dp + C\|\mathcal{O}_1\psi\|^2 \leq C \int (u^2 + p^2)^2 \|\psi_{u,p}\|_\tau^2 du dp + \tilde{C}\|\psi\|^2$$

so that:

$$\int (u^2 + p^2)^2 \|\psi_{u,p}\|_\tau^2 du dp \leq C\varepsilon^{-1/2}\|\psi\|^2 \quad (4.26)$$

and the conclusion easily follows (since (4.26) implies the control of the norm of ψ in $B^2(\mathbb{R}_\sigma)$). \blacksquare

We will need a last microlocalization proposition.

Proposition 4.9 *For all $N \geq 1$, $m \geq 0$, there exist $C > 0, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\psi \in \mathfrak{E}_N(\varepsilon)$, we have:*

$$\|\tau^m \sigma^3 \psi\| + \|\tau^m D_\sigma^3 \psi\| + \|\tau^m \sigma^2 D_\sigma \psi\| + \|\tau^m D_\sigma^2 \sigma \psi\| \leq C\varepsilon^{-1/2}\|\psi\|. \quad (4.27)$$

Proof: We prove (4.27) when ψ is an eigenfunction associated with $\lambda = \lambda_n(\varepsilon)$, with $n \in \{1, \dots, N\}$. Thanks to Lemma 4.4 and Propositions 4.6 and 4.7, we just have to deal with the case $m = 0$. Thanks to Proposition 4.7, we deduce from (4.23) that:

$$\int_{|\varepsilon^{1/2}u|+|\varepsilon^{1/2}p|\leq\eta_0} (u^2 + p^2)^3 \|\psi_{u,p}\|_\tau^2 du dp \leq C\varepsilon^{-1}\|\psi\|^2. \quad (4.28)$$

We need again to provide an estimate on the region $|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \geq \eta_0$. In (4.20) we replace ψ by $a^*\psi$ and we obtain:

$$\Re\langle P_\varepsilon^2 a^* \psi, (aa^*)^2 a^* \psi \rangle = \|P_\varepsilon(a(a^*)^2 \psi)\|^2 - \|[aa^*, P_\varepsilon]a^* \psi\|^2 + \Re\langle P_\varepsilon a^* \psi, [[P_\varepsilon, aa^*], aa^*]a^* \psi \rangle.$$

Let us analyze the last two terms. We have:

$$\|[aa^*, P_\varepsilon]a^* \psi\|^2 \leq C\varepsilon^{1/2}\|\psi\|^2$$

and:

$$|\Re\langle P_\varepsilon a^* \psi, [[P_\varepsilon, aa^*], aa^*]a^* \psi \rangle| \leq C\varepsilon^{1/4}\|\psi\|^2.$$

Let us consider $\Re\langle P_\varepsilon^2 a^* \psi, (aa^*)^2 a^* \psi \rangle$:

$$\Re\langle P_\varepsilon^2 a^* \psi, (aa^*)^2 a^* \psi \rangle = \langle a^* P_\varepsilon^2 \psi, (aa^*)^2 a^* \psi \rangle + \langle [P_\varepsilon^2, a^*] \psi, (aa^*)^2 a^* \psi \rangle.$$

We have:

$$[P_\varepsilon^2, a^*] = P_\varepsilon [P_\varepsilon, a^*] - [a^*, P_\varepsilon] P_\varepsilon.$$

and then the following estimate holds:

$$|\langle [P_\varepsilon^2, a^*]\psi, (aa^*)^2 a^* \psi \rangle| \leq C\varepsilon^{1/4} \|\mathcal{O}_3 \psi\|^2 + \varepsilon^{1/4} \|\psi\|^2.$$

We infer that:

$$\lambda \|a(a^*)^2 \psi\|^2 \geq \mathfrak{Q}_\varepsilon(a(a^*)^2 \psi) - C\varepsilon^{1/4} \|\mathcal{O}_3 \psi\|^2 - C\varepsilon^{1/4} \|\psi\|^2.$$

Then, we replace ψ by $a^* \psi$ in (4.22) to get:

$$\lambda \|a(a^*)^2 \psi\|^2 \geq \int \mathfrak{Q}_{u,p,\varepsilon} \left(\frac{u^2 + p^2}{2} \frac{u - ip}{\sqrt{2}} \psi_{u,p} \right) du dp - C\varepsilon^{1/4} \|\mathcal{O}_3 \psi\|^2 - C\varepsilon^{1/4} \|\psi\|^2 + C\varepsilon^{-1/2} \|\psi\|^2.$$

As previously, we deduce that:

$$\int_{|\varepsilon^{1/2}u| + |\varepsilon^{1/2}p| \geq \eta_0} (u^2 + p^2)^3 \|\psi_{u,p}\|_\tau^2 du dp \leq C\varepsilon^{1/4} \|\psi\|^2 + C\varepsilon^{1/4} \|\mathcal{O}_3 \psi\|^2 + C\varepsilon^{-1/2} \|\psi\|^2. \quad (4.29)$$

We infer that:

$$\int (u^2 + p^2)^3 \|\psi_{u,p}\|_\tau^2 du dp \leq C\varepsilon^{-1} \|\psi\|^2 \quad (4.30)$$

and the conclusion follows. \blacksquare

5 Reduction to perturbation theory

Thanks to Propositions 4.6, 4.7 and 4.9, we have reduced our problem to the perturbation theory. We introduce the Feshbach projection:

$$\Pi_0 \psi(\sigma, \tau) = \langle \psi, u_0 \rangle_\tau u_0(\tau)$$

and the corrected Feshbach projection:

$$\Pi_0^\varepsilon \psi = \Pi_0 \psi(\sigma, \tau) + \varepsilon^{1/2} (-D_\sigma \langle \psi, u_0 \rangle_\tau (\partial_\xi u_{\alpha,\xi})_{\alpha_0, \xi_0}(\tau) + \sigma \langle \psi, u_0 \rangle_\tau (\partial_\alpha u_{\alpha,\xi})_{\alpha_0, \xi_0}(\tau)).$$

Proposition 5.1 *For all $N \geq 1$, $m \geq 0$, there exist $C > 0$, $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $\psi \in \mathfrak{E}_N(\varepsilon)$, we have:*

$$\|\psi - \Pi_0 \psi\| \leq C\varepsilon^{1/2} \|\psi\|, \quad (5.1)$$

$$\|\sigma(\psi - \Pi_0 \psi)\| \leq C\varepsilon^{1/4} \|\psi\|, \quad \|D_\sigma(\psi - \Pi_0 \psi)\| \leq C\varepsilon^{1/4} \|\psi\|, \quad (5.2)$$

$$\|\tau^m \sigma(\psi - \Pi_0 \psi)\| \leq C\varepsilon^{1/8} \|\psi\|, \quad \|\tau^m D_\sigma(\psi - \Pi_0 \psi)\| \leq C\varepsilon^{1/8} \|\psi\|. \quad (5.3)$$

Proof: It is enough to establish these approximation results when ψ is an eigenfunction. Let us consider an eigenpair (λ, Ψ) such that $|\lambda - \mu_0| \leq D\varepsilon$ (see Corollary 2.14). We have:

$$(\mathcal{L}_0 - \mu_0)\Psi = (\lambda - \mu_0)\Psi - \varepsilon^{1/2} \mathcal{L}_1 \psi - \varepsilon \mathcal{L}_2 \Psi$$

so that, with the microlocalization properties, we infer:

$$\|(\mathcal{L}_0 - \mu_0)\Psi\| \leq C\varepsilon^{1/2} \|\Psi\|.$$

and thus:

$$\|(\mathcal{L}_0 - \mu_0)(\Psi - \Pi_0 \Psi)\| \leq C\varepsilon^{1/2} \|\Psi\|.$$

Since $\langle \Psi - \Pi_0 \Psi, u_0 \rangle_\tau = 0$, the spectral theorem provides (5.1) when ψ is an eigenfunction. Then, we write:

$$(\mathcal{L}_0 - \mu_0)(\sigma \Psi) = (\lambda - \mu_0)\sigma \Psi - \varepsilon^{1/2}\sigma \mathcal{L}_1 \Psi - \varepsilon \sigma \mathcal{L}_2 \Psi.$$

With the microlocalization properties, we find:

$$\|(\mathcal{L}_0 - \mu_0)\sigma \Psi\| \leq C\varepsilon^{1/4}\|\Psi\|.$$

In the same way, we get:

$$\|(\mathcal{L}_0 - \mu_0)D_\sigma \Psi\| \leq C\varepsilon^{1/4}\|\Psi\|.$$

We deduce (5.2) for $m = 0$. For $m \geq 1$, we notice that:

$$\begin{aligned} \int \tau^{2m} |\Psi - \Pi_0 \Psi|^2 d\sigma d\tau &\leq \|\Psi - \Pi_0 \Psi\| \|\tau^{4m}(\Psi - \Pi_0 \Psi)\| \leq C\varepsilon^{1/2}\|\Psi\|^2, \\ \int \tau^{2m} |\sigma(\Psi - \Pi_0 \Psi)|^2 d\sigma d\tau &\leq \|\sigma(\Psi - \Pi_0 \Psi)\| \|\tau^{4m}\sigma(\Psi - \Pi_0 \Psi)\| \leq C\varepsilon^{1/4}\|\Psi\|^2, \\ \int \tau^{2m} |D_\sigma(\Psi - \Pi_0 \Psi)|^2 d\sigma d\tau &\leq \|D_\sigma(\Psi - \Pi_0 \Psi)\| \|\tau^{4m}D_\sigma(\Psi - \Pi_0 \Psi)\| \leq C\varepsilon^{1/4}\|\Psi\|^2, \end{aligned}$$

where we have used Proposition 4.6. ■

Proposition 5.2 *For all $N \geq 1$, there exist $C > 0$, $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $\psi \in \mathfrak{E}_N(\varepsilon)$, we have:*

$$\|\psi - \Pi_0^\varepsilon \psi\| \leq C\varepsilon^{1/2+1/8}\|\psi\|.$$

Proof: Let us consider an eigenpair (λ, Ψ) such that $|\lambda - \mu_0| \leq D\varepsilon$ (see again Corollary 2.14). We have:

$$\mathcal{L}_\varepsilon \Psi = \lambda \Psi. \tag{5.4}$$

We write:

$$\mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon^{1/2}\mathcal{L}_1 + \varepsilon \mathcal{L}_2 \tag{5.5}$$

and:

$$\Psi = \psi_0 + \varepsilon^{1/2}\psi_1 + R_\varepsilon, \tag{5.6}$$

where $\psi_0 = \Pi_0 \psi(\sigma, \tau)$ and $\psi_1 = -D_\sigma \langle \psi, u_0 \rangle_\tau \partial_\xi u(\tau) + \sigma \langle \psi, u_0 \rangle_\tau \partial_\alpha u(\tau)$. We want to provide a bound for R_ε . Let us already notice that, for all $\sigma \in \mathbb{R}$:

$$\langle R_\varepsilon, u_0 \rangle_\tau = 0. \tag{5.7}$$

We recall that, by construction, we have the relations:

$$(\mathcal{L}_0 - \mu_0)\psi_0 = 0, \quad (\mathcal{L}_0 - \mu_0)\psi_1 = -\mathcal{L}_1 \psi_0.$$

Therefore, with (5.5) and (5.6), Equation (5.4) becomes:

$$(\mathcal{L}_0 - \mu_0)R_\varepsilon = (\lambda - \mu_0)\Psi - \varepsilon \mathcal{L}_1 \psi_1 - \varepsilon^{1/2}\mathcal{L}_1 R_\varepsilon - \varepsilon \mathcal{L}_2 \Psi.$$

We shall provide an upper bound for the right-hand-side. We have:

$$\|\mathcal{L}_1 \psi_1\| \leq C \|\langle S(\sigma, D_\sigma)\Psi, u_0 \rangle_\tau\|_\sigma = C \|S(\sigma, D_\sigma)\Psi\|,$$

where $S \in \mathbb{C}_2[X, Y]$. Therefore, we get:

$$\|\mathcal{L}_1\psi_1\| \leq C\varepsilon^{-1/4}\|\Psi\|^2.$$

Then, we have:

$$\mathcal{L}_1R_\varepsilon = \mathcal{L}_1(\Psi - \psi_0 - \varepsilon^{1/2}\psi_1)$$

so that:

$$\|\mathcal{L}_1R_\varepsilon\| \leq \|\mathcal{L}_1(\Psi - \psi_0)\| + \varepsilon^{1/2}\|\mathcal{L}_1\psi_1\|.$$

But, with Proposition 5.1, we can write:

$$\|\mathcal{L}_1(\Psi - \psi_0)\| \leq C\varepsilon^{1/8}\|\Psi\|^2.$$

We get:

$$\|\mathcal{L}_2\Psi\| \leq \varepsilon^{-1/4}\|\Psi\|.$$

We deduce that:

$$\|(\lambda - \mu_0)\Psi - \varepsilon\mathcal{L}_1\psi_1 - \varepsilon^{1/2}\mathcal{L}_1R_\varepsilon - \varepsilon\mathcal{L}_2\psi_0 - \varepsilon^{3/2}\mathcal{L}_2\psi_1 - \varepsilon\mathcal{L}_2R_\varepsilon\| \leq C\varepsilon^{1/2+1/8}\|\Psi\|$$

so that:

$$\|(\mathcal{L}_0 - \mu_0)R_\varepsilon\| \leq C\varepsilon^{1/2+1/8}\|\Psi\|.$$

Combining (5.7) with the spectral theorem, we infer that:

$$\|R_\varepsilon(\Psi)\| \leq \tilde{C}\varepsilon^{1/2+1/8}\|\Psi\|.$$

■

Proof of Proposition 2.18 Let us estimate $\mathfrak{Q}_\varepsilon(\psi)$ for $\psi \in \mathfrak{E}_N(\varepsilon)$. We have:

$$\mathfrak{Q}_\varepsilon(\psi) = \langle \mathcal{L}_0\psi, \psi \rangle + \varepsilon^{1/2}\langle \mathcal{L}_1\psi, \psi \rangle + \varepsilon\langle \mathcal{L}_2\psi, \psi \rangle.$$

Let us bound from below the different terms. For the first term, we have:

$$\langle \mathcal{L}_0\psi, \psi \rangle \geq \langle \mathcal{L}_0\Pi_0^\varepsilon\psi, \Pi_0^\varepsilon\psi \rangle + 2\langle \mathcal{L}_0\Pi_0^\varepsilon\psi, R_\varepsilon \rangle.$$

We notice that:

$$\langle \mathcal{L}_0\Pi_0^\varepsilon\psi, R_\varepsilon \rangle = \varepsilon^{1/2}\langle \mathcal{L}_0\psi_1, R_\varepsilon \rangle = \varepsilon^{1/2}\langle \psi_1, \mathcal{L}_0R_\varepsilon \rangle.$$

But we have:

$$\|\psi_1\| \leq C\|\psi\|, \quad \|\mathcal{L}_0R_\varepsilon\| \leq C\varepsilon^{1/2+1/8}\|\psi\|$$

so that:

$$|\langle \psi_1, \mathcal{L}_0R_\varepsilon \rangle| \leq C\varepsilon^{1/2+1/8}\|\psi\|^2$$

and thus:

$$\langle \mathcal{L}_0\psi, \psi \rangle \geq \langle \mathcal{L}_0\Pi_0^\varepsilon\psi, \Pi_0^\varepsilon\psi \rangle - C\varepsilon^{1+1/8}\|\psi\|^2.$$

For the second term, we write:

$$\langle \mathcal{L}_1\psi, \psi \rangle \geq \langle \mathcal{L}_1\Pi_0^\varepsilon\psi, \Pi_0^\varepsilon\psi \rangle + 2\langle \mathcal{L}_1\Pi_0^\varepsilon\psi, R_\varepsilon \rangle + \langle \mathcal{L}_1R_\varepsilon, R_\varepsilon \rangle.$$

We get:

$$|\langle \mathcal{L}_1R_\varepsilon, R_\varepsilon \rangle| \leq \|\mathcal{L}_1R_\varepsilon\|\|R_\varepsilon\|.$$

With the microlocalization properties (see Propositions 4.6, 4.7 and 4.9), we can estimate:

$$\|\mathcal{L}_1 R_\varepsilon\| \leq C\varepsilon^{1/8}\|\psi\|$$

so that, we infer:

$$|\langle \mathcal{L}_1 R_\varepsilon, R_\varepsilon \rangle| \leq C\varepsilon^{1/8}\varepsilon^{1/2+1/8}\|\psi\|^2 = C\varepsilon^{3/4}\|\psi\|^2.$$

In the same way, we find:

$$|\langle \mathcal{L}_1 \Pi_0^\varepsilon \psi, R_\varepsilon \rangle| \leq C\varepsilon^{1/2+1/8}\|\psi\|^2.$$

For the third term, we get:

$$\langle \mathcal{L}_2 \psi, \psi \rangle \geq \langle \mathcal{L}_2 \Pi_0^\varepsilon \psi, \Pi_0^\varepsilon \psi \rangle + 2\langle \mathcal{L}_2 \Pi_0^\varepsilon \psi, R_\varepsilon \rangle + \langle \mathcal{L}_2 R_\varepsilon, R_\varepsilon \rangle.$$

We find:

$$|\langle \mathcal{L}_2 R_\varepsilon, R_\varepsilon \rangle| \leq C\varepsilon^{1/8}\varepsilon^{1/8}\|\psi\|^2 = C\varepsilon^{1/4}\|\psi\|^2$$

and:

$$|\langle \mathcal{L}_2 \Pi_0^\varepsilon \psi, R_\varepsilon \rangle| \leq C\varepsilon^{1/8}\|\psi\|^2.$$

Therefore, we deduce that:

$$\mathfrak{Q}_\varepsilon(\psi) \geq \langle \mathcal{L}_0 \Pi_0^\varepsilon \psi, \Pi_0^\varepsilon \psi \rangle + \varepsilon^{1/2} \langle \mathcal{L}_1 \Pi_0^\varepsilon \psi, \Pi_0^\varepsilon \psi \rangle + \varepsilon \langle \mathcal{L}_2 \Pi_0^\varepsilon \psi, \Pi_0^\varepsilon \psi \rangle - C\varepsilon^{1+1/8}\|\psi\|^2.$$

Then, we get:

$$\langle \mathcal{L}_0 \Pi_0^\varepsilon \psi, \Pi_0^\varepsilon \psi \rangle = \langle \mathcal{L}_0 \psi_0, \psi_0 \rangle + \varepsilon \langle \mathcal{L}_0 \psi_1, \psi_1 \rangle, \quad (5.8)$$

$$\langle \mathcal{L}_1 \Pi_0^\varepsilon \psi, \Pi_0^\varepsilon \psi \rangle = \langle \mathcal{L}_1 \psi_0, \psi_0 \rangle + 2\varepsilon^{1/2} \langle \mathcal{L}_1 \psi_0, \psi_1 \rangle + \varepsilon \langle \mathcal{L}_1 \psi_1, \psi_1 \rangle, \quad (5.9)$$

$$\langle \mathcal{L}_2 \Pi_0^\varepsilon \psi, \Pi_0^\varepsilon \psi \rangle = \langle \mathcal{L}_2 \psi_0, \psi_0 \rangle + 2\varepsilon^{1/2} \langle \mathcal{L}_2 \psi_0, \psi_1 \rangle + \varepsilon \langle \mathcal{L}_2 \psi_1, \psi_1 \rangle. \quad (5.10)$$

Using the same micolocalization properties as previously, we obtain:

$$\mathfrak{Q}_\varepsilon(\psi) \geq \langle \mathcal{L}_0 \psi_0, \psi_0 \rangle + \varepsilon \langle \mathcal{L}_0 \psi_1, \psi_1 \rangle + 2\varepsilon \langle \mathcal{L}_1 \psi_0, \psi_1 \rangle + \varepsilon \langle \mathcal{L}_2 \psi_0, \psi_0 \rangle - C\varepsilon^{1+1/8}\|\psi\|^2.$$

Using the relations:

$$\langle \mathcal{L}_0 \psi_1, \psi_1 \rangle + \langle \mathcal{L}_1 \psi_0, \psi_1 \rangle = \mu_0 \|\psi_1\|^2,$$

$$\langle \mathcal{L}_0 \psi_0, \psi_0 \rangle = \mu_0 \|\psi_0\|^2,$$

we get:

$$\mathfrak{Q}_\varepsilon(\psi) \geq \mu_0(\|\psi_0\|^2 + \varepsilon\|\psi_1\|^2) + \varepsilon \langle \mathcal{L}_1 \psi_0, \psi_1 \rangle + \varepsilon \langle \mathcal{L}_2 \psi_0, \psi_0 \rangle - C\varepsilon^{1+1/8}\|\psi\|^2.$$

Using the orthogonality of ψ_0 and ψ_1 , it follows:

$$\mathfrak{Q}_\varepsilon(\psi) \geq \mu_0 \|\psi_0 + \varepsilon^{1/2} \psi_1\|^2 + \varepsilon \langle \mathcal{L}_1 \psi_0, \psi_1 \rangle + \varepsilon \langle \mathcal{L}_2 \psi_0, \psi_0 \rangle - C\varepsilon^{1+1/8}\|\psi\|^2.$$

We notice that:

$$\|\psi_0 + \varepsilon^{1/2} \psi_1 + R_\varepsilon\|^2 = \|\psi_0 + \varepsilon^{1/2} \psi_1\|^2 + \|R_\varepsilon\|^2 + 2\varepsilon^{1/2} \langle \psi_1, R_\varepsilon \rangle.$$

We again deduce with the microlocalization estimates and the bound on R_ε :

$$\mathfrak{Q}_\varepsilon(\psi) \geq \mu_0 \|\psi\|^2 + \varepsilon \langle \mathcal{L}_1 \psi_0, \psi_1 \rangle + \varepsilon \langle \mathcal{L}_2 \psi_0, \psi_0 \rangle - C\varepsilon^{1+1/8}\|\psi\|^2.$$

We infer that:

$$\left(\varepsilon^{-1}(\lambda_N(\varepsilon) - \mu_0) + C\varepsilon^{1/8}\right) \|\psi\|^2 \geq \langle \mathcal{L}_1\psi_0, \psi_1 \rangle + \langle \mathcal{L}_2\psi_0, \psi_0 \rangle = Q^{\text{Harm}}(\langle \psi, u_0 \rangle_\tau),$$

where Q^{Harm} is the quadratic form associated with $\mathcal{H}^{\text{Harm}}$. Using Proposition 5.1, we get:

$$\|\psi\|^2 \leq (1 + C\varepsilon^{1/2})\|\psi_0\|^2 = (1 + C\varepsilon^{1/2})\|\langle \psi, u_0 \rangle_\tau\|^2$$

so that:

$$(1 + C\varepsilon^{1/2}) \left(\varepsilon^{-1}(\lambda_N(\varepsilon) - \mu_0) + C\varepsilon^{1/8}\right) \|\langle \psi, u_0 \rangle_\tau\|^2 \geq Q^{\text{Harm}}(\langle \psi, u_0 \rangle_\tau).$$

By applying the min-max principle to the quadratic form Q^{Harm} and the space $\langle \mathfrak{E}_N(\varepsilon), u_0 \rangle_\tau$ which is of dimension N (thanks to Proposition 5.1), we get:

$$(1 + C\varepsilon^{1/2}) \left(\varepsilon^{-1}(\lambda_N(\varepsilon) - \mu_0) + C\varepsilon^{1/8}\right) \geq \nu_N^{\text{Harm}}.$$

This ends the proof of Proposition 2.18.

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