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# Spherically symmetric models: separating expansion from contraction in models with anisotropic pressures

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**Abstract.** We investigate spherically symmetric spacetimes with an anisotropic fluid and discuss the existence and stability of a dividing shell separating expanding and collapsing regions. We find that the dividing shell is defined by a relation between the pressure gradients, both isotropic and anisotropic, and the strength of the fields induced by the Misner-Sharpe mass inside the separating shell and by the pressure fluxes. This balance is a generalization of the Tolman-Oppenheimer-Volkoff equilibrium condition which defines a local equilibrium condition, but conveys also a non-local character given the definition of the Misner-Sharpe mass. We present a particular solution with dust and radiation that provides an illustration of our results.

**Keywords:** Spherical Collapse, Structure Formation, Expanding Universe, Lemaître-Tolman-Bondi

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The universe close to us exhibits structures below certain scales that seem to be immune to the overall expansion of the universe. This reflects two different gravitational behaviors, and usually the dynamics corresponding to the structures that have undergone non-linear collapse is treated under the approximation that Newton's gravitational theory is valid without residual acceleration. However, this approach does not tell us with exactitude what is the critical scale where the latter approximation starts to be valid, nor does it explain in a non-perturbative way how the collapse of over-dense patches decouples from the large-scale expansion. For this purpose one requires a fully general relativistic approach where an exact solution exhibiting the two competing behaviors and allowing us to characterize how the separation between them comes about. It is the understanding of this interplay between collapsing and expanding regions within the theory of general relativity (GR) that we aim to address here.

In previous works we have investigated the present issue in models with spherical symmetry and with a perfect fluid [1, 2]. Here we briefly report our findings when we overcome the limitations of a perfect fluid description of the non-equilibrium setting under focus. We thus consider here an anisotropic fluid.

We resort to a  $3+1$  splitting, and assess the existence and stability of a dividing shell separating expanding and collapsing regions, in a gauge invariant way.

In the generalized Painlevé-Gullstrand coordinates [4] spherically symmetric metrics can be cast as

$$ds^2 = -\alpha(t,r)^2 dt^2 + \frac{1}{1+E(t,r)} (\beta(t,r)dt + dr)^2 + R^2(r,t) d\Omega^2. \quad (1)$$

In the latter expression  $\alpha(t,r)$  is the lapse function and  $\beta(t,r)$  the shift function. Notice that the areal radius  $R$  differs, in principle, from the  $r$  coordinate to account for additional degrees of freedom that are required to cope with both a general fluid that includes anisotropic stresses and heat fluxes.

We consider an energy-momentum tensor

$$T^{ab} = \rho n^a n^b + P h^{ab} + \Pi^{ab} + 2j^{(a} n^{b)}, \quad (2)$$

where  $n^a = \alpha^{-1}(1, \beta, 0, 0)$  is the flow vector,  $h^{ab} = g^{ab} + n^a n^b$  is the metric of the hypersurfaces orthogonal to it,  $\rho$  is the energy density,  $P$  is the pressure,  $\Pi^{ab}$  is the anisotropic stress tensor and  $q^a$  is the heat flux vector.  $\Pi^{ab} n_b = 0$  and  $\Pi^a_a = 0$ , i.e., the anisotropic stress  $\Pi^{ab}$  is orthogonal to  $n^a$  and traceless, and  $j^a = j(t,r)(\beta, 1, 0, 0)$  represents the heat flux which is also orthogonal to the matter flow.

Introducing the Misner-Sharp mass  $M$  and following [4, 5]

$$\frac{\partial}{\partial r} M = 4\pi \left( \rho \frac{\partial}{\partial r} R + j \mathcal{L}_n R \right) R^2 \quad (3)$$

it is possible to derive from the Einstein field equations expressions for  $(\mathcal{L}_n R)^2$  and for  $\mathcal{L}_n^2 R$ . The simultaneous vanishing of the latter quantities defines, on the one hand, the turning point condition  $((\mathcal{L}_n R)^2 = 0)$  and, on the other hand, the generalized local conditions for the existence of a separating shell:  $(\mathcal{L}_n^2 R = 0)$ . This yields

$$(\mathcal{L}_n R)^2 = \frac{2M}{R} + (1+E) \left( \frac{\partial R}{\partial r} \right)^2 - 1 = 0 \quad (4)$$

and

$$-\mathcal{L}_n^2 R = \frac{M}{R^2} + 4\pi(P - 2\Pi)R - \frac{1+E}{\alpha} \frac{\partial \alpha}{\partial r} \frac{\partial R}{\partial r} = 0. \quad (5)$$

This allows us to extend the generalization of the TOV function made in [1, 2], which we called gTOV, to the case where anisotropic stresses are present, since  $\text{gTOV} = -\mathcal{L}_n^2 R$ . In what follows case we shall ignore the heat fluxes (it is then possible to restrict to  $R = r$ , but we will keep it  $R(r,t)$  for the sake of generality). We have

$$-\frac{1}{\alpha} \frac{\partial \alpha}{\partial r} = \frac{1}{(\rho + P - 2\Pi)} \left[ \frac{\partial}{\partial r} (P - 2\Pi) - \frac{6\Pi}{R} \left( \frac{\partial R}{\partial r} \right) \right] \quad (6)$$

and Eqn. (5) becomes

$$\frac{M}{R^2} + 4\pi(P - 2\Pi)R + \frac{1+E}{(\rho + P - 2\Pi)} \left[ \frac{\partial}{\partial r} (P - 2\Pi) - \frac{6\Pi}{R} \left( \frac{\partial R}{\partial r} \right) \right] \frac{\partial R}{\partial r} = 0. \quad (7)$$

which is the gTOV=0 equation of state for the stationarity of the separating shell, and it is immediately apparent that, when going from the isotropic perfect fluid to the case of an anisotropic content, we have to replace  $P$  by  $P - 2\Pi$  in the equations. This means of course that the anisotropic stresses play a fundamental role in defining the pressure gradients that promote the local conditions for the separability of the sign of  $\mathcal{L}_n R$ .

It is possible to relate  $\mathcal{L}_n R$  with the expansion and shear scalars, respectively,  $\Theta = n^a_{;a}$ , and  $a$ , where  $a$  can be defined from the shear tensor  $\sigma_{ij}$  as  $\sigma_{ij} = a(t, r) P_{ij}$  where we use the fact that, from the spherical symmetry, all the quantities  $X_{ij} = h_i^a h_j^a X_{ab}$  share the same spatial eigendirections characterized by the traceless 3-tensor  $P^i_j = \text{diag}[-2, 1, 1]$ . We have

$$\left(\frac{\theta}{3} + a\right) = \frac{\mathcal{L}_n R}{R} \quad (8)$$

and we see that the turning point condition (4) does imply neither the vanishing of the expansion nor of the shear, but it rather means that these quantities should satisfy  $\theta_* = 3a_*$  at the separating shell  $r = r_*$ . If either  $\theta$  or  $a$  were to vanish at this locus we would then have the other quantity vanishing as well. This limit case corresponds to a static separating shell.

Given Eqn. (8) it is interesting to relate the condition (4) to the Hamiltonian constraint that generalizes the Friedman equation

$$\left(\frac{\theta}{3} + a\right)^2 = \frac{8\pi\rho}{3} - \frac{{}^3R}{6} + 2a \left(\frac{\theta}{3} + a\right). \quad (9)$$

We conclude that

$$\frac{2M}{R} + (1 + E) \left(\frac{\partial R}{\partial r}\right)^2 - 1 = \frac{8\pi\rho}{3} - \frac{{}^3R}{6} + 2a \left(\frac{\theta}{3} + a\right). \quad (10)$$

which allows us to emphasize that the stationarity condition for the existence of a separating shell (4) involves non-local quantities namely  $M$ , and  $E$ , while the right-hand side of the expression just derived, (10), only involves local quantities. The non-locality of the conditions (4) and (7) is consistent with the findings of Herrera and co-workers [7] who have studied the ‘‘cracking’’ of compact objects in astrophysics using small anisotropic perturbations around spherically symmetric homogeneous fluids in equilibrium.

We now turn our attention to an exact solution derived by Sussman and Pavón for a spherically symmetric model with a combination of dust and radiation exhibiting anisotropic stress, but no heat fluxes [8]. The metric is written in the Lemaître-Tolman-Bondi (LTB) form

$$ds^2 = -dT^2 + \frac{(\partial_r R)^2}{1 + E(T, r)} dr^2 + R^2 d\Omega^2, \quad (11)$$

and it is assumed that the flow is geodesic to keep ourselves as close as possible to the case where dust is the only component which is present, i.e., the original LTB case. The latter hypothesis implies in GPG coordinates that  $-\frac{1}{\alpha} \frac{\partial \alpha}{\partial r} = 0$ , and hence  $(P - 2\Pi)' + 6\Pi \frac{R'}{R} = 0$

where the prime stands for differentiation with respect to  $r$ . On the other hand the absence of heat fluxes makes  $E = E(r)$ , independent of  $T$ . The condition (4) amounts to

$$\frac{2M(r)}{R} + \frac{2W(r)}{R^2} + E = 0, \quad (12)$$

and equating the gTOV condition reduces to

$$\frac{M(r)}{R^2} + \frac{W(r)}{R^3} + 4\pi(P - 2\Pi)R = 0. \quad (13)$$

The latter conditions defining the separating shell reduce to a differential equation that  $E(r)$  must satisfy, and in [5] we show that it is possible to specify such an  $E$ .

Thus, in the present work we find that the existence of shells separating expanding and contracting domains of the areal radius is defined by two relations: an energy balance yielding a stationarity condition, and a relation between the pressure gradients, both isotropic and anisotropic, and the strength of the fields induced by the Misner-Sharpe mass inside the separating shell and by the pressure fluxes. This balance is a generalization of the Tolman-Oppenheimer-Volkoff equilibrium condition which defines a local equilibrium condition, but simultaneously has a non-local character given the definition of the Misner-Sharpe mass  $M$  (and incidentally that of the function  $E$  as well). We wish also to emphasize that the consideration of anisotropic stresses is most important to guarantee the fulfillment of this gTOV and has an impact of the propagation of the shear. A more detailed and complete discussion of this subject can be found in [5].

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