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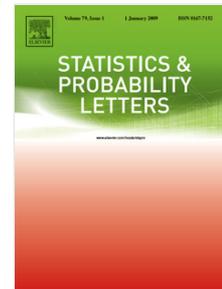
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A note on Marcinkiewicz laws for strictly stationary φ -mixing sequences

Zbigniew S. Szewczak *

30 May 2011

Abstract

A connection between boundedness in probability and laws of large numbers is established for φ -mixing strictly stationary sequences.

Key words: Marcinkiewicz-Zygmund law, boundedness in probability, φ -mixing, strictly stationary sequences, Ibragimov's conjecture

Mathematics Subject Classification (2000): 60F15, 60F05

1 Introduction and results

Let $\{X_k\}_{k \in \mathbf{Z}}$, $\mathbf{Z} = \{\dots, -1, 0, 1, 2, \dots\}$, be a strictly stationary sequence defined on a probability space (Ω, \mathcal{F}, P) taking values on the real line \mathbf{R} , and \mathcal{F}_j^k be the σ -field generated by X_j, X_{j+1}, \dots, X_k , $j, k \in \mathbf{Z}$, $j \leq k$. Set $S_n = \sum_{k=1}^n X_k$, $n \in \mathbf{N} = \{1, 2, \dots\}$ and define

$$\varphi_n = \sup\{|P(B|A) - P(B)|; P(A) > 0, A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_n^\infty\}.$$

If $\lim_{n \rightarrow \infty} \varphi_n = 0$, then $\{X_k\}$ is said to be uniformly strong mixing or φ -mixing.

The main purpose of this note is to shed some light on the connection between boundedness in probability and laws of large numbers for dependent random variables. Recall that a sequence $\{Y_n\}$ is bounded in probability (b.i.p.) or stochastically bounded, if for each $\epsilon > 0$ one can find $M > 0$ with $\sup_n P[|Y_n| > M] < \epsilon$. Our first result states the Marcinkiewicz-Zygmund law in terms of b.i.p. (cf. Marcinkiewicz and Zygmund, 1937, Théorème 9).

Theorem 1 *Suppose $r \in (1, 2)$ is a real number and $\{X_k\}$ is a strictly stationary φ -mixing sequence. Then $n^{-\frac{1}{r}} S_n \rightarrow 0$ almost surely iff $\{n^{-\frac{1}{r}} S_n\}$ is b.i.p. and $E[|X_1|^r] < \infty$.*

If moments do not exist, our next theorem gives a similar characterization (see also Szewczak, 2010, Theorem 1, Theorem 2 and Remark 14). Define

$$c_n = c_n(r) = \sup\{x > 0; x^{-r} E[|X_1|^r I_{|X_1| \leq x}] \geq \frac{1}{n}\}.$$

Since $x^{-r} E[|X_1|^r I_{|X_1| \leq x}] \rightarrow 0$ as $x \rightarrow \infty$, thus c_n is finite and by Lemma 7 in Szewczak (2010) $n E[|X_1|^r I_{|X_1| \leq c_n}] \sim c_n^r$.

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Theorem 2 Suppose $\{X_k\}$ is a strictly stationary φ -mixing sequence and $E[|X_1|^r] = \infty$ for some real $r \in (1, 2)$. If $\{c_n^{-1}S_n\}$ is b.i.p. and $E[|X_1|^r I_{\{|X_1| \leq x\}}]$ is slowly varying in the sense of Karamata, then $c_n^{-1}S_n \rightarrow 0$ in probability. Conversely, if $c_n^{-1}S_n \rightarrow 0$ in probability and $\varphi_1 < 1$, then $E[|X_1|^r I_{\{|X_1| \leq x\}}]$ is slowly varying and $E[X_1] = 0$.

For $\{X_k\}$ independent and identically distributed (i.i.d.) with $E[|X_1|^r I_{\{|X_1| \leq x\}}]$ slowly varying, $r \in (1, 2]$, $\{c_n^{-1}S_n\}$ is b.i.p. iff $E[X_1] = 0$ (see Remark 1 and Lemma 2). In the case of sums of dependent random variables a tool for verifying b.i.p. is von Bahr–Esseen’s inequality (cf. Bahr and Esseen, 1965, Theorem 2). Namely, if $\{X_k\}$ is (not necessarily stationary) random sequence such that $E[|X_k|^r] < \infty$, $k = 1, \dots, n$, and $E[X_{m+1} | S_m] = 0$ a.s., $m \in \mathbf{N}$, $1 \leq m \leq n-1$, $1 \leq r \leq 2$, then $E[|\sum_{k=1}^n X_k|^r] \leq 2 \sum_{k=1}^n E[|X_k|^r]$. Note that the case $r = 2$, related to the Central Limit Theorem (CLT), is also included here. Another approach is the use of the truncation method together with the convergence rate of φ_n coefficients (cf. Szewczak, 2011, Corollary 1). Yet another inequality of interest is stated in Lemma 8.22 on p. 268 in Bradley, 2007, vol. I (see Remark 1). Unfortunately in our setting it is unknown if $\{c_n^{-1}S_n\}$ is b.i.p. or $n^{-1} \sup_n E[|S_n - E[S_n]|^r] < \infty$, $r \in (1, 2]$, (cf. Bradley, 2007, vol. III, P2 Question, p. 457). This situation corresponds to the still unsolved Ibragimov’s conjecture (cf. Bradley, 2007, vol. III, P1 Ibragimov’s Conjecture, p. 457) and Iosifescu–Peligrad’s conjecture (cf. Bradley, 2007, vol. III, P3 Iosifescu’s Conjecture, p. 457; Peligrad, 1990, Conjecture 1.3) related to the CLT and its invariance principle, respectively. Therefore it is natural to state the following conjecture, this time related to Marcinkiewicz laws of large numbers (see Lemma 2).

Conjecture 1 Suppose $\{X_k\}$ is a stationary random sequence which is φ -mixing and $E[|X_1|^r I_{\{|X_1| \leq x\}}]$, $r \in (1, 2]$, varies slowly. If $E[X_1] = 0$ then, $\{c_n^{-1}S_n\}$ is b.i.p.

Remark 1 It is worth noting that under condition $\lim_n q_n^* < 1$ Conjecture 1 holds. Recall the definition of coefficient q_n^* (cf. Bradley, 2007, vol. I, p. 249)

$$q_n^* = \sup \frac{E[(\sum_{j \in Q} X_j)(\sum_{k \in S} X_k)]}{\|(\sum_{j \in Q} X_j)\|_2 \cdot \|(\sum_{k \in S} X_k)\|_2},$$

where this sup is taken over all pairs of nonempty, disjoint, finite sets $Q, S \subset \mathbf{Z}$ such that $\inf_{j \in Q, k \in S} |j - k| \geq n$ and $\|\cdot\|_2$ is L^2 norm. Since $E[|X_1|^r I_{\{|X_1| \leq x\}}]$ varies slowly, $\frac{x^{r-1} E[|X_1|^r I_{\{|X_1| > x\}}]}{E[|X_1|^r I_{\{|X_1| \leq x\}}]} \rightarrow 0$ as $x \rightarrow \infty$ (cf. Feller, 1971, Theorem 2, p. 283) therefore

$$\lim_n n c_n^{-1} E[|X_1|^r I_{\{|X_1| > c_n\}}] = \lim_n \frac{n c_n^{-1} E[|X_1|^r I_{\{|X_1| > c_n\}}]}{\frac{n}{c_n^r} E[|X_1|^r I_{\{|X_1| \leq c_n\}}]} = 0.$$

On the other hand by Lemma 8.22 on p. 268 in Bradley (2007), vol. I

$$c_n^{-2} E\left[\left(\sum_{k=1}^n X_k I_{\{|X_k| \leq c_n\}} - E[X_k I_{\{|X_k| \leq c_n\}}]\right)^2\right]$$

$$\leq 4m \frac{1+q_m^*}{1-q_m^*} \frac{n}{c_n^2} E[X_1^2 I_{\{|X_1| \leq c_n\}}] \leq 4m \frac{1+q_m^*}{1-q_m^*} \frac{n}{c_n^r} E[|X_1|^r I_{\{|X_1| \leq c_n\}}],$$

for m such that $q_m^* < 1$ and $n > m$. Since $\frac{n}{c_n} E[|X_1|^r I_{\{|X_1| \leq c_n\}}]$ is bounded therefore $\{c_n^{-1} S_n\}$ is b.i.p. by the Chebyshev inequality and the truncation argument. Thus if Bradley's conjecture (cf. Bradley, 2007, vol. III, P10 Conjecture, p. 461) turns out to be true, then Conjecture 1 would follow.

The note is organized in such a way that in Section 2 there are some auxiliary inequalities required for the proofs in Section 3.

2 Auxiliary inequalities

Let $\{X_k\}$ be a (not necessarily strictly stationary) random sequence and $n > m$, $i \in \mathbf{N}$, $p, s, t, u > 0$. Recall the following extension of the Hoffmann-Jørgensen inequality from Szewczak (2010).

$$\begin{aligned} P[\max_{1 \leq k \leq n} |S_k| > s + 2t + u; m \cdot \max_{1 \leq i \leq n} |X_i| \leq u] \\ \leq (\varphi_m + P[\max_{m \leq k \leq n} |S_k| > s]) \cdot P[\max_{1 \leq k \leq n-m} |S_k| > t], \end{aligned} \quad (2.1)$$

where $\varphi_m = \sup_{k \in \mathbf{Z}} \{|P(B|A) - P(B)|; A \in \mathcal{F}_{-\infty}^k, B \in \mathcal{F}_{k+m}^\infty\}$. Similarly, in Szewczak (2010) we gave a dependent version of Lévy's inequality (cf. Marcinkiewicz, 1939, Théorème 3), if the laws of reversed sums $\mathcal{L}(S_n - S_k)$ are symmetric for $n > k \geq 1$

$$\left(\frac{1}{2} - \varphi_m\right) \cdot P[\max_{1 \leq k \leq n-m+1} |S_n| > t] \leq P[|S_n| + (m-1) \cdot \max_{1 \leq i \leq n} |X_i| > t]. \quad (2.2)$$

The following moment inequality can be derived from (2.1) (cf. Szewczak, 2010)

$$E[\max_{1 \leq k \leq n} |S_k|^p] \leq \frac{4^p}{1-\tau} \cdot (m^p E[\max_{1 \leq i \leq n} |X_i|^p] + t_\tau^p), \quad (2.3)$$

where

$$t_\tau = \inf\{t > 0; \varphi_m + P[\max_{m \leq k \leq n} |S_k| > t] \leq \frac{\tau}{4^p}\}, \quad \tau \in (0, 1).$$

Since inequalities (2.1), (2.2) and (2.3) include the term $M_n = \max_{1 \leq i \leq n} |X_i|$ the following inequality for strictly stationary sequence $\{X_k\}$ is useful because it allows replacing dependent maxima satisfying Doeblin's condition (i.e. $\varphi_m < 1$) by the maxima of its associated sequence $\{X_k^*\}$ (i.e. the i.i.d. sequence with $\mathcal{L}(X_1) = \mathcal{L}(X_1^*)$):

$$(1 - \varphi_m) \cdot P[M_{\lfloor \frac{n}{m} \rfloor}^* > t] \leq P[M_n > t] \leq m \cdot (1 + \varphi_m) \cdot P[M_{\lfloor \frac{n}{m} \rfloor + 1}^* > t], \quad t \geq 0, n \geq 1, \quad (2.4)$$

where $M_n^* = \max_{1 \leq i \leq n} |X_i^*|$ (cf. Peligrad, 1990, p. 298). Recall also von Bahr-Esseen's symmetrization inequality (cf. Bahr and Esseen, 1965, Lemma 4)

$$\frac{1}{2} E[|\widehat{X}|^r] \leq E[|X - E[X]|^r] \leq E[|\widehat{X}|^r], \quad r \in [1, 2], \quad (2.5)$$

where $\widehat{X} = X - \tilde{X}$ and \tilde{X} is an independent copy of X .

3 Proofs

Theorems 1 and 2 are implied respectively by Theorem 1 in Szewczak (2011), as well as Theorem 1 and 2 in Szewczak (2010) and the following two lemmas (cf. Szewczak, 2010, Remark 14).

Lemma 1 *Suppose $\{X_k\}$ is a strictly stationary φ -mixing sequence, $r \in (1, 2]$ and $E[|X_1|^r I_{|X_1| \leq x}]$ varies slowly. If $\{c_n^{-1} S_n\}$ is b.i.p. then,*

$$\sup_n c_n^{-r} E\left[\left|\sum_{k=1}^n (X_k I_{|X_k| \leq c_n} - E[X_1 I_{|X_1| \leq c_n}])\right|^r\right] < \infty. \quad (3.6)$$

PROOF OF LEMMA 1

Set $S'_n = \sum_{k=1}^n X_k I_{|X_k| \leq c_n}$. Since $E[|X_1|^r I_{|X_1| \leq x}]$ varies slowly, thus we have $nP[|X_1| > \epsilon c_n] \rightarrow 0$, $\epsilon > 0$ (cf. Feller, 1971, Theorem 2, p. 283). Hence

$$c_n^{-1} \max_{1 \leq i \leq n} |X_i| \rightarrow 0, \quad \text{in probability.} \quad (3.7)$$

Therefore $\{c_n^{-1} S'_n\}$ is b.i.p. Set $\widehat{S}'_n = \sum_{k=1}^n X_k I_{|X_k| \leq c_n} - \widetilde{X}_k I_{|\widetilde{X}_k| \leq c_n}$, where $\{\widetilde{X}_k\}$ is an independent copy of $\{X_k\}$. Hence $\{c_n^{-1} \widehat{S}'_n\}$ is b.i.p. too. Choose m such that $\varphi_m < 1 - \frac{1}{\sqrt{2}}$ so that $\widehat{\varphi}_1 = \varphi_1(\{\widehat{X}_k\}) < \frac{1}{2}$ (cf. Bradley, 2007, vol. I, Theorem 6.6, p.199). Now (2.2) and (3.7) yield that $\{\max_{1 \leq k \leq n} c_n^{-1} |\widehat{S}'_k|\}$ is b.i.p. Moreover, by (2.4)

$$\overline{\lim}_n c_n^{-r} E\left[\max_{1 \leq i \leq n} |X_i|^r I_{|X_i| \leq c_n}\right] \leq 2m \overline{\lim}_n \left[\frac{n}{m}\right] c_n^{-r} E[|X_1|^r I_{|X_1| \leq c_n}] \leq 2$$

thus

$$\sup_n c_n^{-r} E\left[\max_{1 \leq i \leq n} |X_i I_{|X_i| \leq c_n} - \widetilde{X}_i I_{|\widetilde{X}_i| \leq c_n}|^r\right] < \infty.$$

Therefore by (2.3) we get $E[\max_{1 \leq k \leq n} c_n^{-r} |\widehat{S}'_k|^r] < C$ for some $C < \infty$ and every $n \in \mathbf{N}$. Consequently, condition (3.6) follows from (2.5). This completes the proof. \square

It is interesting that the converse of Conjecture 1 holds.

Lemma 2 *Suppose $\{X_k\}$ is a strictly stationary φ -mixing sequence, $r \in (1, 2]$ and $E[|X_1|^r I_{|X_1| \leq x}]$ varies slowly. If $\{c_n^{-1} S_n\}$ is b.i.p., then $E[X_1] = 0$.*

PROOF OF LEMMA 2

Since $r > 1$, and $E[|X_1|^r I_{|X_1| \leq x}]$ varies slowly, thus $E|X_1| < \infty$. By the proof of Lemma 1 we see that $\{c_n^{-1} S'_n\}$ is b.i.p. Moreover, by the Markov inequality and (3.6), the sequence $\{c_n^{-1} (S'_n - nE[X_1 I_{|X_1| \leq c_n}])\}$ is also b.i.p. so that $\{c_n^{-1} nE[X_1 I_{|X_1| \leq c_n}]\}$ is b.i.p. too. Furthermore, by a variant of Karata's theorem (cf. Feller, 1971, Theorem 2, p. 283)

$$c_n^{r-1} \frac{E[|X_1|^r I_{|X_1| > c_n}]}{E[|X_1|^r I_{|X_1| \leq c_n}]} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.8)$$

Since $nE[|X_1|^r I_{|X_1| \leq c_n}] \sim c_n^r$, thus (3.8) yields

$$c_n^{-1} n E[|X_1| I_{|X_1| > c_n}] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore $\{c_n^{-1} n |E[X_1]|\}$ is a bounded sequence. On the other hand the slow variation of $E[|X_1|^r I_{|X_1| \leq x}]$ entails

$$c_n^r n^{-r} \sim n^{1-r} E[|X_1|^r I_{|X_1| \leq c_n}] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus the only possibility is $E[X_1] = 0$ and Lemma 2 follows. \square

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