

Problem Title Description. Problem Name: The Black Riders Author's Name: Tom and Jerry Problem Code: FINDHOLE Alphabet: A

Problem: There are  $N$  hobbits and  $M$  holes. You are given the time it takes for each hobbit to reach each hole. Also, the hole has space for just one hobbit; however, once a hobbit reaches a hole, he can dig at it so that it becomes spacious enough for two. This digging takes  $C$  units of time, and the hole cannot then be further expanded. What is the minimum time required to ensure that atleast  $K$  hobbits are in holes.

Solution: First, lets see if we get any intuition by removing the quantity  $C$  from the picture. Imagine that each hole could hold just one hobbit, and there was no scope for digging. We now have  $N$  hobbits,  $M$  holes, and at our supposed min-time, we have some  $K$  of these hobbits in  $K$  holes. Each hobbit goes to only one hole, if any, and each hole accommodates atmost one hobbit. In some sense, the  $K$  hobbits are *matched* to their respective  $K$  holes.

So if we were to have a bipartite graph of hobbits and holes, then our final assignment would look like a matching of size atleast  $K$ . But this matching is of which graph? Draw an edge from hobbit  $i$  to hole  $j$  if hobbit  $i$  can reach hole  $j$ , right? And we can tell this if it is within a certain threshold time  $T$ . In essence, draw an edge from hobbit  $i$  to hole  $j$  iff  $A[i][j] \leq T$ . ( $A$  is the given matrix of time required for hobbits to reach holes). Lets call this graph  $G(T)$

Which brings us to binary-search. For large enough  $T$ , we definitely have that  $K$  hobbits will fit in  $K$  holes, (since our bipartite graph becomes a complete bipartite graph). For too small  $T$ , it is not possible, since our graph would have no edges. We then ask ourself, given  $T$ , is it possible to have  $K$  hobbits reach holes? Or, given  $T$ , does a max-matching have size atleast  $K$  in the graph  $G(T)$ ? Binary-searching on this predicate would give us our answer *for the case where you cannot dig in a hole*.

How can we modify this idea to allow for digging? If 2 hobbits were to be in a particular hole  $j$  by time  $T$ , one of them would have had to have reached it by time  $T - C$ . In fact, given time  $T$ , if we were to have a separate threshold for  $T - C$  edges, we would be able to tell which holes can be filled and have space for digging.

Hence for this, we make *copies* of all the holes and then  $G(T)$  would be defined by edge from hobbit  $i$  to hole  $j$  iff  $A[i][j] \leq T$ , and edge from hobbit  $i$  to hole-copy  $j$  iff  $A[i][j] \leq T - C$ . A matching in this graph corresponds to a way of assigning the hobbits to holes such that when a hobbit-hole-copy edge in matching, that corresponds to the hobbit reaching and digging, and when a hobbit-hole edge in matching, that corresponds to the hobbit reaching the hole itself. Finally, we check for feasibility by testing if the max-matching

on  $G(T)$  has size atleast  $K$ .

You can binary-search on the whole  $[0, 2 * 10^9]$  time range, or on the specific  $2NM$  time-points ( $A[i][j], A[i][j] + C$  time points).

Time Complexity:  $O(KMN \log(MN))$ .