

Problem Title Description. Problem Name: Gandalf vs the Balrog Author's Name: Attack and Counterattack Problem Code: ATKCTR Alphabet:

Problem: Given a tournament graph (a directed graph such that between any two nodes u, v , you either have a $u \rightarrow v$ edge, or a $v \rightarrow u$ edge), you and your opponent play a game. The graph describes which node can "defeat" which node in the game. First, you pick a node. Then your opponent tries to pick a node that defeats you. If he can't you get a score of 2. If he can, then you *finally* pick a node to defeat his. If this is impossible now, your score is 0, else your score is 1. Determine the maximum score you can get, and if you can get a score of 2, also determine which node you would have chosen in the first choice.

Solution: Given this, note that when both play optimally, *you will never get a score of 0*. Also, you can get a score of 2 *iff* some node has indegree 0. Thus, given this, you only have to test whether there exists such a node (in which case you output 2 u where u is that unique node), or just output 1.

The above basic solution made two assertions, which we shall prove below.

(1) You can get a score of 2 *iff* there exists a node having outdegree $= N - 1$. If such a node exists, then by picking it, no matter what node your opponent picks, your node will defeat it (since it defeats all nodes). If such a node doesn't exist, then no matter what node you pick, some node will be able to defeat it, and hence you cannot get a score of 2.

(2) You will never get a score of 0. There are 2 approaches to convince oneself of this. The first one uses a theorem regarding tournament graphs: In a tournament graph, there exists some node u which is capable of reaching all other nodes by distance atmost 2. (Note that here, if this u can reach all other nodes by distance 1, then you can get a score of 2.) This theorem implies the assertion since if you were to choose u in your first move, and your opponent chose v such $v \rightarrow u$ is an edge, since u can reach v through a distance of atmost 2, there must exist a w such that $w \rightarrow v$ edge exists, and you can then choose this w as your final move. Further, the proof that there exists such a u in the first place is slightly more involved than the second proof which will be described later, and if you were doing the contest, you would either have applied this theorem, or arrived at the assertion through the easier means. The proof that such a u exists, is as follows: Consider u to be any vertex having maximum outdegree (greedily this would be your choice). Now, either $u \rightarrow v$ or $v \rightarrow u$. Consider any $v \rightarrow u$. We need to prove that \exists some w such that $u \rightarrow w, w \rightarrow v$ both exist. If this does not hold, then we have that $v \rightarrow w \forall w$ defeated by u . Plus, $v \rightarrow u$, and hence $\text{outdeg}(v) > \text{outdeg}(u)$, which contradicts the fact that u is a vertex having maximum outdegree.

The easier way to prove the assertion is just by taking two cases: (a)

\exists node u having outdegree $= N - 1$, in which case your score is 2, or (b) outdegree(u) $< N - 1 \forall u$. In which case, whichever node v that your opponent chooses, there exists some node w that defeats it: $w \rightarrow v$. Also, $w \neq u$ since $v \rightarrow u$, but $w \rightarrow v$. Which means, w has still been unpicked, and you can pick it and defeat v .

How one's thought process should lead to solution: The main 2 assertions above need to be reasoned out, and from there it is a trivial code. The first assertion is very easy to see and to prove. The second assertion can be arrived at through playing around with examples. A few examples (even small ones) are enough to suggest that the Balrog can never force Gandalf to get a score of 0; and hence the problem boils down to just deciding between a score of 2 or a score of 1.

A Note on Implementation: Since the input was given to have the integer N and then only the M anomalous pairs, one might be tempted to just check if node N had zero indegree or not. Indeed, if there were no anomalous pairs involving node N , then clearly outdegree(N) $= N - 1$. However, this does not mean that only N had to be the one with indegree of 0, and you needed to check over other nodes as well.

The judging panel saw two approaches to this problem's implementation. One just counted (out)degrees of each node. Initialize indegree(u) to $u - 1$, then decrement/increment indegrees of anomalous u, v pairs. Finally check if some node has indegree 0. Time Complexity $O(N + M)$. The other checked if a node u was *ever* on the "receiving end" (was defeated) of an anomalous pair, and counted if the number of times it was on the "giving end" (was defeating a superior) was equal to the number of nodes above it ($N - u$). Time Complexity: $O(M \log M)$ using a STL map.