

Problem Title Description. Problem Name: The Black Riders Author's Name: Tom and Jerry Problem Code: FINDHOLE Alphabet: A

Problem: There are N hobbits and M holes. You are given the time it takes for each hobbit to reach each hole. Also, the hole has space for just one hobbit; however, once a hobbit reaches a hole, he can dig at it so that it becomes spacious enough for two. This digging takes C units of time, and the hole cannot then be further expanded. What is the minimum time required to ensure that atleast K hobbits are in holes.

Solution: First, lets see if we get any intuition by removing the quantity C from the picture. Imagine that each hole could hold just one hobbit, and there was no scope for digging. We now have N hobbits, M holes, and at our supposed min-time, we have some K of these hobbits in K holes. Each hobbit goes to only one hole, if any, and each hole accommodates atmost one hobbit. In some sense, the K hobbits are *matched* to their respective K holes.

So if we were to have a bipartite graph of hobbits and holes, then our final assignment would look like a matching of size atleast K . But this matching is of which graph? Draw an edge from hobbit i to hole j if hobbit i can reach hole j , right? And we can tell this if it is within a certain threshold time T . In essence, draw an edge from hobbit i to hole j iff $A[i][j] \leq T$. (A is the given matrix of time required for hobbits to reach holes). Lets call this graph $G(T)$

Which brings us to binary-search. For large enough T , we definitely have that K hobbits will fit in K holes, (since our bipartite graph becomes a complete bipartite graph). For too small T , it is not possible, since our graph would have no edges. We then ask ourself, given T , is it possible to have K hobbits reach holes? Or, given T , does a max-matching have size atleast K in the graph $G(T)$? Binary-searching on this predicate would give us our answer *for the case where you cannot dig in a hole*.

How can we modify this idea to allow for digging? If 2 hobbits were to be in a particular hole j by time T , one of them would have had to have reached it by time $T - C$. In fact, given time T , if we were to have a separate threshold for $T - C$ edges, we would be able to tell which holes can be filled and have space for digging.

Hence for this, we make *copies* of all the holes and then $G(T)$ would be defined by edge from hobbit i to hole j iff $A[i][j] \leq T$, and edge from hobbit i to hole-copy j iff $A[i][j] \leq T - C$. A matching in this graph corresponds to a way of assigning the hobbits to holes such that when a hobbit-hole-copy edge in matching, that corresponds to the hobbit reaching and digging, and when a hobbit-hole edge in matching, that corresponds to the hobbit reaching the hole itself. Finally, we check for feasibility by testing if the max-matching

on $G(T)$ has size atleast K .

You can binary-search on the whole $[0, 2 * 10^9]$ time range, or on the specific $2NM$ time-points ($A[i][j], A[i][j] + C$ time points).

Time Complexity: $O(KMN \log(MN))$.