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# Every totally real algebraic integer is a tree eigenvalue

Justin Salez

*Université Paris Diderot & LPMA*

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## Abstract

Graph eigenvalues are examples of totally real algebraic integers, i.e. roots of real-rooted monic polynomials with integer coefficients. Conversely, the fact that every totally real algebraic integer occurs as an eigenvalue of some finite graph is a deep result, conjectured forty years ago by Hoffman, and proved seventeen years later by Estes. This short paper provides an independent and elementary proof of a stronger statement, namely that the graph may actually be chosen to be a tree. As a by-product, our result implies that the atoms of the limiting spectrum of  $n \times n$  symmetric matrices with independent Bernoulli  $\left(\frac{c}{n}\right)$  entries ( $c > 0$  is fixed as  $n \rightarrow \infty$ ) are exactly the totally real algebraic integers. This settles an open problem raised by Ben Arous (2010).

*Keywords:* adjacency matrix, tree, eigenvalue, totally real algebraic integer  
*2010 MSC:* 05C05, 05C25, 05C31, 05C50

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## 1. Introduction

By definition, the *eigenvalues* of a finite graph  $G = (V, E)$  are the roots of its characteristic polynomial  $\Phi_G(x) := \det(xI - A)$ , where  $A = \{A_{i,j}\}_{i,j \in V}$  is the adjacency matrix of  $G$ :

$$A_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Those eigenvalues capture a considerable amount of information about  $G$ . For a detailed account, see e.g. [1, 2]. It follows directly from this definition that any graph eigenvalue is a *totally real algebraic integer*, i.e. a root of some real-rooted monic polynomial with integer coefficients. Remarkably enough,

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*URL:* <http://www.proba.jussieu.fr/~salez/> (Justin Salez)

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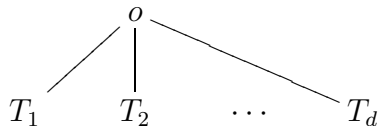
the converse is also true: every totally real algebraic integer is an eigenvalue of some finite graph. This deep result was conjectured forty years ago by Hoffman [3], and established seventeen years later by Estes [4], see also [5]. The present paper provides an independent, elementary proof of a stronger statement, namely that the graph may actually be chosen to be a tree.

**Theorem 1.** *Every totally real algebraic integer is an eigenvalue of some finite tree.*

Trees undoubtedly play a special role in many aspects of graph theory. We therefore believe that the strengthening provided by Theorem 1 may be of independent interest, beyond the fact that it provides a simpler proof of Hoffman's conjecture. In addition, Theorem 1 settles an open problem raised by Ben Arous [6, Problem 14], namely that of determining the set  $\Sigma$  of atoms of the limiting spectral distribution of  $n \times n$  symmetric matrices with independent Bernoulli  $\left(\frac{c}{n}\right)$  entries ( $c > 0$  is fixed and  $n \rightarrow \infty$ ). Indeed, it follows from the log-Hölder continuity of the spectrum of integer matrices at algebraic numbers (see e.g. [7, Section 6]) that  $\Sigma$  is contained in the set of totally real algebraic integers. On the other hand,  $\Sigma$  is easily seen to contain all tree eigenvalues, as noted by Ben Arous. Theorem 1 precisely states that those inner and outer bounds coincide, thereby settling the question.

## 2. Outline of the proof

Let  $T$  be a finite tree with a distinguished vertex  $o$  (the root). Removing  $o$  naturally yields a decomposition of  $T$  into smaller rooted trees  $T_1, \dots, T_d$  ( $d \in \mathbb{N}$ ) as depicted in the following diagram:



To such a rooted tree, let us associate the rational function

$$f_T(x) = 1 - \frac{\Phi_T(x)}{x\Phi_{T \setminus o}(x)} = 1 - \frac{\Phi_T(x)}{x\Phi_{T_1}(x) \cdots \Phi_{T_d}(x)}. \quad (1)$$

Expressed in terms of this function, the classical recursion for the characteristic polynomial of trees (see e.g. [2, Proposition 5.1.1]) simply reads

$$f_T(x) = \frac{1}{x^2} \sum_{i=1}^d \frac{1}{1 - f_{T_i}(x)}, \quad (2)$$

the sum being interpreted as 0 when empty (i.e. when  $T$  is reduced to  $o$ ). Now fix  $\lambda \in \mathbb{C} \setminus \{0\}$ . In view of (1), the problem of finding a (minimal) tree with eigenvalue  $\lambda$  is equivalent to that of finding  $T$  such that  $\mathfrak{f}_T(\lambda) = 1$ . Thanks to the recursion (2), this task boils down to that of generating the number 1 from the initial seed 0 using repeated applications of the maps

$$(\alpha_1, \dots, \alpha_d) \mapsto \frac{1}{\lambda^2} \sum_{i=1}^d \frac{1}{1 - \alpha_i} \quad (d \in \mathbb{N}).$$

As an example, consider the golden ratio  $\lambda = \frac{1+\sqrt{5}}{2}$ . Iterating 4 times the map  $\alpha \mapsto \frac{1}{\lambda^2(1-\alpha)}$  and using the identity  $\lambda^2 = \lambda + 1$  yields successively:

$$0 \longrightarrow \frac{1}{\lambda^2} \longrightarrow \frac{1}{\lambda^2-1} = \frac{1}{\lambda} \longrightarrow \frac{1}{\lambda^2-\lambda} = 1,$$

which shows that  $\lambda$  is an eigenvalue of the linear tree  $T = P_4$ . Remarkably enough, this seemingly specific argument can be extended in a systematic way to any totally real algebraic integer  $\lambda$ , and the numbers that may be produced in this way can be completely determined. To formalize this, let us fix a totally real algebraic integer  $\lambda \neq 0$ , and introduce the rational function

$$\Psi(x) = \frac{1}{\lambda^2(1-x)}. \quad (3)$$

Define  $\mathcal{F} \subseteq \mathbb{R}$  as the smallest set containing 0 and satisfying for all  $d \geq 1$ ,

$$\alpha_1, \dots, \alpha_d \in \mathcal{F} \setminus \{1\} \implies \Psi(\alpha_1) + \dots + \Psi(\alpha_d) \in \mathcal{F}. \quad (4)$$

We will prove that  $\mathcal{F}$  is nothing but the field generated by  $\lambda^2$ :

$$\mathcal{F} = \left\{ \frac{P(\lambda^2)}{Q(\lambda^2)} : P, Q \in \mathbb{Z}[x], Q(\lambda^2) \neq 0 \right\} =: \mathbb{Q}(\lambda^2). \quad (5)$$

In particular,  $1 \in \mathcal{F}$ , and Theorem 1 follows. The remainder of the paper is devoted to the proof of (5). The detailed argument is given in Section 4, while Section 3 provides the necessary background on algebraic numbers.

### 3. Algebraic preliminaries

A number  $\zeta \in \mathbb{C}$  is *algebraic* if there is  $P \in \mathbb{Q}[x]$  such that  $P(\zeta) = 0$ . In that case, such  $P$  are exactly the multiples of a unique monic polynomial  $P_0 \in \mathbb{Q}[x]$ , called the *minimal polynomial* of  $\zeta$ . The algebraic number  $\zeta$  is

- *totally real* if all the complex roots of  $P_0$  are real ;
- *totally positive* if all the complex roots of  $P_0$  are real and positive.

The following Lemma gathers the basic properties of algebraic numbers that will be used in the sequel. These are well-known (see e.g. [8]) and follow directly from the fact that if  $P(x) = \prod_i (x - \alpha_i)$  and  $Q(x) = \prod_j (x - \beta_j)$  have rational coefficients, then so do the polynomials

$$\begin{array}{ccc} \prod_{i,j} (x - \alpha_i - \beta_j), & \prod_i \left( x - \frac{1}{\alpha_i} \right), & \prod_{i,j} (x - \alpha_i \beta_j) \\ \prod_i (x + \alpha_i), & \prod_i (x^2 - \alpha_i^2), & \prod_i (x^2 - \alpha_i). \end{array}$$

**Lemma 1** (Elementary algebraic properties).

- The totally real algebraic numbers form a sub-field of  $\mathbb{R}$ .
- The set of totally positive algebraic numbers is stable under  $+$ ,  $\times$ ,  $\div$ .
- If  $\alpha$  is totally real and  $\beta$  is totally positive, then  $\alpha + n\beta$  is totally positive for all sufficiently large  $n \in \mathbb{N}$ .
- If  $\alpha \neq 0$  is totally real, then  $\alpha^2$  is totally positive.

We shall also use twice the following elementary result.

**Lemma 2.** Let  $\zeta$  be algebraic with minimal polynomial  $P$ . Set  $n = \deg P$ . Then for any  $q_1 > \dots > q_n \in \mathbb{Q}$ , there exist  $m_1, \dots, m_n \in \mathbb{Z}$  such that

$$\frac{m_1}{\zeta - q_1} + \dots + \frac{m_n}{\zeta - q_n} \in \mathbb{N}^+ = \{1, 2, \dots\}, \quad (6)$$

and  $m_k$  has the same sign as  $(-1)^k P(q_k)$  for every  $k \in \{1, \dots, n\}$ .

*Proof.* For  $1 \leq k \leq n$ , consider the rational number

$$r_k := \frac{-P(q_k)}{\prod_{j \neq k} (q_k - q_j)}.$$

Note that  $r_k$  has the same sign as  $(-1)^k P(q_k)$ , since  $q_1 > \dots > q_n$ . Moreover,

$$r_1 \prod_{k \neq 1} (x - q_k) + \dots + r_n \prod_{k \neq n} (x - q_k) = \prod_{k=1}^n (x - q_k) - P(x).$$

Indeed, both sides are polynomials of degree less than  $n$ , and they coincide at the  $n$  points  $q_1, \dots, q_n$ . Evaluating at  $x = \zeta$  gives

$$\frac{r_1}{\zeta - q_1} + \dots + \frac{r_n}{\zeta - q_n} = 1,$$

and multiplying this by a large enough integer yields the result.  $\square$

#### 4. Proof

Before we start, let us make three simple observations which will be used several times in the sequel. First, by (4), we have for any  $k \in \mathbb{N}$ ,

$$\frac{1}{\lambda^2 - k} = \Psi \left( \underbrace{\Psi(0) + \dots + \Psi(0)}_{k \text{ terms}} \right) \in \mathcal{F}. \quad (7)$$

Second,  $\mathcal{F}$  is stable under internal addition:

$$\alpha, \beta \in \mathcal{F} \implies \alpha + \beta \in \mathcal{F}. \quad (8)$$

Indeed, the conclusion is trivial if  $\alpha = 0$  or  $\beta = 0$ . Now if  $\alpha, \beta$  are non-zero elements of  $\mathcal{F}$ , then by construction they are of the form

$$\alpha = \Psi(\alpha_1) + \dots + \Psi(\alpha_n) \quad \text{and} \quad \beta = \Psi(\beta_1) + \dots + \Psi(\beta_m),$$

for some  $n, m \geq 1$  and  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m$  in  $\mathcal{F} \setminus \{1\}$ . But then,

$$\alpha + \beta = \Psi(\alpha_1) + \dots + \Psi(\alpha_n) + \Psi(\beta_1) + \dots + \Psi(\beta_m),$$

which, by (4), shows that  $\alpha + \beta \in \mathcal{F}$ . Third, the field  $\mathbb{Q}(\lambda^2)$  obviously contains 0 and satisfies (4). Since  $\mathcal{F}$  is minimal with this property, we deduce that

$$\mathcal{F} \subseteq \mathbb{Q}(\lambda^2). \quad (9)$$

By part (a) of Lemma 2, we also get that all elements in  $\mathcal{F}$  are totally real.

**Step 1:  $\mathcal{F}$  contains a positive integer**

We may assume that  $1 \notin \mathcal{F}$ , otherwise there is nothing to prove. Consequently, we do not need to worry about divisions by zero when applying  $\Psi$  to an element  $\alpha \in \mathcal{F}$ . Let us first apply Lemma 2 to  $\zeta = \lambda^2$  with  $q_k = k, 1 \leq k \leq \deg \zeta$ . From (7) and (8), it follows that the sum appearing in (6) is the difference of two elements in  $\mathcal{F}$ . In other words, we have found  $\Delta \in \mathbb{N}^+$  and  $\alpha$  with the following property:

$$\alpha \in \mathcal{F} \quad \text{and} \quad \alpha - \Delta \in \mathcal{F}. \quad (10)$$

As already noted,  $\alpha$  is totally real. In fact we may even assume that  $1 - \alpha$  is totally positive, because  $\alpha' = \alpha + \beta$  also satisfies (10) for any  $\beta \in \mathcal{F}$ , and choosing  $\beta = \frac{j}{\lambda^2 - k}$  with  $j, k \in \mathbb{N}$  large enough eventually makes  $1 - \alpha'$  totally positive, by parts (b) and (c) of Lemma 1. In turn, parts (b) and (d) now guarantee that

$$\xi := \lambda^2(1 - \alpha) \quad \text{is totally positive.} \quad (11)$$

Now fix  $j, k \in \mathbb{N}$  and set  $i = (\Delta - 1)j + 1$ . Since  $\mathcal{F}$  is stable under addition, property (10) and the fact that  $\frac{1}{\lambda^2} = \Psi(0) \in \mathcal{F}$  imply that

$$\underbrace{\alpha + \cdots + \alpha}_{i \text{ terms}} + \underbrace{(\alpha - \Delta) + \cdots + (\alpha - \Delta)}_{j \text{ terms}} + \underbrace{\frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^2}}_{k \text{ terms}} \in \mathcal{F}.$$

But this number equals  $1 - (\Delta j + 1)(1 - \alpha) + \frac{k}{\lambda^2}$ , so applying  $\Psi$  gives

$$\frac{1}{(\Delta j + 1)\xi - k} \in \mathcal{F}.$$

Adding up  $\Delta j + 1$  copies of this last number, we finally arrive at

$$\frac{1}{\xi - q} \in \mathcal{F} \quad \text{for any} \quad q \in \mathcal{Q} := \left\{ \frac{k}{\Delta j + 1} : j, k \in \mathbb{N} \right\}. \quad (12)$$

We may now conclude: by (11), the minimal polynomial  $P$  of  $\xi$  has  $n := \deg P$  pairwise distinct positive roots. Since  $\mathcal{Q}$  is dense in  $[0, \infty)$ , one can find  $q_1 > \cdots > q_n$  in  $\mathcal{Q}$  that interleave those roots, in the sense that  $P(q_k)$  has sign  $(-1)^k$  for every  $1 \leq k \leq n$ . Consequently, Lemma 2 provides us with non-negative integers  $m_1, \dots, m_n$  such that

$$\frac{m_1}{\xi - q_1} + \cdots + \frac{m_n}{\xi - q_n} \in \mathbb{N}^+.$$

On the other hand, this sum is in  $\mathcal{F}$  by (12) and (8). Thus,  $\mathcal{F} \cap \mathbb{N}^+ \neq \emptyset$ .

**Step 2:  $(\mathcal{F}, +)$  is a group**

We know that  $\mathcal{F}$  contains some  $d \in \mathbb{N}^+$ . Since  $d + d$  also belongs to  $\mathcal{F}$ , we may assume without loss of generality that  $d \neq 1$  to avoid divisions by zero below. Now fix  $\alpha \in -\mathcal{F}$  with  $\alpha \neq 1$ . Since  $\mathcal{F}$  is stable under addition,

$$d + \underbrace{(-\alpha) + \cdots + (-\alpha)}_{d-1 \text{ terms}} \in \mathcal{F}.$$

Applying  $\Psi$  shows that  $\frac{-1}{\lambda^{2(1-\alpha)(d-1)}} \in \mathcal{F}$ . Adding up  $(d-1)$  copies of this number, we conclude that  $\frac{-1}{\lambda^{2(1-\alpha)}} \in \mathcal{F}$ . We have proved:

$$\alpha \in (-\mathcal{F}) \setminus \{1\} \implies \Psi(\alpha) \in -\mathcal{F}.$$

In other words,  $-\mathcal{F}$  is stable under  $\Psi$ . In view of (8), we deduce that  $-\mathcal{F}$  satisfies (4). By minimality of  $\mathcal{F}$ , we conclude that  $\mathcal{F} \subseteq -\mathcal{F}$ , i.e. that  $\mathcal{F}$  is stable under negation. Thus, the monoid  $(\mathcal{F}, +)$  is a group.

**Step 3:  $\mathcal{F}$  is the field  $\mathbb{Q}(\lambda^2)$ .**

In view of (9), we only need to show that for  $P, Q \in \mathbb{Z}[x]$  with  $Q(\lambda^2) \neq 0$ ,

$$\frac{P(\lambda^2)}{Q(\lambda^2)} \in \mathcal{F}. \tag{13}$$

Since  $\lambda^2$  is an algebraic integer, we may assume that  $Q$  is monic with  $\deg Q > \deg P$  (otherwise, replace  $Q$  with  $Q + P_0^{\deg P}$ , where  $P_0$  denotes the minimal polynomial of  $\lambda^2$ ). Let us prove the claim by induction over  $n = \deg Q$ . The case  $n = 0$  is simply the fact that  $0 \in \mathcal{F}$ . Now, assume that the claim holds for a certain  $n \in \mathbb{N}$ , and consider

$$Q(x) = x^{n+1} + a_n x^n + \cdots + a_0,$$

with  $a_0, \dots, a_n \in \mathbb{Z}$ . Let us first prove (13) in the following two special cases:

- Case 1:  $P(x) = x^n$ . By our induction hypothesis,  $\frac{1}{1+\lambda^{2n}} \in \mathcal{F}$  and hence

$$\frac{1}{\lambda^{2n+2}} + \frac{1}{\lambda^2} = \Psi\left(\frac{1}{1+\lambda^{2n}}\right) \in \mathcal{F}.$$



But  $\mathcal{F}$  also contains  $\frac{1}{\lambda^2}, \dots, \frac{1}{\lambda^{2n}}$  by our induction hypothesis. Since  $(\mathcal{F}, +)$  is a group, one deduces that  $\mathcal{F}$  contains

$$-\left(\frac{a_n}{\lambda^2} + \dots + \frac{a_0}{\lambda^{2n+2}}\right) = 1 - \frac{Q(\lambda^2)}{\lambda^{2n+2}}.$$

Finally, applying  $\Psi$  shows that  $\frac{\lambda^{2n}}{Q(\lambda^2)} = \frac{P(\lambda^2)}{Q(\lambda^2)} \in \mathcal{F}$ , as desired.

- Case 2:  $P$  is monic of degree  $n$  with  $P(0) = 1$ . Then

$$R(x) := P(x) - \frac{Q(x) - Q(0)P(x)}{x}$$

is a polynomial over  $\mathbb{Z}$  with  $\deg R < n$ . Thus, our induction hypothesis guarantees that  $\mathcal{F}$  contains  $\frac{R(\lambda^2)}{P(\lambda^2)}$ , hence  $\frac{R(\lambda^2)}{P(\lambda^2)} - \frac{Q(0)}{\lambda^2}$  and hence also

$$\Psi\left(\frac{R(\lambda^2)}{P(\lambda^2)} - \frac{Q(0)}{\lambda^2}\right) = \frac{P(\lambda^2)}{Q(\lambda^2)}.$$

For the general case, note that every monomial  $x^k$  ( $0 \leq k \leq n$ ) may be written as a signed sum of polynomials  $P$  of the form considered in the two special cases above. Since  $(\mathcal{F}, +)$  is a group, we conclude that (13) holds for all  $P \in \mathbb{Z}[x]$  with  $\deg P \leq n$ , and the induction is complete.

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