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Every totally real algebraic integer is a tree eigenvalue

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Abstract

Graph eigenvalues are examples of totally real algebraic integers, i.e. roots of real-rooted monic polynomials with integer coefficients. Conversely, the fact that every totally real algebraic integer occurs as an eigenvalue of some finite graph is a deep result, conjectured forty years ago by Hoffman, and proved seventeen years later by Estes. This short paper provides an independent and elementary proof of a stronger statement, namely that the graph may actually be chosen to be a tree. As a by-product, our result implies that the atoms of the limiting spectrum of $n \times n$ symmetric matrices with independent Bernoulli $\left(\frac{c}{n}\right)$ entries ($c > 0$ is fixed as $n \rightarrow \infty$) are exactly the totally real algebraic integers. This settles an open problem raised by Ben Arous (2010).

Keywords: adjacency matrix, tree, eigenvalue, totally real algebraic integer
2010 MSC: 05C05, 05C25, 05C31, 05C50

1. Introduction

By definition, the *eigenvalues* of a finite graph $G = (V, E)$ are the roots of its characteristic polynomial $\Phi_G(x) := \det(xI - A)$, where $A = \{A_{i,j}\}_{i,j \in V}$ is the adjacency matrix of G :

$$A_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Those eigenvalues capture a considerable amount of information about G . For a detailed account, see e.g. [1, 2]. It follows directly from this definition that any graph eigenvalue is a *totally real algebraic integer*, i.e. a root of some real-rooted monic polynomial with integer coefficients. Remarkably enough,

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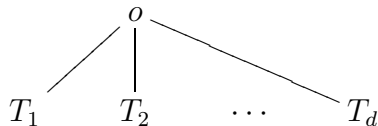
the converse is also true: every totally real algebraic integer is an eigenvalue of some finite graph. This deep result was conjectured forty years ago by Hoffman [3], and established seventeen years later by Estes [4], see also [5]. The present paper provides an independent, elementary proof of a stronger statement, namely that the graph may actually be chosen to be a tree.

Theorem 1. *Every totally real algebraic integer is an eigenvalue of some finite tree.*

Trees undoubtedly play a special role in many aspects of graph theory. We therefore believe that the strengthening provided by Theorem 1 may be of independent interest, beyond the fact that it provides a simpler proof of Hoffman's conjecture. In addition, Theorem 1 settles an open problem raised by Ben Arous [6, Problem 14], namely that of determining the set Σ of atoms of the limiting spectral distribution of $n \times n$ symmetric matrices with independent Bernoulli $\left(\frac{c}{n}\right)$ entries ($c > 0$ is fixed and $n \rightarrow \infty$). Indeed, it follows from the log-Hölder continuity of the spectrum of integer matrices at algebraic numbers (see e.g. [7, Section 6]) that Σ is contained in the set of totally real algebraic integers. On the other hand, Σ is easily seen to contain all tree eigenvalues, as noted by Ben Arous. Theorem 1 precisely states that those inner and outer bounds coincide, thereby settling the question.

2. Outline of the proof

Let T be a finite tree with a distinguished vertex o (the root). Removing o naturally yields a decomposition of T into smaller rooted trees T_1, \dots, T_d ($d \in \mathbb{N}$) as depicted in the following diagram:



To such a rooted tree, let us associate the rational function

$$f_T(x) = 1 - \frac{\Phi_T(x)}{x\Phi_{T \setminus o}(x)} = 1 - \frac{\Phi_T(x)}{x\Phi_{T_1}(x) \cdots \Phi_{T_d}(x)}. \quad (1)$$

Expressed in terms of this function, the classical recursion for the characteristic polynomial of trees (see e.g. [2, Proposition 5.1.1]) simply reads

$$f_T(x) = \frac{1}{x^2} \sum_{i=1}^d \frac{1}{1 - f_{T_i}(x)}, \quad (2)$$

the sum being interpreted as 0 when empty (i.e. when T is reduced to o). Now fix $\lambda \in \mathbb{C} \setminus \{0\}$. In view of (1), the problem of finding a (minimal) tree with eigenvalue λ is equivalent to that of finding T such that $\mathfrak{f}_T(\lambda) = 1$. Thanks to the recursion (2), this task boils down to that of generating the number 1 from the initial seed 0 using repeated applications of the maps

$$(\alpha_1, \dots, \alpha_d) \mapsto \frac{1}{\lambda^2} \sum_{i=1}^d \frac{1}{1 - \alpha_i} \quad (d \in \mathbb{N}).$$

As an example, consider the golden ratio $\lambda = \frac{1+\sqrt{5}}{2}$. Iterating 4 times the map $\alpha \mapsto \frac{1}{\lambda^2(1-\alpha)}$ and using the identity $\lambda^2 = \lambda + 1$ yields successively:

$$0 \longrightarrow \frac{1}{\lambda^2} \longrightarrow \frac{1}{\lambda^2-1} = \frac{1}{\lambda} \longrightarrow \frac{1}{\lambda^2-\lambda} = 1,$$

which shows that λ is an eigenvalue of the linear tree $T = P_4$. Remarkably enough, this seemingly specific argument can be extended in a systematic way to any totally real algebraic integer λ , and the numbers that may be produced in this way can be completely determined. To formalize this, let us fix a totally real algebraic integer $\lambda \neq 0$, and introduce the rational function

$$\Psi(x) = \frac{1}{\lambda^2(1-x)}. \quad (3)$$

Define $\mathcal{F} \subseteq \mathbb{R}$ as the smallest set containing 0 and satisfying for all $d \geq 1$,

$$\alpha_1, \dots, \alpha_d \in \mathcal{F} \setminus \{1\} \implies \Psi(\alpha_1) + \dots + \Psi(\alpha_d) \in \mathcal{F}. \quad (4)$$

We will prove that \mathcal{F} is nothing but the field generated by λ^2 :

$$\mathcal{F} = \left\{ \frac{P(\lambda^2)}{Q(\lambda^2)} : P, Q \in \mathbb{Z}[x], Q(\lambda^2) \neq 0 \right\} =: \mathbb{Q}(\lambda^2). \quad (5)$$

In particular, $1 \in \mathcal{F}$, and Theorem 1 follows. The remainder of the paper is devoted to the proof of (5). The detailed argument is given in Section 4, while Section 3 provides the necessary background on algebraic numbers.

3. Algebraic preliminaries

A number $\zeta \in \mathbb{C}$ is *algebraic* if there is $P \in \mathbb{Q}[x]$ such that $P(\zeta) = 0$. In that case, such P are exactly the multiples of a unique monic polynomial $P_0 \in \mathbb{Q}[x]$, called the *minimal polynomial* of ζ . The algebraic number ζ is

- *totally real* if all the complex roots of P_0 are real ;
- *totally positive* if all the complex roots of P_0 are real and positive.

The following Lemma gathers the basic properties of algebraic numbers that will be used in the sequel. These are well-known (see e.g. [8]) and follow directly from the fact that if $P(x) = \prod_i (x - \alpha_i)$ and $Q(x) = \prod_j (x - \beta_j)$ have rational coefficients, then so do the polynomials

$$\begin{array}{ccc} \prod_{i,j} (x - \alpha_i - \beta_j), & \prod_i \left(x - \frac{1}{\alpha_i} \right), & \prod_{i,j} (x - \alpha_i \beta_j) \\ \prod_i (x + \alpha_i), & \prod_i (x^2 - \alpha_i^2), & \prod_i (x^2 - \alpha_i). \end{array}$$

Lemma 1 (Elementary algebraic properties).

- The totally real algebraic numbers form a sub-field of \mathbb{R} .
- The set of totally positive algebraic numbers is stable under $+$, \times , \div .
- If α is totally real and β is totally positive, then $\alpha + n\beta$ is totally positive for all sufficiently large $n \in \mathbb{N}$.
- If $\alpha \neq 0$ is totally real, then α^2 is totally positive.

We shall also use twice the following elementary result.

Lemma 2. Let ζ be algebraic with minimal polynomial P . Set $n = \deg P$. Then for any $q_1 > \dots > q_n \in \mathbb{Q}$, there exist $m_1, \dots, m_n \in \mathbb{Z}$ such that

$$\frac{m_1}{\zeta - q_1} + \dots + \frac{m_n}{\zeta - q_n} \in \mathbb{N}^+ = \{1, 2, \dots\}, \quad (6)$$

and m_k has the same sign as $(-1)^k P(q_k)$ for every $k \in \{1, \dots, n\}$.

Proof. For $1 \leq k \leq n$, consider the rational number

$$r_k := \frac{-P(q_k)}{\prod_{j \neq k} (q_k - q_j)}.$$

Note that r_k has the same sign as $(-1)^k P(q_k)$, since $q_1 > \dots > q_n$. Moreover,

$$r_1 \prod_{k \neq 1} (x - q_k) + \dots + r_n \prod_{k \neq n} (x - q_k) = \prod_{k=1}^n (x - q_k) - P(x).$$

Indeed, both sides are polynomials of degree less than n , and they coincide at the n points q_1, \dots, q_n . Evaluating at $x = \zeta$ gives

$$\frac{r_1}{\zeta - q_1} + \dots + \frac{r_n}{\zeta - q_n} = 1,$$

and multiplying this by a large enough integer yields the result. \square

4. Proof

Before we start, let us make three simple observations which will be used several times in the sequel. First, by (4), we have for any $k \in \mathbb{N}$,

$$\frac{1}{\lambda^2 - k} = \Psi \left(\underbrace{\Psi(0) + \dots + \Psi(0)}_{k \text{ terms}} \right) \in \mathcal{F}. \quad (7)$$

Second, \mathcal{F} is stable under internal addition:

$$\alpha, \beta \in \mathcal{F} \implies \alpha + \beta \in \mathcal{F}. \quad (8)$$

Indeed, the conclusion is trivial if $\alpha = 0$ or $\beta = 0$. Now if α, β are non-zero elements of \mathcal{F} , then by construction they are of the form

$$\alpha = \Psi(\alpha_1) + \dots + \Psi(\alpha_n) \quad \text{and} \quad \beta = \Psi(\beta_1) + \dots + \Psi(\beta_m),$$

for some $n, m \geq 1$ and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m$ in $\mathcal{F} \setminus \{1\}$. But then,

$$\alpha + \beta = \Psi(\alpha_1) + \dots + \Psi(\alpha_n) + \Psi(\beta_1) + \dots + \Psi(\beta_m),$$

which, by (4), shows that $\alpha + \beta \in \mathcal{F}$. Third, the field $\mathbb{Q}(\lambda^2)$ obviously contains 0 and satisfies (4). Since \mathcal{F} is minimal with this property, we deduce that

$$\mathcal{F} \subseteq \mathbb{Q}(\lambda^2). \quad (9)$$

By part (a) of Lemma 2, we also get that all elements in \mathcal{F} are totally real.

Step 1: \mathcal{F} contains a positive integer

We may assume that $1 \notin \mathcal{F}$, otherwise there is nothing to prove. Consequently, we do not need to worry about divisions by zero when applying Ψ to an element $\alpha \in \mathcal{F}$. Let us first apply Lemma 2 to $\zeta = \lambda^2$ with $q_k = k, 1 \leq k \leq \deg \zeta$. From (7) and (8), it follows that the sum appearing in (6) is the difference of two elements in \mathcal{F} . In other words, we have found $\Delta \in \mathbb{N}^+$ and α with the following property:

$$\alpha \in \mathcal{F} \quad \text{and} \quad \alpha - \Delta \in \mathcal{F}. \quad (10)$$

As already noted, α is totally real. In fact we may even assume that $1 - \alpha$ is totally positive, because $\alpha' = \alpha + \beta$ also satisfies (10) for any $\beta \in \mathcal{F}$, and choosing $\beta = \frac{j}{\lambda^2 - k}$ with $j, k \in \mathbb{N}$ large enough eventually makes $1 - \alpha'$ totally positive, by parts (b) and (c) of Lemma 1. In turn, parts (b) and (d) now guarantee that

$$\xi := \lambda^2(1 - \alpha) \quad \text{is totally positive.} \quad (11)$$

Now fix $j, k \in \mathbb{N}$ and set $i = (\Delta - 1)j + 1$. Since \mathcal{F} is stable under addition, property (10) and the fact that $\frac{1}{\lambda^2} = \Psi(0) \in \mathcal{F}$ imply that

$$\underbrace{\alpha + \cdots + \alpha}_{i \text{ terms}} + \underbrace{(\alpha - \Delta) + \cdots + (\alpha - \Delta)}_{j \text{ terms}} + \underbrace{\frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^2}}_{k \text{ terms}} \in \mathcal{F}.$$

But this number equals $1 - (\Delta j + 1)(1 - \alpha) + \frac{k}{\lambda^2}$, so applying Ψ gives

$$\frac{1}{(\Delta j + 1)\xi - k} \in \mathcal{F}.$$

Adding up $\Delta j + 1$ copies of this last number, we finally arrive at

$$\frac{1}{\xi - q} \in \mathcal{F} \quad \text{for any} \quad q \in \mathcal{Q} := \left\{ \frac{k}{\Delta j + 1} : j, k \in \mathbb{N} \right\}. \quad (12)$$

We may now conclude: by (11), the minimal polynomial P of ξ has $n := \deg P$ pairwise distinct positive roots. Since \mathcal{Q} is dense in $[0, \infty)$, one can find $q_1 > \cdots > q_n$ in \mathcal{Q} that interleave those roots, in the sense that $P(q_k)$ has sign $(-1)^k$ for every $1 \leq k \leq n$. Consequently, Lemma 2 provides us with non-negative integers m_1, \dots, m_n such that

$$\frac{m_1}{\xi - q_1} + \cdots + \frac{m_n}{\xi - q_n} \in \mathbb{N}^+.$$

On the other hand, this sum is in \mathcal{F} by (12) and (8). Thus, $\mathcal{F} \cap \mathbb{N}^+ \neq \emptyset$.

Step 2: $(\mathcal{F}, +)$ is a group

We know that \mathcal{F} contains some $d \in \mathbb{N}^+$. Since $d + d$ also belongs to \mathcal{F} , we may assume without loss of generality that $d \neq 1$ to avoid divisions by zero below. Now fix $\alpha \in -\mathcal{F}$ with $\alpha \neq 1$. Since \mathcal{F} is stable under addition,

$$d + \underbrace{(-\alpha) + \cdots + (-\alpha)}_{d-1 \text{ terms}} \in \mathcal{F}.$$

Applying Ψ shows that $\frac{-1}{\lambda^{2(1-\alpha)(d-1)}} \in \mathcal{F}$. Adding up $(d-1)$ copies of this number, we conclude that $\frac{-1}{\lambda^{2(1-\alpha)}} \in \mathcal{F}$. We have proved:

$$\alpha \in (-\mathcal{F}) \setminus \{1\} \implies \Psi(\alpha) \in -\mathcal{F}.$$

In other words, $-\mathcal{F}$ is stable under Ψ . In view of (8), we deduce that $-\mathcal{F}$ satisfies (4). By minimality of \mathcal{F} , we conclude that $\mathcal{F} \subseteq -\mathcal{F}$, i.e. that \mathcal{F} is stable under negation. Thus, the monoid $(\mathcal{F}, +)$ is a group.

Step 3: \mathcal{F} is the field $\mathbb{Q}(\lambda^2)$.

In view of (9), we only need to show that for $P, Q \in \mathbb{Z}[x]$ with $Q(\lambda^2) \neq 0$,

$$\frac{P(\lambda^2)}{Q(\lambda^2)} \in \mathcal{F}. \quad (13)$$

Since λ^2 is an algebraic integer, we may assume that Q is monic with $\deg Q > \deg P$ (otherwise, replace Q with $Q + P_0^{\deg P}$, where P_0 denotes the minimal polynomial of λ^2). Let us prove the claim by induction over $n = \deg Q$. The case $n = 0$ is simply the fact that $0 \in \mathcal{F}$. Now, assume that the claim holds for a certain $n \in \mathbb{N}$, and consider

$$Q(x) = x^{n+1} + a_n x^n + \cdots + a_0,$$

with $a_0, \dots, a_n \in \mathbb{Z}$. Let us first prove (13) in the following two special cases:

- Case 1: $P(x) = x^n$. By our induction hypothesis, $\frac{1}{1+\lambda^{2n}} \in \mathcal{F}$ and hence

$$\frac{1}{\lambda^{2n+2}} + \frac{1}{\lambda^2} = \Psi\left(\frac{1}{1+\lambda^{2n}}\right) \in \mathcal{F}.$$

But \mathcal{F} also contains $\frac{1}{\lambda^2}, \dots, \frac{1}{\lambda^{2n}}$ by our induction hypothesis. Since $(\mathcal{F}, +)$ is a group, one deduces that \mathcal{F} contains

$$-\left(\frac{a_n}{\lambda^2} + \dots + \frac{a_0}{\lambda^{2n+2}}\right) = 1 - \frac{Q(\lambda^2)}{\lambda^{2n+2}}.$$

Finally, applying Ψ shows that $\frac{\lambda^{2n}}{Q(\lambda^2)} = \frac{P(\lambda^2)}{Q(\lambda^2)} \in \mathcal{F}$, as desired.

- Case 2: P is monic of degree n with $P(0) = 1$. Then

$$R(x) := P(x) - \frac{Q(x) - Q(0)P(x)}{x}$$

is a polynomial over \mathbb{Z} with $\deg R < n$. Thus, our induction hypothesis guarantees that \mathcal{F} contains $\frac{R(\lambda^2)}{P(\lambda^2)}$, hence $\frac{R(\lambda^2)}{P(\lambda^2)} - \frac{Q(0)}{\lambda^2}$ and hence also

$$\Psi\left(\frac{R(\lambda^2)}{P(\lambda^2)} - \frac{Q(0)}{\lambda^2}\right) = \frac{P(\lambda^2)}{Q(\lambda^2)}.$$

For the general case, note that every monomial x^k ($0 \leq k \leq n$) may be written as a signed sum of polynomials P of the form considered in the two special cases above. Since $(\mathcal{F}, +)$ is a group, we conclude that (13) holds for all $P \in \mathbb{Z}[x]$ with $\deg P \leq n$, and the induction is complete.

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