HAL
open science

Existence of global strong solution for Korteweg system with large infinite energy initial data

Boris Haspot

To cite this version:

Boris Haspot. Existence of global strong solution for Korteweg system with large infinite energy initial data. Journal of Mathematical Analysis and Applications, 2016, 438, pp.395-443. hal-00789782v2

HAL Id: hal-00789782

https://hal.science/hal-00789782v2

Submitted on 19 Feb 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Existence of global strong solution for Korteweg system with large infinite energy initial data

Boris Haspot *

Abstract

This work is devoted to the study of the initial boundary value problem for a general isothermal model of capillary fluids derived by J.E Dunn and J.Serrin (1985) (see [16]), which can be used as a phase transition model. We will prove the existence of local and global (under a condition of smallness on the initial data) strong solutions with discontinuous initial density where $\ln \rho_0$ belongs to the Besov space $B^{\frac{N}{2}}_{2,\infty}(\mathbb{R}^N)$. In passing our result allow to consider initial data with discontinuous interfaces (up our knowledge in the literature the results of existence of strong solutions require generally continuous initial density), it implies also that we can work with initial data space of infinite energy. Our result relies on the fact that the density can be written as the sum of the solution ρ_L associated to linear system and a remainder density $\bar{\rho}$ which is more regular than ρ_L by taking into account the regularizing effects induced on the bilinear convection term. The main difficulty concerns the proof of new estimate of maximum principle type for the linear system associated to the Korteweg system, the proof is based on a characterization of the Besov space in terms of the semi group associated to this linear system. Let also mention that we prove the existence of global strong solution with a smallness hypothesis which is subcritical in terms of the scaling of the equations, it allows us to exhibit a family of large energy initial data for the scaling of the equations providing global strong solution. In particular for the first time up our knowledge we show the existence of global strong solution for some large energy initial data when $N = 2$.
We finish this paper by introducing the notion of quasi-solutions for the Korteweg's system (a tool which has been developed in the framework of the compressible Navier-Stokes equations [25, 26, 20, 28, 29]) which enables us to improve the previous result and by obtaining the existence of global strong solution with large initial velocity in $B^{\frac{N}{2}-1}_{2,\infty}$ (in particular when $N = 2$ it implies the existence of global strong solution with large energy data). As a corollary, we get global existence (and uniqueness) for highly compressibleKorteweg system when $N \geq 2$. It means that for any large initial data (under an irrotational condition on the initial velocity) we have the existence of global strong solution provided that the pressure is sufficiently highly compressible.

1 Introduction

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by Van der Waals and Korteweg

*Université Paris Dauphine, place du Maréchal De Lattre De Tassigny 75775 Paris cedex 16 (France), haspot@ceremade.dauphine.fr

[43, 35] and was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [16, 33, 42]. The first investigations begin with the Young-Laplace theory which claims that the phases are separated by a hypersurface and that the jump in the pressure across the hypersurface is proportional to the curvature of the hypersurface. The main difficulty consists in describing the location and the movement of the interfaces.

Another major problem is to understand whether the interface behaves as a discontinuity in the state space (sharp interface) or whether the phase boundary corresponds to a more regular transition (diffuse interface, DI). The diffuse interface models have the advantage to consider only one set of equations in a single spatial domain (the density takes into account the different phases) which considerably simplifies the mathematical and numerical study (indeed in the case of sharp interfaces, we have to treat a problem with free boundary).

Another approach corresponds to determine equilibrium solutions which classically consists in the minimization of the free energy functional. Unfortunately this minimization problem has an infinity of solutions, and many of them are physically wrong (some details are given later). In order to overcome this difficulty, Van der Waals in the XIX-th century was the first to add a term of capillarity to select the physically correct solutions, modulo the introduction of a diffuse interface. This theory is widely accepted as a thermodynamically consistent model for equilibria.

Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but also on the gradient of the density. Alternatively, another way to penalize the high density variations consists in applying a zero order but non-local operator to the density gradient (see [38], [39], [40]). For more results on non local Korteweg system, we refer also to [9, 10, 11, 21, 23].

Let us now consider a fluid of density $\rho \geq 0$, velocity field $u \in \mathbb{R}^N$, we are now interested in the following compressible capillary fluid model, which can be derived from a Cahn-Hilliard like free energy (see the pioneering work by J.- E. Dunn and J. Serrin in [16] and also in [1, 6, 18]). The conservation of mass and of momentum write:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)\mathrm{D(u)}) - \nabla\big(\lambda(\rho)\mathrm{divu}\big) + \nabla \mathrm{P}(\rho) = \mathrm{divK}, \end{cases} \tag{1.1}$$

where the Korteweg tensor read as following:

$$\mathrm{div}K = \nabla\big(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa^{'}(\rho))|\nabla\rho|^2\big) - \mathrm{div}\big(\kappa(\rho)\nabla\rho \otimes \nabla\rho\big). \tag{1.2}$$

κ is the coefficient of capillarity and is a regular function. The term $\mathrm{div}K$ allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid-vapor. P is a general increasing pressure term. $D(u) = (\nabla u +^t \nabla u)$ defines the stress tensor, μ and λ are the two Lamé viscosity coefficients depending on the density ρ and satisfying:

$$\mu > 0 \ \text{ and } \ 2\mu + N\lambda \geq 0.$$

Here we are interested in investigating the Cauchy problem of the existence of global strong solution for the system (1.1) when we authorize jump in the pressure across the

interfaces. In order to obtain a such result, we are going to show the existence of strong solutions in critical space (it means in spaces as large as possible) for the scaling of the equations with an initial density which is not necessary continuous.
Before detailing this program, let us recall briefly the classical energy estimates for the system (1.1). Let $\bar{\rho} > 0$ be a constant reference density (in the sequel we will assume that $\bar{\rho} = 1$), and let Π be defined by:

$$\Pi(s) = s\Big(\int_{\bar{\rho}}^{s} \frac{P(z)}{z^2} dz - \frac{P(\bar{\rho})}{\bar{\rho}}\Big),$$

so that $P(s) = s\Pi^{'}(s) - \Pi(s)\,,\, \Pi^{'}(\bar{\rho}) = 0$. Multiplying the equation of momentum conservation in the system (1.1) by u and integrating by parts over $(0,t)\times\mathbb{R}^N$, we obtain the following estimate:

$$\begin{aligned}\int_{\mathbb{R}^N} \big(\frac{1}{2}\rho|u|^2 + (\Pi(\rho) - \Pi(\bar{\rho})) + \frac{1}{2\kappa\rho}|\nabla\rho|^2\big)(t)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^N} \mu(\rho)|D(u)|^2 dxdt \\ \leq \int_{\mathbb{R}^N} \big(\frac{|m_0|^2}{2\rho} + (\Pi(\rho_0) - \Pi(\bar{\rho})) + \frac{1}{2\kappa\rho_0}|\nabla\rho_0|^2\big)dx.\end{aligned} \tag{1.3}$$

It follows that assuming that the initial total energy is finite:

$$\mathcal{E}_0 = \int_{\mathbb{R}^N} \big(\frac{|m_0|^2}{2\rho} + (\Pi(\rho_0) - \Pi(\bar{\rho})) + \frac{\kappa(\rho_0)}{2}|\nabla\rho_0|^2\big)dx < +\infty\,,$$

then we have the a priori following bounds when $P(\rho) = a\rho^\gamma$ with $\gamma > 1$:

$$(\rho - 1) \in L^\infty(L^\gamma_2), \;\text{ and }\; \rho|u|^2 \in L^1(0,\infty, L^1(\mathbb{R}^N)),$$

$$\sqrt{\kappa(\rho)}\nabla\rho \in L^\infty(0,\infty, L^2(\mathbb{R}^N))^N, \;\text{ and }\; \nabla u \in L^2(0,\infty,\mathbb{R}^N)^{N^2}.$$

We refer to [37] for the definition of the Orlicz spaces. Let us now recall the notion of scaling for the Korteweg's system (1.1). Such an approach is now classical for incompressible Navier-Stokes equation and yields local well-posedness (or global well-posedness for small data) in spaces with minimal regularity. In our situation we can easily check that, if (ρ, u) solves (1.1), then $(\rho_\lambda, u_\lambda)$ solves also this system:

$$\rho_\lambda(t,x) = \rho(\lambda^2 t, \lambda x)\,,\; u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

provided the pressure laws P have been changed into $\lambda^2 P$.

Definition 1.1 *We say that a functional space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation:*

$$(\rho, u) \longrightarrow (\rho_\lambda, u_\lambda)$$

(up to a constant independent of λ).

This suggests us to choose initial data (ρ_0, u_0) in spaces whose norm is invariant for all $\lambda > 0$ by the transformation $(\rho_0, u_0) \longrightarrow (\rho_0(\lambda\cdot), \lambda u_0(\lambda\cdot))$. A natural candidate is the Besov space (see the section 2 for some definitions of Besov spaces) $B^{N/2}_{2,\infty} \times (B^{N/2-1}_{2,\infty})^N$,

however since $B^{N/2}_{2,\infty}$ is not included in L^∞, we cannot expect to get a priori L^∞ estimate on the density when $\rho_0 \in B^{N/2}_{2,\infty}$ (in particular it makes the study of the non linear term delicate since it appears impossible to use composition theorems) . Typically except in the case of specific pressure we are not able to deal with this last one. An other difficulty concerns the control of the vacuum or more precisely of the L^∞ norm of $\frac{1}{\rho}$, indeed it is crucial to avoid vacuum in order to involve the parabolicity of the momentum equation. An other candidate of initial data space would be $(B^{N/2}_{2,\infty} \cap L^\infty) \times (B^{N/2-1}_{2,\infty})^N$, it will be typically in this last type of space that we shall prove some result of global strong solution by considering specific physical coefficients. Indeed controlling the L^∞ norm of the density for general physical coefficients is generally an hard task in fluid mechanics, even if the Korteweg system allows to obtain regularizing effects on the density. This is the reason why in the literature the authors consider initial density which are in Banach spaces imbedded in L^∞ and such that the regularity is propagate via the parabolic structure of the system. Let us briefly mention that the existence of strong solutions for $N \geq 2$ is known since the works by H. Hattori and D. Li [31, 32]. R. Danchin and B. Desjardins in [15] improve this result by working in critical spaces for the scaling of the equations, more precisely the initial data $(q_0, \rho_0 u_0)$ belong to $B^{\frac{N}{2}}_{2,1} \times B^{\frac{N}{2}-1}_{2,1}$ (as mentioned previously, let us emphasize on the fact that $B^{\frac{N}{2}}_{2,1}$ is embedded in $L^\infty$and it plays a crucial role in order to control the vacuum and the L^∞ norm of the density). In [36], M. Kotschote showed the existence of strong solution for the isothermal model in bounded domain by using Doreâ-Venni Theory and $\mathcal{H}^\infty$ calculus. In [31], we generalize the results of [15] in the case of non isothermal Korteweg system with physical coefficients depending on the density and the temperature.

1.1 Mathematical results

We now are going to state our main results. As we explained previously, one of the main difficulty to obtain strong solutions for Korteweg's system in very general Besov spaces consists in dealing with Besov space $B^{\frac{N}{2}}_{2,\infty}$ which are not embedded in L^∞, it implies that in order to ensure the parabolicity of the momentum equation we must control the norm $\frac{1}{\rho}$ in L^∞ by an other way than simply propagating the regularity $B^{\frac{N}{2}}_{2,\infty}$ on the density. To do this we shall precisely understand the structure of the equations by developing new arguments related to the maximum principle.
Before developing this program let us start with observing specific structure on the Korteweg system when $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\kappa > 0$, indeed it is possible to rewrite the system (1.1) in a simple way by introducing an effective velocity (we refer to [34, 19] where the authors prove the existence of global weak solutions).

Remark 1 *Let us give some explanations on this choice of capillarity $\kappa(\rho) = \frac{\kappa}{\rho}$, indeed this regime flows exhibits particular phenomena in the case of the compressible Korteweg Euler system (which is called quantum compressible Euler system when $\kappa(\rho) = \frac{\kappa}{\rho}$). At least heuristically, the system is equivalent via the Madelung transform to the Gross-Pitaevskii equations which are globally well-posed for large initial data in dimension $N = 1, 2, 3$ (we refer to [17]). One of the main difficulty to pass from Gross-Pitaevskii to Quantic Euler consists in dealing with the vacuum. This is one of the reasons why the*

mathematical community is interested in building solitons for this type of problem (one of the main other reasons correspond to give a negative answer to the problem of scattering and to study the stability of the soliton, we refer to [4]). We would also like to mention very interesting results of global weak solutions for the compressible quantic Euler equation with a regime $\kappa(\rho) = \frac{1}{\rho}$ due to Antonelli and Marcati (see [2]). To finish let also mention that we prove in [12] that it exits global strong solutions in one dimension for the system (1.1) when $\kappa(\rho) = \frac{\mu^2}{\rho}$ and $\mu(\rho) = \mu\rho$, $\lambda(\rho) = 0$ and furthermore these solutions converge to a global weak entropy solution of the compressible Euler system when μ goes to 0. It shows in particular that the Korteweg system is relevant to select the physical solution of the compressible Euler system via a viscosity-capillarity vanishing process.

When $\kappa(\rho) = \frac{\kappa}{\rho}$, we rewrite the capillarity tensor as follows (see the appendix for more details on the computations):

$$K(\rho) = \kappa\rho(\nabla\Delta(\ln\rho) + \frac{1}{2}\nabla(|\nabla\ln\rho|^2)).$$

We now want to consider the eulerian form of the system (1.1) when $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\kappa > 0$, we obtain then by dividing by ρ the momentum equation the following system:

$$\begin{cases} \partial_t \ln\rho + u\cdot\nabla\ln\rho + \mathrm{div}u = 0, \\ \partial_t u + u\cdot\nabla u - \frac{1}{\rho}\mathrm{div}(2\mu(\rho)Du) - \frac{1}{\rho}\nabla(\lambda(\rho)\mathrm{div}u) + \nabla F(\rho) = \kappa\nabla\Delta(\ln\rho) + \frac{\kappa}{2}\nabla(|\nabla\ln\rho|^2), \\ (\ln\rho, u)_{\ t=0} = (\ln\rho_0, u_0). \end{cases} \tag{1.4}$$

with $F(\rho)$ defined by $\frac{F'(\rho)}{\rho} = P'(\rho)$. In the sequel we will use the following definition.

Definition 1.2 *We now set:*

$$q = \ln\rho.$$

One can now state the main results of the paper. In the first theorem we prove the existence of global strong solution for (1.4) with *small* initial data and of strong solution in finite time with large initial data when we choose specific viscosity coefficients and pressure terms. More precisely we shall deal with the shallow water viscosity coefficients:

$$\mu(\rho) = \mu\rho,\ \lambda(\rho) = \lambda\rho,$$

with $\mu > 0$ and $2\mu + \lambda > 0$. It leads to the following system:

$$\begin{cases} \partial_t q + u\cdot\nabla q + \mathrm{div}u = 0, \\ \partial_t u + u\cdot\nabla u - \mu\Delta u - 2\mu\nabla q\cdot D(u) - (\lambda+\mu)\nabla\mathrm{div}u - \lambda\mathrm{div}u\nabla q + \nabla F(\rho) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \kappa\nabla\Delta q + \frac{\kappa}{2}\nabla(|\nabla q|^2), \\ (q, u)_{\ t=0} = (\ln\rho_0, u_0). \end{cases} \tag{1.5}$$

Let us now state our main results, we refer to the section 2 for the definitions of the Besov space and the Hybrid Besov spaces.

Theorem 1.1 *Let $N \geq 2$. Assume that $\mu(\rho) = \mu\rho$, $\lambda(\rho) = \lambda\rho$ with $\mu > 0$, $2\mu + \lambda > 0$ and $P(\rho) = K\rho$ with $K > 0$. We also suppose that:*

$$q_0 \in B^{\frac{N}{2}}_{2,\infty} \text{ and } u_0 \in B^{\frac{N}{2}-1}_{2,\infty}.$$

There exists a time T such that (1.5) has a unique solution (q, u) on $(0, T)$ with:

$$q \in \widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,\infty}), \text{ and } u \in \widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty}).$$

Furthermore it exists ε_0 such that if in addition $q_0 \in B^{\frac{N}{2}-1}_{2,\infty}$ and:

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}} + \|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \leq \varepsilon_0.$$

then the solution (q, u) is global with:

$$q \in \widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty}), \text{ and } u \in \widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty}). \tag{1.6}$$

If in addition with the previous smallness hypothesis we assume that (q_0, u_0) belongs in $B^{\frac{N}{2}}_{2,2} \times B^{\frac{N}{2}-1}_{2,2}$ and that $q_0 \in L^\infty$ then there exists a unique solution (ρ, u) of the system (1.1) with $q = \ln \rho$. Furthermore for any T it exists C_T such that:

$$\|\frac{1}{\rho}\|_{L^\infty_T(L^\infty)} + \|\rho\|_{L^\infty_T(L^\infty)} \leq C_T.$$

Remark 2 *We want to point out that in the first part of the theorem 1.1, we solve the system (1.5) and not the system (1.1). Indeed we do not assume any control on q_0 in L^∞, it means that we have no information on the vacuum of the density. So it is not clear that a solution from (1.5) is also a solution from (1.1) when there is vacuum. This result proves in a certain way that the good variable to consider is not the density ρ but rather $\ln \rho$. Let us emphasize on the fact that this result is the first (up our knowledge) for compressible system where we are able to work with such general critical initial data (in particular $u_0 \in B^{\frac{N}{2}-1}_{2,\infty}$, on the contrary this regularity is classical for incompressible Navier-Stokes equations, see [8]).*
In the second part of theorem we assume additional hypothesis on the density since q_0 belongs also in $L^\infty \cap B^{\frac{N}{2}}_{22}$ and $u_0 \in B^{\frac{N}{2}-1}_{22}$, in counter part we are able to show that with such initial data we have existence of strong solution for the "real" Korteweg system (1.1). The main difficulty consists in proving L^∞ estimates on q by splitting the solution q in the sum of the solution of the linearized system q_L and a remainder term $\bar{q}$ taking into account the non linear terms. We combine some maximum principle results on q_L and regularizing effect on $\bar{q}$ on the third index of the Besov space to show that q is well controlled in L^∞. This part is quite technical and involves new ideas for developing some results of maximum principle type, to do this we show in particular a characterization of Besov spaces in terms of the semi group associated to the linear system related to (1.1).

Let us finish by observing that in this theorem we authorize that the initial density is not necessary continuous as in [15], this is one of the main motivation of this paper. Indeed we can deal with initial density which have some jump, we observe that the density is immediately regularized in the sense that ρ is in $C^\infty((0, T), \mathbb{R}^N)$ for any $T > 0$. This is due to the fact that the interfaces are diffuse.

Remark 3 *The second interest of this theorem it to obtain the existence of global strong solution with subcritical smallness on the initial data. More precisely we assume that* (q_0, u_0) *belongs in* $(B^{\frac{N}{2}-1}_{2,\infty} \cap B^{\frac{N}{2}}_{2,2} \cap L^\infty) \times B^{\frac{N}{2}-1}_{2,2}$ *but we require only a smallness assumption in* $\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}$*. In particular it allows us to prove the existence of global strong solution with large initial data in the energy space when* $N = 2$*, this is up our knowledge the first result of global strong solution with large initial energy data for the Korteweg system when* $N = 2$*. We give a example of such initial data in the corollary 3 where* u_{0,ε,l_0} *can be chose as small as possible in* $B^{\frac{N}{2}-1}_{2,\infty}$ *but very large in* $B^{\frac{N}{2}-1}_{2,2}$*. Let us recall that when* $N = 2$*,* $B^{\frac{N}{2}-1}_{2,2}$ *corresponds to* L^2 *which is the energy space for* u_0 *(see (1.3)).*

An other interesting point is that compared with [15], we can choose initial density with a large L^∞ *norm.*

Remark 4 *We now want to point out the specificity of the different physical coefficients, it is typically the case for the pressure and the viscosity coefficients where* $P(\rho) = K\rho$ *and* $\mu(\rho) = \mu\rho$*. Indeed under this form we can check that the variable* $\ln\rho$ *appears naturally everywhere but also that there is no non-linear terms in the density such that it would be necessary to have* L^∞ *control on the density for dealing with these terms. Let us mention that it is not possible to obtain the same type of result for compressible Navier-Stokes equations as we have no regularizing effects on the density. Indeed in this case it would be not possible to deal with the following term* $\nabla \ln\rho \cdot \nabla u$ *and to prove that it remains in* $\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})$*.*
In the following theorem we are going to extend the previous analysis to general physical coefficients.

Remark 5 *We would like to mention that we could easily extend the result of strong solution in finite time to the framework of Besov spaces constructed on general* L^p *spaces when the initial data verify:*

$$q_0 \in B^{\frac{N}{p}}_{p,\infty} \ \text{ and } \ u_0 \in B^{\frac{N}{p}-1}_{p,\infty},$$

Concerning the existence of global strong solution we shall assume that q_0 *is in* $\widetilde{B}^{\frac{N}{2}-1\frac{N}{p}}_{2,p,\infty}$ *and* $u_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,\infty}$*, the main difficulty would concern the proof of estimate for the linear system associated to the Korteweg system with such initial data (we refer to [24] for a such study in the framework of the compressible Navier Stokes equations).*

The previous theorem takes into account in an essential way the structure of the viscosity, capillary and pressure coefficients, indeed it allows us to assume a smallness hypothesis only on initial data in $\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty} \times B^{\frac{N}{2}-1}_{2,\infty}$ in order to obtain global strong solution. Furthermore with such physical coefficients we are able to prove the existence of global strong solution for the system (1.4) by assuming only that (q_0, u_0) belongs in $\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty} \times B^{\frac{N}{2}-1}_{2,\infty}$ with a smallness assumption in this space. We would like to extend this result to general physical coefficients and in particular dealing with the case of the constant capillary coefficient in order to improve the results of [15]. Let us state a new theorem dealing with general pressure and with the capillary coefficient such that $\kappa(\rho) = \frac{\kappa}{\rho}$ or κ with $\kappa > 0$.

Theorem 1.2 *Let $N \geq 2$. Assume that $\mu(\rho) = \mu\rho$ or μ, $\lambda(\rho) = \lambda\rho$ or λ with $\mu > 0$, $2\mu + \lambda > 0$ and P a regular function such that $P^{'}(1) > 0$. Furthermore we suppose that $\rho_0 = 1 + h_0$:*

$$h_0 \in \widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-2} \cap L^\infty \ \ \text{and} \ \ u_0 \in B_{2,1}^{\frac{N}{2}-2} \cap B_{2,2}^{\frac{N}{2}-1}.$$

There exists ε_0 such that if:

$$\|h_0\|_{\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-2} \cap L^\infty} + \|u_0\|_{B_{2,1}^{\frac{N}{2}-2} \cap B_{2,2}^{\frac{N}{2}-1}} \leq \varepsilon_0.$$

then it exists a global unique solution (ρ, u) of the system (1.1) with $\rho = 1 + h$ and:

$$\begin{aligned} h \in \widetilde{L}^\infty(\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-2}) \cap \widetilde{L}^1(\widetilde{B}_{2,2}^{\frac{N}{2}+1,\frac{N}{2}+2} \cap B_{2,1}^{\frac{N}{2}}) \cap L^\infty(L^\infty) \\ \text{and} \ \ u \in \widetilde{L}^\infty(B_{2,2}^{\frac{N}{2}-1} \cap B_{2,1}^{\frac{N}{2}-2}) \cap \widetilde{L}^1(B_{2,2}^{\frac{N}{2}+1} \cap B_{2,1}^{\frac{N}{2}}). \end{aligned} \tag{1.7}$$

Remark 6 *We would like to mention that this theorem improve the results of [15] in terms of rough regularity on the initial data. Here we assume only that u_0 belongs in $B_{2,2}^{\frac{N}{2}-1}$ instead $B_{2,1}^{\frac{N}{2}-1}$, the main task as in the previous theorem 1.1 consists in getting a control of the density in L^∞ without assuming $h_0 \in B_{2,1}^{\frac{N}{2}}$ as it is the case in [15]. To do this we have to study precisely the linearized system of (1.1) and involving some results of maximum principle type, in addition the solution are the sum of solution for the linear system associated to (1.1) and remainder terms taking into account the non linearities. These remainders terms benefit from regularizing effects on the third index of the Besov space which will ensure a L^∞ control on the density. For more details we refer to the proof of theorem 1.2.*
Let us mention that for the first time up my knowledge we can choose initial density which are not continuous as in [15], in particular we authorize some jump on the initial density which is immediately regularized.

Remark 7 *Let us point out that in the previous theorem 1.2 we ask additional regularity in low frequencies on (h_0, u_0), indeed we assume that (h_0, u_0) are in $B_{2,1}^{\frac{N}{2}-2} \times (B_{2,1}^{\frac{N}{2}-2})^N$. This is essentially due to the fact that we have shown maximum principle for the system (N) p18 and not for the system $(N1)$ p 20 (see the proposition 3.93). We think that it would be possible to extend the proposition 3.93 to the system $(N1)$. In particular it will allow to avoid this additional regularity on the initial data in low frequencies. For more explanations we refer to the remark 20.*

We are now interested in dealing with the specific case $\kappa = \mu^2$ where we can exhibit a specific structure on the system (1.1), we obtain from system (1.5) (when $\lambda = 0$) the following simplified model by assuming that $v = u + \mu\nabla \ln \rho$ (we refer to the appendix for more details on the computation or [19])):

$$\begin{cases} \partial_t \rho - \mu\Delta\rho = -\mathrm{div}(\rho v), \\ \rho\partial_t v + \rho u \cdot \nabla v - \mathrm{div}(\mu\rho\,\nabla \mathrm{v}) + \nabla \mathrm{P}(\rho) = 0, \end{cases} \tag{1.8}$$

which is equivalent to the following system if we control the vacuum on the density ρ with $q = \ln\rho$:

$$\begin{cases} \partial_t q - \mu\Delta q + v\cdot\nabla q = -\mathrm{div} v + \mu|\nabla q|^2, \\ \partial_t v + u\cdot\nabla v - \mu\Delta v - \mu\nabla q\cdot\nabla v + \nabla F(\rho) = 0, \end{cases} \tag{1.9}$$

In this particular case we are able to use a new tool developed in [28, 29, 25, 26, 20] called the quasi-solutions. More precisely we can check that it exists a particular solution of the following system (where we have canceled out the pressure P):

$$\begin{cases} \partial_t \rho - \mu\Delta\rho = -\mathrm{div}(\rho v), \\ \rho\partial_t v + \rho u\cdot\nabla v - \mathrm{div}(\mu\rho\,\nabla \mathrm{v}) = 0, \end{cases} \tag{1.10}$$

Indeed we verify that $(\rho_1, -\mu\nabla\ln\rho_1)$ is a particular solution of (1.10) if the density ρ_1 verifies the following heat equation:

$$\partial_t\rho_1 - \mu\Delta\rho_1 = 0. \tag{1.11}$$

The idea will consists in working around this quasi-solution which allows us to prove the existence of global strong solution with small initial data in subcritical norms. Indeed we are going to search solution under the form:

$$q = \ln\rho = \ln\rho_1 + h_2 \ \text{ with } \rho_1 = 1 + h_1 \ \text{ and } \ u = -\mu\nabla\ln\rho_1 + u_2.$$

We deduce from (1.9) that (h_2, u_2) verifies the following system:

$$\begin{cases} \partial_t h_2 + \mathrm{div} u_2 - \mu\nabla\ln\rho_1\cdot\nabla h_2 + u_2\cdot\nabla\ln\rho_1 = F(h_2,u_2), \\ \partial_t u_2 - \mu\Delta u_2 - \mu\nabla\mathrm{div} u_2 - \kappa\nabla\Delta h_2 + K\nabla h_2 - 2\mu\nabla\ln\rho_1\cdot Du_2 - 2\mu\nabla h_2\cdot Du_1 \\ \qquad\qquad + u_1\cdot\nabla u_2 + u_2\cdot\nabla u_1 - \mu^2\nabla(\nabla\ln\rho_1\cdot\nabla h_2) = G(h_2,u_2), \\ (h_2(0,\cdot), u_2(0,\cdot)) = (h_0^2, u_0^2). \end{cases} \tag{1.12}$$

with:

$$\begin{aligned} F(h_2,u_2) &= -u_2\cdot\nabla h_2, \\ G(h_2,u_2) &= -u_2\cdot\nabla u_2 + 2\mu\nabla h_2\cdot Du_2 - K\nabla\ln\rho_1 + \frac{\mu^2}{2}\nabla(|\nabla h_2|^2). \end{aligned} \tag{1.13}$$

Let us state our main theorem for the system (1.12).

Theorem 1.3 *Let $N \geq 2$. Assume that $\mu(\rho) = \mu\rho$, $\kappa(\rho) = \frac{\mu^2}{\rho}$ and $\lambda(\rho) = 0$ with $\mu > 0$ and $P(\rho) = K\rho$ with $K > 0$. Furthermore we suppose that $u_0 = -\mu\nabla[\ln\rho_1^0] + u_2^0$ and $\ln\rho_0 = \ln(\rho_1^0) + h_2^0$ such that $\rho_1^0 = 1 + h_1 0$ and it exits $c_1 > 0$ such that $\rho_1^0 \geq c_1 > 0$. In addition we assume that:*

$$h_1^0 \in \widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}} \ \ h_2^0 \in \widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}} \ \ \textit{and} \ \ u_2 0 \in B_{2,1}^{\frac{N}{2}-1}.$$

Furthermore it exists ε_0, $C > 0$ and two regular function g, g_1 such that if:

$$\begin{aligned} &Cg(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}-2}} \exp\left(Cg_1(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}})\right) \leq \frac{1}{2}, \\ &\|h_2^0\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_2^0\|_{B_{2,2}^{\frac{N}{2}-1}} \leq \varepsilon_0. \end{aligned} \tag{1.14}$$

then it exists a global unique solution (ρ, u) of the system (1.1) with: $u = -\mu\nabla \ln \rho_1 + u_2$ and $\ln \rho = \ln \rho_1 + h_2$ with $\rho_1 = 1 + h_1$ verifying the following system:

$$\begin{cases} \partial_t \rho_1 - \mu\Delta\rho_1 = 0, \\ \rho_1(0,\cdot) = \rho_1^0 = 1 + h_1^0. \end{cases}$$

Furthermore we have:

$$h_2 \in \widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}) \cap \widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1}) \ \text{ and } \ u_2 \in \widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,1}) \cap \widetilde{L}^1(B^{\frac{N}{2}+1}_{2,1}). \tag{1.15}$$

Remark 8 *Let us mention that in this theorem by a very accurate study on the linear system associated to the system (1.12) (see the proposition 3.9), it would be probably possible to improve the initial condition on h_1^0 by assuming only h_1^0 in $B^{\frac{N}{2}-2}_{2,1} \cap B^{\frac{N}{2}}_{2,2-\varepsilon} \cap L^\infty$ with $\varepsilon > 0$ (see the remark 19). In particular it would allow to choose initial density not necessary continuous.*

Remark 9 *Let us mention than the main interest of this theorem is to prove the existence of global strong solution with large initial data for the scaling of the equation which is completely new up our knowledge. Indeed it suffices to choose $h_1^0(x) = \varphi(\lambda x)$ with $\varphi \in \widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}}_{2,1}$ such that $1 + \varphi \geq c > 0$ we then verify easily that:*

$$\begin{cases} \|h_1^0\|_{B^{\frac{N}{2}}_{2,1}} = \|\varphi\|_{B^{\frac{N}{2}}_{2,1}}, \\ \|\rho_1^0\|_{L^\infty} = \|1+\varphi\|_{L^\infty}, \\ \|\dfrac{1}{\rho_1^0}\|_{L^\infty} = \|\dfrac{1}{1+\varphi}\|_{L^\infty}, \\ \|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}} = \dfrac{1}{\lambda^2}\|\varphi\|_{B^{\frac{N}{2}-2}_{2,1}}. \end{cases} \tag{1.16}$$

It implies that h_1^0 verifies (1.14) by choosing λ large enough. In particular it implies that by taking φ large in $B^{\frac{N}{2}}_{2,1}$ that our initialdensity h_1^0 is large in the critical Besov space $B^{\frac{N}{2}}_{2,1}$ for the scaling of the initial data. It is also possible to choose φ large in $B^1_{2,2}$ which shows that there is existence of global strong solution for large initial data in the energy space when $N = 2$. This is certainly the main novelty of this paper.
We could also to choose $h_1^0(x) = \ln(\lambda)\varphi(\lambda x)$ with $\lambda > 0$ which shall still improve the previous estimates in term of large initial data in $B^{\frac{N}{2}}_{2,2}$. let us also emphasize that we can take φ large in $B^{\frac{N}{2}}_{2,\infty}$, in particular it implies that h_1^0 is large in $B^{\frac{N}{2}}_{2,\infty}$ however we obtain the existence of global strong solution, it improves from this point of view the results of theorem 1.1. We would like also to point out that the condition (1.14) is in the same spirit that smallness condition in [13, 14] for incompressible Navier-Stokes equations.

Remark 10 *We think that we could improve the condition on initial density $h_1^0 \in B^{\frac{N}{2}-2}_{2,1}$ by working around the quasi solution only in high frequencies and by working in low frequencies directly with $\rho - 1$ in the spirit of [24]. It would be then possible to assume that h_1^0 belongs only in $B^{\frac{N}{2}-1}_{2,1}$.*

Remark 11 *It may be also probably possible to work with general regular pressure, it would make the proof technically more difficult.*

We are going to finish by presenting a result of global strong solution with large initial data when we assume the system (1.1) highly compressible.

Corollary 1 *Let $N \geq 2$. Assume that $\mu(\rho) = \mu\rho$, $\kappa(\rho) = \frac{\mu^2}{\rho}$ and $\lambda(\rho) = 0$ with $\mu > 0$ and $P(\rho) = K\rho$ with $K > 0$. Furthermore we suppose that $u_0 = -\mu\nabla[\ln \rho_1^0] + u_2^0$ and $\ln \rho_0 = \ln(\rho_1^0) + h_2^0$ such that $\rho_1^0 = 1 + h_1^0$ and it exits $c_1 > 0$ such that $\rho_1^0 \geq c_1 > 0$. In addition we suppose that:*

$$h_1^0 \in \widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}} \quad h_2^0 \in \widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}} \quad \text{and} \quad u_2^0 \in B_{2,1}^{\frac{N}{2}-1}.$$

Furthermore it exists $\varepsilon_0 > 0$ (depending on h_1^0 and the viscosity coefficient μ) such that for any $K \leq \varepsilon_0$ it exits $\varepsilon_1 > 0$ such that if

$$\|h_2^0\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_2^0\|_{B_{2,1}^{\frac{N}{2}-1}} \leq \varepsilon_1. \tag{1.17}$$

then it exists a global unique solution (ρ, u) of the system (1.1) with: $u = -\mu\nabla \ln \rho_1 + u_2$ and $\ln \rho = \ln \rho_1 + h_2$ with $\rho_1 = 1 + h_1$ verifying the following system:

$$\begin{cases} \partial_t \rho_1 - \mu\Delta\rho_1 = 0, \\ \rho_1(0,\cdot) = \rho_1^0 = 1 + h_1^0. \end{cases}$$

Furthermore we have:

$$h_2 \in \widetilde{L}^\infty(\widetilde{B}_{2,1}^{\frac{N}{2}-1,N}) \cap \widetilde{L}^1(\widetilde{B}_{2,1}^{\frac{N}{2}+1,\frac{N}{2}+2}) \quad \text{and} \quad u_2 \in \widetilde{L}^\infty(B_{2,1}^{\frac{N}{2}-1}) \cap \widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1}). \tag{1.18}$$

Remark 12 *The main interest of this theorem is to prove the existence of global strong solution for any large initial data provided that K is sufficiently small with $P(\rho) = K\rho$. In other terms we get global existence (and uniqueness) for highly compressible fluids in any dimension $N \geq 2$. Up my knowledge it is the first time that we have a result of global strong solution with large initial data in dimension 3 (under a condition of course of high compressibility, which means that K must be small in function of the initial data and in function of μ). Roughly speaking K tends to be very small when $\|h_1^0\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}}}$ is very large.*

This article is structured in the following way, first of all we recall in the section 2 some definitions an theorems related to the Littlewood-Paley theory. Next we will concentrate in the section 3 on the proof of theorem 1.1 and 1.2. In section 4, we shall prove the theorem 1.3 by introducing the notion of quasi solution which will play a crucial role. In the section 4 we show the corollary 1. We postpone in appendix (see section 6) some technical computation on the capillarity tensor and some extensions on the previous results.

2 Littlewood-Paley theory and Besov spaces

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A \lesssim B$ means that $A \leq CB$. For all Banach space X, we denote by $C([0,T],X)$ the set of continuous functions on $[0,T]$ with values in X. For $p \in [1,+\infty]$, the notation $L^p(0,T,X)$ or $L^p_T(X)$ stands for the set of measurable functions on $(0,T)$ with values in X such that $t \to \|f(t)\|_X$ belongs to $L^p(0,T)$. Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. We can use for instance any $\varphi \in C^\infty(\mathbb{R}^N)$ and $\chi \in C^\infty(\mathbb{R}^N)$, supported respectively in $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and $B(0, \frac{4}{3})$ such that:

$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1 \ \text{ if } \ \xi \neq 0,$$

and:

$$\chi(\xi) + \sum_{l \in \mathbb{N}} \varphi(2^{-l}\xi) = 1 \ \text{ if } \ \forall \xi.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\Delta_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x-y) dy \ \text{ and } \ S_l u = \sum_{k \leq l-1} \Delta_k u \,.$$

Formally, one can write that:

$$u = \sum_{k \in \mathbb{Z}} \Delta_k u \,.$$

This decomposition is called homogeneous Littlewood-Paley decomposition. Let us observe that the above formal equality does not hold in $\mathcal{S}'(\mathbb{R}^N)$ for two reasons:

1. The right hand-side does not necessarily converge in $\mathcal{S}'(\mathbb{R}^N)$.
2. Even if it does, the equality is not always true in $\mathcal{S}'(\mathbb{R}^N)$ (consider the case of the polynomials).

For the non homogeneous decomposition, we define the dyadic blocks as follows:

$$\begin{aligned}
&\dot{\Delta}_l u = 0 \ \text{ for } \ l \leq -2, \\
&\dot{\Delta}_{-1} u = \chi(D)u, \\
&\dot{\Delta}_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x-y) dy, \ \text{ for } \ l \geq 0.
\end{aligned}$$

Formally, one can write that:

$$u = \sum_{k \in \mathbb{Z}} \Delta_k u \,.$$

This decomposition is called non homogeneous Littlewood-Paley decomposition.

2.1 Homogeneous and non homogeneous Besov spaces and first properties

Definition 2.3 *We denote by $\mathcal{S}_h^{'}$ the space of temperate distribution u such that:*

$$\lim S_j u_{j\to+\infty} = 0 \ \ in \ \ \mathcal{S}^{'}.$$

Definition 2.4 *For $s \in \mathbb{R},\ p \in [1,+\infty],\ q \in [1,+\infty],$ and $u \in \mathcal{S}^{'}(\mathbb{R}^N)$ we set:*

$$\|u\|_{B^s_{p,q}} = (\sum_{l\in\mathbb{Z}} (2^{ls}\|\Delta_l u\|_{L^p})^q)^{\frac{1}{q}}.$$

The homogeneous Besov space $B^s_{p,q}$ is the set of distribution u in $\mathcal{S}_h^{'}$ such that $\|u\|_{B^s_{p,q}} < +\infty$.

Remark 13 *The above definition is a natural generalization of the homogeneous Sobolev and Hölder spaces: one can show that $B^s_{\infty,\infty}$ is the homogeneous Hölder space C^s and that $B^s_{2,2}$ is the homogeneous space H^s.*

Proposition 2.1 *The following properties holds:*

1. *there exists a constant universal C such that:*
   $C^{-1}\|u\|_{B^s_{p,r}} \leq \|\nabla u\|_{B^{s-1}_{p,r}} \leq C\|u\|_{B^s_{p,r}}$.

2. *If $p_1 < p_2$ and $r_1 \leq r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(1/p_1-1/p_2)}_{p_2,r_2}$.*

3. *$B^{s^{'}}_{p,r_1} \hookrightarrow B^s_{p,r}$ if $s^{'} > s$ or if $s = s^{'}$ and $r_1 \leq r$.*

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [5, 3]).

Proposition 2.2 *We have the following laws of product:*

- *For all $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ we have:*

$$\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}})\,. \tag{2.19}$$

- *Let $(p,p_1,p_2,r,\lambda_1,\lambda_2) \in [1,+\infty]^2$ such that:$\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \leq \lambda_2$, $p_2 \leq \lambda_1$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1}$ and $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2}$. We have then the following inequalities:*
  if $s_1 + s_2 + N\inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$, $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$ then:

$$\|uv\|_{B^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}, \tag{2.20}$$

  when $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ (resp $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$) we replace $\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}}$) by $\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,r}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}\cap L^\infty}$), if $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ we take

$r=1$.
If $s_1+s_2=0$, $s_1\in(\frac{N}{\lambda_1}-\frac{N}{p_2},\frac{N}{p_1}-\frac{N}{\lambda_2}]$ *and* $\frac{1}{p_1}+\frac{1}{p_2}\leq 1$ *then:*

$$\|uv\|_{B^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,\infty}}\lesssim\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,\infty}}. \tag{2.21}$$

If $|s|<\frac{N}{p}$ *for* $p\geq 2$ *and* $-\frac{N}{p'}<s<\frac{N}{p}$ *else, we have:*

$$\|uv\|_{B^s_{p,r}}\leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}. \tag{2.22}$$

Remark 14 *In the sequel* p *will be either* p_1 *or* p_2 *and in this case* $\frac{1}{\lambda}=\frac{1}{p_1}-\frac{1}{p_2}$ *if* $p_1\leq p_2$, *resp* $\frac{1}{\lambda}=\frac{1}{p_2}-\frac{1}{p_1}$ *if* $p_2\leq p_1$.

Corollary 2 *Let* $r\in[1,+\infty]$, $1\leq p\leq p_1\leq+\infty$ *and* s *such that:*

- $s\in(-\frac{N}{p_1},\frac{N}{p_1})$ *if* $\frac{1}{p}+\frac{1}{p_1}\leq 1$,
- $s\in(-\frac{N}{p_1}+N(\frac{1}{p}+\frac{1}{p_1}-1),\frac{N}{p_1})$ *if* $\frac{1}{p}+\frac{1}{p_1}>1$,

then we have if $u\in B^s_{p,r}$ *and* $v\in B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty$*:*

$$\|uv\|_{B^s_{p,r}}\leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}.$$

The study of non stationary PDE's requires space of type $L^\rho(0,T,X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Payley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0,T,X)$ (except if $r=p$). We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

Definition 2.5 *Let* $\rho\in[1,+\infty]$, $T\in[1,+\infty]$ *and* $s_1\in\mathbb{R}$. *We set:*

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}=\Big(\sum_{l\in\mathbb{Z}}2^{lrs_1}\|\Delta_l u(t)\|^r_{L^\rho(L^p)}\Big)^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *as the set of temperate distribution* u *over* $(0,T)\times\mathbb{R}^N$ *such that* $\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}<+\infty$.

We set $\widetilde{C}_T(\widetilde{B}^{s_1}_{p,r})=\widetilde{L}^\infty_T(\widetilde{B}^{s_1}_{p,r})\cap\mathcal{C}([0,T],B^{s_1}_{p,r})$. Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\leq\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{ if }\ r\geq\rho,\quad \|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\geq\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{ if }\ r\leq\rho. \tag{2.23}$$

Remark 15 *It is easy to generalize propositions 2.2, 2 to* $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *spaces. The indices* s_1, p, r *behave just as in the stationary case whereas the time exponent* ρ *behaves according to Hölder inequality.*

In the sequel we will need of composition lemma in $\widetilde{L}^\rho_T(B^s_{p,r})$ spaces (we refer to [3] for a proof).

Proposition 2.3 *Let $s>0$, $(p,r)\in[1,+\infty]$ and $u\in\widetilde{L}^\rho_T(B^s_{p,r})\cap L^\infty_T(L^\infty)$.*

1. *Let $F\in W^{[s]+2,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0)=0$. Then $F(u)\in\widetilde{L}^\rho_T(B^s_{p,r})$. More precisely there exists a function C depending only on s, p, r, N and F such that:*

$$\|F(u)\|_{\widetilde{L}^\rho_T(B^s_{p,r})}\leq C(\|u\|_{L^\infty_T(L^\infty)})\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

2. *Let $F\in W^{[s]+3,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0)=0$. Then $F(u)-F'(0)u\in\widetilde{L}^\rho_T(B^s_{p,r})$. More precisely there exists a function C depending only on s, p, r, N and F such that:*

$$\|F(u)-F'(0)u\|_{\widetilde{L}^\rho_T(B^s_{p,r})}\leq C(\|u\|_{L^\infty_T(L^\infty)})\|u\|^2_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

Now we give some result on the behavior of the Besov spaces via some pseudodifferential operator (see [3]).

Definition 2.6 *Let $m\in\mathbb{R}$. A smooth function function $f:\mathbb{R}^N\to\mathbb{R}$ is said to be a $\mathcal{S}^m$ multiplier if for all muti-index α, there exists a constant C_α such that:*

$$\forall\xi\in\mathbb{R}^N,\ \ |\partial^\alpha f(\xi)|\leq C_\alpha(1+|\xi|)^{m-|\alpha|}.$$

Proposition 2.4 *Let $m\in\mathbb{R}$ and f be a $\mathcal{S}^m$ multiplier. Then for all $s\in\mathbb{R}$ and $1\leq p,r\leq+\infty$ the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

Let us now give some estimates for the heat equation:

Proposition 2.5 *Let $s\in\mathbb{R}$, $(p,r)\in[1,+\infty]^2$ and $1\leq\rho_2\leq\rho_1\leq+\infty$. Assume that $u_0\in B^s_{p,r}$ and $f\in\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_{p,r})$. Let u be a solution of:*

$$\begin{cases}\partial_t u-\mu\Delta u=f\\ u_{t=0}=u_0\,.\end{cases}$$

Then there exists $C>0$ depending only on N,μ,ρ_1 and ρ_2 such that:

$$\|u\|_{\widetilde{L}^{\rho_1}_T(\widetilde{B}^{s+2/\rho_1}_{p,r})}\leq C\big(\|u_0\|_{B^s_{p,r}}+\mu^{\frac{1}{\rho_2}-1}\|f\|_{\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_{p,r})}\big)\,.$$

If in addition r is finite then u belongs to $C([0,T],B^s_{p,r})$.

Hybrid Besov spaces

The homogeneous Besov spaces fail to have nice inclusion properties: owing to the low frequencies, the embedding $B^s_{p,1} \hookrightarrow B^t_{p,1}$ does not hold for $s > t$. Still, the functions of $B^s_{p,1}$ are locally more regular than those of $B^t_{p,1}$: for any $\phi \in C^\infty_0$ and $u \in B^s_{p,1}$, the function $\phi u \in B^t_{p,1}$. This motivates the definition of Hybrid Besov spaces introduced by R. Danchin (see a definition in [3]) where the growth conditions satisfied by the dyadic blocks and the coefficient of integrability are not the same for low and high frequencies. Hybrid Besov spaces have been used by R. Danchin in order to prove global well-posedness for compressible gases in critical spaces (we refer to [3] for an elegant proof of this result). We generalize here a little bit the definition by allowing for different Lebesgue norms in low and high frequencies.

Definition 2.7 *Let $l_0 \in \mathbb{N}$, $s, t, \in \mathbb{R}$, $(r, r_1) \in [1, +\infty]^2$ and $(p, q) \in [1, +\infty]$. We set:*

$$\|u\|_{\widetilde{B}^{s,t}_{p,q,1}} = \sum_{l \leq l_0} 2^{ls} \|\Delta_l u\|_{L^p} + \sum_{l > l_0} 2^{lt} \|\Delta_l u\|_{L^q},$$

and:

$$\|u\|_{\widetilde{B}^{s,t}_{(p,r),(q,r_1)}} = \big(\sum_{l \leq l_0} (2^{ls} \|\Delta_l u\|_{L^p})^r \big)^{\frac{1}{r}} + \big(\sum_{l > l_0} (2^{lt} \|\Delta_l u\|_{L^q})^{r_1} \big)^{\frac{1}{r_1}}.$$

Remark 16 *It will be important in the sequel to chose l_0 big enough.*

Notation 1 *We will often use the following notation:*

$$u_{BF} = \sum_{l \leq l_0} \Delta_l u \quad \text{and} \quad u_{HF} = \sum_{l > l_0} \Delta_l u.$$

Remark 17 *We have the following properties:*

- *We have $\widetilde{B}^{s,s}_{p,p,1} = B^s_{p,1}$.*
- *If $s_1 \geq s_3$ and $s_2 \geq s_4$ then $\widetilde{B}^{s_3,s_2}_{p,q,1} \hookrightarrow \widetilde{B}^{s_1,s_4}_{p,q,1}$.*

Remark 18 *In the sequel we shall often use this hybrid Besov space in order to distinguish the behavior of our solution in low and high frequencies, in particular we would like to mention that we can prove results analogous to propositions 2.2 and 2.*

We shall conclude this section by some example of initial data verifying the theorem 1.1 with large energy initial data when $N = 2$.More generally we are interested in defining initial data which are small in $\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}$ but large in $B^{\frac{N}{2}}_{2,2}$. It implies in particular that your initial data $\nabla\sqrt{\rho_0}$ have large energy data when $N = 2$, furthermore $\ln \rho_0$ is large in $B^{\frac{N}{2}}_{2,2}$ which is a scaling invariant space (it improves in particular the results of [15] where the initial density is assumed small in $B^{\frac{N}{2}}_{2,1}$).
Let us start with recalling a classical example of function in $B^s_{p,\infty}$, by sake of completness we are going to recall the proof (see also [3]).

Proposition 2.6 *Let $\sigma \in]0, N[$. For any $p \in [1, +\infty]$, the function $|\cdot|^{-\sigma}$ belongs to $B^{\frac{N}{p}-\sigma}_{p,\infty}$.*

Proof: By proposition 2.1 it suffices to show that $u_\sigma = |\cdot|^{-\sigma}$ belongs to $B^{N-\sigma}_{1,\infty}$. Let us introduce a smooth compactly supported function χ which is identically equal to 1 near the unit ball and such that u is splitting as follows:

$$u_\sigma = u_0 + u_1 \quad \text{with} \quad u_0(x) = \chi(x)|x|^{-\sigma} \quad \text{and} \quad u_1(x) = (1-\chi(x))|x|^{-\sigma}.$$

Clearly u_0 is in L^1 and u_1 belongs in L^q whenever $q > \frac{N}{\sigma}$. The homogeneity of the function u_σ gives via a change of variable:

$$\begin{aligned} \Delta_j u_\sigma &= 2^{jN} u_\sigma * h(2^j \cdot) \\ &= 2^{j(N+\sigma)} u_\sigma(2^j \cdot) * h(2^j \cdot) \\ &= 2^{j\sigma} (\Delta_0 u_\sigma)(2^j \cdot). \end{aligned} \tag{2.24}$$

Therefore, $2^{j(N-\sigma)} \|\Delta_j u_\sigma\|_{L^1} = \|\Delta_0 u_\sigma\|_{L^1}$, it reduces the problem to show that $\Delta_0 u_\sigma$ is in L^1. As u_0 is in L^1, $\Delta_0 u_0$ is also in L^1 according the continuity of the operator Δ_0 on Lebesgue spaces. By Bernstein inequalities, we have:

$$\|\Delta_0 u_1\|_{L^1} \leq C_k \|D^k \Delta_0 u_1\|_{L^1} \leq C_k \|D^k u_1\|_{L^1}.$$

Leibniz's formula ensures that $D^k u_1 - (1-\chi) D^k u_\sigma$ is a smooth compactly supported function. We then complete the proof by choosing k such that $k > N - \sigma$. □

We can now deduce from the previous proposition suitable functions verifying the theorem 1.1.

Corollary 3 *Let us consider:*

$$u_{0,\varepsilon,l_0}(x) = \mathcal{S}\left(\frac{1}{|x|^{1-\varepsilon}}\right),$$

with $\mathcal{S}$ defined by $\mathcal{S}\hat{u}(\xi) = 1_{\{\xi \in \mathbb{R}^N \sigma B(0,2^{l_0})}(\xi)\hat{u}(\xi)$. Then for all $r \in [1, +\infty[$, for all $M >$ for all $\varepsilon_1 > 0$ it exits $\varepsilon > 0$, it exits $l_0 > 0$ such that:

$$\begin{aligned} &\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \leq \varepsilon_1, \\ &\|u_0\|_{B^{\frac{N}{2}-1}_{2,r}} \geq M. \end{aligned} \tag{2.25}$$

Proof: In the sequel in order to simplify the notation we shall write u_0 for u_{0,ε,l_0}. Let us denote by u_σ the function $\frac{1}{|x|^\sigma}$ with $\sigma \in]0, N[$. By (2.24) we observe that:

$$\|\Delta_l u_\sigma\|_{L^2} = 2^{l(\sigma - \frac{N}{2})} \|\Delta_0 u_\sigma\|_{L^2}. \tag{2.26}$$

It implies that for $l \geq l_0$ we have for M_0 independent on ε when $\varepsilon \leq \frac{1}{10}$:

$$2^{l(\frac{N}{2}-1+\varepsilon)} \|\Delta_l u_0\|_{L^2} = \|\Delta_0 u_{1-\varepsilon}\|_{L^2} \leq M_0.$$

Indeed using the same arguments than in the proof of the proposition 2.6 we have by Bernstein inequality for $C > 0$:

$$\begin{aligned}\|\Delta_0 u_{1-\varepsilon}\|_{L^2} &\leq C\|\Delta_0 u_{1-\varepsilon}\|_{L^1},\\ &\leq C(\|\chi u_{1-\varepsilon}\|_{L^1} + C^k\|D^k((1-\chi)u_{1-\varepsilon})\|_{L^1}),\\ &\leq M_0.\end{aligned}$$

According to (2.26) we deduce that for all $l \geq l_0$:

$$2^{l(\frac{N}{2}-1)}\|\Delta_l u_0\|_{L^2} = 2^{-l\varepsilon}\|\Delta_0 u_\sigma\|_{L^2} \leq M_0 2^{-l_0\varepsilon}.$$

It implies in particular since $\Delta_l u_0 = 0$ for $l < l_0$ that:

$$\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \leq M_0 2^{-l_0\varepsilon}. \tag{2.27}$$

Let us now estimate the norm of u_0 in $B^{\frac{N}{2}-1}_{2,r}$ with $r \in [1, +\infty[$:

$$\begin{aligned}\|u_0\|_{B^{\frac{N}{2}-1}_{2,r}} &= (\sum_{l\in\mathbb{Z}} 2^{rl(\frac{N}{2}-1)}\|\Delta_l u_0\|^r_{L^2})^{\frac{1}{r}},\\ &= (\sum_{l\geq l_0} 2^{-rl\varepsilon} M^r_\varepsilon)^{\frac{1}{r}},\\ &= M_\varepsilon 2^{-l_0\varepsilon}(\frac{1}{1-2^{-r\varepsilon}})^{\frac{1}{r}},\end{aligned} \tag{2.28}$$

It implies that for all $M > 0$, for all $r > 0$ it exits $\varepsilon > 0$ small enough such that:

$$(\frac{1}{1-2^{-r\varepsilon}})^{\frac{1}{r}} \geq M. \tag{2.29}$$

We fix now l_0 such that:

$$M_0 2^{-l_0\varepsilon} = \varepsilon_1. \tag{2.30}$$

Let us prove now that when $\varepsilon \leq \frac{1}{10}$ it exist $\alpha > 0$ such that $M_\varepsilon \geq \alpha > 0$. Assume by the absurd that this is wrong. It implies that it exists a sequel $(\varepsilon_n)_{n\in\mathbb{N}}$ which converges to 0 when n goes to infinity such that $M_{\varepsilon_n} = \|\Delta_0 u_{1-\varepsilon_n}\|_{L^2}$ goes to 0 when n goes to infinity. By Plancherel theorem and the fact that we know the Fourier transform of $|\cdot|^{-1+\varepsilon}$ (see [3] p 23) it implies that:

$$\|\mathcal{F}\Delta_0 u_{1-\varepsilon_n}\|_{L^2} = c_{N,1-\varepsilon_n}\|\varphi|\cdot|^{1-\varepsilon_n-N}\|_{L^2} \to_{n\to+\infty} 0.$$

Since we can bound by below $\|\varphi|\cdot|^{1-\varepsilon_n-N}\|_{L^2}$ independently of n, it implies that $c_{N,1-\varepsilon_n}$ goes to 0 when n goes to infinity and this is absurd.
We obtain finally from (2.27), (2.30) and (2.28), (2.29) that:

$$\begin{aligned}&\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \leq \varepsilon_1,\\ &\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \geq \frac{M\varepsilon_1\alpha}{M_0}.\end{aligned} \tag{2.31}$$

It concludes the proof of the corollary. □

3 Proof of theorems 1.1 and 1.2

In this part we are interested in proving the theorems 1.1 and 1.2 of existence of strong solutions in critical space for the scaling of the equations. We want to point out that in the theorem 1.1 the viscosity and the capillarity coefficients are chosen with a very specific structure. This fact will be crucial in the sequel of the proof in order to obtain estimates on the density without assuming any control on the vacuum or on the L^∞ norm of the density for the theorem 1.1. Indeed when $\mu(\rho) = \mu\rho$ and $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\kappa > 0$ then the system depends only on the unknown $\ln\rho$ in a linear way and provides a new entropy (see [19] for more details).
In a second time in order to prove the second part of the theorem 1.1, we are going to estimate the vacuum (which means the L^∞ norm of $\frac{1}{\rho}$) and the L^∞ norm of the density. More precisely we are going to write a solution (q, u) under the form:

$$(q, u) = (q_L, u_L) + (\bar{q}, \bar{u}),$$

with (q_L, u_L) the solution of the linearized part of the system (1.5). We shall combine maximum principle arguments on q_L in order to bound q_L in L^∞ norm and regularizing effect on the third index of Besov space for $\bar{q}$. Indeed we will prove that $\bar{q}$ is in $\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,1})$ for any $T > 0$ which is embedded in $L^\infty_T(L^\infty)$. Let us mention that in order to prove that q_L is bounded in L^∞ norm we shall prove a very accurate characterization of the Besov space in term of the semi group associated to the linearized part of the system (1.5) (see the proposition 3.11). For more details on this part which is the main difficulty of the proof we refer to the subsection 3.4.1 and 3.4.2.
As a first step of the proof of theorems 1.1 and 1.2, let us start with studying the linear part of the system (1.5) (when we are interested in proving the existence of strong solution in finite time with large initial data, in particular it implies that we do not consider the low frequencies terms coming from the pressure) which corresponds to the following system:

$$(N) \qquad \begin{cases} \partial_t q + \mathrm{div} u = F, \\ \partial_t u - a\Delta u - b\nabla \mathrm{div} u - c\nabla\Delta q = G, \end{cases}$$

3.1 Study of the linearized equation

We want to prove a priori estimates in Chemin-Lerner spaces for system (N) with the following hypotheses on a, b, c which are constant:

$$0 < a < \infty,\ 0 < a + b < \infty \ \text{ and } \ 0 < c < \infty.$$

This system has been studied by Danchin and Desjardins (see [15]) in the framework of the Besov space $B^s_{2,1}$, the following proposition uses exactly the same type of arguments used in [15] excepted that we extend the result to general Besov spaces $B^s_{2,r}$ with $r \in [1, +\infty]$. By sake of completness we are going to show the following proof.

Proposition 3.7 *Let $1 \leq r \leq +\infty$, $s \in \mathbb{R}$, and we assume that (q_0, u_0) belongs in $B^{\frac{N}{2}+s}_{2,r} \times (B^{\frac{N}{2}-1+s}_{2,r})^N$ with the source terms (F, G) in $\widetilde{L}^1_T(B^{\frac{N}{2}+s}_{2,r}) \times (\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r}))^N$.*

Let $(q,u)\in(\widetilde{L}^1_T(B^{\frac{N}{2}+s+2}_{2,r})\cap\widetilde{L}^\infty_T(B^{\frac{N}{2}+s}_{2,r}))\times\big(\widetilde{L}^1_T(B^{\frac{N}{2}+s+1}_{2,r})\cap\widetilde{L}^\infty_T(B^{\frac{N}{2}+s-1}_{2,r})\big)^N$ *be a solution of the system* (N)*, then there exists a universal constant* C *such that for any* $T>0$*:*

$$\|(\nabla q,u)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1+s}_{2,r})\cap\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s}_{2,r})}\leq C(\|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1+s}_{2,2}}+\|(\nabla F,G)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r})}).$$

Proof: As we mentioned previously, we are going to follow the arguments developed in [15]. Let us apply to the system (N) the operator Δ_l which gives:

$$\partial_t q_l+\mathrm{div}u_l=F_l \tag{3.32}$$

$$\partial_t u_l-\mathrm{div}(a\nabla u_l)-\nabla(b\,\mathrm{div}u_l)-c\nabla\Delta q_l=G_l \tag{3.33}$$

Performing integrations by parts and using (3.32) we have:

$$\begin{aligned}-c\int_{\mathbb{R}^N}u_l\cdot\nabla\Delta q_l dx&=c\int_{\mathbb{R}^N}\mathrm{div}u_l\,\Delta q_l dx,\\&=-c\int_{\mathbb{R}^N}\partial_t q_l\,\Delta q_l dx+c\int_{\mathbb{R}^N}F_l\,\Delta q_l dx,\\c&=\int_{\mathbb{R}^N}\partial_t\nabla q_l\cdot\nabla q_l dx-c\int_{\mathbb{R}^N}\nabla F_l\cdot\nabla q_l dx,\\&=\frac{c}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|\nabla q_l|^2dx-c\int_{\mathbb{R}^N}\nabla q_l.\nabla F_l\,dx.\end{aligned}$$

Next, we take the inner product of (3.33) with u_l and using the previous equality, it yields:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|u_l\|^2_{L^2}+c\int_{\mathbb{R}^N}|\nabla q_l|^2dx\big)+\int_{\mathbb{R}^N}(a|\nabla u_l|^2+b|\mathrm{div}u_l|^2)dx&\\=\int_{\mathbb{R}^N}G_l.u_l\,dx+c\int_{\mathbb{R}^N}\nabla q_l.\nabla F_l\,dx\,.&\end{aligned} \tag{3.34}$$

In order to recover some terms in Δq_l we take the inner product of the gradient of (3.32) with u_l, the inner product scalar of (3.33) with ∇q_l and we sum, we obtain then:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N}\nabla q_l.u_l dx+c\int_{\mathbb{R}^N}(\Delta q_l)^2dx=\int_{\mathbb{R}^N}(G_l.\nabla q_l+|\mathrm{div}u_l|^2+u_l.\nabla F_l&\\-a\nabla u_l:\nabla^2q_l-b\Delta q_l\mathrm{div}u_l)dx.&\end{aligned} \tag{3.35}$$

Let $\alpha>0$ small enough. We define k_l by:

$$k_l^2=\|u_l\|^2_{L^2}+c\|\nabla q_l\|^2_{L^2}+2\alpha\int_{\mathbb{R}^N}\nabla q_l.u_l dx\,. \tag{3.36}$$

By using (3.34), (3.35) and the Young inequalities, we have by summing and the fact that α is chosen small enough:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}k_l^2+\frac{1}{2}\int_{\mathbb{R}^N}(a|\nabla u_l|^2+b|\mathrm{div}u_l|^2+2\alpha c|\Delta q_l|^2)dx\lesssim\|G_l\|_{L^2}(2\alpha\|\nabla q_l\|_{L^2}+\|u_l\|_{L^2})&\\+\|\nabla F_l\|_{L^2}(2\alpha\|u_l\|_{L^2}+\|\nabla q_l\|_{L^2}).&\end{aligned} \tag{3.37}$$

For small enough α and applying the Young inequality, we have according (3.36):

$$\frac{1}{2}k_l^2 \leq \|u_l\|^2 + c\|\nabla q_l\|_{L^2}^2 \leq \frac{3}{2}k_l^2 \ . \tag{3.38}$$

Hence according to (3.37) and (3.38) there exists $K > 0$ small enough, $C > 0$ such that:

$$\frac{1}{2}\frac{d}{dt}k_l^2 + K2^{2l}k_l^2 \leq C\, k_l\, (\|G_l\|_{L^2} + \|\nabla F_l\|_{L^2}).$$

By integrating with respect to the time, we obtain:

$$k_l(t) \leq e^{-K2^{2l}t}k_l(0) + C\int_0^t e^{-K2^{2l}(t-\tau)}(\|\nabla F_l(\tau)\|_{L^2} + \|G_l(\tau)\|_{L^2})d\tau \ .$$

Using convolution inequalities we get for $1 \leq \rho_1 \leq \rho \leq +\infty$:

$$\|k_l\|_{L^\rho([0,T])} \lesssim \big(2^{-\frac{2l}{\rho}}k_l(0) + 2^{-2l(1+\frac{1}{\rho}-\frac{1}{\rho_1})}\|(\nabla F_l, G_l)\|_{L_T^{\rho_1}(L^2)}\big). \tag{3.39}$$

Moreover since we have:

$$C^{-1}\, k_l \leq \|\nabla q_l\|_{L^2} + \|u_l\|_{L^2} \leq C\, k_l,$$

multiplying by $2^{(\frac{N}{2}-1+s+\frac{2}{\rho})l}$, taking the l^r norm and using (3.38), we end up with:

$$\|(\nabla q, u)\|_{L_T^\rho(B_{2,r}^{\frac{N}{2}-1+s+\frac{2}{\rho}})} \leq \|(\nabla F, G)\|_{\widetilde{L}_T^{\rho_1}(B_{2,r}^{\frac{N}{2}-3+s+\frac{2}{\rho_1}})} + \|(\nabla q_0, u_0)\|_{B_{2,r}^{\frac{N}{2}-1+s}}.$$

It conclude the proof of the proposition. □

Let us extend the result of the proposition 3.7 to the case where we include the pressure term inside of the linearized system. This is necessary when we are interested in dealing with the existence of global strong solution with small initial data. Indeed in this case it is very important to take into account the behavior in low frequencies of the density and in particular the pressure term which provides the behavior of the density in low frequencies. More precisely we will consider the following linear system:

$$(N1) \qquad \begin{cases} \partial_t q + \mathrm{div} u = F, \\ \partial_t u - a\Delta u - b\nabla \mathrm{div} u - c\nabla\Delta q + d\nabla q = G, \end{cases}$$

We now want to prove a priori estimates in Chemin-Lerner spaces for system $(N1)$ with the following hypotheses on a, b, c, d which are constant:

$$0 < a < \infty,\ 0 < a + b < \infty,\ 0 < c < \infty \ \text{ and } \ 0 < d < \infty.$$

This system has also been studied by Danchin and Desjardins in [15] in the framework of Besov spaces of the type $B_{2,1}^s$ with $s \in \mathbb{R}$, the following proposition extends the results of [15] to the case of general Besov spaces $B_{2,r}^s$ with $r \in [1, +\infty]$ by using similar arguments.

Proposition 3.8 *Let $1 \leq r \leq +\infty$, $s \in \mathbb{R}$, and we assume that (q_0, u_0) belongs in $\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r} \times (B^{\frac{N}{2}-1+s}_{2,r})^N$. Furthermore we suppose that the source terms (F, G) are in $\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r}) \times (\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r}))^N$.*
Let $(q, u) \in (\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+s+1,\frac{N}{2}+s+2}_{2,r}) \cap \widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r})) \times \big(\widetilde{L}^1_T(B^{\frac{N}{2}+s+1}_{2,r}) \cap \widetilde{L}^\infty_T(B^{\frac{N}{2}+s-1}_{2,r})\big)^N$ be a solution of the system $(N1)$, then there exists a universal constant C such that for any $T > 0$:

$$\|q\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1+s,\frac{N}{2}+2+s}_{2,r})} + \|q\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r})} + \|u\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1+s}_{2,r})} + \|u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s}_{2,r})}$$
$$\leq C\big(\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r}} + \|u_0\|_{B^{\frac{N}{2}+s}_{2,r}} + \|F\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r})} + \|G\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r})}\big).$$

Proof: It suffices to follow exactly the same lines as the proof of proposition 3.7 except that we have to consider the following k_l:

$$k_l^2 = \|u_l\|^2_{L^2} + c\|\nabla q_l\|^2_{L^2} + d\|q_l\|^2_{L^2} + 2\alpha \int_{\mathbb{R}^N} \nabla q_l \cdot u_l dx.$$

Now choosing α suitably small , we deduce that:

$$\frac{1}{2}k_l^2 \leq \|u_l\|^2_{L^2} + c\|\nabla q_l\|^2_{L^2} + d\|q_l\|^2_{L^2} \leq \frac{3}{2}k_l^2. \tag{3.40}$$

By combining energy estimates in frequencies space, we show as in [15] that it exists $c, C > 0$ such that:

$$\frac{1}{2}\frac{d}{dt}k_l^2 + c2^{2l}k_l^2 \leq Ck_l(\|G_l\|_{L^2} + \|(\nabla F_l, F_l)\|_{L^2}).$$

As in the proof of proposition 3.7 routine computations yield proposition 3.8. □

We are now interested in studying the following system with $\mu > 0$ and $\kappa > 0$:

$$\begin{cases} \partial_t h_2 + \mathrm{div} u_2 - \mu\nabla \ln\rho_1 \cdot \nabla h_2 + u_2 \cdot \nabla \ln\rho_1 = F, \\ \partial_t u_2 - \mu\Delta u_2 - \mu\nabla \mathrm{div} u_2 - \kappa\nabla\Delta h_2 + K\nabla h_2 - 2\mu\nabla\ln\rho_1 \cdot Du_2 - 2\mu\nabla h_2 \cdot Du_1 \\ \qquad\qquad + u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 - \mu^2\nabla(\nabla\ln\rho_1 \cdot \nabla h_2) = G, \\ (h_2(0,\cdot), u_2(0,\cdot)) = (h_0^2, u_0^2). \end{cases} \tag{3.41}$$

Here (h_2, u_2) are the unknowns, ρ_1 and u_1 corresponds to some functions defined in suitable Chemin Lerner Besov space $\widetilde{L}^\rho(B^s_{p,r})$ that we will precise below in the proposition 3.9 and F, G are source term (we will also precise their regularity). In order to prove the theorem 1.3 we will need precise estimates on the solution (h_2, u_2) of (3.44) in terms of Chemin Lerner Besov spaces, to do this we are going to prove the following proposition.

Proposition 3.9 *Let $(h_0^2, u_0^2) \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1} \times B^{\frac{N}{2}-1}_{2,1}$ and we assume that $\ln\rho_1$ belongs in $\widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})$ and u_1 in $\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,1}) \cap \widetilde{L}^1(\mathbb{R}^+, B^{\frac{N}{2}+1}_{2,1})$.*

Furthermore (F,G) *are in* $\widetilde{L}^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})\times\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1})$. *Let* (h_2,u_2) *the solution of the linear system (3.44) then it exists* $C>0$ *such that* (h_2,u_2) *verify:*

$$\begin{aligned}
&\|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}+\|h_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,1})}+\|u_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,1})}\\
&\leq C\big(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}+\|u_0^2\|_{B^{\frac{N}{2}}_{2,1}}+\|F\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|G\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\big)\\
&\times\exp\bigg(C\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\bigg).
\end{aligned}\tag{3.42}$$

Remark 19 *Let us mention that in the proposition 3.9 the main estimates (3.42) involves a control of* $\nabla\ln\rho_1$ *in* $L^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})\cap L^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})$ *and of* u_1 *in* $L^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})\cap L^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})$. *By Minkowski inequality (see (2.23)) and since* l^p *is embedded in* l^∞ *for any* $p\geq 1$, *we know that for example:*

$$\|\nabla\ln\rho_1\|_{L^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})\cap L^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\leq\|\nabla\ln\rho_1\|_{\widetilde{L}^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\frac{4}{3}})\cap\widetilde{L}^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\frac{4}{3}})}\tag{3.43}$$

Of course when we assume that $\ln\rho_1$ *belongs in* $\widetilde{L}^\infty(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})\cap\widetilde{L}^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})$, *it implies by interpolation that* $\nabla\ln\rho_1$ *is in* $\widetilde{L}^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\frac{4}{3}})\cap\widetilde{L}^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\frac{4}{3}})$. *In particular it proves that with such assumptions we control the right hand side of the inequality (3.42).*
Let us mention that since in the theorem 1.3 we will have $u_1=-\mu\nabla\ln\rho_1$ *with* ρ_1 *verifying a heat equation, in particular we could improve the condition on the initial data* $\rho_1^0=h_1^0+1$ *of theorem 1.3 by assuming only* $h_0^1\in B^{\frac{N}{2}-2}_{2,1}\cap B^{\frac{N}{2}}_{2,\frac{4}{3}}\cap L^\infty$ *and* $\rho_0^1\geq c>0$. *Indeed with such condition* $\nabla\ln\rho_1$ *would verify exactly the quantity on the right hand side of (3.43) which is sufficient to control the right hand side on (3.42).*
In fact a more accurate proof of the proposition 3.9 by using critical interpolation estimate would show that $h_0^1\in B^{\frac{N}{2}-2}_{2,1}\cap B^{\frac{N}{2}}_{2,2-\varepsilon}\cap L^\infty$ *(with* $\varepsilon>0$*) and* $\rho_0^1\geq c>0$ *is sufficient for the theorem 1.3.*

Proof: We observe that (h_2,u_2) are solution of the following system:

$$\begin{cases}\partial_t h_2+\mathrm{div}u_2=F(h_2,u_2),\\ \partial_t u_2-\mu\Delta u_2-\mu\nabla\mathrm{div}u_2-\kappa\nabla\Delta h_2+K\nabla h_2=G(h_2,u_2),\\ (h_2(0,\cdot),u_2(0,\cdot))=(h_0^2,u_0^2).\end{cases}\tag{3.44}$$

with:

$$\begin{aligned}
&F(h_2,u_2)=F+\mu\nabla\ln\rho_1\cdot\nabla h_2-u_2\cdot\nabla\ln\rho_1,\\
&G(h_2,u_2)=G+2\mu\nabla\ln\rho_1\cdot Du_2+2\mu\nabla h_2\cdot Du_1-u_1\cdot\nabla u_2-u_2\cdot\nabla u_1+\mu^2\nabla(\nabla\ln\rho_1\cdot\nabla h_2).
\end{aligned}$$

By applying the proposition 3.8, we have:

$$\begin{aligned}
&\|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})} + \|h_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} + \|u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,1})} + \|u_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,1})} \\
&\leq C\big(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}} + \|u_0^2\|_{B^{\frac{N}{2}}_{2,1}} + \|F(h_2,u_2)\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} + \|G(h_2,u_2)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\big).
\end{aligned} \tag{3.45}$$

We have only to deal with the right hand side of (3.45), we have in particular:

$$\begin{aligned}
&\|\nabla \ln\rho_1\cdot\nabla h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} = \int_{\mathbb{R}^+}\|\nabla \ln\rho_1\cdot\nabla h_2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}(s)ds, \\
&\lesssim \int_{\mathbb{R}^+}\big(\|\nabla h_2\|_{\widetilde{B}^{\frac{N}{2}-\frac12,\frac{N}{2}+\frac12}_{2,1}}\|\nabla\ln\rho_1\|_{B^{\frac{N}{2}-\frac12}_{2,\infty}} + \|\nabla h_2\|_{\widetilde{B}^{\frac{N}{2}-\frac32,\frac{N}{2}-\frac12}_{2,1}}\|\nabla\ln\rho_1\|_{B^{\frac{N}{2}+\frac12}_{2,\infty}}\big)(s)ds.
\end{aligned} \tag{3.46}$$

By interpolation see the proposition 2.1 we have:

$$\begin{aligned}
&\|\nabla h_2\|_{\widetilde{B}^{\frac{N}{2}-\frac32,\frac{N}{2}-\frac12}_{2,1}} \lesssim \|\nabla h_2\|^{\frac34}_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}-1}_{2,1}}\|\nabla h_2\|^{\frac14}_{\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}_{2,1}}, \\
&\|\nabla h_2\|_{B^{\frac{N}{2}-\frac12,\frac{N}{2}+\frac12}_{2,1})} \leq \|\nabla h_2\|^{\frac14}_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}-1}_{2,1}}\|\nabla h_2\|^{\frac34}_{\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}_{2,1}}.
\end{aligned} \tag{3.47}$$

By combining (3.47), (3.48) and Young inequality we have for any $\varepsilon>0$:

$$\begin{aligned}
&\|\nabla \ln\rho_1\cdot\nabla h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} = \int_{\mathbb{R}^+}\|\nabla \ln\rho_1\cdot\nabla h_2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}(s)ds, \\
&\lesssim \int_{\mathbb{R}^+}\big(\|\nabla h_2\|^{\frac14}_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}-1}_{2,1}}\|\nabla h_2\|^{\frac34}_{\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}_{2,1}}\|\nabla\ln\rho_1\|_{B^{\frac{N}{2}-\frac12}_{2,\infty}} \\
&\qquad\qquad + \|\nabla h_2\|^{\frac34}_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}-1}_{2,1}}\|\nabla h_2\|^{\frac14}_{\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}_{2,1}}\|\nabla\ln\rho_1\|_{B^{\frac{N}{2}+\frac12}_{2,\infty}}\big)(s)ds. \\
&\lesssim \int_{\mathbb{R}^+}\big(2\varepsilon\|\nabla h_2\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}_{2,1}} + C_\varepsilon\|\nabla\ln\rho_1\|^4_{B^{\frac{N}{2}-\frac12}_{2,\infty}}\|\nabla h_2\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}-1}_{2,1}} \\
&\qquad\qquad + C_\varepsilon\|\nabla h_2\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}-1}_{2,\infty}}\|\nabla\ln\rho_1\|^{\frac43}_{B^{\frac{N}{2}+\frac12}_{2,\infty}}\big)(s)ds.
\end{aligned} \tag{3.48}$$

In a similar way by interpolation we have:

$$\begin{aligned}
&\|u_2\cdot\nabla\ln\rho_1\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} = \int_{\mathbb{R}^+}\|u_2\cdot\nabla\ln\rho_1\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}(s)ds, \\
&\lesssim \int_{\mathbb{R}^+}\big(\|\nabla\ln\rho_1\|_{\widetilde{B}^{\frac{N}{2}-\frac12,\frac{N}{2}+\frac12}_{2,\infty}}\|u_2\|_{B^{\frac{N}{2}-\frac12}_{2,1}} + \|\nabla\ln\rho_1\|_{\widetilde{B}^{\frac{N}{2}-\frac32,\frac{N}{2}-\frac12}_{2,\infty}}\|u_2\|_{B^{\frac{N}{2}+\frac12}_{2,1}}\big)(s)ds. \\
&\lesssim \int_{\mathbb{R}^+}\big(\|\nabla\ln\rho_1\|_{\widetilde{B}^{\frac{N}{2}-\frac12,\frac{N}{2}+\frac12}_{2,\infty}}\|u_2\|^{\frac34}_{B^{\frac{N}{2}-1}_{2,1}}\|u_2\|^{\frac14}_{B^{\frac{N}{2}+1}_{2,1}} \\
&\qquad\qquad + \|\nabla\ln\rho_1\|_{\widetilde{B}^{\frac{N}{2}-\frac32,\frac{N}{2}-\frac12}_{2,\infty}}\|u_2\|^{\frac14}_{B^{\frac{N}{2}-1}_{2,1}}\|u_2\|^{\frac34}_{B^{\frac{N}{2}+1}_{2,1}}\big)(s)ds,
\end{aligned} \tag{3.49}$$

Applying Young inequality it yields:

$$\|u_2\cdot\nabla\ln\rho_1\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}\lesssim\int_{\mathbb{R}^+}\big(2\varepsilon\|u_2\|_{B^{\frac{N}{2}+1}_{2,1}}+C_\varepsilon\|\nabla\ln\rho_1\|^4_{\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty}}\|u_2\|_{B^{\frac{N}{2}-1}_{2,1}}$$
$$+C_\varepsilon\|u_2\|_{B^{\frac{N}{2}-1}_{2,\infty}}\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds. \tag{3.50}$$

Let us proceed in a similar way for $\|G(h_2,u_2)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}$, we are going only to treat only two terms (the other one will be left to the reader). In a similar way, we have:

$$\begin{aligned}
&\|\nabla\ln\rho_1\cdot Du_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}=\int_{\mathbb{R}^+}\|\nabla\ln\rho_1\cdot Du_2\|_{B^{\frac{N}{2}-1}_{2,1}}(s)ds,\\
&\lesssim\int_{\mathbb{R}^+}\big(\|\nabla\ln\rho_1\|_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}\|Du_2\|_{B^{\frac{N}{2}-\frac{1}{2}}_{2,1}}+\|\nabla\ln\rho_1\|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\|Du_2\|_{B^{\frac{N}{2}-\frac{3}{2}}_{2,1}}\big)(s)ds,\\
&\lesssim\int_{\mathbb{R}^+}\big(2\varepsilon\|u_2\|_{B^{\frac{N}{2}+1}_{2,1}}+C_\varepsilon\|\nabla\ln\rho_1\|^4_{\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty}}\|u_2\|_{B^{\frac{N}{2}-1}_{2,1}}\\
&\qquad+C_\varepsilon\|u_2\|_{B^{\frac{N}{2}-1}_{2,\infty}}\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds.
\end{aligned}\tag{3.51}$$

and:

$$\begin{aligned}
&\|u_1\cdot\nabla u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}=\int_{\mathbb{R}^+}\|u_1\cdot\nabla u_2\|_{B^{\frac{N}{2}-1}_{2,1}}(s)ds,\\
&\lesssim\int_{\mathbb{R}^+}\big(2\varepsilon\|u_2\|_{B^{\frac{N}{2}+1}_{2,1}}+C_\varepsilon\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}\|u_2\|_{B^{\frac{N}{2}-1}_{2,1}}+C_\varepsilon\|u_2\|_{B^{\frac{N}{2}-1}_{2,1}}\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds.
\end{aligned}\tag{3.52}$$

Finally combining (3.45), (3.48), (3.50), (3.51) and (3.52) we obtain that for $C>0$ and $\varepsilon>0$ small enough such that $C\varepsilon\leq\frac{1}{2}$ that:

$$\begin{aligned}
&\|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}+\|h_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,1})}+\|u_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,1})}\\
&\leq C\bigg(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}+\|u_0^2\|_{B^{\frac{N}{2}}_{2,1}}+\|F\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|G\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\\
&+\int_{\mathbb{R}^+}\bigg(\varepsilon\big(\|u_2\|_{B^{\frac{N}{2}+1}_{2,1}}+\|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}\big)+C_\varepsilon\big(\|u_2\|_{B^{\frac{N}{2}-1}_{2,1}}+\|h_2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}\big)\\
&\qquad\times\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}\big)\bigg)(s)ds\bigg).
\end{aligned}\tag{3.53}$$

By a bootstrap argument and the Gronwall lemma, we get for $C>0$ large enough:

$$\begin{aligned}
&\|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}+\|h_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,1})}+\|u_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,1})}\\
&\leq C\big(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}+\|u_0^2\|_{B^{\frac{N}{2}}_{2,1}}+\|F\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|G\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\big)\\
&\times\exp\bigg(C\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}\big)(s)ds\bigg).
\end{aligned}\tag{3.54}$$

It concludes the proof of the proposition 3.9. □

3.2 Existence of local solutions for system (1.5)

We now are going to prove the existence of strong solutions in finite time with large initial data verifying the hypothesis of theorem 1.1 for the system (1.5). More precisely we assume that (q_0, u_0) belong in $B^{\frac{N}{2}}_{2,\infty} \times B^{\frac{N}{2}-1}_{2,\infty}$.

Existence of solutions

The existence part of the theorem is proved by an iterative method. We define a sequence (q^n, u^n) as follows:

$$q^n = q_L + \bar{q}^n,\ u^n = u_L + \bar{u}^n,$$

where (q_L, u_L) stands for the solution of:

$$\begin{cases} \partial_t q_L + \mathrm{div} u_L = 0, \\ \partial_t u_L - \mathcal{A} u_L - \kappa \nabla(\Delta q_L) = 0, \end{cases} \tag{3.55}$$

supplemented with initial data:

$$q_L(0) = q_0\ ,\ u_L(0) = u_0.$$

Here $\mathcal{A}$ define the Lamé operator $\mathcal{A}u = \mu\Delta u + (\lambda+\mu)\nabla \mathrm{div} u$. Using the proposition 3.7, we obtain the following estimates on (q_L, u_L) for all $T > 0$:

$$q_L \in \widetilde{C}([0,T], B^{\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,\infty}) \ \text{ and } \ u_L \in \widetilde{C}([0,T], B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty}).$$

Setting $(\bar{q}^0, \bar{u}^0) = (0,0)$ we now define $(\bar{q}_n, \bar{u}_n)$ as the solution of the following system:

$$(N_1) \qquad \begin{cases} \partial_t \bar{q}^n + \mathrm{div}(\bar{u}^n) = F_{n-1}, \\ \partial_t \bar{u}_n - \mathcal{A}\bar{u}_n - \kappa\nabla(\Delta \bar{q}^n) = G_{n-1}, \\ (\bar{q}_n, \bar{u}_n)_{t=0} = (0,0), \end{cases}$$

where:

$$F_{n-1} = -\, u^{n-1} \cdot \nabla q^{n-1},$$
$$G_{n-1} = -\,(u^{n-1})^*.\nabla u^{n-1} + 2\mu \nabla q^{n-1} \cdot Du^{n-1} + \lambda \nabla q^{n-1}\, \mathrm{div} u^{n-1} + \frac{\kappa}{2}\nabla(|\nabla q^{n-1}|^2) - K\nabla q^{n-1}.$$

1) First Step , Uniform Bound

Let ε be a small parameter and choose T small enough such that according to the proposition 3.7 we have:

$$(\mathcal{H}_\varepsilon) \qquad \begin{aligned} &\|u_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} + \|q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,\infty})} \le \varepsilon, \\ &\|u_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})} + \|q_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,\infty})} \le CA_0, \end{aligned}$$

with $A_0 = \|q_0\|_{B^{\frac{N}{2}}_{2,\infty}} + \|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}}$. We are going to show by induction that:

$$(\mathcal{P}_n), \qquad \|(\bar{q}^n, \bar{u}^n)\|_{F_T} \leq \sqrt{\varepsilon}.$$

for ε small enough with:

$$F_T = \big(\widetilde{C}([0,T], B^{\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,\infty})\big) \times \big(\widetilde{C}([0,T], B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})\big)^N.$$

As $(\bar{q}^0, \bar{u}^0) = (0,0)$ the result is true for $n = 0$. We now suppose $(\mathcal{P}_{n-1})$ (with $n \geq 1$) true and we are going to show $(\mathcal{P}_n)$. Applying proposition 3.7 we have:

$$\|(\bar{q}^n, \bar{u}^n)\|_{F_T} \leq C\|(\nabla F_{n-1}, G_{n-1})\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})}. \tag{3.56}$$

Bounding the right-hand side of (3.56) may be done by applying proposition 2.2, lemma 2.3 and corollary 2. We begin with treating the case of $\|F_{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})}$, let us recall that:

$$F_{n-1} = -u_L \cdot \nabla q_L - \bar{u}^{n-1} \cdot \nabla q_L - u^L \cdot \nabla \bar{q}^{n-1} - \bar{u}^{n-1} \cdot \nabla \bar{q}^{n-1}. \tag{3.57}$$

We are going to bound each term of we have then:

$$\begin{aligned} \|u_L \cdot \nabla q_L\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} \leq \|u_L\|_{\widetilde{L}^1_T(B^{N/2+1}_{2,\infty})}\|q_L\|_{\widetilde{L}^\infty_T(B^{N/2}_{2,\infty})} \\ + \|q_L\|_{\widetilde{L}^1_T(B^{N/2+2}_{2,\infty})}\|u_L\|_{\widetilde{L}^\infty_T(B^{N/2-1}_{2,\infty})}. \end{aligned} \tag{3.58}$$

Similarly we obtain:

$$\begin{aligned} \|u_L \cdot \nabla \bar{q}^{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} \leq \|u_L\|_{\widetilde{L}^1_T(B^{N/2+1}_{2,\infty})}\|\bar{q}^{n-1}\|_{\widetilde{L}^\infty_T(B^{N/2}_{2,\infty})} \\ + \|\nabla \bar{q}^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u_L\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}, \end{aligned} \tag{3.59}$$

$$\begin{aligned} \|\bar{u}^{n-1} \cdot \nabla q_L\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} \leq \|\bar{u}^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q_L\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\ + \|q_L\|_{\widetilde{L}^1_T(B^{N/2+2}_{2,\infty})}\|\bar{u}^{n-1}\|_{\widetilde{L}^\infty_T(B^{N/2-1}_{2,\infty})}, \end{aligned} \tag{3.60}$$

and:

$$\begin{aligned} \|\bar{u}^{n-1} \cdot \nabla \bar{q}^{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} \leq \|\bar{u}^{n-1}\|_{\widetilde{L}^1_T(B^{N/2+1}_{2,\infty})}\|\bar{q}^{n-1}\|_{\widetilde{L}^\infty_T(B^{N/2}_{2,\infty})} \\ + \|\bar{q}^{n-1}\|_{\widetilde{L}^1_T(B^{N/2+2}_{2,\infty})}\|\bar{u}^{n-1}\|_{\widetilde{L}^\infty_T(B^{N/2-1}_{2,\infty})}. \end{aligned} \tag{3.61}$$

By using the previous inequalities (3.58), (3.59), (3.60), (3.61), $(\mathcal{P}_{n-1})$ and by interpolation, we obtain that for $C > 0$ large enough:

$$\|F_n\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} \leq C\sqrt{\varepsilon}(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}} + \sqrt{\varepsilon}(1 + A_0) + \varepsilon). \tag{3.62}$$

Next we want to control $\|G_n\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}$. According to propositions 2.2, corollary 2 and 3.7, we have:

$$
\begin{aligned}
&\|(u^{n-1})^*.\nabla u^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} \lesssim \|u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})},\\
&\|\nabla(|\nabla q^{n-1}|^2)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} \lesssim \||\nabla q^{n-1}|^2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}}_{2,\infty})},\\
&\qquad\qquad \lesssim \|\nabla q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})},\\
&\qquad\qquad \lesssim \|q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{3}{2}}_{2,\infty})}\|q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})},\\
&\|\nabla q^{n-1}\cdot Du^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} \lesssim \|\nabla q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\\
&\qquad\qquad + \|u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})},\\
&\|\nabla q^{n-1}\mathrm{div}u^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} \lesssim \|\nabla q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\\
&\qquad\qquad + \|u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})},\\
&\|\nabla q^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} \le T\|q^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,\infty})}.
\end{aligned}
\tag{3.63}
$$

Let us recall that we have by interpolation and the condition $(\mathcal{P}_{n-1})$:

$$
\begin{aligned}
&\|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \le \|u^{n-1}\|^{\frac{3}{4}}_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})}\|u^{n-1}\|^{\frac{1}{4}}_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} \le A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+\sqrt{\varepsilon},\\
&\|u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})} \le \|u^{n-1}\|^{\frac{1}{4}}_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})}\|u^{n-1}\|^{\frac{3}{4}}_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} \le A_0^{\frac{1}{4}}\varepsilon^{\frac{3}{4}}+\sqrt{\varepsilon}.
\end{aligned}
\tag{3.64}
$$

Using (3.56), (3.62), (3.63) and (3.64) we obtain for a certain $C>0$ large enough:

$$
\begin{aligned}
\|(\bar{q}^n,\bar{u}^n)\|_{F_T} &\le C\sqrt{\varepsilon}(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+\sqrt{\varepsilon}(1+A_0)+\varepsilon)+C(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+\sqrt{\varepsilon})(A_0^{\frac{1}{4}}\varepsilon^{\frac{3}{4}}+\sqrt{\varepsilon})+T(A_0+\sqrt{\varepsilon}),\\
&\le C\sqrt{\varepsilon}(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+2\sqrt{\varepsilon}(1+A_0)+A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+A_0^{\frac{1}{4}}\varepsilon^{\frac{3}{4}}+\varepsilon)+T(A_0+\sqrt{\varepsilon}).
\end{aligned}
$$

By choosing T and ε small enough the property $(\mathcal{P}_n)$ is verified, so we have shown by induction that (q^n,u^n) is bounded in F_T.

Second Step: Convergence of the sequence

We will show that (q^n,u^n) is a Cauchy sequence in the Banach space F_T, hence converges to some $(q,u)\in F_T$. Let:

$$\delta q^n = q^{n+1}-q^n,\ \delta u^n = u^{n+1}-u^n.$$

The system verified by $(\delta q^n,\delta u^n)$ reads:

$$
\begin{cases}
\partial_t\delta q^n + \mathrm{div}\delta u^n = F_n - F_{n-1},\\
\partial_t\delta u^n - \mu\Delta\delta u^n - (\lambda+\mu)\nabla\mathrm{div}\delta u^n - \kappa\nabla\Delta\delta q^n = G_n - G_{n-1},\\
\delta q^n(0)=0\ ,\ \delta u^n(0)=0,
\end{cases}
$$

Applying propositions 3.7, we obtain:

$$\|(\delta q^n, \delta u^n)\|_{F_T} \leq C(\|F_n - F_{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} + \|G_n - G_{n-1}\|_{\widetilde{L}^1_T(B^{N/2-1}_{2,\infty})}). \tag{3.65}$$

Tedious calculus ensure that:

$$F_n - F_{n-1} = -\delta u^{n-1} \cdot \nabla q^n - u^{n-1} \cdot \nabla \delta q^{n-1},$$

$$G_n - G_{n-1} = -u^n \cdot \nabla \delta u^{n-1} - \delta u^{n-1} \cdot \nabla u^{n-1} + \mu \nabla q^n \cdot D\delta u^{n-1} + \mu \nabla \delta q^{n-1} \cdot Du^{n-1}$$
$$+ \lambda \nabla q^n \mathrm{div} \delta u^{n-1} + \lambda \nabla \delta q^{n-1} \mathrm{div} u^{n-1} - K \nabla \delta q^{n-1} + \nabla\big(\nabla q^n \cdot \nabla \delta q^{n-1} + \nabla \delta q^{n-1} \cdot \nabla q^{n-1}\big).$$

It remains only to estimate the terms on the right hand side of (3.65) by using the same type of estimates than in the previous section and the property $(\mathcal{P}_n)$. More precisely we have via the proposition 2.2 and $(\mathcal{P}_n)$, it exists $C > 0$ such that:

$$\begin{aligned} \|F_n - F_{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} &\leq \|\delta u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})} \|\nabla q^n\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\ &+ \|\delta u^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})} \|\nabla q^n\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} + \|u^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} \|\nabla \delta q^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})} \\ &+ \|\nabla \delta q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})} \|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\ &\leq C(A_0^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} + A_0^{\frac{1}{4}} \varepsilon^{\frac{3}{4}} + \sqrt{\varepsilon} + \varepsilon) \|(\delta q^{n-1}, \delta u^{n-1})\|_{F_T}. \end{aligned} \tag{3.66}$$

In a similar way we show that it exists $C > 0$ large enough such that:

$$\|G_n - G_{n-1}\|_{\widetilde{L}^1_T(B^{N/2-1}_{2,\infty})} \leq C(A_0^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} + A_0^{\frac{1}{4}} \varepsilon^{\frac{3}{4}} + \sqrt{\varepsilon} + \varepsilon + T) \|(\delta q^{n-1}, \delta u^{n-1})\|_{F_T}. \tag{3.67}$$

By combining (3.65), (3.66) and (3.67) , we get for $C > 0$ large enough:

$$\|(\delta q^n, \delta u^n)\|_{F_T} \leq C(A_0^{\frac{3}{4}} \varepsilon^{\frac{1}{4}} + A_0^{\frac{1}{4}} \varepsilon^{\frac{3}{4}} + \sqrt{\varepsilon} + \varepsilon + T) \|(\delta q^{n-1}, \delta u^{n-1})\|_{F_T}.$$

It implies that choosing ε and T small enough (q^n, u^n) is a Cauchy sequence in F_T which is a Banach. It provides that (q^n, u^n) converges to (q, u) is in F_T. The verification that the limit (q, u) is solution of $(NHV1)$ in the sense of distributions is a straightforward application of proposition 2.2.

Third step: Uniqueness

Now, we are going to prove the uniqueness of the solution in F_TSuppose that (q_1, u_1) and (q_2, u_2) are solutions with the same initial conditions and belonging in F_T where (q_1, u_1) corresponds to the previous solution. We set:

$$\delta q = q_2 - q_1 \quad \text{and} \quad \delta u = u_2 - u_1.$$

We deduce that $(\delta q, \delta u)$ satisfy the following system:

$$\begin{cases} \partial_t \delta q + \mathrm{div} \delta u = F_2 - F_1, \\ \partial_t \delta u - \mu \Delta \delta u - (\lambda + \mu) \nabla \mathrm{div} \delta u - \kappa \nabla \Delta \delta q = G_1 - G_2, \\ \delta q(0) = 0 \ , \ \delta u(0) = 0. \end{cases}$$

We now apply proposition 3.7 to the previous system, and by using the same type of estimates than in the previous part, we show that:

$$\|(\delta q,\delta u)\|_{\widetilde{F}^{\frac{N}{2}}_{T_1}} \lesssim (\|q_1\|_{\widetilde{L}^2T_1(B^{\frac{N}{2}+1}_{2,\infty})} + \|q_2\|_{\widetilde{L}T_1{}^2(B^{\frac{N}{2}+1}_{2,\infty})} + \|u_1\|_{\widetilde{L}^2T_1(B^{\frac{N}{2}+1}_{2,\infty})} + \|u_2\|_{\widetilde{L}T_1{}^2(B^{\frac{N}{2}}_{2,\infty})}) \times \|(\delta q,\delta u)\|_{\widetilde{F}^{\frac{N}{2}}_{T_1}}.$$

We have then for T_1 small enough: $(\delta q,\delta u) = (0,0)$ on $[0,T_1]$ and by connectivity we finally conclude that:

$$q_1 = q_2,\ u_1 = u_2 \ \text{ on } \ [0,T].$$

□

3.3 Global solution near equilibrium for system (1.5)

We are now interested in proving the existence of global strong solution with small initial data for the system (1.5). The main difference with the previous proof consists essentially in dealing with the behavior of the density in low frequencies, to do this we shall use the proposition 3.7. More precisely we are going to use a contracting mapping argument for the function ψ defined as follows:

$$\psi(q,u) = W(t,\cdot) * \begin{pmatrix} q_0 \\ u_0 \end{pmatrix} + \int_0^t W(t-s) \begin{pmatrix} F(q,u) \\ G(q,u) \end{pmatrix} ds\ . \tag{3.68}$$

where W is the semi group associated to the linear system $(N1)$ with $a=\mu$, $b=\lambda+\mu$, $c=\kappa$ and $d=K$. The non linear terms F,G are defined as follows:

$$\begin{aligned} F(q,u) &= -\,u\cdot\nabla q, \\ G(q,u) &= -\,u.\nabla u + 2\mu\nabla q\cdot Du + \lambda \mathrm{div} u\,\nabla q + \frac{\kappa}{2}\nabla(|\nabla q|^2). \end{aligned} \tag{3.69}$$

We are going to check that we can apply a fixed point theorem for the function ψ in $E^{\frac{N}{2}}$ defined below, the proof is divided in two step the stability of ψ for a ball $B(0,R)$ in $E^{\frac{N}{2}}$ and the contraction property. We define $E^{\frac{N}{2}}$ by:

$$E^{\frac{N}{2}} = \big(\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})\big) \times \big(\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})\big)^N.$$

1) First step, stability of $B(0,R)$:

Let:

$$\eta = \|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}} + \|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}}\ .$$

We are going to show that ψ maps the ball $B(0,R)$ into itself if R is small enough. According to proposition 3.8, we have:

$$\|W(t,\cdot) * \begin{pmatrix} q_0 \\ u_0 \end{pmatrix}\|_{E^{\frac{N}{2}}} \le C\eta\ . \tag{3.70}$$

According to the proposition 3.8 it implies also that:

$$\|\psi(q,u)\|_{E^{\frac{N}{2}}} \leq C\big(\eta + \|F(q,u)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})} + \|G(q,u)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}\big). \tag{3.71}$$

The main task consists in using the propositions 2.2 and corollary 2 to obtain estimates on

$$\|F(q,u)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})},\ \|G(q,u)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}.$$

Let us first estimate $\|F(q,u)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}$. According to proposition 2.2, we have:

$$\begin{aligned}\|u\cdot\nabla q\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})} \lesssim & \|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u\|_{\widetilde{L}^4(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\ & + \|\nabla q\|_{\widetilde{L}^4(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\|u\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}.\end{aligned} \tag{3.72}$$

Let us now estimate $\|G(q,u)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}$. Hence by proposition 2.2 it yields:

$$\|u\cdot\nabla u\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})} \lesssim \|u\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}. \tag{3.73}$$

In the same way we have by proposition 2.2 and the fact that $\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty} \hookrightarrow B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}$, $\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty} \hookrightarrow B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}$ (see the remark 3.3):

$$\begin{aligned}\|\nabla q\cdot Du\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})} \lesssim & \|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|Du\|_{\widetilde{L}^4(B^{\frac{N}{2}-\frac{3}{2}}_{2,\infty})} \\ & + \|\nabla q\|_{\widetilde{L}^4(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\|Du\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}. \\ \|\nabla q\,\mathrm{div}u\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})} \lesssim & \|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\mathrm{div}u\|_{\widetilde{L}^4(B^{\frac{N}{2}-\frac{3}{2}}_{2,\infty})} \\ & + \|\nabla q\|_{\widetilde{L}^4(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\|\mathrm{div}u\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}.\end{aligned} \tag{3.74}$$

It now remains only to deal with the capillary terms:

$$\begin{aligned}\|\nabla(|\nabla q|^2)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})} \lesssim & \||\nabla q|^2\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,\infty})}, \\ \lesssim & \|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q\|_{\widetilde{L}^4(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}.\end{aligned} \tag{3.75}$$

We have previously used the fact that $\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty} \hookrightarrow B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}$ and $\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty} \hookrightarrow B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}$. We are now going to assume that (q,u) belongs in the ball $B(0,R)$ of $E^{\frac{N}{2}}$ with $R>0$. Combining the estimates (3.72), (3.73), (3.74) and (3.75) we get:

$$\|\psi(q,u)\|_{E^{\frac{N}{2}}} \leq C((C+1)\eta + R)^2. \tag{3.76}$$

By choosing R and η small enough we have:

$$C((C+1)\eta + R)^2 \leq R. \tag{3.77}$$

It implies that the ball $B(0,R)$ of $E^{\frac{N}{2}}$ is stable under ψ, indeed we have:

$$\psi(B(0,R)) \subset B(0,R)\ ,$$

2) Second step: Property of contraction

We consider (q_1,u_1), (q_2,u_2) in $B(0,R)$ and we are interested in verifying that ψ is a contraction. According to the proposition 3.8 we have:

$$\begin{aligned}\|\psi(q_2,u_2)-\psi(q_1,u_1)\|_{E^{\frac{N}{2}}} \leq C\big(&\|F(q_2,u_2)-F(q_1,u_1)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}\\ &+\|G(q_2,u_2)-G(q_1,u_1)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}\big).\end{aligned} \tag{3.78}$$

We set:

$$\delta q = q_2-q_1 \quad \text{and} \quad \delta u = u_2-u_1.$$

We have:

$$F(q_2,u_2)-F(q_1,u_1) = -\delta u\cdot\nabla q_2 - u_1\cdot\nabla\delta q.$$

$$\begin{aligned}G(q_2,u_2)-G(q_1,u_1) = &-u_2\cdot\nabla\delta u-\delta u\cdot\nabla u_1+\mu\nabla q_2\cdot D\delta u+\mu\nabla\delta q\cdot Du_1\\ &+\lambda\nabla q_2\mathrm{div}\delta u+\lambda\nabla\delta q\mathrm{div}u_1+\nabla\big(\nabla q_2\cdot\nabla\delta q+\nabla\delta q\cdot\nabla q_1\big).\end{aligned}$$

Let us first estimate $\|F(q_2,u_2)-F(q_1,u_1)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}$. We have by proposition 2.2 and the remark :

$$\begin{aligned}\|F(q_2,u_2)-F(q_1,u_1)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})} &\lesssim \|\delta u\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q_2\|_{\widetilde{L}^4_T(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\\ &+\|\delta u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})}\|\nabla q_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}_{2,\infty})}+\|u_1\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})}\|\nabla\delta q\|_{\widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}-1}_{2,\infty})}\\ &+\|\nabla\delta q\|_{\widetilde{L}^{\frac{4}{3}}_T(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u_1\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\\ &\leq C\big(2\|(q_2,u_2)\|_{E^{\frac{N}{2}}}+2\|(q_1,u_1)\|_{E^{\frac{N}{2}}}\big)\|(\delta q,\delta u)\|_{E^{\frac{N}{2}}}.\end{aligned} \tag{3.79}$$

Next, we have to bound $\|G(q_2,u_2)-G(q_1,u_1)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}$. We treat only one typical term, the others are of the same form.

$$\begin{aligned}\|-u_2\cdot\nabla\delta u-\delta u\cdot\nabla u_1\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})} &\lesssim \|\nabla\delta u\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\|u_2\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\\ &+\|\nabla\delta u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-2}_{2,\infty})}\|u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})}+\|\nabla u_1\|_{\widetilde{L}^1_T(B^{\frac{N}{2}}_{2,\infty})}\|\delta u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})}\\ &+\|\delta u\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla u_1\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{3}{2}}_{2,\infty})}\\ &\leq C\big(2\|(q_2,u_2)\|_{E^{\frac{N}{2}}}+2\|(q_1,u_1)\|_{E^{\frac{N}{2}}}\big)\|(\delta q,\delta u)\|_{E^{\frac{N}{2}}}.\end{aligned} \tag{3.80}$$

We can bound the other terms of $\|G(q_2,u_2)-G(q_1,u_1)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}$. in a similar way and this work is left to the reader. Finally by combining (3.78), (3.79) and (3.80) we obtain for $C>0$ large enough:

$$\|\psi(q_2,u_2)-\psi(q_1,u_1)\|_{E^{\frac{N}{2}}} \leq C\,\|(\delta q,\delta u)\|_{E^{\frac{N}{2}}}\big(\|(q_1,u_1)\|_{E^{\frac{N}{2}}}+\|(q_2,u_2)\|_{E^{\frac{N}{2}}}\|_{E^{\frac{N}{2}}}\big).$$

If one chooses R small enough such that $RC \leq \frac{3}{4}$, we end up with using the previous estimate which yields:

$$\|\psi(q_2,u_2)-\Psi(q_1,u_1)\|_{E^{\frac{N}{2}}} \leq \frac{3}{4}\|(\delta q,\delta u)\|_{E^{\frac{N}{2}}}.$$

We thus have the property of contraction and so by the fixed point theorem, we have the existence of a global solution (q,u) to the system $(NHV1)$. Indeed we can see easily that $E^{\frac{N}{2}}$ is a Banach space.
Concerning the uniqueness of this solution, it suffices to apply the arguments of the third step of the result of uniqueness in finite time (see the previous section). More precisely if (q,u) is the previous solution and (q_1,u_1) an other solution in $E^{\frac{N}{2}}$ then by setting $\delta q = q - q_1$ and $\delta u = u - u_1$ we show that for T small enough:

$$\delta q = 0 \quad \text{and} \quad \delta u = 0.$$

on $[0,T]$ and we conclude after by connectivity to get the uniqueness on $\mathbb{R}^+$. $\square$

3.4 Global solution near equilibrium for system (1.1)

We are now interested in proving the existence of global strong solution for the original system (1.1), as explained in the remarka 2 the main difficulty consists in observing that a priori the system (1.1) is not equivalent to the system (1.5). Indeed we need at least to control the L^∞ norm on the density ρ and on the vacuum $\frac{1}{\rho}$ in order to propagate on the density ρ the regularity proved in theorem 1.1 for the unknown $q = \ln\rho$. Furthermore we shall easily verify that with such regularity on (ρ,u) then (ρ,u) verify the system (1.1) and is a unique solution of (1.1). To do this we are going to assume additional hypothesis on (q_0,u_0) since now (q_0,u_0) is in $\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,2}$ with $q_0 = \ln\rho_0 \in L^\infty$ (this last condition implies in particular that ρ_0 and $\frac{1}{\rho_0}$ belong in L^∞).
The first part of the proof consists in getting L^∞ estimates for the linear system (N), and the second part consists in splitting the solution (q,u) under the following form:

$$(q,u) = (q_L,u_L) + (\bar{q},\bar{u}),$$

with (q_L,u_L) solution of (N) with $F = G = 0$ and $(q_L(0,\cdot),u_L(0,\cdot) = (q_0,u_0)$. The key points will be to show that q_L is belonging in $L^\infty_T(L^\infty)$ for any $T>0$ and that $\bar{q}$ is more regular that q_L. More precisely by a regularizing effect on the third index of the Besov space we shall prove that $\bar{q}$ is in $\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,1})$ for any $T>0$ with is embedded in $L^\infty_T(L^\infty)$. It will then be sufficient to deduce a control of $\ln\rho$ in $L^\infty_T(L^\infty)$ and to propagate the regularity of q on ρ via the proposition 2.3.

3.4.1 Result of maximum principle type for the linear system (N)

Let us start with studying the following system:

$$\begin{cases} \partial_t q + \mathrm{div} u = 0, \\ \partial_t u - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div} u - \kappa\nabla\Delta q = 0, \\ (q(0,\cdot),u(0,\cdot)) = (q_0,u_0), \end{cases} \tag{3.81}$$

with $\mu > 0$ and $\lambda + 2\mu > 0$. We are interesting in characterizing the Besov space in term of the semi group $B(t)$ associated to the system (3.81), it shall be useful in order to obtain L^∞ estimates for q. More precisely we have the following proposition.

Proposition 3.10 *Let s be a positive real number and $(p,r) \in [1,+\infty]^2$. Let (q,u) the solution of (3.81) with $(q,u)(t) = e^{B(t)}(q_0,u_0)$ and with the following notation:*

$$(\nabla q, u)(t) = e^{B(t)}(\nabla q_0, u_0).$$

Then there exists a constant $C > 0$ which satisfies:

$$\| \, \|t^s e^{B(t)}(\nabla q_0, u_0)\|_{L^p} \, \|_{L^r(\mathbb{R}^+, \frac{dt}{t})} \leq C\|(\nabla q_0, u_0)\|_{B^{-2s}_{p,r}} \quad \forall (\nabla q_0, u_0) \in B^{-2s}_{p,r}. \tag{3.82}$$

Proof Apply operator Δ to the first equation of (3.81) and operators div and curl to the second one; we obtain the following system with $\nu = 2\mu + \lambda$:

$$\begin{cases} \partial_t \Delta q + \Delta \mathrm{div} u = 0, \\ \partial_t \mathrm{div} u - \nu \Delta \mathrm{div} u - \kappa \Delta^2 q = 0, \\ \partial_t \mathrm{curl} u - \mu \Delta \mathrm{curl} u = 0, \\ (q(0,\cdot), u(0,\cdot)) = (q_0, u_0). \end{cases} \tag{3.83}$$

We observe that the third equation is a heat equation and we know via lemma 2.4 p 54 in [3] that it exists $C, c > 0$ such that:

$$\|e^{\mu t \Delta} \Delta_l \mathrm{curl} u_0\|_{L^p} \leq C e^{-c 2^{2l} \mu} \|\Delta_l \mathrm{curl} u_0\|_{L^p} \quad \forall p \in [1,+\infty]. \tag{3.84}$$

Let us study now the following system:

$$\begin{cases} \partial_t c + \Delta v = 0, \\ \partial_t v - \mu \Delta v - \kappa \Delta c = 0, \\ (c(0,\cdot), v(0,\cdot)) = (c_0, v_0). \end{cases} \tag{3.85}$$

where $c = \Delta q$ and $v = \mathrm{div} u$. Denoting by $U(t)$ the semi-group associated to (3.85), we deduce from Duhamel's formula that:

$$\partial_t \begin{pmatrix} \hat{q}(t,\xi) \\ \hat{d}(t,\xi) \end{pmatrix} = U(t) \begin{pmatrix} q_0 \\ u_0 \end{pmatrix}$$

We are now interested in proving estimate of the same form than (3.84) for $e^{U(t)}(\Delta_l c, \Delta_l v)$, and to do this we are going to prove the following lemma which is a direct consequence of the lemma 3 of [15]. For the sake of completeness we are going to recall its proof.

Lemma 1 *For any $p \in [1,+\infty]$ and any $t > 0$ we have:*

$$\|e^{U(t)}(\Delta_q c_0, \Delta_q v_0)\|_{L^p} \leq C e^{-c \min(1, \frac{4\kappa}{\nu^2}) 2^{2q} \nu t} (\|\Delta_q c_0\|_{L^p} + \|\Delta_q v_0\|_{L^p}). \tag{3.86}$$

Proof: Simple calculus show that $U(t) = e^{-tA(D)}$ with:

$$A(\xi) = \begin{pmatrix} 0 & -|\xi|^2 \\ \kappa|\xi|^2 & \mu|\xi|^2 \end{pmatrix}.$$

Following [15] we show that:

$$e^{-tA(\xi)} = e^{-\frac{t\nu|\xi|^2}{2}} \begin{pmatrix} h_1(t,\xi) + \frac{\nu}{2}h_2(t,\xi) & h_2(t,\xi) \\ -\kappa h_2(t,\xi) & h_1(t,\xi) - \frac{\nu}{2}h_2(t,\xi) \end{pmatrix},$$

with:

$$h_1(t,\xi) = \cos(\nu^{'}|\xi|^2 t), \quad h_2(t,\xi) = \frac{\sin(\nu^{'}|\xi|^2 t)}{\nu^{'}}, \text{if } \mu^2 < 4\kappa.$$
$$h_1(t,\xi) = 1, \quad h_2(t,\xi) = t|\xi|^2; \text{if } \nu^2 = 4\kappa.$$
$$h_1(t,\xi) = \cos(\nu^{'}|\xi|^2 t), \quad h_2(t,\xi) = \frac{\sin(\nu^{'}|\xi|^2 t)}{\mu^{'}}, \text{if } \nu^2 > 4\kappa.$$

and $\nu^{'} = \sqrt{|\kappa - \frac{\nu^2}{4}|}$. Let φ defined as in the definition of Littlewood-Paley theory, then denoting $a_{ij}(t,\xi)$ the coefficients of the matrix $e^{-tA(\xi)}$ and:

$$\Delta_q b_{ij}(t,x) = \Delta_q\big(\mathcal{F}^{-1}(a_{ij})(t,x)\big) = (2\pi)^{-N}\int e^{ix\cdot\xi}a_{ij}(t,\xi)\varphi(2^{-q}\xi)d\xi.$$

Let us show now that:

$$\|\Delta_q(b_{ij})(t,\cdot)\|_{L^1} \leq Ce^{-c\min(1,\frac{4\kappa}{\nu^2})2^{2q}\nu t}. \tag{3.87}$$

where c depends only on μ, κ and c is a universal constant. We first remark that $\|\Delta_q b_{ij}\|_{L^1} = \|h_{ijq}\|_{L^1}$ with:

$$h_{ijq}(t,y) = (2\pi)^{-N}\int e^{iy\cdot\eta}a_{ij}(t,2^q\eta)\varphi(\eta)d\eta.$$

Let us observe that the functions h_{ijq} can be rewritten under the form:

$$h_q(t,x) = \int e^{iy\cdot\xi}f(2^q|xi|^2t)\varphi(\xi)d\xi. \tag{3.88}$$

with $f \in C^\infty(\mathbb{R}^+)$. By integration by parts and Leibniz' formula, we obtain for all $\alpha \in \mathbb{N}^N$,

$$(-ix)^\alpha h_q(x) = \sum_{\beta\leq\alpha}\tbinom{\alpha}{\beta}\int e^{ix\cdot\xi}\partial^\beta f(2^q|xi|^2t)\partial^{\alpha-\beta}\varphi(\xi)d\xi. \tag{3.89}$$

Next, from Faà-di-Bruno's formula, we deduce that:

$$\partial^\beta f(2^q|xi|^2t) = \sum_{\gamma_1+\cdots+\gamma_m=\beta,|\gamma_i|\geq 1} f^{(m)}(2^q|xi|^2t)(2^{2q}t)^m(Pi_{j=1}^m\partial^{\gamma_j}(|xi|^2)). \tag{3.90}$$

Let us suppose first that $\nu^2 > 4\kappa$. Then, we just have to prove that:

$$\|h_q\|_{L^1} \leq Ce^{-c2^{2q}\nu t}, \tag{3.91}$$

for $f(u) = e^{i\mu' u} e^{-\mu u/2}$. We have:

$$f^{(m)}(u) = (i\nu' - \frac{\nu}{2})^m e^{i\nu' u} e^{-\nu u/2},$$

so that $|f^{(m)}(u)| \leq (\nu' + \frac{\nu}{2})^m e^{-\mu u/2}$. Using (3.89), (3.90), we prove the existence of constant $C_{\alpha,\beta,m}$ such that:

$$|x^\alpha h_q(x)| \leq \sum_{\beta \leq \alpha} \sum_{m=1}^{|\beta|} C_{\alpha,\beta,m} (2^{2q} t)^m e^{-\nu t 2^{2q}/8}.$$

For any constant $c < 1$ and $m \in \mathbb{N}$, there exists C_m such that $u^m e^{-u} \leq C_m e^{-cu}$. This clearly yields (3.91).
When $\nu^2 = 4\kappa$, it suffices to verify (3.91) for $f(u) = ue^{-\nu u/2}$ and $f(u) = e^{-\nu u/2}$. This is a direct consequence of (3.90) and Leibniz' formula. When $\nu^2 > 4\kappa$, we have to check (3.91) for:

$$f(u) = \exp(\frac{\nu}{2}(1 + \sqrt{1 - \frac{4\kappa}{\nu^2}})u) \text{ and } f(u) = \exp(\frac{\nu}{2}(1 - \sqrt{1 - \frac{4\kappa}{\nu^2}})u).$$

Using again (3.90) we have:

$$|x^\alpha h_q(x)| \leq C \max(e^{-c\nu t 2^{2q}(1+\sqrt{1-\frac{4\kappa}{\nu^2}})}, e^{-c\nu t 2^{2q}(1-\sqrt{1-\frac{4\kappa}{\nu^2}})}) \leq C e^{-c(\frac{\kappa}{\nu})2^{2q} t}$$

and we conclude to (3.87).
We obtain finally by using (3.87) and the Young inequality for the convolution:

$$\|e^{U(t)}(\Delta_q c_0, \Delta_q v_0)\|_{L^p} \leq C e^{-c\min(1,\frac{4\kappa}{\nu^2})2^{2q}\mu t}(\|\Delta_q c_0\|_{L^p} + \|\Delta_q v_0\|_{L^p}).$$

It proves the lemma 1. □

Via the Bernstein lemma, the lemma 2.4, (3.84) and (3.86) we obtain that $\forall l \in \mathbb{Z}$:

$$\|e^{B(t)}(\Delta_l \nabla q_0, \Delta_l u_0)\|_{L^p} \leq C e^{-c\min(1,\frac{4\kappa}{\nu^2})2^{2l}\nu t}(\|\Delta_l \nabla q_0\|_{L^p} + \|\Delta_l u_0\|_{L^p}). \tag{3.92}$$

According the estimate (3.92) we have by setting $V = (\nabla q_0, u_0)$ for $c, C > 0$:

$$\|t^s \Delta_l e^{B(t)} V\|_{L^p} \leq C t^s 2^{2ls} e^{-ct2^{2l}} 2^{-2ls} \|\Delta_l V\|_{L^p}.$$

We are now going to adapt a criterion to define some Besov space in terms of the heat kernel (see theorem 2.34 in [3]) to our case, it means to the semi-group $B(t)$.
Since V belongs in $\mathcal{S}_h'$ and the definition of the homogeneous Besov semi norm we have:

$$\begin{aligned}\|t^s e^{tB(t)} V\|_{L^p} &\leq \sum_{l\in\mathbb{Z}} \|t^s \Delta_l e^{B(t)} V\|_{L^p}, \\ &\leq C\|V\|_{B^{-2s}_{p,r}} \sum_{l\in\mathbb{Z}} t^s 2^{2ls} e^{-ct2^{2l}} c_{r,l}\end{aligned}$$

where $(c_{rl})_{l\in\mathbb{Z}}$ is a element of the unit sphere of $l^r(\mathbb{Z})$. If $r = +\infty$, we easily show (3.86) by using the following lemma (we left to the reader the proof of this last one).

Lemma 1 *For any s, we have:*

$$\sup_{t>0}\sum_{l\in\mathbb{Z}} t^s 2^{2ls} e^{-ct2^{2l}} < +\infty.$$

Let us deal now with the case $r < +\infty$, combining Hölder's inequality with the weight $2^{2ls}e^{-ct2^{2l}}$ and lemma 1, we obtain:

$$\begin{aligned}\int_0^{+\infty} t^{rs}\|e^{B(t)}V\|_{L^p}^r\frac{dt}{t} &\leq C\|V\|_{B^{-2s}_{p,r}}^r\int_0^{+\infty}(\sum_{l\in\mathbb{Z}} t^s 2^{2ls}e^{-ct2^{2l}}c_{rl})^r\frac{dt}{t},\\ &\leq C\|V\|_{B^{-2s}_{p,r}}^r\int_0^{+\infty}(\sum_{l\in\mathbb{Z}} t^s 2^{2ls}e^{-ct2^{2l}})^{r-1}(\sum_{l\in\mathbb{Z}} t^s 2^{2ls}e^{-ct2^{2l}}c_{rl}^r)\frac{dt}{t},\\ &\leq C\|V\|_{B^{-2s}_{p,r}}^r\int_0^{+\infty}(\sum_{l\in\mathbb{Z}} t^s 2^{2ls}e^{-ct2^{2l}}c_{rl}^r)\frac{dt}{t}\end{aligned}$$

Using Fubini's theorem and the change of variable $u = ct2^j$, we have:

$$\begin{aligned}\int_0^{+\infty} t^{rs}\|e^{B(t)}V\|_{L^p}^r\frac{dt}{t} &\leq C\|V\|_{B^{-2s}_{p,r}}^r\sum_{l\in\mathbb{Z}} c_{rl}^r\int_0^{+\infty} t^s 2^{2ls}e^{-ct2^{2l}}\frac{dt}{t},\\ &\leq C\Gamma(s)\|V\|_{B^{-2s}_{p,r}}^r,\end{aligned}$$

with $\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}dt$. The proposition 3.10 is thus proved. □

Let us prove now L^∞ estimate for the density solution q of the system (3.81). We recall that in the sequel q will correspond roughly speaking to $\ln\rho$, in particular L^∞ estimate on q shall also provide L^∞ estimate on ρ and $\frac{1}{\rho}$.

Proposition 3.11 *Let $q_0 \in B^{\frac{N}{2}}_{2,2}$, $u_0 \in B^{N-1}_{2,2}$ and $q_0 \in L^\infty$. Let (q,u) the solution of the system 3.81, then we have for all $T > 0$:*

$$\begin{aligned}&\sup_{t\in[0,T]}(\sqrt{t}\|(\nabla q,u)(t,\cdot)\|_{L^\infty}) \leq C\|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1}_{2,2}}.\\ &\|q\|_{L^\infty_T(L^\infty)} \leq \|q_0\|_{L^\infty} + C\|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1}_{2,2}}.\end{aligned} \tag{3.93}$$

Proof: The first estimate in (3.93) is a direct application of the proposition 3.10 applied to $p = +\infty$, $r = +\infty$, $s = \frac{1}{2}$ and using the fact that $B^{\frac{N}{2}-1}_{2,2}$ is embedded in $B^{-1}_{\infty,\infty}$.
Let $\mathcal{E}$ the fundamental solution of the Laplacian operator, and we define the operator $(\Delta)^{-1}$ by the convolution operator $(\Delta)^{-1}f = \mathcal{E} * f$ with $f \in \mathcal{S}'(\mathbb{R}^N)$. By applying the operator $(\Delta)^{-1}\mathrm{div}$ to the second equation of (3.81) and using the fact that $\Delta c = \mathrm{div}u$, we obtain the following system with $c = (\Delta)^{-1}\mathrm{div}u$:

$$\begin{cases}\partial_t q - \dfrac{\kappa}{\mu}\Delta q = -\dfrac{1}{\mu}\partial_t c,\\ \partial_t c - \mu\Delta c - \kappa\Delta q = 0.\end{cases}$$

Let us prove that q belongs in $L^\infty_T(L^\infty)$ for any $T>0$; by Duhamel formula we have:

$$q(t,x)=e^{\frac{\kappa}{\mu}t\Delta}q_0-\frac{1}{\mu}\int_0^t e^{\frac{\kappa}{\mu}(t-s)\Delta}\partial_t c(s)ds. \tag{3.94}$$

By the maximum principle for the heat equation, we deduce that:

$$\|e^{\frac{\kappa}{\mu}t\Delta}q_0\|_{L^\infty(L^\infty)}\leq\|q_0\|_{L^\infty(\mathbb{R}^N)}. \tag{3.95}$$

Next we are going to consider $\int_0^t e^{\frac{\kappa}{\mu}(t-s)\Delta}\partial_s c(s)ds$, we recall that:

$$\begin{aligned}
e^{\frac{\kappa}{\mu}(t-s)\Delta}\partial_s c(s)&=K(\frac{\cdot}{\sqrt{t-s}})*_x\partial_s(\Delta)^{-1}\mathrm{div}u(s,\cdot),\\
&=K(\frac{\cdot}{\sqrt{t-s}})*_x\big(\partial_s[(\mathcal{E}*_x\mathrm{div}u(s,\cdot)]\big),\\
&=K(\frac{x}{\sqrt{t-s}})*_x\big([\sum_i\partial_i\mathcal{E}*_x\partial_s u_i(s,\cdot)]\big),\\
&=\sum_i\big(K(\frac{x}{\sqrt{t-s}})*_x\partial_i\mathcal{E}\big)*_x\partial_s u_i(s,\cdot)]\big),\\
&=\sum_i\big(\partial_i[K(\frac{\cdot}{\sqrt{t-s}})]*_x\mathcal{E}\big)*_x\partial_s u_i(s,\cdot)\\
&=\sum_i\partial_i[K(\frac{\cdot}{\sqrt{t-s}})]*_x\big(\mathcal{E}*_x\partial_s u_i(s,\cdot)\big)\\
&=\sum_i\partial_i[K(\frac{\cdot}{\sqrt{t-s}})]*_x(\Delta)^{-1}\partial_s u_i(s,\cdot),
\end{aligned} \tag{3.96}$$

with $*_x$ the convolution in space and setting $\bar\mu=\frac{\kappa}{\mu}$:

$$K(\frac{x}{\sqrt{t}})=\frac{1}{(4\pi\bar\mu t)^{\frac{N}{2}}}e^{-\frac{|x|^2}{4\bar\mu t}}.$$

We deduce that:

$$\partial_i[K(\frac{\cdot}{\sqrt{t-s}})](x)=\frac{-2x_i}{\pi^{\frac{N}{2}}}\frac{1}{(4\bar\mu(t-s))^{\frac{N}{2}+1}}e^{-\frac{|x|^2}{4\bar\mu(t-s)}}$$

We easily check by a change of variable $u=\frac{x}{\sqrt{4\bar\mu(t-s)}}$ that:

$$\|\partial_i[K(\frac{\cdot}{\sqrt{t-s}})]\|_{L^1}\leq\frac{C}{\sqrt{t-s}}.$$

Then by Young's inequality we have for $0<s<t$:

$$\|\partial_i[K(\frac{\cdot}{\sqrt{t-s}})]*_x(\Delta)^{-1}\partial_s u_i(s,\cdot)\|_{L^\infty_x}\leq\frac{C}{\sqrt{t-s}}\frac{1}{\sqrt{s}}\sup_{0<s<t}\sqrt{s}\|(\Delta)^{-1}\partial_s u(s)\|_{L^\infty} \tag{3.97}$$

with $C>0$. By applying the operator $(\Delta)^{-1}$ to the second equation of (3.81) we show that:

$$\partial_t(\Delta)^{-1}u=\mu u+\kappa\nabla q. \tag{3.98}$$

We conclude as follows by using (3.96), (3.97) and (3.98), then there exists $C > 0$ such that:

$$\begin{aligned}\|\int_0^t e^{\frac{\kappa}{\mu}(t-s)\Delta}\partial_s c(s)ds\|_{L^\infty(L^\infty)} &\leq (\sup_{0<s<t}\sqrt{s}\|\Delta^{-1}\partial_s u(s)\|_{L^\infty})\int_0^t \frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}ds,\\ &\leq \pi \sup_{0<s<t}\sqrt{s}\|\mu u(s,\cdot)+\kappa\nabla q(s,\cdot)\|_{L^\infty},\\ &\leq C\pi\sup_{0<s<t}\sqrt{s}\|(u(s,\cdot),\nabla q(s,\cdot))\|_{L^\infty}.\end{aligned} \tag{3.99}$$

because:

$$\int_0^t \frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}ds = \pi.$$

By using the first estimate in (3.93) we deduce that by using (3.99) and (3.95) we obtain that for any $T > 0$, it exists $C > 0$:

$$\|q\|_{L^\infty_T(L^\infty)} \leq \|q_0\|_{L^\infty} + C\|(\nabla q_0, u_0)\|_{B^{\frac{N}{2}-1}_{2,2}}.$$

It achieves the proof of the proposition 3.93. □

3.4.2 The unique solution of (1.5) verifies $\rho \in L^\infty_T(L^\infty)$

Regularizing effect on the third index of the Besov spaces

Let us start by recalling that we are concerned now with initial data such that (q_0, u_0) belong in $(B^{\frac{N}{2}}_{2,2}\cap L^\infty)\times B^{\frac{N}{2}-1}_{2,2}$. This additional regularity assumptions on the initial data will be crucial in order to prove L^∞ estimates on $\ln\rho$. We are going to begin with proving additional regularity assumption on the solution (q,u) of the system (1.5) which verify $(q,u)\in E$ with E defined as follows:

$$E = \big(\widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})\cap\widetilde{L}^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})\big)\times\big(\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,\infty})\cap\widetilde{L}^1(\mathbb{R}^+, B^{\frac{N}{2}+1}_{2,\infty})\big)^N.$$

Indeed we are interested in splitting the unique solution (q,u) constructed in the subsection 3.3 as the following sum:

$$(q,u) = (q_L, u_L) + (\bar{q},\bar{u}).$$

with (q_L, u_L) the solution of the system $(N1)$ with $F = G = 0$ $a = \mu$, $b = \lambda+\mu$, $c = \kappa$, $d = K$ and with initial data $(\ln\rho_0, u_0)$. Compared with the subsection 3.3, we are going to show that the remainder term $(\bar{q},\bar{u})$ is more regular than (q_L, u_L) and is in the space F defined as follows:

$$F = \big(\widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})\cap\widetilde{L}^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})\big)\times\big(\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,1}\cap\widetilde{L}^1(\mathbb{R}^+, B^{\frac{N}{2}+1}_{2,1})\big)^N.$$

This type of regularizing effect on the remainder $(\bar{q},\bar{u})$ in term of the third index on the Besov spaces has been observed for the first time by Cannone and Planchon in [8] for the

incompressible Navier-Stokes equations. We are going to use similar ideas in your case than [8], to do this let us use the proposition 3.8 which ensures:

$$\begin{aligned}\|q_L\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,2}\cap B^{\frac{N}{2}}_{2,2})}+\|q_L\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,2}\cap B^{\frac{N}{2}+2}_{2,2})}+\|u_L\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,2})}\\ +\|u_L\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,2})}\lesssim\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,2}}+\|u_0\|_{B^{\frac{N}{2}-1}_{2,2}}.\end{aligned}\tag{3.100}$$

As in the subsection 3.3, we know that $(\bar{q},\bar{u})$ verify the following system:

$$\begin{cases}\partial_t\bar{q}+\mathrm{div}\bar{u}=F(\bar{q},\bar{u}),\\ \partial_t\bar{u}-\mu\Delta\bar{u}-(\lambda+\mu)\nabla\mathrm{div}\bar{u}-\kappa\nabla\Delta\bar{q}+K\nabla q=G(\bar{q},\bar{u}),\\ (\bar{q},\bar{u})(0,\cdot)=(0,0),\end{cases}\tag{3.101}$$

with:

$$\begin{aligned}&F(\bar{q},\bar{u})=-u\cdot\nabla q,\\ &G(\bar{q},\bar{u})=-u\cdot\nabla u+2\mu\nabla q\cdot Du+\lambda\mathrm{div}u\,\nabla q+\frac{\kappa}{2}\nabla(|\nabla q|^2).\end{aligned}$$

We are going to apply the proposition 3.8 to $(\bar{q},\bar{u})$, it implies that:

$$\begin{aligned}\|\bar{q}\|_{\widetilde{L}^\infty(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|\bar{q}\|_{\widetilde{L}^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1})}\\ \lesssim\|F(\bar{q},\bar{u})\|_{\widetilde{L}^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|G(\bar{q},\bar{u})\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1})}.\end{aligned}\tag{3.102}$$

It remains only to bound the terms $F(\bar{q},\bar{u})$ and $G(\bar{q},\bar{u})$ on the right hand side of (3.102). Let us start with $F(\bar{q},\bar{u})$and $u\cdot\nabla q$ which that we rewrite under the following form:

$$u\cdot\nabla q=\bar{u}\cdot\nabla q+u_L\cdot\nabla\bar{q}+u_L\cdot\nabla q_L.$$

According to proposition 2.2 we have:

$$\begin{aligned}\|\bar{u}\cdot\nabla q\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}&\lesssim\|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}_T(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\bar{u}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,1})}+\|\nabla q\|_{\widetilde{L}^4_T(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\\ &\quad\times\|\bar{u}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,1})},\\ \|u_L\cdot\nabla\bar{q}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}}_{2,1})}&\lesssim\|\nabla\bar{q}\|_{\widetilde{L}^{\frac{4}{3}}_T(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,1})}\|u_L\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}+\|\nabla\bar{q}\|_{\widetilde{L}^4_T(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,1})}\\ &\quad\times\|u_L\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})},\\ \|u_L\cdot\nabla q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}}_{2,1})}&\lesssim\|\nabla q_L\|_{\widetilde{L}^{\frac{4}{3}}_T(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,2})}\|u_L\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,2})}+\|\nabla q_L\|_{\widetilde{L}^4_T(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,2})}\\ &\quad\times\|u_L\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,2})}.\end{aligned}\tag{3.103}$$

Let us deal now with the term $G(\bar{q},\bar{u})$ and in particular the term $u\cdot\nabla u$, we have then:

$$u\cdot\nabla u=\bar{u}\cdot\nabla u+u_L\cdot\nabla\bar{u}+u_L\cdot\nabla u_L.$$

We have then by proposition 2.2:

$$\|\bar{u}\cdot\nabla u\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|\bar{u}\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,1})}\|\nabla u\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,\infty})}+\|\nabla u\|_{\widetilde{L}^2(B^{\frac{N}{2}-1}_{2,\infty})}\|\bar{u}\|_{\widetilde{L}^2(B^{\frac{N}{2}}_{2,1})}$$

$$\|u_L\cdot\nabla\bar{u}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}\|\nabla\bar{u}\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,1})}+\|\nabla\bar{u}\|_{\widetilde{L}^2(B^{\frac{N}{2}-1}_{2,1})}\|u_L\|_{\widetilde{L}^2(B^{\frac{N}{2}}_{2,\infty})}$$

$$\|u_L\cdot\nabla u_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,2})}\|\nabla u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,2})}+\|\nabla u_L\|_{\widetilde{L}^2(B^{\frac{N}{2}-1}_{2,2})}\|u_L\|_{\widetilde{L}^2(B^{\frac{N}{2}}_{2,2})}. \tag{3.104}$$

We can proceed similarly for the terms $\nabla q\cdot Du$ and $\mathrm{div}u\,\nabla q$. Let us treat the last term $\nabla|\nabla q|^2$ we have then:

$$|\nabla q|^2=\nabla q\cdot\nabla\bar{q}+\nabla\bar{q}\cdot\nabla q_L+|\nabla q_L|^2.$$

By proposition 2.2, we get:

$$\|\nabla(|\nabla q|^2)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\||\nabla q|^2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}}_{2,1})},$$

$$\||\nabla q_L|^2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|\nabla q_L\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,2})}\|\nabla q_L\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,2})},$$

$$\|\nabla q\cdot\nabla\bar{q}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla\bar{q}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,1})}+\|\nabla\bar{q}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,1})}\|\nabla q\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}, \tag{3.105}$$

By collecting the estimates (3.103), (3.104) and (3.105) and by interpolation we obtain that it exists $C>0$ such that:

$$\begin{aligned}
&\|\bar{q}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1}\cap B^{\frac{N}{2}}_{2,1})}+\|\bar{q}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1}\cap B^{\frac{N}{2}+2}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1})}\\
&\leq C\big(\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}+\|u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})}+\|q_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})}\\
&+\|u\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}+\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}+\|u\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})}+\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})}\big)\\
&\times(\|\bar{q}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|\bar{q}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1})})\\
&+C\big(\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,2})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,2})}+\|u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,2})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,2})}\big)^2.
\end{aligned} \tag{3.106}$$

Let us recall that via the first part of the theorem 1.1 and the proposition 3.8 (see the proof in the subsection 3.3) we have for a $C>0$ and $M>0$ large enough:

$$\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}+\|u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})}\leq C\varepsilon_0,$$

$$\|u\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}+\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}+\|u\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})}+\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})}\leq M\varepsilon_0,$$

with:

$$\varepsilon_0=\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}}+\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}}. \tag{3.107}$$

By choosing ε_0 small enough we can apply a bootstrap argument in (3.106) which implies that it exists $C>0$ such that

$$\begin{aligned}&\|\bar{q}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1}\cap B^{\frac{N}{2}}_{2,1})}+\|\bar{q}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1}\cap B^{\frac{N}{2}+2}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1})}\\&\leq C\big(\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,2})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,2})}+\|u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,2})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,2})}\big)^2.\end{aligned}\tag{3.108}$$

It proves finally that $(\bar{q},\bar{u})$ is in F and achieves this subsection.

q is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ for any $T>0$

In order to show that q is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ for any $T>0$ we shall use the proposition 3.93. Indeed we have seen in the previous subsection that:

$$(q,u)=(q_L,u_L)+(\bar{q},\bar{u}),$$

with (q_L,u_L) solution of the system $(N1)$ with $F=G=0$ $a=\mu$, $b=\lambda+\mu$, $c=\kappa$, $d=K$ and with initial data $(\ln\rho_0,u_0)$. In particular it implies via the definition of the semi group $B(t)$ in the proposition 3.10 and the Duhamel formula that:

$$(q_L(t),u_L(t))=e^{B(t)}(q_0,u_0)+\int_0^t e^{B(t-s)}(0,K\nabla q_L)(s)ds.$$

By using propositions 3.7, 3.93 and the embedding of $B^{\frac{N}{2}}_{2,1}$ in L^∞ we deduce that for any $T>0$ it exists $C>0$ independent on T such that (here $[\cdot]_1$ defines the first coordinate of the vector field):

$$\begin{aligned}\|q_L\|_{L^\infty_T(L^\infty)}&\leq\|[e^{B(t)}(q_0,u_0)]_1\|_{L^\infty_T(L^\infty)}+\|[\int_0^t e^{B(t-s)}(0,K\nabla q_L)(s)ds]_1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,1})},\\&\leq C(\|\ln\rho_0\|_{L^\infty}+\|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1}_{2,2}}+\|\nabla q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}).\end{aligned}\tag{3.109}$$

From proposition 3.8, we know that it exists $C>0$ such that:

$$\|q_L\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,2})}+\|q_L\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,2})}\leq C(\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,2}}+\|u_0\|_{B^{\frac{N}{2}-1}_{2,2}}).\tag{3.110}$$

Let us deal now with the term $\|\nabla q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}$ on the right hand side of (3.109), we have by interpolation for a constant $C>0$:

$$\|q_L(t)\|_{B^{\frac{N}{2}}_{2,1}}\leq C\|q_L(t)\|^{\frac{1}{2}}_{B^{\frac{N}{2}-1}_{2,2}}\|q_L(t)\|^{\frac{1}{2}}_{B^{\frac{N}{2}+1}_{2,2}}.$$

It implies that by Hölder's inequality and (2.23) that it exists $C,C_1>0$ large enough such that:

$$\begin{aligned}\|q_L(t)\|_{L^1_T(B^{\frac{N}{2}}_{2,1})}&\leq C\|q_L\|^{\frac{1}{2}}_{L^1_T(B^{\frac{N}{2}-1}_{2,2})}\|q_L\|^{\frac{1}{2}}_{L^1_T(B^{\frac{N}{2}+1}_{2,2})},\\&\leq CT^{\frac{3}{4}}\|q_L\|^{\frac{1}{2}}_{L^\infty_T(B^{\frac{N}{2}-1}_{2,2})}\|q_L\|^{\frac{1}{2}}_{L^2_T(B^{\frac{N}{2}+1}_{2,2})},\\&\leq C_1T^{\frac{3}{4}}(\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,2}}+\|u_0\|_{B^{\frac{N}{2}-1}_{2,2}}).\end{aligned}\tag{3.111}$$

Combining (3.109) and (3.111) we get:

$$\|q_L\|_{L^\infty_T(L^\infty)} \leq \|[e^{B(t)}(q_0,u_0)]_1\|_{L^\infty_T(L^\infty)} + \|[\int_0^t e^{B(t-s)}(0,K\nabla q_L)(s)ds]_1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,1})},$$
$$\leq C(\|\ln\rho_0\|_{L^\infty} + \|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1}_{2,2}} + C_1T^{\frac{3}{4}}(\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,2}} + \|u_0\|_{B^{\frac{N}{2}-1}_{2,2}}). \tag{3.112}$$

We have then proved that q_L belongs in $L^\infty_{lot}(L^\infty)$. Since we have seen that $(\bar{q},\bar{u})$ is in F, it implies in particular that $\bar{q}$ belongs in $\widetilde{L}^\infty(B^{\frac{N}{2}}_{2,1})$ which is embedded in $L^\infty(L^\infty)$. We deduce that $q = q_L + \bar{q}$ belongs in $L^\infty_{loc}(L^\infty)$.
It remains only to prove that $(\rho,u) = (\exp(q),u)$ is a global strong solution of (1.1) and we define $h = \exp(q) - 1$ with $\rho = 1 + h$. By proposition 2.3 and the fact that ρ and $\frac{1}{\rho}$ belong in $L^\infty_{loc}(L^\infty)$ we easily show that h is for any $T > 0$ in:

$$H_T = \big(\widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})\big).$$

With such regularity on h and u, the verification that (ρ,u) is a solution of (1.1) in the sense of distribution is a straightforward application of propositions 2.2 and 2.3. The uniqueness follows also the same line than in the proof of subsection 3.2. □

3.5 Proof of the theorem 1.2

In the theorem 1.2 we are interested in extending the results of theorem 1.1 to the case of general pressure P and also to the case of constant viscosity and capillary coefficients. It implies in particular that the eulerian form of the system 1.1 does not only depend on the unknown $q = \ln\rho$. Typically the pressure term $\frac{\nabla P(\rho)}{\rho}$ is non linear in terms of $q = \ln\rho$, that is why we need to control the L^∞ norm of the density in order to estimate this term in Besov space, indeed otherwise we could have a loss of regularity on this term. For this reason it seems impossible to hope a result of global strong solution involving only a smallness hypothesis on $\|q_0\|_{B^{\frac{N}{2}}_{2,\infty}}$ and $\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}}$ as it is the case under some specific physical condition in the theorem 1.1.
We are going now to explain how to adapt the previous arguments of the proof of theorem 1.1 to this new situation. We are only dealing with the case $\mu(\rho) = \mu$, $\lambda(\rho) = \lambda$, $\kappa(\rho) = \kappa$ and P a regular function such that $P^{'}(1) > 0$, here $\mu > 0$ $2\mu+\lambda > 0$ and $\kappa > 0$ (the other case are similar to treat). The system 1.1 is equivalent to the following system where we set in this section $\rho = 1 + h$. We can then rewrite the system (1.1) as follows:

$$\begin{cases} \partial_t h + \mathrm{div} u = F(h,u), \\ \partial_t u - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}u + F^{'}(1)\nabla h - \kappa\nabla\Delta h = G(h,u), \\ (h,u)_{\ t=0} = (h_0,u_0), \end{cases} \tag{3.113}$$

with $F^{'}(\rho) = \frac{P^{'}(\rho)}{\rho}$ and:

$$\begin{aligned} F(h,u) &= -u\cdot\nabla h - h\mathrm{div}u, \\ G(h,u) &= (F^{'}(1) - F^{'}(\rho))\nabla h + (\frac{\mu}{\rho} - \mu)\Delta u + (\frac{\mu+\lambda}{\rho} - \mu - \lambda)\nabla\mathrm{div}u - u\cdot\nabla u. \end{aligned} \tag{3.114}$$

Let (h_L, u_L) the solution of $(N1)$ with $a = \mu$, $b = \lambda + \mu$, $c = \kappa$, $d = P^{'}(1)$, $F = G = 0$ and the initial data (h_0, u_0). In the sequel we will consider solution under the form:

$$(h, u) = (h_L, u_L) + (\bar{h}, \bar{u}).$$

In order to prove theorem 1.2 we shall use a fixed point theorem, more precisely we are going to consider the following functional:

$$\psi(\bar{h}, \bar{u}) = \int_0^t W(t-s) \begin{pmatrix} F(h,u) \\ G(h,u) \end{pmatrix} ds \ . \tag{3.115}$$

$W(t)$ is the semi group associated to $(N1)$ with the following conditions $a = \mu$, $b = \lambda + \mu$, $c = \kappa$, $d = P^{'}(1)$. The proof is divided in two step the stability of ψ for a ball $B(0, R)$ in $E^{\frac{N}{2}}$ defined below and the contraction property. We consider $E^{\frac{N}{2}}$ defined as follows:

$$E^{\frac{N}{2}} = \big(\widetilde{L}^{\infty}(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}) \cap \widetilde{L}^{1}(\widetilde{B}_{2,1}^{\frac{N}{2}+1,\frac{N}{2}+2})\big) \times \big(\widetilde{L}^{\infty}(B_{2,1}^{\frac{N}{2}-1}) \cap \widetilde{L}^{1}(B_{2,1}^{\frac{N}{2}+1})\big)^N.$$

1) First step, stability of $B(0, R)$:

By using the proposition 3.8 we have: for $C > 0$:

$$\begin{aligned} \|h_L\|_{\widetilde{L}_T^{\infty}(\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}})} + \|h_L\|_{\widetilde{L}_T^{1}(\widetilde{B}_{2,2}^{\frac{N}{2}+1,\frac{N}{2}+2})} + \|u_L\|_{\widetilde{L}_T^{\infty}(B_{2,2}^{\frac{N}{2}-1})} + \|u_L\|_{\widetilde{L}_T^{1}(B_{2,2}^{\frac{N}{2}+1})} \\ \leq C(\|h_0\|_{\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{B_{2,2}^{\frac{N}{2}-1}}). \end{aligned} \tag{3.116}$$

By combining the proposition 3.93 and 3.8, we have for any $T > 0$ and $C > 0$ independent on T:

$$\|h_l\|_{L_T^{\infty}(L^{\infty})} \leq C(\|h_0\|_{L^{\infty}} + +\|h_0\|_{B_{2,1}^{\frac{N}{2}-2}} + \|u_0\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-2}}). \tag{3.117}$$

Remark 20 *Let us mention that the condition $(h_0, u_0) \in B_{2,1}^{\frac{N}{2}-2} \times (B_{2,1}^{\frac{N}{2}-2})^N$ plays only a role in the previous estimate (3.117) in order to bound the term $P^{'}(1)\nabla h_L$ that we consider as a remainder term for the system (N). Indeed we are interested in applying proposition 3.93 and the Duhamel formula. It would be possible to avoid this additional regularity by extending the proposition 3.93 to the system $(N1)$.*

It remains only to apply the proposition 3.8 in order to get a priori estimates on $(\bar{h}, \bar{u})$, indeed we have for $C > 0$:

$$\|\psi(\bar{h}, \bar{u})\|_{E^{\frac{N}{2}}} \leq C(\|F(h,u)\|_{\widetilde{L}_T^{1}(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} + \|G(h,u)\|_{\widetilde{L}_T^{1}(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})}). \tag{3.118}$$

By applying proposition 2.2 and lemma 2.3 (indeed we have seen via the estimate (3.116) and the fact that $\bar{h}$ is in $\widetilde{L}^{\infty}(B_{2,1}^{\frac{N}{2}})$ that h belongs in L^{∞} and $\rho = 1 + h \geq c > 0$) we have

as in (3.106) for a function $C>0$:

$$\begin{aligned}
&\|\psi(\bar{h},\bar{u})\|_{E^{\frac{N}{2}}} \leq C(\|h\|_{L^\infty})\big(\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})} + \|h_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})} + \|h_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})} \\
&+ \|h_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})} + \|u\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})} + \|h\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})} + \|u\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})} + \|h\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})}\big) \\
&\times (\|\bar{h}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} + \|\bar{h}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})} + \|\bar{u}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1})} + \|\bar{u}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1})}) \\
&+ C(\|h\|_{L^\infty})\big(\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,2})} + \|h_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,2})} + \|u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,2})} + \|h_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,2})}\big)^2.
\end{aligned} \tag{3.119}$$

We set

$$\varepsilon_0 = \|h_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,2}} + \|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} + \|h_0\|_{L^\infty} + \|h_0\|_{B^{\frac{N}{2}-2}_{2,1}} + \|u_0\|_{B^{\frac{N}{2}-2}_{2,1}}. \tag{3.120}$$

By choosing $R = M\varepsilon_0$ with $M>2$, it suffices to choose ε_0 small enough such that (3.119) ensures that:

$$\|\psi(\bar{h},\bar{u})\|_{E^{\frac{N}{2}}} \leq R.$$

The proof of the contraction follows the same lines as in the proof of the theorem 1.1 and is left to the reader. It achieves the proof of the theorem 1.2. □

4 Proof of the theorem 1.3

We are now interested in proving the theorem 1.3 where we assume that $\kappa(\rho) = \frac{\mu^2}{\rho}$, $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$. We have observed that it exists quasi solution, it means a solution of the form $(\rho_1, -\mu\nabla\ln\rho_1)$ with:

$$\begin{cases} \partial_t\rho_1 - \mu\Delta\rho_1 = 0, \\ (\rho_1)_{t=0} = (\rho_1)_0. \end{cases}$$

This quasi solution $(\rho_1, -\mu\nabla\ln\rho_1)$ verifies the following system:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u\otimes u) - \mathrm{div}(2\mu\rho \mathrm{D(u)}) - \mu^2\rho\nabla\Delta\ln\rho - \dfrac{\mu^2}{2}\nabla(|\nabla\ln\rho|^2) = 0, \\ (\rho,u)_{t=0} = \big((\rho_1)_0, -\mu\nabla\ln(\rho_1)_0\big). \end{cases} \tag{4.121}$$

We are now interested in working around this quasi solution, more precisely we search global solution of (1.1) under the form:

$$q = \ln\rho = \ln\rho_1 + h_2 \ \text{ with } \rho_1 = 1 + h_1 \ \text{ and } \ u = -\mu\nabla\ln\rho_1 + u_2.$$

We deduce from (1.9) that (h_2, u_2) verifies the following system:

$$\begin{cases} \partial_t h_2 + \mathrm{div} u_2 - \mu\nabla\ln\rho_1\cdot\nabla h_2 + u_2\cdot\nabla\ln\rho_1 = F(h_2,u_2), \\ \partial_t u_2 - \mu\Delta u_2 - \mu\nabla\mathrm{div}u_2 - \kappa\nabla\Delta h_2 + K\nabla h_2 - 2\mu\nabla\ln\rho_1\cdot Du_2 - 2\mu\nabla h_2\cdot Du_1 \\ \qquad\qquad + u_1\cdot\nabla u_2 + u_2\cdot\nabla u_1 - \mu^2\nabla(\nabla\ln\rho_1\cdot\nabla h_2) = G(h_2,u_2), \\ (h_2(0,\cdot), u_2(0,\cdot)) = (h_0^2, u_0^2). \end{cases} \tag{4.122}$$

with:

$$
\begin{aligned}
&F(h_2,u_2) = -u_2\cdot\nabla h_2,\\
&G(h_2,u_2) = -u_2\cdot\nabla u_2 + 2\mu\nabla h_2\cdot Du_2 - K\nabla\ln\rho_1 + \frac{\mu^2}{2}\nabla(|\nabla h_2|^2).
\end{aligned}
\tag{4.123}
$$

We have now to solve the previous system (4.122) and to do this we are going to apply a fixed point theorem. We start with defining the following map ψ:

$$
\psi(h,u) = W_1(t,\cdot)*\begin{pmatrix} h_0^2\\ u_0^2\end{pmatrix} + \int_0^t W_1(t-s)\begin{pmatrix} F(h,u)\\ G(h,u)\end{pmatrix}\,ds\,. \tag{4.124}
$$

where W_1 is the semi group associated to the following linear system (4.125):

$$
\begin{cases}
\partial_t h_2 + \mathrm{div} u_2 - \mu\nabla\ln\rho_1\cdot\nabla h_2 + u_2\cdot\nabla\ln\rho_1 = 0,\\
\partial_t u_2 - \mu\Delta u_2 - \mu\nabla\mathrm{div}u_2 - \kappa\nabla\Delta h_2 + K\nabla h_2 - 2\mu\nabla\ln\rho_1\cdot Du_2 - 2\mu\nabla h_2\cdot Du_1\\
\qquad\qquad + u_1\cdot\nabla u_2 + u_2\cdot\nabla u_1 - \mu^2\nabla(\nabla\ln\rho_1\cdot\nabla h_2) = 0.\\
(h_2(0,\cdot),u_2(0,\cdot)) = (h_0^2,u_0^2).
\end{cases}
\tag{4.125}
$$

The non linear terms F,G are defined in (4.123). We are going to check that we can apply a fixed point theorem for the function ψ, the proof is divided in two step the stability of ψ for a ball $B(0,R)$ in $E^{\frac{N}{2}}$ and the contraction property. We define now $E^{\frac{N}{2}}$ by:

$$
E^{\frac{N}{2}} = \big(\widetilde{L}^\infty(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})\cap\widetilde{L}^1(\widetilde{B}_{2,1}^{\frac{N}{2}+1,\frac{N}{2}+2})\big)\times\big(\widetilde{L}^\infty(B_{2,1}^{\frac{N}{2}-1})\cap\widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1})\big)^N.
$$

1) First step, stability of $B(0,R)$:

Let:

$$
\eta = \|h_0^2\|_{\widetilde{B}_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0^2\|_{B_{2,\infty}^{\frac{N}{2}-1}}\,.
$$

We are going to show that ψ maps the ball $B(0,R)$ into itself if R is small enough. By using the proposition 3.9 we have:

$$
\begin{aligned}
&\|\psi(h,u)\|_{E^{\frac{N}{2}}} \le C\big(\eta + \|F(h,u)\|_{\widetilde{L}^1_T(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} + \|G(h,u)\|_{\widetilde{L}^1_T(B_{2,1}^{\frac{N}{2}-1})}\big)\\
&\times\exp\bigg(C\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B_{2,\infty}^{\frac{N}{2}-\frac12}} + \|u_1\|^{\frac43}_{B_{2,\infty}^{\frac{N}{2}+\frac12}} + \|\nabla\ln\rho_1\|^{\frac43}_{B_{2,\infty}^{\frac{N}{2}+\frac12}} + \|\nabla\ln\rho_1\|^{\frac43}_{B_{2,\infty}^{\frac{N}{2}+\frac12}}\big)(s)ds\bigg).
\end{aligned}
\tag{4.126}
$$

The main task consists in using the propositions 2.2 and corollary 2 to obtain estimates on

$$
\|F(h,u)\|_{\widetilde{L}^1(\widetilde{B}_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}})},\ \|G(h,u)\|_{\widetilde{L}^1(B_{2,\infty}^{\frac{N}{2}-1})}.
$$

Let us first estimate $\|F(h,u)\|_{\widetilde{L}^1(\widetilde{B}_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}})}$. According to proposition 2.2, we have:

$$
\begin{aligned}
\|u\cdot\nabla h\|_{\widetilde{L}^1(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} &\lesssim \|\nabla h\|_{\widetilde{L}^{\frac43}(\widetilde{B}_{2,1}^{\frac{N}{2}-\frac12,\frac{N}{2}+\frac12})}\|u\|_{\widetilde{L}^4(B_{2,1}^{\frac{N}{2}-\frac12})}\\
&\quad + \|\nabla h\|_{\widetilde{L}^4(\widetilde{B}_{2,1}^{\frac{N}{2}-\frac32,\frac{N}{2}-\frac12})}\|u\|_{\widetilde{L}^{\frac43}(B_{2,1}^{\frac{N}{2}+\frac12})}.
\end{aligned}
\tag{4.127}
$$

Similarly for $\|G(h,u)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}$ we have:

$$\begin{aligned}\|u\cdot\nabla u\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})} \lesssim \|u\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}+\frac{1}{2}}_{2,1})}\|\nabla u\|_{\widetilde{L}^4(B^{\frac{N}{2}-\frac{3}{2}}_{2,1})} \\ + \|u\|_{\widetilde{L}^4(B^{\frac{N}{2}-\frac{1}{2}}_{2,1})}\|\nabla u\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{1}{2}}_{2,1})}.\end{aligned} \tag{4.128}$$

The most important term is certainly $K\nabla\ln\rho_1$ which belongs in $\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})$ since we have assume that $\ln\rho_1^0$ belongs in $B^{\frac{N}{2}-2}_{2,1}\cap L^\infty$ and the fact that ρ_1 verifies a heat equation (1.11). Using propositions 2.3, 2.5 and the maximum principle it exists a regular function g such that:

$$\begin{aligned}\|\nabla\ln\rho_1\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})} &\lesssim \|\ln\rho_1\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,1})}, \\ &\lesssim g(\|(\rho_1,\frac{1}{\rho_1})\|_{L^\infty})\|h_1\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,1})}, \\ &\lesssim g(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}.\end{aligned} \tag{4.129}$$

We deal with the others terms in a similar way and combining (4.126), (4.127), (4.128) and (4.129) we obtain for a $C>0$ large enough and using the fact that (h,u) is in the ball $B(0,R)$ of $E^{\frac{N}{2}}$:

$$\begin{aligned}&\|\psi(h,u)\|_{E^{\frac{N}{2}}} \leq C\big(\eta+R^2+g(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}\big) \\ &\times\exp\bigg(C\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\bigg).\end{aligned} \tag{4.130}$$

Let us recall that ρ_1 verifies a heat equation (1.11), we deduce from propositions 2.5 and the maximum principle that it exists a regular function g_1 such that:

$$\begin{aligned}\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds \\ \lesssim g_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}.\end{aligned} \tag{4.131}$$

From (4.131) we deduce that for a C large enough we have:

$$\begin{aligned}&g(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}\times\exp\bigg(C\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} \\ &\qquad+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\bigg) \\ &\leq g(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}\big).\end{aligned} \tag{4.132}$$

In particular it implies that:

$$C\eta\exp\Big(C\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\Big)$$
$$\leq C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}). \tag{4.133}$$

Let us prove now the stability of the functional ψ, from label (4.130) let us choose:

$$R=4C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}). \tag{4.134}$$

and suppose that:

$$g(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}\leq C\eta. \tag{4.135}$$

Now combining (4.130), (4.134) and (4.135) it yields:

$$\|\psi(h,u)\|_{E^{\frac{N}{2}}}\leq 2C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})+CR^2\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}). \tag{4.136}$$

From (4.136) we need to assume that:

$$R^2\leq 2C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}). \tag{4.137}$$

Indeed combining (4.137), (4.136) and (4.134) shows that:

$$\|\psi(h,u)\|_{E^{\frac{N}{2}}}\leq R. \tag{4.138}$$

This concludes the proof of the stability except that it remains to choose η and to verify that it implies the condition 1.14 of theorem 1.3. The condition 4.137 via (4.134) is equivalent to:

$$16C^2\eta^2\exp(2Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\leq 2C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}). \tag{4.139}$$

and:

$$8C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\leq 1. \tag{4.140}$$

From (4.135) let us choose:

$$\eta=\frac{1}{C}g(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}} \tag{4.141}$$

The condition (4.140) implies by using (4.141) that:

$$8g(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\leq 1. \tag{4.142}$$

Let us point out that this condition corresponds exactly to the condition (1.14) of the theorem 1.3. It concludes the proof of the stability of the functional ψ. Let us prove now some contraction properties of the functional ψ.

2) Second step, contraction properties:

Consider two element (h,u) and $(h^{'},u^{'})$ in $B(0,R)$, according to proposition 3.9 we have:

$$\|\psi(h,u)-\psi(h^{'},u^{'})\|_{E^{\frac{N}{2}}}\leq C\big(\|F(h,u)-F(h^{'},u^{'})\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|G(h,u)-G(h^{'},u^{'})\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\big)$$
$$\times\exp\bigg(C\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\bigg). \tag{4.143}$$

Let us deal with the term $\|F(h,u)-F(h^{'},u^{'})\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}$, we have then by proposition 2.2 and by denoting $\delta h=h-h^{'}$ and $\delta u=u-u^{'}$:

$$\begin{aligned}\|F(h^{'},u^{'})-F(h,u)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}&\lesssim\|\delta u\|_{\widetilde{L}^4_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla h^{'}\|_{\widetilde{L}^4_T(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\\&+\|\delta u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})}\|\nabla h^{'}\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}_{2,\infty})}+\|u\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})}\|\nabla\delta h\|_{\widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}-1}_{2,\infty})}\\&+\|\nabla\delta h\|_{\widetilde{L}^{\frac{4}{3}}_T(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\\&\leq C\big(2\|(h,u)\|_{E^{\frac{N}{2}}}+2\|(h^{'},u^{'})\|_{E^{\frac{N}{2}}}\big)\|(\delta h,\delta u)\|_{E^{\frac{N}{2}}}.\end{aligned} \tag{4.144}$$

We proceed similarly for the term $\|G(h,u)-G(h^{'},u^{'})\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}$ (let us mention only that the delicate term $K\nabla\ln\rho_1$ disappears by the subtraction). We get finally for a $C>0$ large enough by using (4.131):

$$\begin{aligned}&\|\psi(h,u)-\psi(h^{'},u^{'})\|_{E^{\frac{N}{2}}}\leq C\big(2\|(h,u)\|_{E^{\frac{N}{2}}}+2\|(h^{'},u^{'})\|_{E^{\frac{N}{2}}}\big)\|(\delta h,\delta u)\|_{E^{\frac{N}{2}}}\\&\times\exp\bigg(C\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\bigg),\\&\leq 4CR\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\big)\|(\delta h,\delta u)\|_{E^{\frac{N}{2}}},\end{aligned} \tag{4.145}$$

From (4.134) and (4.141) we have with Clarge enough:

$$\begin{aligned}&4CR\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\big)=16C\eta\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\big)\\&=16Cg(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\big).\end{aligned} \tag{4.146}$$

In particular we ensure the contraction property via (4.145) if:

$$4CR\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\big)\leq\frac{1}{2}. \tag{4.147}$$

By using (4.146), the previous condition (4.147) is realized if:

$$16Cg(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\big)\leq\frac{1}{2}. \tag{4.148}$$

Let us mention that the condition (5.150) is exactly the condition (1.14) of the theorem 1.3. It achieves the proof of the contraction property of ψ and by the fixed point theorem we have proved the existence of global strong solution under the hypothesis (1.14). It concludes the proof of theorem 1.3.

5 Proof of corollary 1

It suffices to apply the same proof than for the theorem 1.3, in particular to apply a fixed point theorem for the function ψ previously defined. The main difference concerns the way to deals with the remainder term $K\nabla \ln \rho_1$. Following the same arguments than for the estimate (4.129) we have for a regular function g and $C > 0$:

$$\begin{aligned} \|K\nabla \ln \rho_1\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})} &\lesssim \|\ln \rho_1\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,1})}, \\ &\leq CKg(\|(\rho_1, \frac{1}{\rho_1})\|_{L^\infty})\|h_1\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,1})}, \\ &\leq CKg(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}. \end{aligned} \tag{5.149}$$

By using exactly the same arguments than in the previous proof we need the following smallness hypothesis with $C > 0$ and g, g_1 regular function in order to ensure the stability of the functional ψ for a ball $B(0, R)$ with R defined as previously:

$$16KCg(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}} \exp\big(Cg_1(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}})\big) \leq \frac{1}{2}. \tag{5.150}$$

If we choose K small enough this last condition will be verified, the rest of the proof follows the same lines as in the proof of theorem 1.3 which achieves the proof of corollary 1.

6 Appendix

In this appendix, we just give a technical lemma on the computation of the capillarity tensor.

Lemma 2 *When $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\kappa > 0$ then:*

$$\mathrm{div} K = \kappa\rho(\nabla\Delta \ln \rho + \frac{1}{2}\nabla(|\nabla \ln \rho|^2)).$$

and:

$$\mathrm{div} K = \kappa \mathrm{div}(\rho\nabla\nabla \ln \rho).$$

Proof: We recall that:

$$\mathrm{div} K = \nabla\big(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa^{'}(\rho))|\nabla\rho|^2\big) - \mathrm{div}\big(\kappa(\rho)\nabla\rho \otimes \nabla\rho\big).$$

When $\kappa(\rho) = \frac{\kappa}{\rho}$, we have:

$$\mathrm{div} K = \kappa \nabla \Delta \rho - \kappa \mathrm{div}(\frac{1}{\rho} \nabla \rho \otimes \nabla \rho). \tag{6.151}$$

But as:

$$\Delta \rho = \rho \Delta \ln \rho + \frac{1}{\rho} |\nabla \rho|^2,$$

we have by injecting this expression in (6.151):

$$\mathrm{div} K = \kappa \rho \nabla \Delta \ln \rho + \kappa \nabla \rho \Delta \ln \rho + \kappa \nabla (\frac{1}{\rho} |\nabla \rho|^2) - \kappa \mathrm{div}(\frac{1}{\rho} \nabla \rho \otimes \nabla \rho). \tag{6.152}$$

As we have:

$$\kappa \mathrm{div}(\frac{1}{\rho} \nabla \rho \otimes \nabla \rho) = \kappa \Delta \ln \rho \nabla \rho + \nabla (\frac{1}{\rho} |\nabla \ln \rho|^2) - \frac{\kappa}{2} \rho \nabla (|\nabla \ln \rho|^2).$$

It concludes the first part of the lemma.

We now want to prove that we can rewrite the capillarity tensor under the form of a viscosity tensor. To see this, we have:

$$\begin{aligned}
\mathrm{div}(\rho \nabla (\nabla \ln \rho))_j &= \sum_i \partial_i (\rho \partial_{ij} \ln \rho), \\
&= \sum_i [\partial_i \rho \partial_{ij} \ln \rho + \rho \partial_{iij} \ln \rho], \\
&= \rho (\Delta \nabla \ln \rho)_j + \sum_i \rho \partial_i \ln \rho \partial_j \partial_i \ln \rho), \\
&= \rho (\Delta \nabla \ln \rho)_j + \frac{\rho}{2} (\nabla (|\nabla \ln \rho|^2))_j, \\
&= \mathrm{div} K.
\end{aligned}$$

We have then:

$$\mathrm{div} K = \kappa \mathrm{div}(\rho \nabla \nabla \ln \rho) = \kappa \mathrm{div}(\rho D(\nabla \ln \rho)).$$

□

References

[1] D.M. Anderson, G.B McFadden and A.A. Wheller. Diffuse-interface methods in fluid mech. In Annal review of fluid mechanics, Vol. 30, pages 139-165. Annual Reviews, Palo Alto, CA, 1998.

[2] P. Antonelli and P. Marcati, On the finite energy weak solutions to a system in quantum fluid dynamics, *Comm. Maths. Phys* 2009.

[3] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, *Springer*, Berlin (to appear).

[4] F. Bethuel, P. Gravejat and J-C Saut, Travelling waves for the Gross-Pitaevskii equation II, *Comm. Math. Phys.*, 285, 2, 2009, 567-651.

[5] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Annales Scientifiques de l'école Normale Supérieure 14 (1981) 209-246.

[6] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system, I. Interfacial free energy, J. Chem. Phys. 28 (1998) 258-267.

[7] M. Cannone, Y. Meyer and F. Planchon. Solutions auto-similaires des équations de Navier-Stokes. *Séminaire sur les équations aux dérivées partielles, 1993-1994*, exp. No12 pp. École polytech, palaiseau, 1994.

[8] M. Cannone and F. Planchon, On the Regularity of the Bilinear Term for Solutions to the Incompressible Navier-Stokes Equations, *Revista Matematica Iberoamericana*, vol 16, no 1, 2000.

[9] F. Charve, Convergence of a low order non-local Navier-Stokes-Korteweg system: the order-parameter model, preprint (hal 2013).

[10] F. Charve and B. Haspot, Convergence of capillary fluid models: from the non-local to the local Korteweg model, *Indiana University Mathematics Journal* **6** (2011) 2021-2060.

[11] F. Charve and B. Haspot, On a Lagrangian method for the convergence of a non-local to a local Korteweg capillary fluid model, *Preprint*.

[12] F. Charve and B. Haspot, Existence of global strong solution and vanishing capillarity-viscosity limit in one dimension for the Korteweg system, to appear in *SIAM Journal on Mathematical Analysis*.

[13] J.-Y. Chemin and I. Gallagher, On the global wellposedness of the 3-D incompressible Navier-Stokes equations, *Annales Scientifiques de l'Ecole Normale Supérieure de Paris*, 39 (2006), 679-698.

[14] J.-Y. Chemin and I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in $\mathbb{R}^3$, *Annales de l'Institut H. Poincaré*, Analyse non Linéaire, 26 (2009), no. 2, 599624.

[15] R. Danchin and B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, Annales de l'IHP,Analyse non linéaire 18,97-133 (2001)

[16] J.E. Dunn and J. Serrin, On the thermomechanics of interstitial working , Arch. Rational Mech. Anal. 88(2) (1985) 95-133.

[17] P. Gérard. The Cauchy problem for the Gross-Pitaevskii equation. *Ann. Inst. Henri Poincaré, Analyse Non Linéaire*, 23(5): 765-779, 2006.

[18] M. E. Gurtin, D. Poligone and J. Vinals, Two-phases binary fluids and immiscible fluids described by an order parameter, Math. Models Methods Appl. Sci.. 6(6) (1996) 815–831.

[19] B. Haspot. New entropy for Kortewegs system, existence of global weak solution and Prodi-Serrin theorem, *preprint*.

[20] B. Haspot, Porous media, Fast diffusion equations and the existence of global weak solution for the quasi-solution of compressible Navier-Stokes equations, *preprint* hal (January 2013).

[21] B. Haspot, Cauchy problem for viscous shallow water equations with a term of capillarity, *M3AS*, 20 (7) (2010), 1049-1087.

[22] B. Haspot, Existence of solutions for compressible fluid models of Korteweg type, *Annales Mathématiques Blaise Pascal* **16**, 431-481 (2009).

[23] B. Haspot, Existence of global weak solution for compressible fluid models with a capillary tensor for discontinuous interfaces, *Differential and Integral Equations*, Volume 23 Numbers 9-10, September/October 2010.

[24] B. Haspot, Existence of global strong solutions in critical spaces for barotropic viscous fuids, *Archive for Rational Mechanics and Analysis*, Volume 202, Issue 2 (2011), Pages 427-460.

[25] B. Haspot, Global existence of strong solution for shallow water system with large initial data on the irrotational part, *preprint* arXiv:1201.5456.

[26] B. Haspot, Global existence of strong solution for shallow water system with large initial data on the rotational part, *preprint and submitted*.

[27] B. Haspot, Existence of strong global solutions for the shallow-water equations with large initial data, *preprint* arXiv:1110.6100.

[28] B. Haspot, Global existence of strong solution for the Saint-Venant system with large initial data on the irrotational part, *C. R. Math. Acad. Sci. Paris,* Vol 350 - N 5-6 - mars 2012 229-332.

[29] B. Haspot, Existence of global strong solutions for the barotropic Navier Stokes system system with large initial data on the rotational part of the velocity, *C. R. Math. Acad. Sci. Paris.* , 350 (2012), pp. 487-492 .

[30] B. Haspot, On the porous medium equations, fast diffusion equations and compressible Navier-Stokes equations, new results on the quasi-solutions and on the scaling of the equations, submitted to *C. R. Math. Acad. Sci. Paris.*

[31] H. Hattori and D.Li, The existence of global solutions to a fluid dynamic model for materials for Korteweg type, J. Partial Differential Equations 9(4) (1996) 323-342.

[32] H. Hattori and D. Li, Global Solutions of a high-dimensional system for Korteweg materials, J. Math. Anal. Appl. 198(1) (1996) 84-97.

[33] D. Jamet, O. Lebaigue, N. Coutris and J.M. Delhaye, The second gradient method for the direct numerical simulation of liquid-vapor flows with phase change. J. Comput. Phys, 169(2): 624–651, (2001).

[34] A. Jüngel, Global weak solutions to compressible Navier-Stokes equations for quantum fluids, *SIAM journal on mathematical analysis*, 2011, vol. 42, no3, pp. 1025-1045.

[35] D.J. Korteweg. Sur la forme que prennent les équations du mouvement des fluides si l'on tient compte des forces capillaires par des variations de densité. Arch. Néer. Sci. Exactes Sér. II, 6 :1-24, 1901.

[36] M. Kotschote. Strong solutions for a compressible fluid model of Korteweg type. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, Volume 25, Issue 4, July-August 2008, 679-696

[37] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol 2, Compressible models, Oxford University Press (1996)

[38] C. Rohde, A local and low-order Navier-Stokes-Korteweg system. Preprint-Reihe IANS, Universität Stuttgart (2009).

[39] C. Rohde, On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions. *ZAMM Z. Angew. Math. Mech.* **85** (2005), no. 12, 839-857.

[40] C. Rohde, Approximation of Solutions of Conservation Laws by Non- Local Regularization and Discretization, *Habilitation Thesis*, University of Freiburg (2004).

[41] J.S. Rowlinson, Translation of J.D van der Waals "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density". J.Statist. Phys., 20(2): 197-244, 1979.

[42] C. Truedell and W. Noll. The nonlinear field theories of mechanics. Springer-Verlag, Berlin, second edition, 1992.

[43] J.F Van der Waals, Thermodynamische Theorie der Kapillarität unter Voraussetzung stetiger Dichteänderung, *Phys. Chem.* **13**, 657-725 (1894).