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The determinant of the Malliavin matrix and the determinant of the covariance matrix for multiple integrals

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Abstract

A well-known problem in Malliavin calculus concerns the relation between the determinant of the Malliavin matrix of a random vector and the determinant of its covariance matrix. We give an explicit relation between these two determinants for couples of random vectors of multiple integrals.

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Key words: multiple stochastic integrals, Wiener chaos, Malliavin matrix, covariance matrix, existence of density.

1 Introduction

The original motivation of the Malliavin calculus was to study the existence and the regularity of the densities of random variables. In this research direction, the determinant of the so-called Malliavin matrix plays a crucial role.

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We give here an explicit formula that connects the determinant of the Malliavin matrix and the determinant of the covariance matrix of a couple of multiple stochastic integrals. This is related to two open problems stated in [1]. In this reference, the authors showed that, if $F = (F_1, \dots, F_d)$ is a random vector whose components belong to a finite sum of Wiener chaoses, then the law of F is not absolutely continuous with respect to the Lebesgue measure if and only if $E \det \Lambda = 0$. Here Λ denotes the Malliavin matrix of the vector F . In particular, they proved that a couple of multiple integrals of order 2 either admits a density or its components are proportional.

They stated two open questions (Questions 6.1 and 6.2 in [1]): if C is the covariance matrix and Λ the Malliavin matrix of a vector of multiple stochastic integrals,

- is there true that $E \det \Lambda \geq c \det C$, with $c > 0$ an universal constant?
- is there true that the law of a vector of multiple integrals with components in the same Wiener chaos is either absolutely continuous with respect to the Lesque measure or its components are proportional?

We make a first step in order to answer to these two open problems. The basic idea is to write the Malliavin matrix as a sum of squares and to compute the dominant term of its determinant.

We organized our paper as follows. Section 2 contains some preliminaries on analysis on Wiener chaos. Section 3 is devoted to express the Malliavin matrix as the sum of the squares of some random variables and in Section 4 we derive an explicit formula for the determinant of Λ which also involves the determinant of the covariance matrix.

2 Preliminaries

We briefly describe the tools from the analysis on Wiener space that we will need in our work. For complete presentations, we refer to [4] or [2]. Let H be a real and separable Hilbert space and consider $(W(h), h \in H)$ an isonormal process. That is, $(W(h), h \in H)$ is a family of centered Gaussian random variables on the probability space (Ω, \mathcal{F}, P) such that $EW(h)W(g) = \langle f, g \rangle_H$ for every $h, g \in H$. Assume that the σ -algebra \mathcal{F} is generated by W .

Denote, for $n \geq 0$, by \mathcal{H}_n the n th Wiener chaos generated by W . That is, \mathcal{H}_n is the vector subspace of $L^2(\Omega)$ generated by $(H_n(W(h)), h \in H, \|h\| = 1)$ where H_n the Hermite polynomial of degree n . For any $n \geq 1$, the mapping $I_n(h^{\otimes n}) = H_n(W(h))$ can be extended to an isometry between the Hilbert space $H^{\otimes n}$ endowed with the

norm $\sqrt{n!} \|\cdot\|_{H^{\otimes n}}$ and the n th Wiener chaos \mathcal{H}_n . The random variable $I_n(f)$ is called the multiple Wiener Itô integral of f with respect to W .

Consider $(e_j)_{j \geq 1}$ a complete orthonormal system in H and let $f \in H^{\otimes n}$, $g \in H^{\otimes m}$ be two symmetric functions with $n, m \geq 1$. Then

$$f = \sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} e_{j_1} \otimes \dots \otimes e_{j_n} \quad (1)$$

and

$$g = \sum_{k_1, \dots, k_m \geq 1} \beta_{k_1, \dots, k_m} e_{k_1} \otimes \dots \otimes e_{k_m} \quad (2)$$

where the coefficients λ_i and β_j satisfy $\lambda_{j_{\sigma(1)}, \dots, j_{\sigma(n)}} = \lambda_{j_1, \dots, j_n}$ and $\beta_{k_{\pi(1)}, \dots, k_{\pi(m)}} = \beta_{k_1, \dots, k_m}$ for every permutation σ of the set $\{1, \dots, n\}$ and for every permutation π of the set $\{1, \dots, m\}$. Actually $\lambda_{j_1, \dots, j_n} = \langle f, e_{j_1} \otimes \dots \otimes e_{j_n} \rangle$ and $\beta_{k_1, \dots, k_m} = \langle g, e_{k_1} \otimes \dots \otimes e_{k_m} \rangle$ in (1) and (2). Note that, throughout the paper we will use the notation $\langle \cdot, \cdot \rangle$ to indicate the scalar product in $H^{\otimes k}$, independently of k .

If $f \in H^{\otimes n}$, $g \in H^{\otimes m}$ are symmetric given by (1), (2) respectively, then the contraction of order r of F and g is given by

$$\begin{aligned} f \otimes_r g &= \sum_{i_1, \dots, i_r \geq 1} \sum_{j_1, \dots, j_{n-r} \geq 1} \sum_{k_1, \dots, k_{m-r} \geq 1} \lambda_{i_1, \dots, i_r, j_1, \dots, j_{n-r}} \beta_{i_1, \dots, i_r, k_1, \dots, k_{m-r}} \\ &\quad \times (e_{j_1} \otimes \dots \otimes e_{j_{n-r}}) \otimes (e_{k_1} \otimes \dots \otimes e_{k_{m-r}}) \end{aligned} \quad (3)$$

for every $r = 0, \dots, m \wedge n$. In particular $f \otimes_0 g = f \otimes g$. Note that $f \otimes_r g$ belongs to $H^{\otimes(m+n-2r)}$ for every $r = 0, \dots, m \wedge n$ and it is not in general symmetric. We will denote by $f \tilde{\otimes}_r g$ the symmetrization of $f \otimes_r g$. In the particular case when $H = L^2(T, \mathcal{B}, \mu)$ where μ is a sigma-finite measure without atoms, (3) becomes

$$\begin{aligned} &(f \otimes_r g)(t_1, \dots, t_{m+n-2r}) \\ &= \int_{T^r} d\mu(u_1) \dots d\mu(u_r) f(u_1, \dots, u_r, t_1, \dots, t_{n-r}) g(u_1, \dots, u_r, t_{n-r+1}, \dots, t_{m+n-2r}) \end{aligned} \quad (4)$$

An important role will be played by the following product formula for multiple Wiener-Itô integrals: if $f \in H^{\otimes n}$, $g \in H^{\otimes m}$ are symmetric, then

$$I_n(f) I_m(g) = \sum_{r=0}^{m \wedge n} r! C_m^r C_n^r I_{m+n-2r}(f \tilde{\otimes}_r g). \quad (5)$$

We will need the concept of Malliavin derivative D with respect to W , but we will use only its action on Wiener chaos. In order to avoid too many details, we will just

say that, if f is given by (1) and $I_n(f)$ denotes its multiple integral of order n with respect to W , then

$$DI_n(f) = n \sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} I_{n-1}(e_{j_2} \otimes \dots \otimes e_{j_n}) e_{j_1}.$$

If F, G are two random variables which are differentiable in the Malliavin sense, we will denote throughout the paper by C the covariance matrix and by Λ the Malliavin matrix of the random vector (F, G) . That is,

$$\Lambda = \begin{pmatrix} \|DF\|_H^2 & \langle DF, DG \rangle_H \\ \langle DF, DG \rangle_H & \|DG\|_H^2 \end{pmatrix}.$$

3 The Malliavin matrix as a sum of squares

In this section we will express the determinant of the Malliavin matrix of a random couple as a sum of squares of certain random variables. This will be useful in order to derive the exact formula for the determinant of the Malliavin matrix and its connection with the determinant of the covariance matrix for a given random vector of dimension 2.

Let $f \in H^{\otimes n}$ and $g \in H^{\otimes m}$ be given by (1) and (2) respectively, with $n, m \geq 1$. Let $F = I_n(f), G = I_m(g)$ denote the multiple Wiener-Itô integrals of f and g with respect to W respectively. Then

$$I_n(f) = \sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} I_n(e_{j_1} \otimes \dots \otimes e_{j_n}) \quad (6)$$

and

$$I_m(g) = \sum_{k_1, \dots, k_m \geq 1} \beta_{k_1, \dots, k_m} I_m(e_{k_1} \otimes \dots \otimes e_{k_m}). \quad (7)$$

From (6) and (7) we have

$$DF = n \sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} I_{n-1}(e_{j_2} \otimes \dots \otimes e_{j_n}) e_{j_1}$$

and

$$DG = m \sum_{k_1, \dots, k_m \geq 1} \beta_{k_1, \dots, k_m} I_{m-1}(e_{k_2} \otimes \dots \otimes e_{k_m}) e_{k_1}.$$

This implies

$$\|DF\|_H^2 = n^2 \sum_{i \geq 1} \sum_{j_2, \dots, j_n \geq 1} \sum_{k_1, \dots, k_n \geq 1} \lambda_{i, j_2, \dots, j_n} \lambda_{i, k_2, \dots, k_n} I_{n-1}(e_{j_2} \otimes \dots \otimes e_{j_n}) I_{n-1}(e_{k_2} \otimes \dots \otimes e_{k_n})$$

and

$$\|DG\|_H^2 = m^2 \sum_{l \geq 1} \sum_{l, j_2, \dots, j_n \geq 1} \sum_{l, k_2, \dots, k_m \geq 1} \beta_{l, j_2, \dots, j_n} \beta_{l, k_2, \dots, k_m} I_{m-1}(e_{j_2} \otimes \dots \otimes e_{j_n}) I_{m-1}(e_{k_2} \otimes \dots \otimes e_{k_m})$$

and

$$\langle DF, DG \rangle_H = nm \sum_{i \geq 1} \sum_{j_2, \dots, j_n \geq 1} \sum_{k_1, \dots, k_m \geq 1} \lambda_{i, j_2, \dots, j_n} \beta_{i, j_1, \dots, j_m} I_{n-1}(e_{j_2} \otimes \dots \otimes e_{j_n}) I_{m-1}(e_{k_2} \otimes \dots \otimes e_{k_m}).$$

Let us make the following notation. For every $i \geq 1$, let

$$S_{i,f} = n \sum_{i \geq 1} \sum_{j_2, \dots, j_n \geq 1} \lambda_{i, j_2, \dots, j_n} I_{n-1}(e_{j_2} \otimes \dots \otimes e_{j_n}) \quad (8)$$

and

$$S_{i,g} = m \sum_{i \geq 1} \sum_{i, k_2, \dots, k_m \geq 1} \beta_{i, k_2, \dots, k_m} I_{m-1}(e_{k_2} \otimes \dots \otimes e_{k_m}). \quad (9)$$

We can write

$$\|DF\|_H^2 = \sum_{i \geq 1} S_{i,f}^2, \quad \|DG\|_H^2 = \sum_{l \geq 1} S_{l,g}^2, \quad \langle DF, DG \rangle_H = \sum_{i \geq 1} S_{i,f} S_{i,g}$$

and

$$\det(\Lambda) = \|DF\|_H^2 \|DG\|_H^2 - \langle DF, DG \rangle_H^2 = \sum_{i, l \geq 1} S_{i,f}^2 S_{l,g}^2 - \left(\sum_{i \geq 1} S_{i,f} S_{i,g} \right)^2.$$

A key observation is that

$$\sum_{i, l \geq 1} S_{i,f}^2 S_{l,g}^2 - \left(\sum_{i \geq 1} S_{i,f} S_{i,g} \right)^2 = \frac{1}{2} \sum_{i, l \geq 1} (S_{i,f} S_{l,g} - S_{l,f} S_{i,g})^2. \quad (10)$$

We obtained

Proposition 1 *The determinant of the Malliavin matrix Λ of the vector $(F, G) = (I_n(f), I_m(g))$ can be expressed as*

$$\det \Lambda = \frac{1}{2} \sum_{i, l \geq 1} (S_{i,f} S_{l,g} - S_{l,f} S_{i,g})^2$$

where $S_{i,f}, S_{i,g}$ are given by (8) and (9) respectively.

Corollary 1 *The determinant of the Malliavin matrix Λ of the vector $(F, G) = (I_n(f), I_m(g))$ can be expressed as*

$$\det \Lambda = \frac{1}{2} \sum_{i,l \geq 1} (\langle DF, e_i \rangle \langle DG, e_l \rangle - \langle DF, e_l \rangle \langle DG, e_i \rangle)^2$$

Proof: *This comes from Proposition 1 and the relations*

$$S_{i,f} = \langle DF, e_i \rangle, \quad S_{i,g} = \langle DG, e_i \rangle$$

for every $i \geq 1$. ■

4 The determinant of the Malliavin matrix on Wiener chaos

Fix $n, m \geq 1$ and f, g in $H^{\otimes n}, H^{\otimes m}$ respectively defined by (1) and (2). Consider the random vector $(F, G) = (I_n(f), I_m(g))$ and denote by Λ its Malliavin matrix and by C its covariance matrix.

Let us compute $E \det \Lambda$. Denote, for every $i, l \geq 1$

$$s_{i,f} = n \sum_{j_2, \dots, j_m \geq 1} \lambda_{i,j_2, \dots, j_n} e_{j_2} \otimes \dots \otimes e_{j_n} \quad (11)$$

and

$$s_{l,g} = m \sum_{k_2, \dots, k_m \geq 1} \beta_{l,k_2, \dots, k_m} e_{k_2} \otimes \dots \otimes e_{k_m}. \quad (12)$$

Clearly, for every $i, l \geq 1$

$$S_{i,f} = I_{n-1}(s_{i,f}), \quad S_{i,g} = I_{m-1}(s_{i,g}). \quad (13)$$

The following lemma plays a key role in our construction.

Lemma 1 *If $f \in H^{\otimes n}$ and $g \in H^{\otimes m}$ are given by (1) and (2) respectively and $s_{i,f}, s_{i,g}$ by (11), (12) respectively, then for every $r = 0, \dots, n-1$*

$$f \otimes_{r+1} g = \frac{1}{nm} \sum_{i \geq 1} (s_{i,f} \otimes_r s_{i,g}).$$

Proof: Consider first $r = 0$. Clearly, by (3)

$$\begin{aligned} f \otimes_1 g &= \sum_{i \geq 1} \sum_{j_2, \dots, j_n \geq 1} \sum_{k_2, \dots, k_m \geq 1} \lambda_{i, j_2, \dots, j_n} \beta_{i, k_2, \dots, k_m} e_{j_2} \otimes \dots \otimes e_{j_n} \otimes e_{k_2} \otimes \dots \otimes e_{k_m} \\ &= \frac{1}{nm} \sum_{i \geq 1} (s_{i, f} \otimes s_{i, g}). \end{aligned}$$

The same argument applies for every $r = 1, \dots, n - 1$. Indeed,

$$\begin{aligned} &f \otimes_{r+1} g \\ &= \left(\sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} e_{j_1} \otimes \dots \otimes e_{j_n} \right) \otimes_r \left(\sum_{k_1, \dots, k_m \geq 1} \beta_{k_1, \dots, k_m} e_{k_1} \otimes \dots \otimes e_{k_m} \right) \\ &= \sum_{i_1, \dots, i_{r+1}} \sum_{j_{r+2}, \dots, j_n} \sum_{k_{r+2}, \dots, k_m} \lambda_{i_1, \dots, i_{r+1}, j_{r+2}, \dots, j_n} \beta_{i_1, \dots, i_{r+1}, k_{r+2}, \dots, k_m} (e_{j_{r+2}} \otimes \dots \otimes e_{j_n}) \otimes (e_{k_{r+2}}, \dots, e_{k_m}) \end{aligned}$$

and by (3) again

$$\begin{aligned} &\sum_{i \geq 1} s_{i, j} \otimes_r s_{i, g} \\ &= nm \sum_{i \geq 1} \left(\sum_{j_2, \dots, j_n \geq 1} \lambda_{i, j_2, \dots, j_n} e_{j_2} \otimes \dots \otimes e_{j_n} \right) \otimes_r \left(\sum_{k_2, \dots, k_m \geq 1} \beta_{i, k_2, \dots, k_m} e_{k_2} \otimes \dots \otimes e_{k_m} \right) \\ &= nm \sum_{i \geq 1} \sum_{i_2, \dots, i_{r+1}} \sum_{j_{r+2}, \dots, j_n} \sum_{k_{r+2}, \dots, k_m} \lambda_{i, i_2, \dots, i_{r+1}, j_{r+2}, \dots, j_n} \beta_{i, i_2, \dots, i_{r+1}, k_{r+2}, \dots, k_m} \\ &\quad \times (e_{j_{r+2}} \otimes \dots \otimes e_{j_n}) \otimes (e_{k_{r+2}}, \dots, e_{k_m}) \\ &= nm \sum_{i_1, \dots, i_{r+1}} \sum_{j_{r+2}, \dots, j_n} \sum_{k_{r+2}, \dots, k_m} \lambda_{i_1, \dots, i_{r+1}, j_{r+2}, \dots, j_n} \beta_{i_1, \dots, i_{r+1}, k_{r+2}, \dots, k_m} (e_{j_{r+2}} \otimes \dots \otimes e_{j_n}) \otimes (e_{k_{r+2}}, \dots, e_{k_m}) \\ &= f \otimes_{r+1} g. \end{aligned}$$

■

We make a first step to compute $E \det \Lambda$.

Lemma 2 *Let $f \in H^{\otimes n}, g \in H^{\otimes m}$ be symmetric and denote by Λ the Malliavin matrix of the vector $(F, G) = (I_n(f), I_m(g))$. Then we have*

$$E \det \Lambda = \sum_{k=0}^{(n-1) \wedge (m-1)} T_k$$

where we denote, for $k = 0, \dots, (m-1) \wedge (n-1)$,

$$T_k := \frac{1}{2} \sum_{i,l \geq 1} k!^2 (C_{m-1}^k)^2 (C_{m-1}^k)^2 (m+n-2-2k)! \|s_{i,f} \tilde{\otimes}_k s_{l,g} - s_{l,f} \tilde{\otimes}_k s_{i,g}\|^2 \quad (14)$$

and $s_{i,f}, s_{i,g}$ are given by (11), (12) for $i \geq 1$.

Proof: By Proposition 1 and relation (13)

$$\begin{aligned} 2 \det \Lambda &= \sum_{i,l \geq 1} (I_{n-1}(s_{i,f}) I_{m-1}(s_{l,g}) - I_{n-1}(s_{l,f}) I_{m-1}(s_{i,g}))^2 \\ &= \sum_{i,l \geq 1} \left(\sum_{k=0}^{(m-1) \wedge (n-1)} k! C_{m-1}^k C_{n-1}^k I_{m+n-2-2k} (s_{i,f} \tilde{\otimes}_k s_{l,g} - s_{l,f} \tilde{\otimes}_k s_{i,g}) \right)^2 \end{aligned}$$

where we used the the product formula (5). Consequently, from the isometry of multiple stochastic integrals,

$$\begin{aligned} E \det \Lambda &= \frac{1}{2} \sum_{i,l \geq 1} \sum_{k=0}^{(n-1) \wedge (m-1)} k!^2 (C_{m-1}^k)^2 (C_{n-1}^k)^2 (m+n-2-2k)! \|s_{i,f} \tilde{\otimes}_k s_{l,g} - s_{l,f} \tilde{\otimes}_k s_{i,g}\|^2 \\ &= \sum_{k=0}^{(n-1) \wedge (m-1)} T_k. \end{aligned}$$

■

For every $n, m \geq 1$ let us denote by

$$R_{n,m} := \sum_{k=1}^{(n-1) \wedge (m-1)} T_k, \quad R_n := R_{n,n}. \quad (15)$$

Remark 1 Obviously all the terms T_k above are positive, for $k = 0, \dots, (n-1) \wedge (n-1)$.

We will need two more auxiliary lemmas.

Lemma 3 Assume $f_1, f_3 \in H^{\otimes n}$ and $f_2, f_4 \in H^{\otimes m}$ are symmetric functions. Then for every $r = 0, \dots, (m-1) \wedge (n-1)$ we have

$$\langle f_1 \otimes_{n-r} f_3, f_2 \otimes_{m-r} f_4 \rangle = \langle f_1 \otimes_r f_2, f_3 \otimes_r f_4 \rangle.$$

Proof: The case $r = 0$ is trivial, so assume $r \geq 1$. Without any loss of the generality, assume that H is $L^2(T; \mu)$ where μ is a sigma-finite measure without atoms. Then, by (4)

$$\begin{aligned}
& \langle f_1 \otimes_{n-r} f_3, f_2 \otimes_{m-r} f_4 \rangle \\
& \int_{T^r} d\mu^r(t_1, \dots, t_r) \int_{T^r} d\mu^r(s_1, \dots, s_r) \\
& \left(\int_{T^{n-r}} d\mu^{n-r}(u_1, \dots, u_{n-r}) f_1(u_1, \dots, u_{n-r}, t_1, \dots, t_r) f_3(u_1, \dots, u_{n-r}, s_1, \dots, s_r) \right) \\
& \left(\int_{T^{m-r}} d\mu^{m-r}(v_1, \dots, v_{m-r}) f_2(v_1, \dots, v_{m-r}, t_1, \dots, t_r) f_4(v_1, \dots, v_{m-r}, s_1, \dots, s_r) \right) \\
& = \int_{T^{n-r}} d\mu^{n-r}(u_1, \dots, u_{n-r}) \int_{T^{m-r}} d\mu^{m-r}(v_1, \dots, v_{m-r}) \\
& (f_1 \otimes_r f_2)(u_1, \dots, u_{n-r}, v_1, \dots, v_{m-r}) (f_3 \otimes_r f_4)(u_1, \dots, u_{n-r}, v_1, \dots, v_{m-r}) \\
& = \langle f_1 \otimes_r f_2, f_3 \otimes_r f_4 \rangle.
\end{aligned}$$

■

Lemma 4 Suppose $f_1, f_4 \in H^{\otimes n}, f_2, f_3 \in H^{\otimes m}$ are symmetric functions. Then

$$\langle f_1 \tilde{\otimes} f_2, f_3 \tilde{\otimes} f_4 \rangle = \frac{m!n!}{(m+n)!} \sum_{r=0}^{m \wedge n} C_n^r C_m^r \langle f_1 \otimes_r f_3, f_4 \otimes_r f_2 \rangle.$$

Proof: This has been stated and proven in [3] in the case $m = n$. Exactly the same lines of the proofs apply for $m \neq n$. ■

We first compute the term T_0 obtained for $k = 0$ in (14).

Proposition 2 Let T_0 be given by (14) with $k = 0$.

$$T_0 = \sum_{r=0}^{(n-1) \wedge (m-1)} mn m! n! C_{n-1}^r C_{m-1}^r [\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2].$$

Proof: From (14),

$$\begin{aligned}
T_0 & = \frac{1}{2}(m+n-2)! \sum_{i,l \geq 1} \|s_{i,f} \tilde{\otimes} s_{l,g} - s_{l,f} \tilde{\otimes} s_{i,g}\|^2 \\
& = \frac{1}{2}(m+n-2)! \sum_{i,l \geq 1} [\|s_{i,f} \tilde{\otimes} s_{l,g}\|^2 + \|s_{l,f} \tilde{\otimes} s_{i,g}\|^2 - 2\langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{l,f} \tilde{\otimes} s_{i,g} \rangle].
\end{aligned}$$

Let us apply Lemma 4 to compute these norms and scalar products. We obtain, by letting $f_1 = s_{i,f} = f_4$ and $f_2 = s_{l,g} = f_3$ (note that $s_{i,f}, s_{i,g}$ are symmetric functions in $H^{\otimes n}, H^{\otimes m}$ respectively)

$$\begin{aligned}
(m+n-2)! \langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{i,f} \tilde{\otimes} s_{l,g} \rangle &= (m+n-2)! \langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{l,g} \tilde{\otimes} s_{i,f} \rangle \\
&= (m-1)!(n-1)! \sum_{r=0}^{(n-1) \wedge (m-1)} C_{n-1}^r C_{m-1}^r \langle s_{i,f} \otimes_r s_{l,g}, s_{i,f} \otimes_r s_{l,g} \rangle \\
&= (m-1)!(n-1)! \sum_{r=0}^{(n-1) \wedge (m-1)} C_{n-1}^r C_{m-1}^r \|s_{i,f} \otimes_r s_{l,g}\|^2.
\end{aligned}$$

Analogously, for $f_1 = s_{l,f} = f_4$ and $f_2 = s_{i,g} = f_3$ in Lemma 4 we get

$$\begin{aligned}
(m+n-2)! \langle s_{l,f} \tilde{\otimes} s_{i,g}, s_{l,f} \tilde{\otimes} s_{i,g} \rangle \\
= \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \|s_{l,f} \otimes_r s_{i,g}\|^2.
\end{aligned}$$

Next, with $f_1 = s_{i,f}, f_2 = s_{l,g}, f_4 = s_{l,f}, f_3 = s_{i,g}$

$$\begin{aligned}
(m+n-2)! \langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{l,f} \tilde{\otimes} s_{i,g} \rangle \\
= (m+n-2)! \langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{i,g} \tilde{\otimes} s_{l,f} \rangle \\
= \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \langle s_{i,f} \otimes_r s_{i,g}, s_{l,f} \otimes_r s_{l,g} \rangle.
\end{aligned}$$

Then

$$\begin{aligned}
& \sum_{i,l \geq 1} (m+n-2)! \sum_{i,l \geq 1} \|s_{i,f} \tilde{\otimes} s_{l,g} - s_{l,f} \tilde{\otimes} s_{i,g}\|^2 \\
= & \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \\
& \sum_{i,l \geq 1} [\|s_{l,f} \otimes_r s_{i,g}\|^2 + \|s_{i,f} \otimes_r s_{l,g}\|^2 - 2\langle s_{i,f} \otimes_r s_{i,g}, s_{l,f} \otimes_r s_{l,g} \rangle] \\
= & 2 \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \sum_{i,l \geq 1} [\|s_{i,f} \otimes_r s_{l,g}\|^2 - \langle s_{i,f} \otimes_r s_{i,g}, s_{l,f} \otimes_r s_{l,g} \rangle] \\
= & 2 \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \\
& \times \left[\sum_{i,l \geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 - \left\langle \sum_{i \geq 1} s_{i,f} \otimes_r s_{i,g}, \sum_{l \geq 1} s_{l,f} \otimes_r s_{l,g} \right\rangle \right] \\
= & 2 \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \left[\sum_{i,l \geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 - \left\| \sum_{i \geq 1} s_{i,f} \otimes_r s_{i,g} \right\|^2 \right]. \quad (16)
\end{aligned}$$

Notice that, by Lemma 1, for every $r = 0, \dots, n-1$

$$\left\| \sum_{i \geq 1} s_{i,f} \otimes_r s_{i,g} \right\|^2 = n^2 m^2 \|f \otimes_{r+1} g\|^2. \quad (17)$$

We apply now Lemma 3 and we get

$$\begin{aligned}
\sum_{i,l \geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 &= \sum_{i,l \geq 1} \langle s_{i,f} \otimes_r s_{l,g}, s_{i,f} \otimes_r s_{l,g} \rangle \\
&= \sum_{i,l \geq 1} \langle s_{i,f} \otimes_{n-1-r} s_{i,f}, s_{l,g} \otimes_{m-1-r} s_{l,g} \rangle \\
&= \left\langle \sum_{i \geq 1} \langle s_{i,f} \otimes_{n-1-r} s_{i,f}, \sum_{l \geq 1} s_{l,g} \otimes_{m-r-1} s_{l,g} \rangle \right.
\end{aligned}$$

and by Lemma 1 and Lemma 3, this equals

$$\begin{aligned}
\sum_{i,l \geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 &= n^2 m^2 \langle f \otimes_{n-r} f, g \otimes_{m-r} g \rangle \\
&= n^2 m^2 \|f \otimes_r g\|^2. \quad (18)
\end{aligned}$$

By replacing (17) and (18) in (16) we obtain

$$\begin{aligned} T_0 &= \frac{1}{2}(m+n-2)! \sum_{i,l \geq 1} \|s_{i,f} \tilde{\otimes} s_{l,g} - s_{l,f} \tilde{\otimes} s_{i,g}\|^2 \\ &= \sum_{r=0}^{(n-1) \wedge (m-1)} mnm!n!C_{n-1}^r C_{m-1}^r [\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2]. \end{aligned}$$

■

As a consequence of the above proof, we obtain a very interesting result with multiple consequences on the Fourth Moment Theorem and on the existence of densities of random variables.

Proposition 3 *For every $r = 0, \dots, (m-1) \wedge (n-1)$ and if $s_{i,f}, s_{i,g}$ are given by (11), (12), it holds that*

$$n^2 m^2 \sum_{r=0}^{n-1} C_{m-1}^r C_{m-1}^r [\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2] = \sum_{i,l \geq 1} \|s_{i,f} \tilde{\otimes} s_{l,g} - s_{l,f} \tilde{\otimes} s_{i,g}\|^2.$$

As a consequence, for every $r = 0, \dots, (m-1) \wedge (n-1)$ we have

$$\sum_{r=0}^{n-1} C_{m-1}^r C_{m-1}^r [\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2] \geq 0. \quad (19)$$

Proof: It is a consequence of the proof of Proposition 2. ■

Let us state the main results of this section.

Theorem 1 *Let $f \in H^{\otimes n}, g \in H^{\otimes m} (n, m \geq 1)$ be symmetric and denote by Λ the Malliavin matrix of the vector $(F, G) = (I_n(f), I_m(g))$. Then*

$$\det \Lambda = \sum_{r=0}^{(n-1) \wedge (m-1)} mnm!n!C_{n-1}^r C_{m-1}^r [\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2] + R_{n,m}$$

where for every $n, m \geq 1$, $R_{n,m}$ is given by (15). Note that $R_{n,m} \geq 0$ for every $n, m \geq 1$.

Proof: It follows from Proposition 2 and Lemma 2. ■

In the case when the two multiple integrals live in the same Wiener chaos, we have a nicer expression.

Theorem 2 Under the same assumptions as in Theorem 1 but with $m = n$, we have

$$\det \Lambda = m^2 \det C + (mm!)^2 \sum_{r=1}^{\lfloor \frac{m-1}{2} \rfloor} ((C_{m-1}^r)^2 - (C_{m-1}^{r-1})^2) (\|f \otimes_r g\|^2 - \|f \otimes_{n-r} g\|^2) + R_m$$

with R_m given by (15). Here $[x]$ denotes the integer part of x .

Proof: Suppose $n \leq m$ and that m is odd. The case m even is similar. From we have

$$\begin{aligned} \det \Lambda &= (mm!)^2 \left[\sum_{r=0}^{(m-1)} (C_{m-1}^r)^2 \|f \otimes_r g\|^2 - \sum_{r=0}^{(m-1)} (C_{m-1}^r)^2 \|f \otimes_{r+1} g\|^2 \right] \\ &= (mm!)^2 \left[\sum_{r=0}^{\lfloor \frac{m-1}{2} \rfloor} (C_{m-1}^r)^2 \|f \otimes_r g\|^2 - \sum_{r=\frac{m-1}{2}}^{(m-1)} (C_{m-1}^r)^2 \|f \otimes_{r+1} g\|^2 \right] \\ &\quad + (mm!)^2 \left[\sum_{r=\frac{m-1}{2}}^{m-1} (C_{m-1}^r)^2 \|f \otimes_r g\|^2 - \sum_{r=0}^{\frac{m-1}{2}} (C_{m-1}^r)^2 \|f \otimes_{r+1} g\|^2 \right] \\ &= (mm!)^2 \sum_{r=0}^{\frac{m-1}{2}} (C_{m-1}^r)^2 [\|f \otimes_r g\|^2 - \|f \otimes_{n-r} g\|^2] \\ &\quad + (mm!)^2 \sum_{r=1}^{\frac{m-1}{2}} (C_{m-1}^r)^2 [\|f \otimes_{n-r} g\|^2 - \|f \otimes_r g\|^2] \end{aligned}$$

where we made the change of index $r' = n - 1 - r$ in the second and third sum above. Finally, noticing that for $r = 0$ we have

$$m^2 m!^2 (C_{m-1}^0)^2 [\|f \otimes_0 g\|^2 - \|f \otimes_n g\|^2] = m^2 \det C$$

we obtain the conclusion. ■

Example 1 Suppose $m = n = 2$. Then

$$\begin{aligned} \det \Lambda &= 16 [\|f \otimes g\|^2 - \|f \otimes_2 g\|^2] + R_2 \\ &= 4 \det C + R_2. \end{aligned}$$

We retrieve the formula in [1] with $R_2 = 32 (\|f \otimes_1 g\|^2 - \|f \tilde{\otimes}_1 g\|^2)$.

Assume $m = n = 3$. Then

$$\begin{aligned}\det \Lambda &= 3 \times 3 \times 3! \times 3! \left[(\|f \otimes g\|^2 - \|f \otimes_3 g\|^2) + 9 (\|f \otimes_1 g\|^2 - \|f \otimes_2 g\|^2) \right] + R_3 \\ &= 9 \det C + 9 (\|f \otimes_1 g\|^2 - \|f \otimes_2 g\|^2) + R_3.\end{aligned}$$

Suppose $m = n = 4$. Then

$$\det \Lambda = 16 \det C + 9 \times 4! \times 4! (\|f \otimes_1 g\|^2 - \|f \otimes_3 g\|^2) + R_4.$$

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