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# The determinant of the Malliavin matrix and the determinant of the covariance matrix for multiple integrals

Ciprian A. Tudor \*

Laboratoire Paul Painlevé, Université de Lille 1

F-59655 Villeneuve d'Ascq, France

and

Academy of Economical Studies, Bucharest, Romania

tudor@math.univ-lille1.fr

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## Abstract

A well-known problem in Malliavin calculus concerns the relation between the determinant of the Malliavin matrix of a random vector and the determinant of its covariance matrix. We give an explicit relation between these two determinants for couples of random vectors of multiple integrals. In particular, if the multiple integrals are of the same order, we show that the expectation of the determinant of the Malliavin matrix is bigger than the determinant of the covariance matrix multiplied by a positive constant. As a consequence we prove that two random variables in the same Wiener chaos either admit a joint density, either are proportional and that the result is not true for random variables in Wiener chaoses of different orders. We will also relax a condition in the Fourth Moment Theorem proven by Nualart and Peccati in [5].

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**Key words:** multiple stochastic integrals, Wiener chaos, Malliavin matrix, covariance matrix, existence of density.

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# 1 Introduction

The original motivation of the Malliavin calculus was to study the existence and the regularity of the densities of random variables. In this research direction, the determinant of the so-called Malliavin matrix plays a crucial role.

We give here an explicit formula that connects the determinant of the Malliavin matrix and the determinant of the covariance matrix of a couple of multiple stochastic integrals. This is related to two open problems stated in [1]. In this reference, the authors showed that, if  $F = (F_1, \dots, F_d)$  is a random vector whose components belong to a finite sum of Wiener chaoses, then the law of  $F$  is not absolutely continuous with respect to the Lebesgue measure if and only if  $E \det \Lambda = 0$ . Here  $\Lambda$  denotes the Malliavin matrix of the vector  $F$ . In particular, they proved that a couple of multiple integrals of order 2 either admits a density or its components are proportional.

They stated two open questions (Questions 6.1 and 6.2 in [1]): if  $C$  is the covariance matrix and  $\Lambda$  the Malliavin matrix of a vector of multiple stochastic integrals,

- is there true that  $E \det \Lambda \geq c \det C$ , with  $c > 0$  an universal constant?
- is there true that the law of a vector of multiple integrals with components in the same Wiener chaos is either absolutely continuous with respect to the Lesque measure or its components are proportional?

We give in this paper a positive answer to these two open problems. The basic idea is to write the Malliavin matrix as a sum of squares and to compute the dominant term of its determinant. Besides, our proofs put in light other interesting facts. In particular, by proving that the norm of the contraction of order  $k$  of two symmetric functions is bigger than the norm of the contraction of order  $k + 1$  of the same functions, we relax a condition in the Fourth Moment Theorem in [5].

We organized our paper as follows. Section 2 contains some preliminaries on analysis on Wiener chaos. Section 3 is devoted to express the Malliavin matrix as the sum of the squares of some random variables and in Section 4 we derive an explicit formula for the determinant of  $\Lambda$  which also involves the determinant of the covariance matrix. In Sections 5 and 6 we obtain some consequences on the existence of the density of random vectors in Wiener chaos and on the Fourth Moment Theorem.

## 2 Preliminaries

We briefly describe the tools from the analysis on Wiener space that we will need in our work. For complete presentations, we refer to [4] or [2]. Let  $H$  be a real and

separable Hilbert space and consider  $(W(h), h \in H)$  an isonormal process. That is,  $(W(h), h \in H)$  is a family of centered Gaussian random variables on the probability space  $(\Omega, \mathcal{F}, P)$  such that  $EW(h)W(g) = \langle f, g \rangle_H$  for every  $h, g \in H$ . Assume that the  $\sigma$ -algebra  $\mathcal{F}$  is generated by  $W$ .

Denote, for  $n \geq 0$ , by  $\mathcal{H}_n$  the  $n$ th Wiener chaos generated by  $W$ . That is,  $\mathcal{H}_n$  is the vector subspace of  $L^2(\Omega)$  generated by  $(H_n(W(h)), h \in H, \|h\| = 1)$  where  $H_n$  the Hermite polynomial of degree  $n$ . For any  $n \geq 1$ , the mapping  $I_n(h^{\otimes n}) = H_n(W(h))$  can be extended to an isometry between the Hilbert space  $H^{\otimes n}$  endowed with the norm  $\sqrt{n!} \|\cdot\|_{H^{\otimes n}}$  and the  $n$ th Wiener chaos  $\mathcal{H}_n$ . The random variable  $I_n(f)$  is called the multiple Wiener Itô integral of  $f$  with respect to  $W$ .

Consider  $(e_j)_{j \geq 1}$  a complete orthonormal system in  $H$  and let  $f \in H^{\otimes n}$ ,  $g \in H^{\otimes m}$  be two symmetric functions with  $n, m \geq 1$ . Then

$$f = \sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} e_{j_1} \otimes \dots \otimes e_{j_n} \quad (1)$$

and

$$g = \sum_{k_1, \dots, k_m \geq 1} \beta_{k_1, \dots, k_m} e_{k_1} \otimes \dots \otimes e_{k_m} \quad (2)$$

where the coefficients  $\lambda_i$  and  $\beta_j$  satisfy  $\lambda_{j_{\sigma(1)}, \dots, j_{\sigma(n)}} = \lambda_{j_1, \dots, j_n}$  and  $\beta_{k_{\pi(1)}, \dots, k_{\pi(m)}} = \beta_{k_1, \dots, k_m}$  for every permutation  $\sigma$  of the set  $\{1, \dots, n\}$  and for every permutation  $\pi$  of the set  $\{1, \dots, m\}$ . Actually  $\lambda_{j_1, \dots, j_n} = \langle f, e_{j_1} \otimes \dots \otimes e_{j_n} \rangle$  and  $\beta_{k_1, \dots, k_m} = \langle g, e_{k_1} \otimes \dots \otimes e_{k_m} \rangle$  in (1) and (2). Note that, throughout the paper we will use the notation  $\langle \cdot, \cdot \rangle$  to indicate the scalar product in  $H^{\otimes k}$ , independently of  $k$ .

If  $f \in H^{\otimes n}$ ,  $g \in H^{\otimes m}$  are symmetric given by (1), (2) respectively, then the contraction of order  $r$  of  $F$  and  $g$  is given by

$$\begin{aligned} f \otimes_r g &= \sum_{i_1, \dots, i_r \geq 1} \sum_{j_1, \dots, j_{n-r} \geq 1} \sum_{k_1, \dots, k_{m-r} \geq 1} \lambda_{i_1, \dots, i_r, j_1, \dots, j_{n-r}} \beta_{i_1, \dots, i_r, k_1, \dots, k_{m-r}} \\ &\quad \times (e_{j_1} \otimes \dots \otimes e_{j_{n-r}}) \otimes (e_{k_1} \otimes \dots \otimes e_{k_{m-r}}) \end{aligned} \quad (3)$$

for every  $r = 0, \dots, m \wedge n$ . In particular  $f \otimes_0 g = f \otimes g$ . Note that  $f \otimes_r g$  belongs to  $H^{\otimes(m+n-2r)}$  for every  $r = 0, \dots, m \wedge n$  and it is not in general symmetric. We will denote by  $f \tilde{\otimes}_r g$  the symmetrization of  $f \otimes_r g$ . In the particular case when  $H = L^2(T, \mathcal{B}, \mu)$  where  $\mu$  is a sigma-finite measure without atoms, (3) becomes

$$\begin{aligned} &(f \otimes_r g)(t_1, \dots, t_{m+n-2r}) \\ &= \int_{T^r} d\mu(u_1) \dots d\mu(u_r) f(u_1, \dots, u_r, t_1, \dots, t_{n-r}) g(u_1, \dots, u_r, t_{n-r+1}, \dots, t_{m+n-2r}) \end{aligned} \quad (4)$$

An important role will be played by the following product formula for multiple Wiener-Itô integrals: if  $f \in H^{\otimes n}$ ,  $g \in H^{\otimes m}$  are symmetric, then

$$I_n(f)I_m(g) = \sum_{r=0}^{m \wedge n} r! C_m^r C_n^r I_{m+n-2r}(f \tilde{\otimes}_r g). \quad (5)$$

We will need the concept of Malliavin derivative  $D$  with respect to  $W$ , but we will use only its action on Wiener chaos. In order to avoid too many details, we will just say that, if  $f$  is given by (1) and  $I_n(f)$  denotes its multiple integral of order  $n$  with respect to  $W$ , then

$$DI_n(f) = n \sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} I_{n-1}(e_{j_2} \otimes \dots \otimes e_{j_n}) e_{j_1}.$$

If  $F, G$  are two random variables which are differentiable in the Malliavin sense, we will denote throughout the paper by  $C$  the covariance matrix and by  $\Lambda$  the Malliavin matrix of the random vector  $(F, G)$ . That is,

$$\Lambda = \begin{pmatrix} \|DF\|_H^2 & \langle DF, DG \rangle_H \\ \langle DF, DG \rangle_H & \|DG\|_H^2 \end{pmatrix}.$$

### 3 The Malliavin matrix as a sum of squares

In this section we will express the determinant of the Malliavin matrix of a random couple as a sum of squares of certain random variables. This will be useful in order to derive the exact formula for the determinant of the Malliavin matrix and its connection with the determinant of the covariance matrix for a given random vector of dimension 2.

Let  $f \in H^{\otimes n}$  and  $g \in H^{\otimes m}$  be given by (1) and (2) respectively, with  $n, m \geq 1$ . Let  $F = I_n(f), G = I_m(g)$  denote the multiple Wiener-Itô integrals of  $f$  and  $g$  with respect to  $W$  respectively. Then

$$I_n(f) = \sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} I_n(e_{j_1} \otimes \dots \otimes e_{j_n}) \quad (6)$$

and

$$I_m(g) = \sum_{k_1, \dots, k_m \geq 1} \beta_{k_1, \dots, k_m} I_m(e_{k_1} \otimes \dots \otimes e_{k_m}). \quad (7)$$

From (6) and (7) we have

$$DF = n \sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} I_{n-1}(e_{j_2} \otimes \dots \otimes e_{j_n}) e_{j_1}$$

and

$$DG = m \sum_{k_1, \dots, k_m \geq 1} \beta_{k_1, \dots, k_m} I_{m-1} (e_{k_2} \otimes \dots \otimes e_{k_m}) e_{k_1}.$$

This implies

$$\|DF\|_H^2 = n^2 \sum_{i \geq 1} \sum_{j_2, \dots, j_n \geq 1} \sum_{k_1, \dots, k_n \geq 1} \lambda_{i, j_2, \dots, j_n} \lambda_{i, k_2, \dots, k_n} I_{n-1} (e_{j_2} \otimes \dots \otimes e_{j_n}) I_{n-1} (e_{k_2} \otimes \dots \otimes e_{k_n})$$

and

$$\|DG\|_H^2 = m^2 \sum_{l \geq 1} \sum_{l, j_2, \dots, j_m \geq 1} \sum_{l, k_2, \dots, k_m \geq 1} \beta_{l, j_2, \dots, j_m} \beta_{l, k_2, \dots, k_m} I_{m-1} (e_{j_2} \otimes \dots \otimes e_{j_m}) I_{m-1} (e_{k_2} \otimes \dots \otimes e_{k_m})$$

and

$$\langle DF, DG \rangle_H = nm \sum_{i \geq 1} \sum_{j_2, \dots, j_n \geq 1} \sum_{k_1, \dots, k_m \geq 1} \lambda_{i, j_2, \dots, j_n} \beta_{i, j_1, \dots, j_m} I_{n-1} (e_{j_2} \otimes \dots \otimes e_{j_n}) I_{m-1} (e_{k_2} \otimes \dots \otimes e_{k_m}).$$

Let us make the following notation. For every  $i \geq 1$ , let

$$S_{i,f} = n \sum_{i \geq 1} \sum_{j_2, \dots, j_n \geq 1} \lambda_{i, j_2, \dots, j_n} I_{n-1} (e_{j_2} \otimes \dots \otimes e_{j_n}) \quad (8)$$

and

$$S_{i,g} = m \sum_{i \geq 1} \sum_{i, k_2, \dots, k_m \geq 1} \beta_{i, k_2, \dots, k_m} I_{m-1} (e_{k_2} \otimes \dots \otimes e_{k_m}). \quad (9)$$

We can write

$$\|DF\|_H^2 = \sum_{i \geq 1} S_{i,f}^2, \quad \|DG\|_H^2 = \sum_{l \geq 1} S_{l,g}^2, \quad \langle DF, DG \rangle_H = \sum_{i \geq 1} S_{i,f} S_{i,g}$$

and

$$\det(\Lambda) = \|DF\|_H^2 \|DG\|_H^2 - \langle DF, DG \rangle_H^2 = \sum_{i, l \geq 1} S_{i,f}^2 S_{l,g}^2 - \left( \sum_{i \geq 1} S_{i,f} S_{i,g} \right)^2.$$

A key observation is that

$$\sum_{i, l \geq 1} S_{i,f}^2 S_{l,g}^2 - \left( \sum_{i \geq 1} S_{i,f} S_{i,g} \right)^2 = \frac{1}{2} \sum_{i, l \geq 1} (S_{i,f} S_{l,g} - S_{l,f} S_{i,g})^2. \quad (10)$$

We obtained

**Proposition 1** *The determinant of the Malliavin matrix  $\Lambda$  of the vector  $(F, G) = (I_n(f), I_m(g))$  can be expressed as*

$$\det \Lambda = \frac{1}{2} \sum_{i,l \geq 1} (S_{i,f} S_{l,g} - S_{l,f} S_{i,g})^2$$

where  $S_{i,f}, S_{i,g}$  are given by (8) and (9) respectively.

**Corollary 1** *The determinant of the Malliavin matrix  $\Lambda$  of the vector  $(F, G) = (I_n(f), I_m(g))$  can be expressed as*

$$\det \Lambda = \frac{1}{2} \sum_{i,l \geq 1} (\langle DF, e_i \rangle \langle DG, e_l \rangle - \langle DF, e_l \rangle \langle DG, e_i \rangle)^2$$

**Proof:** *This comes from Proposition 1 and the relations*

$$S_{i,f} = \langle DF, e_i \rangle, \quad S_{i,g} = \langle DG, e_i \rangle$$

for every  $i \geq 1$ . ■

## 4 The determinant of the Malliavin matrix on Wiener chaos

Fix  $n, m \geq 1$  and  $f, g$  in  $H^{\otimes n}, H^{\otimes m}$  respectively defined by (1) and (2). Consider the random vector  $(F, G) = (I_n(f), I_m(g))$  and denote by  $\Lambda$  its Malliavin matrix and by  $C$  its covariance matrix.

Let us compute  $E \det \Lambda$ . Denote, for every  $i, l \geq 1$

$$s_{i,f} = n \sum_{j_2, \dots, j_m \geq 1} \lambda_{i, j_2, \dots, j_n} e_{j_2} \otimes \dots \otimes e_{j_n} \quad (11)$$

and

$$s_{l,g} = m \sum_{k_2, \dots, k_m \geq 1} \beta_{l, k_2, \dots, k_m} e_{k_2} \otimes \dots \otimes e_{k_m}. \quad (12)$$

Clearly, for every  $i, l \geq 1$

$$S_{i,f} = I_{n-1}(s_{i,f}), \quad S_{i,g} = I_{m-1}(s_{i,g}). \quad (13)$$

The following lemma plays a key role in our construction.

**Lemma 1** *If  $f \in H^{\otimes n}$  and  $g \in H^{\otimes m}$  are given by (1) and (2) respectively and  $s_{i,f}, s_{i,g}$  by (11), (12) respectively, then for every  $r = 0, \dots, n-1$*

$$f \otimes_{r+1} g = \frac{1}{nm} \sum_{i \geq 1} (s_{i,f} \otimes_r s_{i,g}).$$

**Proof:** Consider first  $r = 0$ . Clearly, by (3)

$$\begin{aligned} f \otimes_1 g &= \sum_{i \geq 1} \sum_{j_2, \dots, j_n \geq 1} \sum_{k_2, \dots, k_m \geq 1} \lambda_{i, j_2, \dots, j_n} \beta_{i, k_2, \dots, k_m} e_{j_2} \otimes \dots \otimes e_{j_n} \otimes e_{k_2} \otimes \dots \otimes e_{k_m} \\ &= \frac{1}{nm} \sum_{i \geq 1} (s_{i,f} \otimes s_{i,g}). \end{aligned}$$

The same argument applies for every  $r = 1, \dots, n-1$ . Indeed,

$$\begin{aligned} &f \otimes_{r+1} g \\ &= \left( \sum_{j_1, \dots, j_n \geq 1} \lambda_{j_1, \dots, j_n} e_{j_1} \otimes \dots \otimes e_{j_n} \right) \otimes_r \left( \sum_{k_1, \dots, k_m \geq 1} \beta_{k_1, \dots, k_m} e_{k_1} \otimes \dots \otimes e_{k_m} \right) \\ &= \sum_{i_1, \dots, i_{r+1}} \sum_{j_{r+2}, \dots, j_n} \sum_{k_{r+2}, \dots, k_m} \lambda_{i_1, \dots, i_{r+1}, j_{r+2}, \dots, j_n} \beta_{i_1, \dots, i_{r+1}, k_{r+2}, \dots, k_m} (e_{j_{r+2}} \otimes \dots \otimes e_{j_n}) \otimes (e_{k_{r+2}}, \dots, e_{k_m}) \end{aligned}$$

and by (3) again

$$\begin{aligned} &\sum_{i \geq 1} s_{i,j} \otimes_r s_{i,g} \\ &= nm \sum_{i \geq 1} \left( \sum_{j_2, \dots, j_n \geq 1} \lambda_{i, j_2, \dots, j_n} e_{j_2} \otimes \dots \otimes e_{j_n} \right) \otimes_r \left( \sum_{k_2, \dots, k_m \geq 1} \beta_{i, k_2, \dots, k_m} e_{k_2} \otimes \dots \otimes e_{k_m} \right) \\ &= nm \sum_{i \geq 1} \sum_{i_2, \dots, i_{r+1}} \sum_{j_{r+2}, \dots, j_n} \sum_{k_{r+2}, \dots, k_m} \lambda_{i, i_2, \dots, i_{r+1}, j_{r+2}, \dots, j_n} \beta_{i, i_2, \dots, i_{r+1}, k_{r+2}, \dots, k_m} \\ &\quad \times (e_{j_{r+2}} \otimes \dots \otimes e_{j_n}) \otimes (e_{k_{r+2}}, \dots, e_{k_m}) \\ &= nm \sum_{i_1, \dots, i_{r+1}} \sum_{j_{r+2}, \dots, j_n} \sum_{k_{r+2}, \dots, k_m} \lambda_{i_1, \dots, i_{r+1}, j_{r+2}, \dots, j_n} \beta_{i_1, \dots, i_{r+1}, k_{r+2}, \dots, k_m} (e_{j_{r+2}} \otimes \dots \otimes e_{j_n}) \otimes (e_{k_{r+2}}, \dots, e_{k_m}) \\ &= f \otimes_{r+1} g. \end{aligned}$$

■

We make a first step to compute  $E \det \Lambda$ .



**Lemma 2** Let  $f \in H^{\otimes n}, g \in H^{\otimes m}$  be symmetric and denote by  $\Lambda$  the Malliavin matrix of the vector  $(F, G) = (I_n(f), I_m(g))$ . Then we have

$$E \det \Lambda = \sum_{k=0}^{(n-1) \wedge (m-1)} T_k$$

where we denote, for  $k = 0, \dots, (m-1) \wedge (n-1)$ ,

$$T_k := \frac{1}{2} \sum_{i,l \geq 1} k!^2 (C_{m-1}^k)^2 (C_{n-1}^k)^2 (m+n-2-2k)! \|s_{i,f} \tilde{\otimes}_k s_{l,g} - s_{l,f} \tilde{\otimes}_k s_{i,g}\|^2 \quad (14)$$

and  $s_{i,f}, s_{i,g}$  are given by (11), (12) for  $i \geq 1$ .

**Proof:** By Theorem 1 and relation (13)

$$\begin{aligned} 2 \det \Lambda &= \sum_{i,l \geq 1} (I_{n-1}(s_{i,f}) I_{m-1}(s_{l,g}) - I_{n-1}(s_{l,f}) I_{m-1}(s_{i,g}))^2 \\ &= \sum_{i,l \geq 1} \sum_{k=0}^{(m-1) \wedge (n-1)} k! C_{m-1}^k C_{n-1}^k I_{m+n-2-2k} (s_{i,f} \tilde{\otimes}_k s_{l,g} - s_{l,f} \tilde{\otimes}_k s_{i,g}) \end{aligned}$$

where we used the the product formula (5). Consequently, from the isometry of multiple stochastic integrals,

$$\begin{aligned} E \det \Lambda &= \frac{1}{2} \sum_{k=0}^{(n-1) \wedge (m-1)} k!^2 (C_{m-1}^k)^2 (C_{n-1}^k)^2 (m+n-2-2k)! \|s_{i,f} \tilde{\otimes}_k s_{l,g} - s_{l,f} \tilde{\otimes}_k s_{i,g}\|^2 \\ &= \sum_{k=0}^{(n-1) \wedge (m-1)} T_k. \end{aligned}$$

■

For every  $n, m \geq 1$  let us denote by

$$R_{n,m} := \sum_{k=1}^{(n-1) \wedge (m-1)} T_k, \quad R_n := R_{n,n}. \quad (15)$$

**Remark 1** Obviously all the terms  $T_k$  above are positive, for  $k = 0, \dots, (n-1) \wedge (n-1)$ .

We will need two more auxiliary lemmas.

**Lemma 3** Assume  $f_1, f_3 \in H^{\otimes n}$  and  $f_2, f_4 \in H^{\otimes m}$  are symmetric functions. Then for every  $r = 0, \dots, (m-1) \wedge (n-1)$  we have

$$\langle f_1 \otimes_{n-r} f_3, f_2 \otimes_{m-r} f_4 \rangle = \langle f_1 \otimes_r f_2, f_3 \otimes_r f_4 \rangle.$$

**Proof:** The case  $r = 0$  is trivial, so assume  $r \geq 1$ . Without any loss of the generality, assume that  $H$  is  $L^2(T; \mu)$  where  $\mu$  is a sigma-finite measure without atoms. Then, by (4)

$$\begin{aligned} & \langle f_1 \otimes_{n-r} f_3, f_2 \otimes_{m-r} f_4 \rangle \\ & \int_{T^r} d\mu^r(t_1, \dots, t_r) \int_{T^r} d\mu^r(s_1, \dots, s_r) \\ & \left( \int_{T^{n-r}} d\mu^{n-r}(u_1, \dots, u_{n-r}) f_1(u_1, \dots, u_{n-r}, t_1, \dots, t_r) f_3(u_1, \dots, u_{n-r}, s_1, \dots, s_r) \right) \\ & \left( \int_{T^{m-r}} d\mu^{m-r}(v_1, \dots, v_{m-r}) f_2(v_1, \dots, v_{m-r}, t_1, \dots, t_r) f_4(v_1, \dots, v_{m-r}, s_1, \dots, s_r) \right) \\ & = \int_{T^{n-r}} d\mu^{n-r}(u_1, \dots, u_{n-r}) \int_{T^{m-r}} d\mu^{m-r}(v_1, \dots, v_{m-r}) \\ & (f_1 \otimes_r f_2)(u_1, \dots, u_{n-r}, v_1, \dots, v_{m-r}) (f_3 \otimes_r f_4)(u_1, \dots, u_{n-r}, v_1, \dots, v_{m-r}) \\ & = \langle f_1 \otimes_r f_2, f_3 \otimes_r f_4 \rangle. \end{aligned}$$

■

**Lemma 4** Suppose  $f_1, f_4 \in H^{\otimes n}$ ,  $f_2, f_3 \in H^{\otimes m}$  are symmetric functions. Then

$$\langle f_1 \tilde{\otimes} f_2, f_3 \tilde{\otimes} f_4 \rangle = \frac{m!n!}{(m+n)!} \sum_{r=0}^{m \wedge n} C_n^r C_m^r \langle f_1 \otimes_r f_3, f_4 \otimes_r f_2 \rangle.$$

**Proof:** This has been stated and proven in [3] in the case  $m = n$ . Exactly the same lines of the proofs apply for  $m \neq n$ . ■

We first compute the term  $T_0$  obtained for  $k = 0$  in (14).

**Proposition 2** Let  $T_0$  be given by (14) with  $k = 0$ .

$$T_0 = \sum_{r=0}^{(n-1) \wedge (m-1)} mnm!n! C_{n-1}^r C_{m-1}^r [\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2].$$

**Proof:** From (14),

$$\begin{aligned}
T_0 &= \frac{1}{2}(m+n-2)! \sum_{i,l \geq 1} \|s_{i,f} \tilde{\otimes} s_{l,g} - s_{l,f} \tilde{\otimes} s_{i,g}\|^2 \\
&= \frac{1}{2}(m+n-2)! \sum_{i,l \geq 1} [\|s_{i,f} \tilde{\otimes} s_{l,g}\|^2 + \|s_{l,f} \tilde{\otimes} s_{i,g}\|^2 - 2\langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{l,f} \tilde{\otimes} s_{i,g} \rangle].
\end{aligned}$$

Let us apply Lemma 4 to compute these norms and scalar products. We obtain, by letting  $f_1 = s_{i,f} = f_4$  and  $f_2 = s_{l,g} = f_3$  (note that  $s_{i,f}, s_{i,g}$  are symmetric functions in  $H^{\otimes n}, H^{\otimes m}$  respectively)

$$\begin{aligned}
(m+n-2)! \langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{i,f} \tilde{\otimes} s_{l,g} \rangle &= (m+n-2)! \langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{l,g} \tilde{\otimes} s_{i,f} \rangle \\
&= (m-1)!(n-1)! \sum_{r=0}^{(n-1) \wedge (m-1)} C_{n-1}^r C_{m-1}^r \langle s_{i,f} \otimes_r s_{l,g}, s_{i,f} \otimes_r s_{l,g} \rangle \\
&= (m-1)!(n-1)! \sum_{r=0}^{(n-1) \wedge (m-1)} C_{n-1}^r C_{m-1}^r \|s_{i,f} \otimes_r s_{l,g}\|^2.
\end{aligned}$$

Analogously, for  $f_1 = s_{l,f} = f_4$  and  $f_2 = s_{i,g} = f_3$  in Lemma 4 we get

$$\begin{aligned}
&(m+n-2)! \langle s_{l,f} \tilde{\otimes} s_{i,g}, s_{l,f} \tilde{\otimes} s_{i,g} \rangle \\
&= \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \|s_{l,f} \otimes_r s_{i,g}\|^2.
\end{aligned}$$

Next, with  $f_1 = s_{i,f}, f_2 = s_{l,g}, f_4 = s_{l,f}, f_3 = s_{i,g}$

$$\begin{aligned}
&(m+n-2)! \langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{l,f} \tilde{\otimes} s_{i,g} \rangle \\
&= (m+n-2)! \langle s_{i,f} \tilde{\otimes} s_{l,g}, s_{i,g} \tilde{\otimes} s_{l,f} \rangle \\
&= \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \langle s_{i,f} \otimes_r s_{i,g}, s_{l,f} \otimes_r s_{l,g} \rangle.
\end{aligned}$$

Then

$$\begin{aligned}
& \sum_{i,l \geq 1} (m+n-2)! \sum_{i,l \geq 1} \|s_{i,f} \tilde{\otimes} s_{l,g} - s_{l,f} \tilde{\otimes} s_{i,g}\|^2 \\
&= \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \\
& \quad \sum_{i,l \geq 1} [\|s_{l,f} \otimes_r s_{i,g}\|^2 + \|s_{i,f} \otimes_r s_{l,g}\|^2 - 2\langle s_{i,f} \otimes_r s_{i,g}, s_{l,f} \otimes_r s_{l,g} \rangle] \\
&= 2 \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \sum_{i,l \geq 1} [\|s_{i,f} \otimes_r s_{l,g}\|^2 - \langle s_{i,f} \otimes_r s_{i,g}, s_{l,f} \otimes_r s_{l,g} \rangle] \\
&= 2 \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \\
& \quad \times \left[ \sum_{i,l \geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 - \left\langle \sum_{i \geq 1} s_{i,f} \otimes_r s_{i,g}, \sum_{l \geq 1} s_{l,f} \otimes_r s_{l,g} \right\rangle \right] \\
&= 2 \sum_{r=0}^{(n-1) \wedge (m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \left[ \sum_{i,l \geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 - \left\| \sum_{i \geq 1} s_{i,f} \otimes_r s_{i,g} \right\|^2 \right]. \quad (16)
\end{aligned}$$

Notice that, by Lemma 1, for every  $r = 0, \dots, n-1$

$$\left\| \sum_{i \geq 1} s_{i,f} \otimes_r s_{i,g} \right\|^2 = n^2 m^2 \|f \otimes_{r+1} g\|^2. \quad (17)$$

We apply now Lemma 3 and we get

$$\begin{aligned}
\sum_{i,l \geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 &= \sum_{i,l \geq 1} \langle s_{i,f} \otimes_r s_{l,g}, s_{i,f} \otimes_r s_{l,g} \rangle \\
&= \sum_{i,l \geq 1} \langle s_{i,f} \otimes_{n-1-r} s_{i,f}, s_{l,g} \otimes_{m-1-r} s_{l,g} \rangle \\
&= \left\langle \sum_{i \geq 1} \langle s_{i,f} \otimes_{n-1-r} s_{i,f}, \sum_{l \geq 1} s_{l,g} \otimes_{m-r-1} s_{l,g} \rangle \right.
\end{aligned}$$

and by Lemma 1 and Lemma 3, this equals

$$\begin{aligned}
\sum_{i,l \geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 &= n^2 m^2 \langle f \otimes_{n-r} f, g \otimes_{m-r} g \rangle \\
&= n^2 m^2 \|f \otimes_r g\|^2. \quad (18)
\end{aligned}$$

By replacing (17) and (18) in (16) we obtain

$$\begin{aligned} T_0 &= \frac{1}{2}(m+n-2)! \sum_{i,l \geq 1} \|s_{i,f} \tilde{\otimes} s_{l,g} - s_{l,f} \tilde{\otimes} s_{i,g}\|^2 \\ &= \sum_{r=0}^{(n-1) \wedge (m-1)} mnm!n!C_{n-1}^r C_{m-1}^r [\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2]. \end{aligned}$$

■

As a consequence of the above proof, we obtain a very interesting result with multiple consequences on the Fourth Moment Theorem and on the existence of densities of random variables.

**Proposition 3** *For every  $r = 0, \dots, (m-1) \wedge (n-1)$  and if  $s_{i,f}, s_{i,g}$  are given by (11), (12), it holds that*

$$n^2 m^2 [\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2] = \sum_{i,l \geq 1} \|s_{i,f} \otimes s_{l,g} - s_{l,f} \otimes s_{i,g}\|^2.$$

As a consequence, for every  $r = 0, \dots, (m-1) \wedge (n-1)$  we have

$$\|f \otimes_{r+1} g\|^2 \leq \|f \otimes_r g\|^2. \quad (19)$$

**Proof:** It is a consequence of the proof of Proposition 2. ■

**Remark 2** *The inequality (20) is very interesting by itself and as far as we know, not known. It has important consequences on the calculus on Wiener space (see Sections 5 and 6, for example). Actually, it seems very difficult to prove it by usual techniques. It is somehow unexpected (although  $f \otimes_1 g = 0$  a.e. implies  $f \otimes_k g = 0$  a.e. for  $k \geq 1$ ).*

Let us state the main results of this section.

**Theorem 1** *Let  $f \in H^{\otimes n}, g \in H^{\otimes m} (n, m \geq 1)$  be symmetric and denote by  $\Lambda$  the Malliavin matrix of the vector  $(F, G) = (I_n(f), I_m(g))$ . Then*

$$\det \Lambda = \sum_{r=0}^{(n-1) \wedge (m-1)} mnm!n!C_{n-1}^r C_{m-1}^r [\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2] + R_{n,m}$$

where for every  $n, m \geq 1$ ,  $R_{n,m}$  is given by (15). Note that  $R_{n,m} \geq 0$  for every  $n, m \geq 1$ .

**Proof:** It follows from Proposition 2 and Lemma 2. ■

In the case when the two multiple integrals live in the same Wiener chaos, we have a nicer expression. The following result answers in particular the open question 6.1 in [1] for couples of multiple integrals.

**Theorem 2** *Under the same assumptions as in Theorem 1 but with  $m = n$ , we have*

$$\det \Lambda = m^2 \det C + (mm!)^2 \sum_{r=0}^{\lfloor \frac{m-1}{2} \rfloor} ((C_{m-1}^r)^2 - (C_{m-1}^{r-1})^2) (\|f \otimes g_r\|^2 - \|f \otimes_{n-r} g\|^2) + R_m$$

with  $R_m$  given by (15). Here  $\lfloor x \rfloor$  denotes the integer part of  $x$ . As a consequence

$$\det \Lambda \geq m^2 \det C. \quad (20)$$

**Proof:** Suppose  $n \leq m$  and that  $m$  is odd. The case  $m$  even is similar. From we have

$$\begin{aligned} \det \Lambda &= (mm!)^2 \left[ \sum_{r=0}^{(m-1)} (C_{m-1}^r)^2 \|f \otimes_r g\|^2 - \sum_{r=0}^{(m-1)} (C_{m-1}^r)^2 \|f \otimes_{r+1} g\|^2 \right] \\ &= (mm!)^2 \left[ \sum_{r=0}^{\lfloor \frac{m-1}{2} \rfloor} (C_{m-1}^r)^2 \|f \otimes_r g\|^2 - \sum_{r=\frac{m-1}{2}}^{(m-1)} (C_{m-1}^r)^2 \|f \otimes_{r+1} g\|^2 \right] \\ &\quad + (mm!)^2 \left[ \sum_{r=\frac{m-1}{2}}^{m-1} (C_{m-1}^r)^2 \|f \otimes_r g\|^2 - \sum_{r=0}^{\frac{m-1}{2}} (C_{m-1}^r)^2 \|f \otimes_{r+1} g\|^2 \right] \\ &= (mm!)^2 \sum_{r=0}^{\frac{m-1}{2}} (C_{m-1}^r)^2 [\|f \otimes_r g\|^2 - \|f \otimes_{n-r} g\|^2] \\ &\quad + (mm!)^2 \sum_{r=1}^{\frac{m-1}{2}} (C_{m-1}^r)^2 [\|f \otimes_{n-r} g\|^2 - \|f \otimes_r g\|^2] \end{aligned}$$

where we made the change of index  $r' = n - 1 - r$  in the second and third sum above. Finally, noticing that for  $r = 0$  we have

$$m^2 m!^2 (C_{m-1}^0)^2 [\|f \otimes_0 g\|^2 - \|f \otimes_n g\|^2] = m^2 \det C$$

we obtain the conclusion.

Relation (20) follows from Corollary 19 , the fact that  $R_m$  is positive and from the inequality

$$C_{m-1}^r \geq C_{m-1}^{r-1}, \quad \text{if } 1 \leq r \leq \left\lfloor \frac{m-1}{2} \right\rfloor.$$

■

**Example 1** *Suppose  $m = n = 2$ . Then*

$$\begin{aligned} \det \Lambda &= 16 [f \otimes g\|^2 - \|f \otimes_2 g\|^2] + R_2 \\ &= 4 \det C + R_2. \end{aligned}$$

*We retrieve the formula in [1] with  $R_2 = 32 (\|f \otimes g_1\|^2 - \|f \tilde{\otimes}_1 g\|^2)$ .*

*Assume  $m = n = 3$ . Then*

$$\begin{aligned} \det \Lambda &= 3 \times 3 \times 3! \times 3! [(\|f \otimes g\|^2 - \|f \otimes_3 g\|^2) + 9 (\|f \otimes_1 g\|^2 - \|f \otimes_2 g\|^2)] + R_3 \\ &= 9 \det C + 9 (\|f \otimes_1 g\|^2 - \|f \otimes_2 g\|^2) + R_3. \end{aligned}$$

*Suppose  $m = n = 4$ . Then*

$$\det \Lambda = 16 \det C + 9 \times 4! \times 4! (\|f \otimes_1 g\|^2 - \|f \otimes_2 g\|^2) + R_4.$$

## 5 Densities of vectors of multiple integrals

Let us discuss when a couple of multiple stochastic integrals has a law which is absolutely continuous with respect to the Lebesgue measure. The situations when the components of the vector are in the same chaos or in chaoses of different orders need to be separated.

### 5.1 Variables in the same chaos

We first answer the open problem 6.2 in [1].

**Theorem 3** *Let  $f, g \in H^{\otimes m}$  be symmetric. Then the random vector  $(F, G) = (I_m(f), I_m(g))$  does not admit a density if and only if*

$$\det C = 0.$$

*In other words, the vector  $(F, G)$  does not admit a density if and only if its components are proportional.*

**Proof:** Suppose that the vector  $(F, G)$  does not admit a density. Then by [1],  $E \det \Lambda = 0$ . In this case, formula (20) in Theorem 2 implies that  $\det C = 0$  and so  $F$  and  $G$  are proportional.

Suppose  $\det C = 0$ . Then clearly  $F$  and  $G$  are proportional and the vector  $(F, G) = (I_m(f), I_m(g))$  does not admit a density. ■

## 5.2 Variables in chaoses of different orders

Let us understand what happens when the random variables live in chaoses of different orders. Consider  $(F, G) = (I_n(f), I_m(g))$  with  $n \neq m$ . First, let us note that  $E \det \Lambda = 0$  does not imply  $\det C = 0$ . This can be viewed by considering the following example.

**Example 2** Take  $F = I_2(f)$  and  $G = I_2(h^{\otimes 2})$  where  $\|h\| = 1$ . In this case

$$\det C = 2 \text{ and } \det \Lambda = 0.$$

One can also choose  $F = I_n(h^{\otimes n})$  and  $G = I_m(h^{\otimes m})$  with  $m \neq n$  and  $\|h\| = 1$ .

Is there possible to obtain an "if and only if" condition for the existence of the density of the vectors  $(F, G) = (I_n(f), I_m(g))$  with  $n \neq m$ ?

Theorem 1 allows to obtain the following:

**Theorem 4** *If the vector  $(F, G) = (I_n(f), I_m(g))$  with  $n \neq m$  does not admit a density then*

$$\|f \otimes_0 g\| = \|f \otimes_1 g\| = \dots = \|f \otimes_{m \wedge n} g\|. \quad (21)$$

**Proof:** It is a direct consequence of Theorem 1 and of Proposition 19. ■

To obtain the converse direction, we need to compute the term  $T_k$  with  $k \neq 0$  in (14). This seems very tedious in general. But we can compute the last term  $T_k$  and this allows us to treat several particular cases.

**Proposition 4** *Consider the term in (14) with  $k = (m-1) \wedge (n-1)$  in (14). Suppose  $m \leq n$ . Then*

$$\begin{aligned} T_{(n-1) \wedge (m-1)} = T_{m-1} &= \frac{1}{2} (m-1)!^2 (C_{n-1}^{m-1})^2 (n-m)! 2n^2 m^2 (\|f \otimes_{m-1} g\|^2 - \|f \otimes_m g\|^2) \\ &= m! m n^2 (C_{n-1}^{m-1})^2 (n-m)! (\|f \otimes_{m-1} g\|^2 - \|f \otimes_m g\|^2). \end{aligned}$$

**Proof:** We have

$$T_{(n-1) \wedge (m-1)} = T_{m-1} = (m-1)!^2 (C_{n-1}^{m-1})^2 (n-m)! \|s_{i,f} \tilde{\otimes}_{m-1} s_{l,g} - s_{l,f} \tilde{\otimes}_{m-1} s_{i,g}\|^2. \quad (22)$$

Note that  $s_{i,f} \otimes_{m-1} s_{l,g}$  and  $s_{l,f} \otimes_{m-1} s_{i,g}$  are symmetric functions. Therefore

$$\begin{aligned} \|s_{i,f} \tilde{\otimes}_{m-1} s_{l,g} - s_{l,f} \tilde{\otimes}_{m-1} s_{i,g}\|^2 &= \|s_{i,f} \otimes_{m-1} s_{l,g} - s_{l,f} \otimes_{m-1} s_{i,g}\|^2 \\ &= 2n^2 m^2 (\|f \otimes_{m-1} g\|^2 - \|f \otimes_m g\|^2) \end{aligned}$$



and consequently

$$\begin{aligned} T_{(n-1)\wedge(m-1)} = T_{m-1} &= \frac{1}{2}(m-1)!^2 (C_{n-1}^{m-1})^2 (n-m)!2n^2m^2 (\|f \otimes_{m-1} g\|^2 - \|f \otimes_m g\|^2) \\ &= m!mn^2 (C_{n-1}^{m-1})^2 (n-m)! (\|f \otimes_{m-1} g\|^2 - \|f \otimes_m g\|^2). \end{aligned}$$

■

### 5.2.1 The case $(I_n(f), I_1(g))$

In this case there is only one term in (14) obtained for  $k = 0$ . It reads

$$T_0 = nn! [\|f \otimes_2 g\|^2 - \|f \otimes_{r+1} g\|^2].$$

Note that the fact that  $E \det \Lambda = 0$  does not imply that  $\det C = 0$  (see Example 2).

Also note that  $E \det \Lambda = 0$  if and only if  $\|f \otimes g\| = \|f \otimes_1 g\|$  (this has been also observed in [1]).

### 5.3 The case $(I_n(f), I_2(g))$

When  $(F, G) = (I_n(f), I_2(g))$  there are only two terms in (14) obtained for  $k = 0$  and  $k = 1$ . These terms have been computed in Propositions 2 and 4. We have

$$\begin{aligned} T_0 &= 2n2!n! \sum_{r=0}^1 C_{n-1}^r C_1^r [\|f \otimes_2 g\|^2 - \|f \otimes_{r+1} g\|^2] \\ &= 4nn! [\|f \otimes g\|^2 - \|f \otimes_1 g\|^2] \\ &\quad + 4nn!(n-1) [\|f \otimes_1 g\|^2 - \|f \otimes_2 g\|^2] \end{aligned}$$

and

$$T_1 = 4nn!(n-1) [\|f \otimes_1 g\|^2 - \|f \otimes_2 g\|^2].$$

Note also that this last term is positive, because the term  $T_1$  is a norm. Consequently  $E \det \Lambda = 0$  if and only if

$$\|f \otimes g\|^2 = \|f \otimes_1 g\|^2 = \|f \otimes_2 g\|^2.$$

This does not imply  $\det C = 0$  (Example 2).

We conjecture that (21) is an equivalent condition for the non existence of the density of the vector  $(F, G) = (I_n(f), I_m(g))$  when  $n \neq m$ .

## 6 On the Fourth Moment Theorem

In the seminar paper [5], the authors proved the following result, known as the *Fourth Moment Theorem*.

**Theorem 5** *Consider a sequence  $\{F_k = I_n(f_k)\}_{k \geq 1}$  of square integrable random variables in the  $n$ -th Wiener chaos. Assume that*

$$\lim_{k \rightarrow \infty} E[F_k^2] = n! \lim_{k \rightarrow \infty} \|f_k\|_{H^{\otimes n}}^2 = 1. \quad (23)$$

*Then, the following statements are equivalent.*

1.  $\{F_k = I_n(f_k)\}$  converges to the standard normal law in distribution as  $k \rightarrow \infty$ .
2.  $\lim_{k \rightarrow \infty} E[F_k^4] = 3$ .
3.  $\lim_{k \rightarrow \infty} \|f_k \otimes_l f_k\|_{H^{\otimes 2(n-l)}} = 0$  for  $l = 1, 2, \dots, n-1$ .
4.  $\|DF_k\|_H^2$  converges to  $n$  in  $\mathbb{L}^2$  as  $k \rightarrow \infty$ .

In view of Proposition 19, the condition 3. above can be replaced by

$$\lim_{k \rightarrow \infty} \|f_k \otimes_1 f_k\|_{H^{\otimes 2(n-1)}} = 0.$$

In other words, to check that a sequence of multiple integrals converges to the standard normal law, it is enough to check that the the norm of the contractions of order 1 of the kernels converge to zero (instead of all contractions of order  $1, \dots, n-1$ ). The same remark applies to the multidimensional version of the Fourth Moment Theorem in [6] or to the convergence to the gamma distribution (see [2]).

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