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NON-RESONANT NON-PLANAR FREE MOTIONS OF INEXTENSIONAL NON-COMPACT BEAMS

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Non-linear coupling phenomena of non-compact beams with comparable frequencies are studied in the absence of flexural internal resonance. Use is made of approximate, though consistent, partial differential equations which contain the terms of elastic interaction responsible for the stronger non-linear effects involving the torsional component. Upon assuming a one-mode approximation for each spatial variable and applying the Galerkin procedure, three ordinary equations are deduced, and an approximate solution in the absence of all resonances is accomplished via a perturbation technique. An extended investigation of the non-linear coupling is made for beams of open cross-sections with various boundary conditions, by varying some meaningful ratios of the natural frequencies of the beam.

1. INTRODUCTION

In the past decades a certain number of papers have been devoted to the study of non-planar finite motions of beams [1-8]. Due to the complexity of the description of the problem, various simplifying assumptions regarding the kinematics of the beam and the accounting for the various non-linear elastic and inertia terms have been introduced. Within this ambit it is worth noting that the torsional component has been initially disregarded in the study of the spatial flexural oscillations, restricting the analysis to compact beams for which the torsional frequency is considerably larger than the flexural frequencies.

Only in the work by Crespo da Silva and Glynn [4, 5] has the problem been treated rigorously; by referring to the exact kinematics of the beam, general order-three, non-linear equations of motion of inextensional beam were derived in terms of the configuration variables, without neglecting any coupling term among the flexural and torsional components. However, in references [6, 7] attention was still focused on compact beams, for which the non-planar motion can be accurately described by two equations in the two flexural displacements only after eliminating the torsional component. Besides, due to weak coupling between flexural motions, only the oscillations in conditions of internal resonance have been studied deeply.

Herein the interest is mainly devoted to the non-planar vibrations of non-compact inextensional beams which exhibit comparable flexural and torsional frequencies; in this case the problem is accurately described by three equations in the two transversal displacements and torsion, all of the same order [9], and strong coupling is induced by the non-linear terms. It is then interesting to study the dynamic phenomena also in the absence of internal resonance. For the free motion of a fixed-free beam an approximate but laborious solution of the complete equations was used in reference [8] to analyze some cases of coupling. In this paper a wider investigation of these phenomena is

performed; for this purpose, use is made of three simpler equations of motion obtained through a consistent ordering of the mechanical and geometrical parameters of actual beams. They still contain the quadratic and cubic terms of elastic interaction among the flexural and torsional components which are responsible for the stronger non-linear effects. These equations do not contain warping, since its coherent introduction in the non-linear beam model used is far from being straightforward.

Upon assuming a one-mode approximation for each variable of the spatial configuration and applying the Galerkin procedure, a system of ordinary differential equations is obtained, an approximate solution of which is pursued via the perturbation multiple time scale technique [10]. The conditions involving the linear frequencies under which internal resonance occurs at the different orders of the asymptotic solution are examined. Then the solution of the equations in the absence of all resonances is accomplished.

Non-linear coupling phenomena among the three modes are studied for beams with open cross-sections and different boundary conditions, by varying some meaningful ratios of the natural frequencies of the beam and the initial conditions. However, the absence of warping limits the applicability of the solution obtained to some open cross-sections.

2. EQUATIONS OF MOTION AND PERTURBATION SOLUTION

In the reference frame $Ox_1x_2x_3$, one can consider an inextensional shear-indeformable beam aligned with the x_1 axis, with length l , mass per unit length m , torsional mass moment of inertia I_1 , torsional stiffness E_1 , flexural stiffnesses E_2 and E_3 . The dynamic configuration is described by the angle $\theta(s, t)$ around the axis x_1 and the transversal displacements $u_2(s, t)$ and $u_3(s, t)$, s being a curvilinear abscissa and t the time. The non-dimensionalized variables

$$\tilde{s} = s/l, \quad \tilde{t} = \omega t, \quad \tilde{\theta} = \theta/\rho, \quad \tilde{u}_i = u_i/l \quad (1)$$

are introduced, where ω is any real frequency of the beam and ρ is a scaling factor defined in such a way that the flexural and torsional linearized elastic energies assume the same value when $\tilde{\theta} = \tilde{u}$. In addition, the following relations hold:

$$\beta_2 = E_2/E_1, \quad \beta_3 = E_3/E_1, \quad \nu = I_1\omega^2 l^2/E_1, \quad \mu = m\omega^2 l^4/E_1. \quad (2)$$

Order-three flexural-flexural-torsional equations of the free motion of non-compact beams, obtained from reference [9], are

$$\begin{aligned} G_0 &= \rho\{\nu\rho\ddot{\theta} - \rho\theta'' - (\beta_2 - \beta_3)u_2''u_3'' + (\beta_2 - \beta_3)\rho(u_2''^2 - u_3''^2)\theta\} = 0, \\ G_2' &= \{-\beta_3u_2''' + \rho(\beta_2 - \beta_3)(u_3''\theta)' - (\beta_2 - \beta_3)\rho^2(u_2''\theta^2 + 2u_2''\theta\theta')\}' = \mu\ddot{u}_2, \\ G_3' &= \{-\beta_2u_3''' + \rho(\beta_2 - \beta_3)(u_2''\theta)' + (\beta_2 - \beta_3)\rho^2(u_3''\theta^2 + 2u_3''\theta\theta')\}' = \mu\ddot{u}_3, \end{aligned} \quad (3)$$

where the tilde has been omitted. The corresponding boundary conditions are

$$G_2\delta u_2|_0^1 = 0, \quad G_3\delta u_3|_0^1 = 0, \quad H_0\delta\theta|_0^1 = 0, \quad H_2\delta u_2'|_0^1 = 0, \quad H_3\delta u_3'|_0^1 = 0, \quad (4)$$

where G_2 and G_3 are obtained from equations (3) and

$$\begin{aligned} H_0 &= \rho^2\theta', \quad H_2^* = \beta_3u_2'' - (\beta_2 - \beta_3)\rho u_3''\theta + (\beta_2 - \beta_3)\rho^2u_2''\theta^2, \\ H_3^* &= \beta_2u_3'' - (\beta_2 - \beta_3)\rho u_2''\theta - (\beta_2 - \beta_3)\rho^2u_3''\theta^2. \end{aligned} \quad (5)$$

Equations (3) and (4) were obtained from the complete order-three equations of motion through a consistent analysis of the smallness of the terms based on the assumptions of (a) comparable flexural and torsional frequencies of the beam, (b) $(\beta_2 - \beta_3)$ of order $O(\beta)$,

which corresponds to the absence of internal resonance between the flexural components. In these conditions the order of the coefficients in the equations is [9]

$$\beta = O(\lambda^2/\pi^2), \quad \mu = O(\lambda^2\pi^2), \quad \nu = O(\pi^2), \quad \rho^2 = O(\lambda^2), \quad (6)$$

with λ being the slenderness of the beam. Since no ordering scheme was introduced on the three configuration variables which still remain of the same order, equations (3) are correct for studying the phenomena of non-linear interaction among transversal and torsional vibrations of beams.

The degree of non-linearity of this problem and the range of amplitudes over which a perturbational procedure provides a satisfactory solution can be evaluated by comparing the orders of the linear, quadratic and cubic terms in equations (3). According to equations (6), they are, respectively,

$$T^I = O(\pi^2\lambda^2q), \quad T^{II} = O(\pi^2\lambda^3q^2), \quad T^{III} = O(\pi^2\lambda^4q^3), \quad (7)$$

where q is the amplitude of oscillation. These terms become of comparable magnitude when $q = O(1/\lambda)$: that is, for very small values of the amplitudes. This means that the problem is strongly non-linear and the perturbational solution can be correctly utilized in this region of small amplitudes.

It is worth noticing that the problems of either in-plane oscillations or flexural internal resonant motion of inextensional beams already studied in the literature [11, 12] are those with weak non-linear behaviour that is very different from the case considered herein. Indeed, in both those cases the non-linear terms are essentially cubic and become comparable with the linear terms for a high amplitude value, $q = O(1/\pi)$.

A discrete model of the continuum system is obtained by assuming a shape function for each of the three configuration variables θ , u_2 and u_3 and applying the Galerkin procedure. The solution to system (3) is thus sought in the form

$$\theta = f_1(s)q_1(t), \quad u_2 = f_2(s)q_2(t), \quad u_3 = f_3(s)q_3(t), \quad (8)$$

where the $f(s)$ are the eigenfunctions of the linearized problem and the $q(t)$ are unknown time functions. The integral formulation of the equations of motion (3) and boundary conditions (4, 5) is

$$\int_0^1 \{G_\theta \delta\theta + (G'_2 - \mu \ddot{u}_2) \delta u_2 + (G'_3 - \mu \ddot{u}_3) \delta u_3\} ds \\ + (-H_\theta \delta\theta - G_2 \delta u_2 - G_3 \delta u_3 - H_2^* \delta u'_2 - H_3^* \delta u'_3) \Big|_0^1 = 0. \quad (9)$$

By integrating by parts, making use of relationships (8) and imposing separate vanishing of terms multiplying δq_1 , δq_2 and δq_3 , a system of three ordinary equations of motion with quadratic and cubic non-linearities is obtained:

$$m_1 \ddot{q}_1 + k_1 q_1 = -dq_2 q_3 - c_{12} q_1 q_2^2 - c_{13} q_1 q_3^2, \\ m_2 \ddot{q}_2 + k_2 q_2 = -dq_1 q_3 - c_{12} q_1^2 q_2, \quad m_3 \ddot{q}_3 + k_3 q_3 = -dq_1 q_2 - c_{13} q_1^2 q_3. \quad (10)$$

The coefficients in equations (10) depend on the elastic, geometric and inertial properties of the beam through the dimensionless parameters ρ , β , ν and μ given by equations (1) and (2), and on the integrals of products of the shape functions f_i . Their explicit expressions are given in Appendix 1.

To obtain the solution to system (10) the multiple scale method [10] is adopted. A perturbation parameter, ε , of the order of the amplitude, is introduced and the variables q are considered functions of a sequence of independent time scales T_0, T_1, \dots, T_n ,

which are related to t by the expressions $T_n = \varepsilon^n t$. The following expansions of q_i in powers of ε are assumed as solutions to system (10):

$$q_i = \varepsilon q_{i1}(T_0, T_1, T_2) + \varepsilon^2 q_{i2}(T_0, T_1, T_2) + \varepsilon^3 q_{i3}(T_0, T_1, T_2) + O(\varepsilon^4 t). \quad (11)$$

Here three time scales have been considered and the functions q_{ij} ($i, j = 1, 2, 3$) are assumed to be of order $O(1)$. By expressing the time derivatives in terms of the T_n variables, substituting equations (11) into equations (10) and equating coefficients of like powers of ε , a sequence of linear systems with the unknowns q_{ij} is obtained, as follows:

$$\begin{aligned} \text{order } \varepsilon: \quad & m_1 D_{00} q_{11} + k_1 q_{11} = 0, \\ & m_2 D_{00} q_{21} + k_2 q_{21} = 0, \quad m_3 D_{00} q_{31} + k_3 q_{31} = 0; \end{aligned} \quad (12)$$

$$\begin{aligned} \text{order } \varepsilon^2: \quad & m_1 D_{00} q_{12} + k_1 q_{12} = -d q_{21} q_{31} - 2m_1 D_{01} q_{11}, \\ & m_2 D_{00} q_{22} + k_2 q_{22} = -d q_{11} q_{31} - 2m_2 D_{01} q_{21}, \\ & m_3 D_{00} q_{32} + k_3 q_{32} = -d q_{11} q_{21} - 2m_3 D_{01} q_{31}; \end{aligned} \quad (13)$$

$$\begin{aligned} \text{order } \varepsilon^3: \quad & m_1 D_{00} q_{13} + k_1 q_{13} = -c_{12} q_{11} q_{21}^2 - c_{13} q_{11} q_{31}^2 - d(q_{21} q_{32} + q_{31} q_{22}) \\ & \quad - m_1(2D_{01} q_{12} + D_{11} q_{11} + 2D_{02} q_{11}), \\ & m_2 D_{00} q_{23} + k_2 q_{23} = -c_{12} q_{11}^2 q_{21} - d(q_{11} q_{32} + q_{31} q_{12}) - m_2(2D_{01} q_{22} + D_{11} q_{21} + 2D_{02} q_{21}), \\ & m_3 D_{00} q_{33} + k_3 q_{33} = -c_{13} q_{11}^2 q_{31} - d(q_{11} q_{22} + q_{21} q_{12}) - m_3(2D_{01} q_{32} + D_{11} q_{31} + 2D_{02} q_{31}). \end{aligned} \quad (14)$$

Here $D_i = \partial/\partial T_i$ and $D_{ij} = \partial^2/\partial T_i \partial T_j$. The problem is completed with the initial conditions

$$q_j(0) = \varepsilon q_j^0, \quad \dot{q}_j(0) = \varepsilon \dot{q}_j^0, \quad (15)$$

εq_j^0 and $\varepsilon \dot{q}_j^0$ being the initial amplitudes and velocities, from which the conditions to be associated with each system of the sequence (12)–(14) are obtained:

$$\begin{aligned} \text{order } \varepsilon: \quad & q_{j1}(0) = q_j^0, \quad D_0 q_{j1}(0) = \dot{q}_j^0; \\ \text{order } \varepsilon^2: \quad & q_{j2}(0) = 0, \quad D_0 q_{j2}(0) = -D_1 q_{j1}(0); \\ \text{order } \varepsilon^3: \quad & q_{j3}(0) = 0, \quad D_0 q_{j3}(0) = -D_1 q_{j2}(0) - D_2 q_{j1}(0). \end{aligned} \quad (16)$$

In the following, the solution of the equations of motion up to the ε^3 -order is pursued by assuming a generating solution with three non-zero components, in order to study the non-linear coupling phenomena in the absence of internal resonance.

3. SOLUTION IN THE ABSENCE OF INTERNAL RESONANCE

The periodic solution

$$q_{j1} = A_j(T_1, T_2) e^{i\omega_j T_0} + c.c., \quad j = 1, 2, 3, \quad (17)$$

is adopted as the solution (generating solution) to system (12). In equations (17), $\omega_j = \sqrt{k_j/m_j}$ is the frequency of the beam over the dimensionless time scale t , $A_j(T_1, T_2)$ is an unknown complex function and *c.c.* stands for complex conjugate, the *c.c.* of A_j being denoted with an overbar (\bar{A}_j). The dependence of A_j on the time scales T_1 and T_2 will be obtained by imposing zeroing of the secular terms at the higher orders of the perturbation solution.

Substitution of equations (17) into the ε^2 -system (13) gives

$$D_{00} q_{j2} + \omega_j^2 q_{j2} = -d[A_h A_k e^{i(\omega_k + \omega_h)T_0} + \bar{A}_h A_k e^{i(\omega_k - \omega_h)T_0}]/m_j - i2\omega_j D_1 A_j e^{i\omega_j T_0} + c.c. \quad (18)$$

In equations (18) and in the following, the indexes j, h, k are different from each other and must be rotated in the clockwise sense starting from the set 1, 2, 3; δ_{ij} is the Kronecker symbol. Analysis of the system shows that internal resonance occurs at the order ε^2 if the linear frequencies satisfy the conditions

$$\omega_j + \omega_h = \omega_k, \quad (19)$$

involving all three components q_1, q_2 and q_3 .

In the absence of internal resonance, the zeroing of secular terms in equations (18) gives $A_j = A_j(T_2)$, showing that no frequency correction occurs at this order. Then, by solving the system and substituting, the equations at the ε^3 -order are obtained:

$$\begin{aligned} D_{00}q_{j3} + \omega_j^2 q_{j3} = & - \left\{ \sum_{1i}^3 [d_{ji}A_jA_i^2 e^{i(2\omega_i + \omega_j)T_0} + d_{ji}^* \bar{A}_j A_i^2 e^{i(2\omega_i - \omega_j)T_0}] (1 - \delta_{ij}) \right. \\ & + d[(A_h B_k + A_k B_h) e^{i(\omega_k + \omega_h)T_0} + (\bar{A}_h B_k + A_k \bar{B}_h) e^{i(\omega_k - \omega_h)T_0}] \\ & \left. + \left[\sum_{1i}^3 \tilde{d}_{ji} A_j A_i \bar{A}_i (1 - \delta_{ij}) + i2k_j D_2 A_j / \omega_j \right] e^{i\omega_j T_0} \right\} / m_j + c.c. \quad (20) \end{aligned}$$

In equations (20) the homogeneous solution to system (18) appears, the amplitudes B_j of which depend on T_1 and T_2 . The coefficients $d_{ji} = d_{ij}$, $d_{ji}^* = d_{ij}^*$, $\tilde{d}_{ji} = \tilde{d}_{ij}$ ($i \neq j$) are given in Appendix 1. Internal resonance at this order occurs when

$$\omega_j = \omega_k, \quad (21)$$

involving two components of motion only.

It is worthwhile to notice that even if $A_j(T_1, T_2) \equiv 0$, and in the absence of any resonance, the j th component is forced dynamically from the other two at the order ε^2 provided that these components are both different from zero, this being a phenomenon of non-linear modal coupling. Therefore, a motion with two prevailing components having frequencies dependent on the two non-zero amplitudes A_h and A_k occurs, in which the third component is of higher order.

Still in the absence of internal resonance, the general solution for q_1, q_2 and q_3 of the same order is obtained. The zeroing of the secular terms in equations (20), when the polar forms

$$A_k(T_2) = [a_k(T_2)/2] e^{i\varphi_k(T_2)} \quad (22)$$

are introduced and the real and imaginary parts are separated, provides the following differential system with the unknowns a_k, φ_k :

$$a'_k = 0, \quad \varphi'_k = \omega_k \left[\sum_{1i}^3 \tilde{d}_{ki} a_i^2 (1 - \delta_{ki}) \right] / 8k_k. \quad (23)$$

Here the prime denotes the derivative with respect to T_2 . By integrating equations (23) it follows that

$$a_k = \text{constant}, \quad \varphi_k = \hat{\varphi}_k t + \varphi_k^0, \quad (24)$$

where the definition for T_2 has been accounted for and $\hat{\varphi}_k$ depends on the squares of the amplitudes.

Then, by solving equations (20), the functions q_{j3} are obtained which depend on the amplitudes C_j of the homogeneous solution. The latter, as well as B_j , are assumed to have the same dependence on T_1 and T_2 as A_j . The values of A_j, B_j and C_j at $t=0$ are determined by means of the initial conditions (16) at the different orders; they are given in Appendix 2.

To write the temporal laws of motion in circular form, the amplitudes B_j and C_j are rewritten as

$$B_j(T_2) = (b_j/2) e^{i\psi_j(T_2)}, \quad C_j(T_2) = (c_j/2) e^{i\chi_j(T_2)}, \quad (25)$$

where

$$\psi_j = \hat{\phi}_j t + \psi_j^0, \quad \chi_j = \hat{\phi}_j t + \chi_j^0. \quad (26)$$

Substituting q_{ij} in equation (11) and using equations (22) and (25) gives the time laws of motion in circular form as

$$\begin{aligned} q_j = & \varepsilon a_j \cos \Phi_j + \varepsilon^2 \{ -a_h a_k [\Omega_j d \cos(\Phi_h + \Phi_k) + \Omega_j^* d \cos(\Phi_h - \Phi_k)]/2 + b_j k_j \cos \Psi_j \} / k_j \\ & + \varepsilon^3 \left\{ -\sum_{li}^3 a_j a_i^2 [\Omega_{ji} d_{ji} \cos(2\Phi_i + \Phi_j) + \Omega_{ji}^* d_{ji}^* \cos(2\Phi_i - \Phi_j)](1 - \delta_{ij})/4 \right. \\ & - a_h b_k [\Omega_j d \cos(\Psi_k + \Phi_h) + \Omega_j^* d \cos(\Psi_k - \Phi_h)]/2 \\ & \left. - a_k b_h [\Omega_j d \cos(\Psi_h + \Phi_k) + \Omega_j^* d \cos(\Psi_h - \Phi_k)]/2 + c_j k_j \cos X_j \right\} / k_j. \end{aligned} \quad (27)$$

In equations (27) the dynamic amplification factors Ω are defined as

$$\begin{aligned} \Omega_j &= \omega_j^2 / [\omega_j^2 - (\omega_h + \omega_k)^2], & \Omega_j^* &= \omega_j^2 / [\omega_j^2 - (\omega_h - \omega_k)^2], \\ \Omega_{ij} &= \omega_i^2 / [\omega_i^2 - (\omega_i + 2\omega_j)^2], & \Omega_{ij}^* &= \omega_i^2 / [\omega_i^2 - (\omega_i - 2\omega_j)^2] \end{aligned} \quad (28)$$

and the phases are

$$\Phi_k = \bar{\omega}_k t + \varphi_k^0, \quad \Psi_k = \bar{\omega}_k t + \psi_k^0, \quad X_k = \bar{\omega}_k t + \chi_k^0, \quad (29)$$

$\bar{\omega}_k$ being the non-linear frequencies of the three configuration variables:

$$\bar{\omega}_k = \omega_k \left[1 + \sum_{li}^3 \tilde{d}_{kl} a_i^2 \varepsilon^2 (1 - \delta_{kl}) / 8 k_k \right]. \quad (30)$$

Equations (27) reveal deep modifications of the temporal laws with respect to the linear ones. Indeed, the motion of the i th component is described by the superposition of several harmonics with frequencies which are combinations of the three non-linear fundamental frequencies (30); namely, $\bar{\omega}_j$, $\bar{\omega}_h \pm \bar{\omega}_k$, $2\bar{\omega}_h \pm \bar{\omega}_k$ ($j, h, k = 1, 2, 3$; $h \neq k$). Harmonics having the same frequency ($\bar{\omega}_j$, $\bar{\omega}_h \pm \bar{\omega}_k$) but different phases occur; however, such differences reduce to zero if the particular case of zero initial velocity for all components is considered.

Equations (30) show that the non-linear frequency of each variable depends on the squares of the amplitudes of oscillation of the remaining two variables.

4. ANALYTICAL INVESTIGATION

The analytical investigation is made by using the perturbation solution developed, which gives information on the parameters governing the phenomenon. Due to their large number, in the following the description of the dynamic behaviour of beams is made with reference to specific representative cases selected on the basis of these parameters.

The non-linear dynamic coupling depends both on the values of the ratios among the natural frequencies of the beam and on the amount of coupling occurring already in the static range. Within the assumption of cross-sections with comparable flexural rigidities, the latter is stronger, as these are different from each other, and it is described quantitatively just by the ratio β_2/β_3 governing the importance of the non-linear terms in the equilibrium equations. Therefore, in the analytical investigation, with reference to open sections with

torsional rigidity about one hundred times lower than the flexural ones, basically two types of section are considered: (i) with different flexural rigidities ($\beta_2/\beta_3 = 2.60$); (ii) with more nearly equal rigidities ($\beta_2/\beta_3 = 1.19$). The beam frequencies are then varied suitably to analyze the specific effect of the dynamic amplification.

The beams considered have the torsional frequency ω_1 greater than the flexural frequencies (with $\omega_3 > \omega_2$), as generally occurs in technical applications with open sections. Thus, according to equations (27), the ratios playing the more important role in the dynamic problem are $(\omega_2 + \omega_3)/\omega_1$ and ω_1/ω_3 at the quadratic and cubic order, respectively, for section (i), and ω_3/ω_2 for section (ii).

Different beams (B1 to B3) are considered with section (i), which is expected to be more interesting. The relevant values of the frequency ratios appearing in the significant dynamic amplification factors (28) are reported in Table 1: they show how the beams are gradually further from a resonant condition involving all three components, and nearer to the resonance between the torsional and one flexural component. In addition, for the sake of comparison, a beam with closed section (BCS) having the same flexural rigidities as the open sections and torsional rigidity of the same order, is also examined. Since in this case the frequencies are not all of the same order, the relevant dynamic results are obtained with the equations of motion in their complete form [8]. In all the other cases the simplified equations (3), specifically derived for treating beams with frequencies of the same order, are referred to. Preliminary investigations on the beams considered, made using the complete and the simplified equations, validated the ordering introduced to obtain the latter equations.

TABLE 1
Frequency ratios of the beams

	BCS	B1	B2	B3
$(\omega_2 + \omega_3)/\omega_1$	0.14	1.19	1.35	1.48
$(2\omega_1 - \omega_3)/\omega_3$	22.2	1.72	1.38	1.18

In contrast, a unique beam (B4) having section (ii) is considered, with a value of the frequency ratio $(2\omega_2 - \omega_3)/\omega_3$ appearing in the meaningful dynamic amplification factor equal to 1.18.

Some results showing the non-linear coupling among the three components of motion are now presented, with reference to the time laws, equations (27), which allow insight into the effect of coupling not only in terms of maximum amplitudes. Fixed-free and hinged-hinged beams with different sets of initial conditions (i.c.) are considered, in any case with the assumption of zero initial velocities.

The fixed-free beam (FF) is considered first. The temporal laws for the three components q_1 , q_2 and q_3 obtained for BCS and B1 with zero initial conditions for the twist angle q_1 and 0.1 for the two flexural variables are plotted in Figures 1 and 2, respectively. The BCS shows a very low torsion q_1 forced dynamically from q_2 and q_3 , the amplitudes of these latter components remaining unaltered during the motion; q_{1max} reaches only $0.16q_{2max}$. Instead, the forced torsional component is much stronger for B1, with a maximum value equal to $0.90q_{2max}$. This means that the energy content associated with the angle of twist, which is zero initially, increases to a value comparable with that pertaining to a flexural displacement. In addition, B1 also exhibits the effect of coupling between the two flexural variables, due essentially to the forcing effect exerted, in turn, from q_1 .

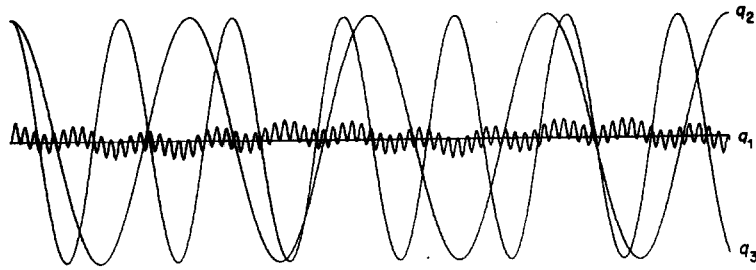


Figure 1. Time laws of the three displacement components for fixed-free beam BCS; $a_1, a_2, a_3 = 0.0, 0.1, 0.1$.

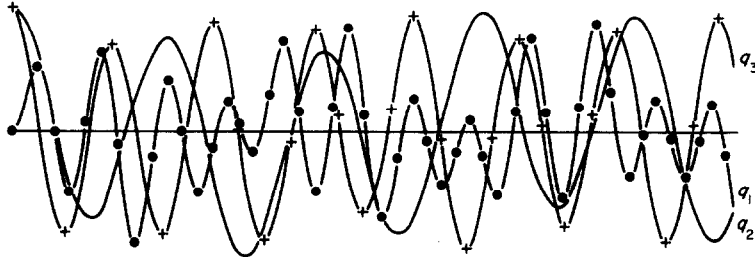
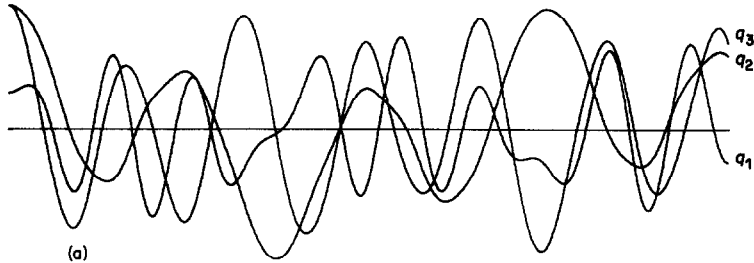


Figure 2. As Figure 1 but for beam B1; $a_1, a_2, a_3 = 0.0, 0.1, 0.1$.

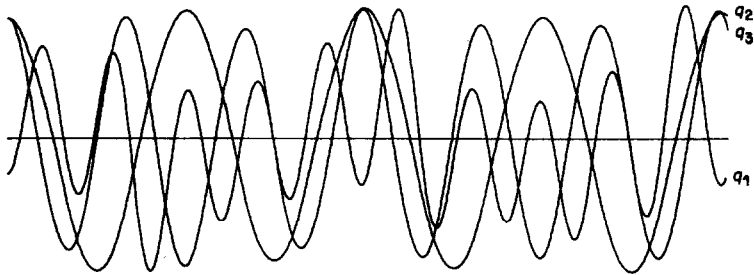
The oscillations of B1 obtained with all three initial conditions different from zero are plotted in Figures 3(a) and (b), where the same amplitude of the torsional component is considered but with opposite sign. The results are quite different from each other: indeed, with $a_1 = 0.03$ the non-linear effect on q_{1max} is weaker than that obtained in the other case and even weaker than with $a_1 = 0$, while the laws appear less regular and more similar to this last case. The first item is associated with the pattern of the non-linear strain energy of the system,

$$\Pi = \frac{1}{2}(k_1 q_1^2 + k_2 q_2^2 + k_3 q_3^2) + d q_1 q_2 q_3 + c_{12} q_1^2 q_2^2 + c_{13} q_1^2 q_3^2, \quad (31)$$

calculated at $t = 0$, having its minimum (see Figure 4) at a value different from zero



(a)



(b)

Figure 3. As Figure 2, but (a) $a_1, a_2, a_3 = 0.03, 0.1, 0.1$, and (b) $a_1, a_2, a_3 = -0.03, 0.1, 0.1$.

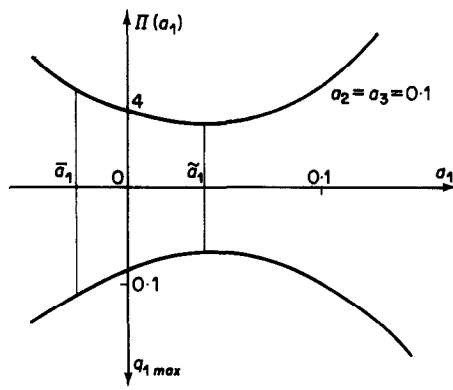


Figure 4. Initial strain energy and maximum response of q_1 versus a_1 for assigned values of a_2 and a_3 . ($\tilde{a}_1 = 0.039$), which is related to the static non-linear coupling of the system; correspondingly, the maximum response of q_1 also attains its minimum at \tilde{a}_1 . The second item, the general regularity of the laws in Figure 3(b), is due to the initial conditions in this case being close to the value $\tilde{a}_1 = -0.027$ for which the harmonic with own frequency does not appear in the q_1 time law.

The dynamic coupling among the three components is strong for B1 due to proximity to the quadratic resonance condition involving all of them. Indeed, if the results obtained with B2—which is further from that condition—are analyzed, minor coupling effects on the twist angle and therefore between the flexural co-ordinates are observed. Nevertheless, q_{1max} still reaches $0.44q_{2max}$, mainly because of static non-linear coupling. The interaction reduces further on passing to B3, the major proximity of which to the resonant condition $\omega_1 = \omega_3$ does not give rise to new coupling phenomena with this set of initial conditions.

The picture is different when finite initial conditions are imposed on torsion and flexure, while the other flexure is either zero or simply perturbed. In Figures 5 and 6 the laws

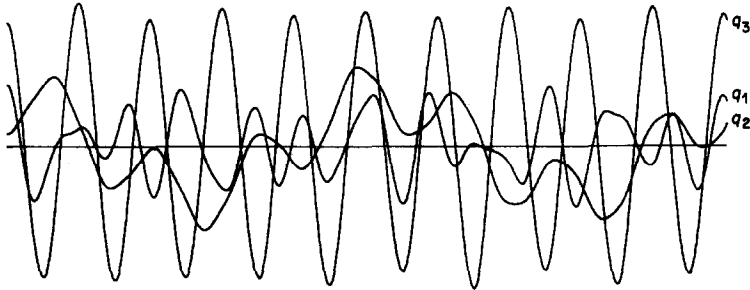


Figure 5. As Figure 2 but for $a_1, a_2, a_3 = 0.05, 0.01, 0.1$.

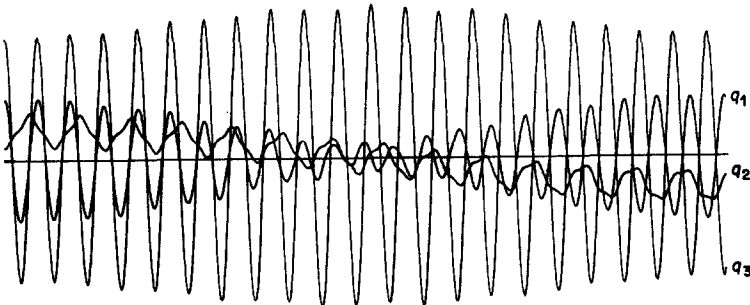


Figure 6. As Figure 5 but for beam B3; $a_1, a_2, a_3 = 0.05, 0.01, 0.1$.

obtained for B1 and B3 with $a_1 = 0.05$, $a_2 = 0.01$ and $a_3 = 0.1$ are plotted. For B1, closer to the condition $\omega_2 + \omega_3 = \omega_1$, the perturbed component q_2 is notably forced, up to 6.28 times its initial value, and vibrates with a law resulting mainly from the combination of the harmonic of non-linear frequency $\bar{\omega}_2$ and that of an amplitude dependent on $a_1 a_3$ and a frequency $\bar{\omega}_3 - \bar{\omega}_1$ almost four times lower. The flexural component q_3 vibrates with nearly constant amplitude, since only cubic harmonics remain in its law due to smallness of the product $a_1 a_2$. Torsion q_1 has a complex law resulting from the combination of several harmonics of different amplitudes and frequencies.

B3 exhibits a different response, being closer to the cubic resonance condition $\omega_1 = \omega_3$, involving just the two components which are assigned finite initial amplitudes. A minor forced component q_2 arises with respect to B1 ($q_{2\max}/a_2 = 3.5$), mainly due to the static coupling. Instead, dynamic interaction and energy exchange occur between q_1 and q_3 but to a minor extent relative to those occurring among the three components for B1, simply because the resonance condition $\omega_1 = \omega_3$ is of cubic order. However, in the case considered, the non-linear frequencies of q_1 and q_3 approach each other ($\bar{\omega}_1/\bar{\omega}_3 = 0.97$), so that the presence of a forced term of frequency $(2\bar{\omega}_3 - \bar{\omega}_1) \approx \bar{\omega}_1$ in the first of equations (27) induces beating on q_1 .

It is also interesting to analyze the response of B3 to the set of initial conditions 0.05, 0.1 and 0.01 (see Figure 7). In this case, in which the almost resonant flexural component q_3 is simply perturbed, the proximity to the resonance condition $\omega_1 = \omega_3$ does not produce notable coupling effects. Indeed, in the absence of resonance, the interaction between two components can develop only with finite initial values for both of them, while in the condition of resonance it occurs with even small perturbations of one of them.

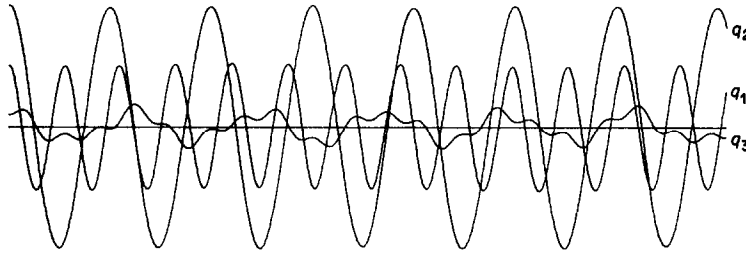


Figure 7. As Figure 6 but $a_1, a_2, a_3 = 0.05, 0.1, 0.01$.

Consider now the hinged-hinged beam with either one (HH1) or both (HH2) ends torsionally constrained. Analysis of the coefficients of the motion equations (10) reveals the different importance of the non-linear terms with the three boundary conditions considered: namely, they are stronger on passing from the FF beam to HH2 and HH1. This is shown by making reference to the case in which non-zero initial amplitudes (a_2, a_3) are assigned to the flexural components only. Owing to the non-linear static coupling, the torsional component $a_1 = (-d/k_1)a_2 a_3$ arises, which in turn gives rise to the main non-linear term in the two flexural equations ($da_1 a_i, i = 3, 2$). In Figure 8 the ratio between the non-linear and the linear terms in the j th equation ($j = 2, 3$) is plotted for beam sections of type (i) in terms of equal flexural amplitudes \hat{a}_i normalized in such a way as to have comparable values of the relevant initial energy for the three boundary conditions examined: i.e., $\hat{a}_i = a_i$ for the FF beam and $\hat{a}_i = 4a_i$ for the HH beams. Strong differences occur among the three boundary conditions. That occurring between FF and HH2 is associated mainly with the value of the coefficient d , which depends on the product of the flexural and torsional eigenfunctions (see Appendix 1): for the FF beam the former

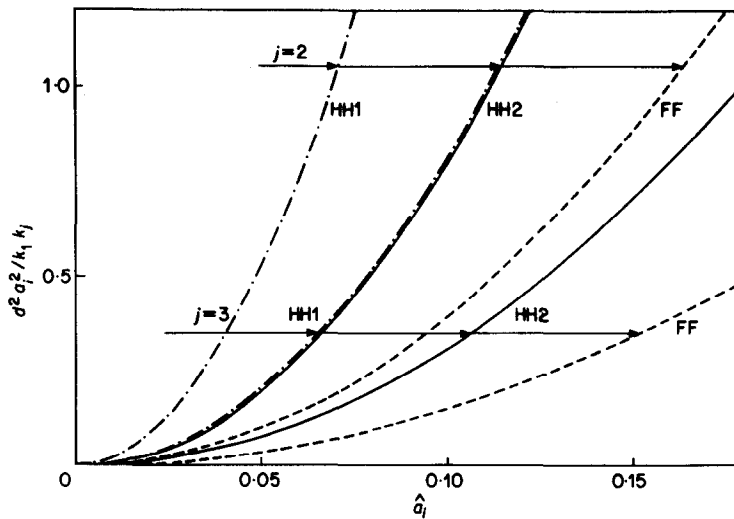


Figure 8. Ratio between non-linear and linear terms in the equations of motion for B1 with different boundary conditions. FF, $\hat{a}_i = a_i$; HH, $\hat{a}_i = 4a_i$.

is maximum where the latter vanishes and *vice versa*, so that a lower degree of non-linearity occurs. Instead, the further strong difference existing between HH2 and HH1 is due mainly to the strong decrease of the torsional rigidity k_1 , which is verified on passing to the HH1 beam.

The different non-linear behaviours of the three beams are shown also in the results of the dynamic analysis performed (see Figure 9 and Table 2). They refer to initial amplitudes \hat{a}_2 , \hat{a}_3 for which the ratio between the non-linear and the linear term is less than unity for the beam with stronger non-linear behaviour (HH1), in order that the perturbation solution will furnish accurate results. The non-dimensional time scale considered in Figure 9 is the same for the three beams, which also have equal values of the frequency ratios, so as to evidence directly the effects of non-linearities. The expected stronger non-linear coupling of the HH1 beam comes out clearly from the figure, while Table 2 shows the occurrence of a maximum value of the forced torsional component, becoming more and more large upon passing from FF to HH2 to HH1, as well as higher frequency corrections with respect to the linear values.

The beam of cross-section (ii) with comparable flexural rigidities (B4), which is somewhat closer to the resonance condition $\omega_2 = \omega_3$, should exhibit weaker non-linear behaviour with respect to the beams closer to a resonance condition involving the torsion, as previously noticed in section 2.

The response of the fixed-free beam is plotted in Figure 10. To account correctly for the major proximity to the flexural resonance condition, it was calculated by using the equations of motion in their complete form [8] rather than equations (10). Use of these latter would lead to an underestimation of the coupling phenomenon already noticeable with the value $\omega_2/\omega_3 = 1.09$ considered but of little interest, due to the low degree of non-linearity of this beam confirmed by the analysis of the response. Indeed, notwithstanding higher values of the initial amplitudes, lower interaction between q_2 and q_3 occurs than that between q_1 and q_3 in the response of B3 (Figure 6), the two beams being equally far from the relevant cubic resonance condition, namely $\omega_2 = \omega_3$ and $\omega_1 = \omega_3$; that is, the former region of resonance is narrower. A not very strong forced torsional component

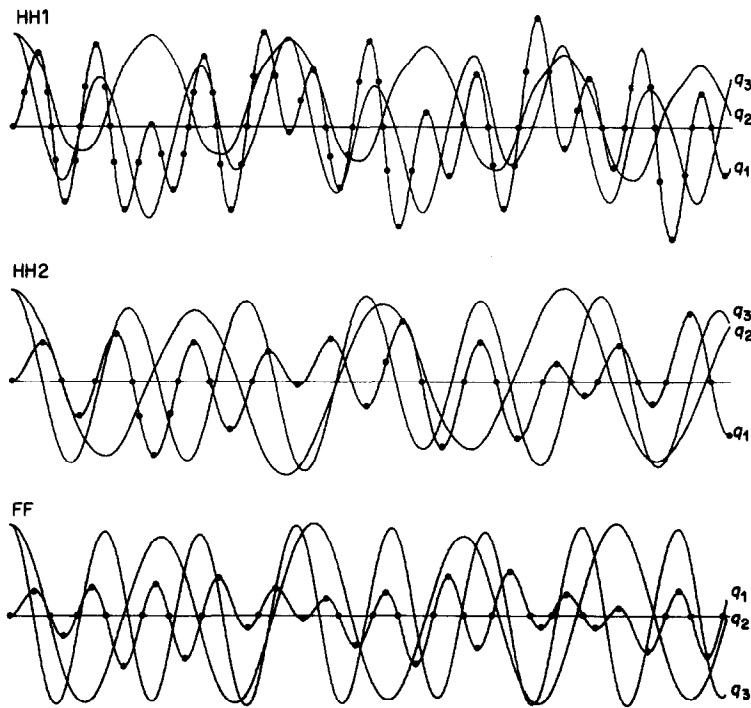


Figure 9. Comparison of non-linear response of beams with different boundary conditions for $a_1 = 0$ and the same flexural amplitudes $\hat{a}_2 = \hat{a}_3 = 0.06$.

TABLE 2
Comparisons of dynamic behaviour of beams FF, HH2 and HH1

	$ q_{1max} / q_{2max} $	$\bar{\omega}_1/\omega_1$	$\bar{\omega}_2/\omega_2$	$\bar{\omega}_3/\omega_3$
FF	0.547	0.961	1.024	1.009
HH2	0.796	0.919	1.050	1.019
HH1	1.227	0.793	1.127	1.049

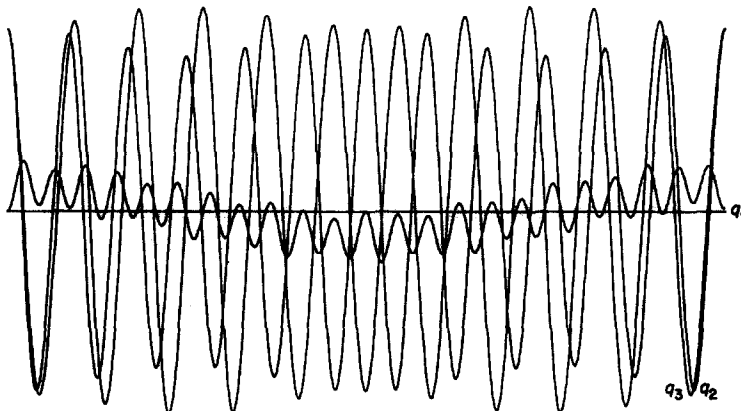


Figure 10. Response of the fixed-free beam B4; $a_1, a_2, a_3 = 0.0, 0.3, 0.3$.

arises in the motion of B4, oscillating with its frequency but modulated on the low time scale by the amplitudes a_2 and a_3 . Of course, when finite initial conditions are given to torsion and one flexure while the other flexure is simply perturbed, only small forcing effects on this latter co-ordinate are observed.

5. CONCLUSIONS

The non-linear dynamic coupling phenomena among flexural and torsional vibrations have been studied for inextensional non-compact beams with all frequencies of the same order and in the absence of internal resonance. Use has been made of accurate equations consistently deduced from more general ones through an ordering of the coefficients, with no assumptions on the configuration variables. The law of motion of the beam has been obtained by means of a third order perturbation analysis.

Some conclusions can be drawn from the investigation performed for beams of open cross-sections with different dynamic characteristics and boundary conditions, selected on the basis of some parameters which govern the phenomenon according to the perturbation solution. It must be underlined that the results presented are obtained within a beam theory in which warping is neglected.

Once the important role played by the non-linear static interaction has been ascertained, the dynamic response is seen to be notably influenced by the values of some meaningful combinations of the linear frequencies, being characterized by different amounts of energy exchange among the co-ordinates. With attention restricted to the combinations involving the torsional frequency—which are the most interesting for analysis within the present context—strong forced torsional motion and associated flexural coupling arise for beams having all comparable frequencies, while very low interaction occurs if the torsional frequency is notably higher than the other ones. In addition, the time response changes notably depending on the resonance condition which is approached and the initial amplitudes considered.

In summary, when a resonance condition involving three components—which is of quadratic order—is approached, even though one component is assigned a zero initial value, it is forced strongly in the motion and in turn modifies the laws of the remaining two. Instead, when a condition—of cubic order—involving two components is approached, and provided that they are given finite initial amplitudes, notable although lower energy exchange occurs between them, while the third component is forced slightly if it is zero or simply perturbed initially; in particular, little interaction occurs where the condition does not involve the torsional frequency.

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APPENDIX 1: EXPRESSIONS FOR THE COEFFICIENTS m, k, c, d

$$\begin{aligned}
 m_1 &= \nu \rho^2 \int_0^1 f_1^2 ds, & m_2 &= \mu \int_0^1 f_2^2 ds, & m_3 &= \mu \int_0^1 f_3^2 ds, \\
 k_1 &= \rho^2 \int_0^1 f_1'^2 ds, & k_2 &= \beta_3 \int_0^1 f_2''^2 ds, & k_3 &= \beta_2 \int_0^1 f_3''^2 ds, \\
 d &= -\rho(\beta_2 - \beta_3) \int_0^1 f_1 f_2'' f_3'' ds, & c_{12} &= \rho^2(\beta_2 - \beta_3) \int_0^1 f_1^2 f_2''^2 ds, \\
 c_{13} &= -\rho^2(\beta_2 - \beta_3) \int_0^1 f_1^2 f_3''^2 ds, \\
 d_{12} &= (c_{12} - \Omega_3 d^2 / k_3), & d_{12}^* &= (c_{12} - \Omega_3^* d^2 / k_3), & \tilde{d}_{12} &= d_{12} + d_{12}^*, \\
 d_{13} &= (c_{13} - \Omega_2 d^2 / k_2), & d_{13}^* &= (c_{13} - \Omega_2^* d^2 / k_2), & \tilde{d}_{13} &= d_{13} + d_{13}^*, \\
 d_{23} &= -\Omega_1 d^2 / k_1, & d_{23}^* &= -\Omega_1^* d^2 / k_1, & \tilde{d}_{23} &= d_{23} + d_{23}^*.
 \end{aligned}$$

APPENDIX 2: EXPRESSIONS FOR THE INITIAL VALUES OF THE AMPLITUDES A_j, B_j, C_j OF THE HOMOGENEOUS SOLUTION

$$\begin{aligned}
 A_j(0) &= \frac{1}{2}(q_j^0 - i \dot{q}_j^0 / \omega_j), \\
 B_j(0) &= (1/2k_j) \{ [\Omega_j d (A_h A_k + \bar{A}_h \bar{A}_k) + \Omega_j^* d (\bar{A}_h A_k + A_h \bar{A}_k)] \\
 &\quad + (1/\omega_j) [(\omega_h + \omega_k) \Omega_j d (A_h A_k - \bar{A}_h \bar{A}_k) + (\omega_k - \omega_h) \Omega_j^* d (\bar{A}_h A_k - A_h \bar{A}_k)] \}_0, \\
 C_j(0) &= (1/2k_j) \{ \Omega_{jh} d_{jh} (A_j A_h^2 + \bar{A}_j \bar{A}_h^2) + \Omega_{jh}^* d_{jh}^* (\bar{A}_j A_h^2 + A_j \bar{A}_h^2) + \Omega_{jk} d_{jk} (A_j A_k^2 + \bar{A}_j \bar{A}_k^2) \\
 &\quad + \Omega_{jk}^* d_{jk}^* (\bar{A}_j A_k^2 + A_j \bar{A}_k^2) + \Omega_j d (A_h B_k + A_k B_h + \bar{A}_h \bar{B}_k + \bar{A}_k \bar{B}_h) \\
 &\quad + \Omega_j^* d (\bar{A}_h B_k + A_k \bar{B}_h + A_h \bar{B}_k + \bar{A}_k B_h) + i(D_2 A_j + D_2 \bar{A}_j) k_j / \omega_j \\
 &\quad + [(\omega_j + 2\omega_h) \Omega_{jh} d_{jh} (A_j A_h^2 - \bar{A}_j \bar{A}_h^2) + (2\omega_h - \omega_j) \Omega_{jh}^* d_{jh}^* (\bar{A}_j A_h^2 - A_j \bar{A}_h^2) \\
 &\quad + (\omega_j + 2\omega_k) \Omega_{jk} d_{jk} (A_j A_k^2 - \bar{A}_j \bar{A}_k^2) + (2\omega_k - \omega_j) \Omega_{jk}^* d_{jk}^* (\bar{A}_j A_k^2 - A_j \bar{A}_k^2) \\
 &\quad + (\omega_h + \omega_k) \Omega_j d (A_h B_k + A_k B_h - \bar{A}_h \bar{B}_k - \bar{A}_k \bar{B}_h) \\
 &\quad + (\omega_k - \omega_h) \Omega_j^* d (\bar{A}_h B_k + A_k \bar{B}_h - A_h \bar{B}_k - \bar{A}_k B_h)] / \omega_j \}_0.
 \end{aligned}$$