Lagrangian fibrations in duality on moduli space of rank two logarithmic connections over the projective line

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Abstract. We study the moduli space of logarithmic connections of rank 2 on $\mathbb{P}^1_{\mathbb{C}}$ minus n points with fixed spectral data. There are two natural Lagrangian maps: one towards apparent singularities of the associated fuchsian scalar equation, and another one towards moduli of parabolic bundles. We show that these are transversal and dual to each other. In case $n = 5$, we recover the beautiful geometry of Del Pezzo surfaces of degree 4.

1. Introduction

In this paper, we investigate the geometry of moduli space of rank 2 logarithmic connections over the Riemann sphere and extend some results obtained together with Carlos Simpson in the 4-point case [**13**]. Precisely, we fix a reduced effective divisor $D = t_1 + \cdots + t_n$ on $X := \mathbb{P}^1_{\mathbb{C}}$ and consider those pairs (E, ∇) where E is a rank 2 vector bundle over X and $\nabla : E \to E \otimes \Omega^1_X(D)$ a connection having simple poles supported by D. At each pole, we have two residual eigenvalues $\{\nu_i^+, \nu_i^-\}$, $i = 1, \ldots, n$; they satisfy Fuchs relation $\sum_i(\nu_i^+ + \nu_i^-) + d = 0$ where $d = \deg(E)$. Moreover, we can naturally introduce parabolic strucures $\boldsymbol{l} = \{l_i\}_{1\le i\le n}$ such that l_i is a one dimensional subspace of $E_{|t_i}$ which corresponds to an eigen space of the residue of ∇ at t_i with the eigenvalue ν_i^+. Note that when $\nu_i^+ \neq \nu_i^-$, the parabolic structure $\boldsymbol{l}$ is determined by the connection (E, ∇). Fixing spectral data $\boldsymbol{\nu} = (\nu_i^{\pm})$ with integral sum $-d$, by introducing the weight $\boldsymbol{w}$ for stablity, we can construct the moduli space $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$ of $\boldsymbol{w}$-stable $\boldsymbol{\nu}$-parabolic connections $(E, \nabla, \boldsymbol{l})$ by Geometric Invariant Theory [**10**] and the moduli space $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$ turns to be a smooth irreducible quasi-projective variety of dimension $2(n-3)$. It is moreover equipped with a natural holomorphic symplectic structure. We note that, when $\sum_{i=1,\ldots,n} \nu_i^{\epsilon_i} \notin \mathbb{Z}$, for any choice $(\epsilon_i) \in \{+, -\}^n$, every parabolic connection $(E, \nabla, \boldsymbol{l})$ is irreducible, and thus stable; the moduli space $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$ does not depend on the choice of weights $\boldsymbol{w}$ in this case. These moduli spaces occur as space of initial conditions for Garnier systems, the case $n = 4$ corresponding to the Painlevé VI equation (cf. [**12, 11**]).

2010 *Mathematics Subject Classification.* Primary 34M55; Secondary 14D20, 32G20, 32G34.

Key words and phrases. Painlevé VI, Garnier systems, Logarithmic connection, Parabolic structure, Projective line, apparent singularity, Okamoto-Painlevé pair, Del Pezzo surface.

This research was partly supported by JSPS Grant-in-Aid for Scientific Research (S)19104002, 24224001, challenging Exploratory Research 23654010.

There are many isomorphisms between these moduli spaces. For instance, twisting by a rank 1 connection (with the same poles), one can translate the spectral data as $(\nu_i^\pm) \mapsto (\nu_i^\pm + \mu_i)$ with the only restriction $\sum_i \mu_i \in \mathbb{Z}$. Also, by using elementary (or Schlesinger) transformations, we may shift each $\nu_i^\pm$ by arbitrary integer, thus freely shifting the degree d of vector bundles. In particular, it is enough for our purpose to consider the case where $\sum_i(\nu_i^+ + \nu_i^-) = 1$, which means by Fuchs relations that $d = \det(E) = -1$.

1.1. Apparent map. There are two natural Lagrangian fibrations on these moduli spaces. One of them is given by the "apparent map"

$$\mathrm{App} : M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu}) \dashrightarrow |\mathcal{O}_X(n-3)| \simeq \mathbb{P}^{n-3}_{\mathbb{C}},$$

which is a rational dominant map towards the projective space of the linear system (see [**8, 11, 19**]). Here, we need to fix degree $d = -1$. The image $\mathrm{App}(E, \nabla)$ is defined by the zero divisor of the composite map

$$\mathcal{O}_X \to E \xrightarrow{\nabla} E \otimes \Omega^1_X(D) \to (E/\mathcal{O}_X) \otimes \Omega^1_X(D).$$

For a generic connection (E, ∇), it is well-known that the underlying bundle is $E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ and the map $\mathcal{O}_X \to E$ is therefore unique up to scalar multiplication; the right-hand-side arrow is just the quotient by the image of $\mathcal{O}_X \to E$. The apparent map is therefore well-defined on a large open set of $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$. Choosing the image of $\mathcal{O}_X$ as a cyclic vector allow to derive a $2nd$ order Fuchsian differential equation; $\mathrm{App}(E, \nabla)$ gives the position of extra apparent singular points arising from this construction, whence the name. The apparent map App has indeterminacy points where $E \not\simeq \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ or $E \simeq \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ and $\mathcal{O}_X$ is invariant under ∇.

1.2. Parabolic fibration. Let $\mathcal{P}_d(\mathbf{t})$ be the moduli stack of undecomposable parabolic bundles of degree d, and denote by $P_d(\mathbf{t})$ its corresponding coarse moduli space. It is known that the natural map $\mathcal{P}_d(\mathbf{t}) \longrightarrow P_d(\mathbf{t})$ is a $\mathbf{G}_m$-gerbe. We assume that $\boldsymbol{\nu}$ is generic such as undecomposable parabolic bundles of degree d coincide with the $\boldsymbol{\nu}$-flat parabolic bundles, i.e, those admitting a connection with prescribed parabolic and spectral data $(\boldsymbol{l}, \boldsymbol{\nu})$ (see Proposition 3.1). Under this assumption, we can define the second fibration, which is more natural, but actually more subtle to define. It is the forgetfull map $(E, \nabla, \boldsymbol{l}) \mapsto (E, \boldsymbol{l})$ towards the coarse moduli space $P_d(\mathbf{t})$ (here we do not need anymore $d = -1$). Always assuming generic spectral data $\boldsymbol{\nu}$, the parabolic structure $\boldsymbol{l} = \{l_i\}$ is the data of the residual eigendirection $l_i \subset E_{|t_i}$ of ∇ with respect to the eigenvalue ν_i^+ for each pole t_i. However, as observed in [**4, 3**], the coarse moduli space $P_d(\mathbf{t})$ is a non Hausdorff topological space, or a nonseparated scheme. To get a nice moduli space, we have to impose a stability condition with respect to weights $\boldsymbol{w} = (w_i) \in [0,1]^n$ (see [**15**]); the moduli space $P_d^{\boldsymbol{w}}(\mathbf{t})$ of $\boldsymbol{w}$-semistable degree d parabolic bundles $(E, \boldsymbol{l})$ is therefore a normal irreducible projective variety; the open subset of $\boldsymbol{w}$-stable bundles is smooth. These moduli spaces actually depend on the choice of weights. For generic weights, $\boldsymbol{w}$-semistable=$\boldsymbol{w}$-stable and we get a smooth projective variety. Precisely, there are finitely many hyperplanes (walls) cutting out $[0,1]^n$ into finitely many chambers and strictly $\boldsymbol{w}$-semistable bundles only occur along walls. The moduli space $P_d^{\boldsymbol{w}}(\mathbf{t})$ is locally constant in each chamber and is either empty, or has expected dimension $n-3$. It becomes singular along walls (or maybe reduced to a single point). A generic $\boldsymbol{w}$-stable connection is also $\boldsymbol{w}'$-stable for any weight $\boldsymbol{w}'$

(for which the corresponding moduli space has the right dimension). We thus get natural birational maps $P_d^{\boldsymbol{w}}(\mathbf{t}) \overset{\sim}{\dashrightarrow} P_d^{\boldsymbol{w}'}(\mathbf{t})$ identifying generic bundles that occur in both moduli spaces. An important fact (see Proposition 3.4) is that a parabolic bundle $(E, \boldsymbol{l})$ is undecomposable if, and only if, it is $\boldsymbol{w}$-stable for a convenient choice of weights. As a consequence, the coarse moduli space $P_d(\mathbf{t})$ of undecomposable parabolic bundles of degree d is covered by those smooth projective varieties $P_d^{\boldsymbol{w}}(\mathbf{t})$ when $\boldsymbol{w}$ runs over all chambers (cf. Proposition 3.6):

$$P_d(\mathbf{t}) = \cup_{i,finite} P_d^{\boldsymbol{w}_i}(\mathbf{t}). \tag{1.1}$$

In fact, the coarse moduli space is obtained by patching together these (non empty) projective charts along (strict) open subsets; each of these projective charts are open and dense in the coarse moduli space. Once we choose one of these charts, we get a rational map $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu}) \dashrightarrow P_d^{\boldsymbol{w}}(\mathbf{t})$ which turns to be Lagrangian, like for App in case $d = -1$. Moreover we can extend this rational map to a rational map $\mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu}) \dashrightarrow P_d(\mathbf{t})$, which turns to be a morphism when $\boldsymbol{\nu}$ is generic.

1.3. Results. Assuming from now on $d = -1$, from the two rational maps App and Bun, we obtain the rational map

$$\mathrm{App} \times \mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu}) \dashrightarrow |\mathcal{O}_X(n-3)| \times P_{-1}^{\boldsymbol{w}}(\mathbf{t}). \tag{1.2}$$

In this paper, we will basically prove that this map is birational provided that $\sum_i \nu_i^- \neq 0$. However we will be able to give more precise information about the rational map (1.2) by introducing a choice of democratic weights $\boldsymbol{w}$ (see (4.1)) and a good open subset $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0 \subset M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$.

For such a choice of weights $\boldsymbol{w}$ in (4.1), $\boldsymbol{w}$-stable parabolic bundles $(E, \boldsymbol{l})$ are precisely those flat bundles for which $E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ and none of the parabolics coincide with the special subbundle: $l_i \notin \mathcal{O}_X$ for all $i = 1, \ldots, n$. We are moreover able to construct a natural isomorphism $P_{-1}^{\boldsymbol{w}}(\mathbf{t}) \xrightarrow{\sim} |\mathcal{O}_X(n-3)|^*$ with the dual of the linear system involved in the apparent map (cf. Proposition 3.7). We therefore introduce the open subset

$$M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0 := \mathrm{Bun}^{-1} P_{-1}^{\boldsymbol{w}}(\mathbf{t}) \ \subset \ M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$$

by imposing the conditions on $(E, \nabla, \boldsymbol{l})$ that $(E, \boldsymbol{l}) \in P_{-1}^{\boldsymbol{w}}(\mathbf{t})$. Then the two rational maps App and Bun now induce a natural morphism

$$\mathrm{App} \times \mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0 \longrightarrow |\mathcal{O}_X(n-3)| \times |\mathcal{O}_X(n-3)|^* \tag{1.3}$$

and both App and Bun are Lagrangian. We can state our result as follows.

THEOREM 1.1. *Under the assumption that $\sum_i \nu_i^- \neq 0$, the morphism* App×Bun *in (1.3) is an open embedding and its image coincides with the complement of the incidence variety $\Sigma \subset |\mathcal{O}_X(n-3)| \times |\mathcal{O}_X(n-3)|^*$ for the duality.*

In order to make the statement of Theorem 1.1 more precise, let us introduce the homogeneous coordinates $\mathbf{a} = (a_0 : \cdots : a_{n-3})$ on $|\mathcal{O}_X(n-3)| \simeq \mathbb{P}_{\mathbf{a}}^{n-3}$, and the dual coordinates $\mathbf{b} = (b_0 : \cdots : b_{n-3})$ on $|\mathcal{O}_X(n-3)|^* \simeq \mathbb{P}_{\mathbf{b}}^{n-3}$. Let $\Sigma \subset \mathbb{P}_{\mathbf{a}}^{n-3} \times \mathbb{P}_{\mathbf{b}}^{n-3}$ be the incidence variety, whose defining equation is given by $\sum_k a_k b_k = 0$. Then the morphism App × Bun induces the isomorphism (see Theorem 1.1)

$$\mathrm{App} \times \mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0 \xrightarrow{\sim} (\mathbb{P}_{\mathbf{a}}^{n-3} \times \mathbb{P}_{\mathbf{b}}^{n-3}) \setminus \Sigma.$$

Setting $\rho := -\sum_i \nu_i^-$, the symplectic structure of $M(\mathbf{t},\boldsymbol{\nu})^0$ is given by

$$\omega = d\eta \quad \text{where} \quad \eta = \rho\frac{a_0 db_0 + \cdots + a_{n-3}db_{n-3}}{a_0 b_0 + \cdots + a_{n-3}b_{n-3}}.$$

When $\rho = 0$, we prove that App $\times$ Bun degenerates: it is dominant onto Σ.

In order to prove Theorem 1.1, we intoduce a good compactification $\overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0} \supset M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0$ (cf. Section 4.2) which turns to be another moduli space:

$$(1.4) \qquad \overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0} := \left\{ \begin{array}{l} (E,\nabla,\lambda\in\mathbb{C},\boldsymbol{l}) \\ \lambda\text{-}\boldsymbol{\nu}\text{-parabolic connection} \end{array} \mid (E,\boldsymbol{l}) \in P_{-1}^{\boldsymbol{w}}(\mathbf{t}) \right\} / \simeq$$

where equivalence $\simeq$ is given by bundle isomorphisms and the natural $\mathbb{C}^*$-action by scalar multiplication. The open subset $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0 \hookrightarrow \overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0}$ is given by those λ-$\boldsymbol{\nu}$-parabolic connections for which $\lambda \neq 0$, and the complement

$$M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0_H := \overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0} \setminus M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0$$

is the moduli space of $\boldsymbol{w}$-stable parabolic Higgs bundles. Now Theorem 1.1 easily follows from the following (cf. Theorem 4.3)

THEOREM 1.2. *If $\sum_i \nu_i^- \neq 0$, the moduli space $\overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0}$ is a smooth projective variety and we can extend the morphism (1.3) as an isomorphism*

$$\text{App}\times\text{Bun} : \overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0} \xrightarrow{\sim} |\mathcal{O}_X(n-3)| \times |\mathcal{O}_X(n-3)|^*$$

Moreover, by restriction, we also obtain the isomorphism

$$\text{App}\times\text{Bun}_{|M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0_H} : M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0_H \xrightarrow{\sim} \Sigma$$

where $\Sigma \subset |\mathcal{O}_X(n-3)| \times |\mathcal{O}_X(n-3)|^*$ is the incidence variety for the duality.

Here we note that the coarse moduli space of $\boldsymbol{w}$-stable λ-$\boldsymbol{\nu}$ parabolic connections without the condition in (1.4) have singularities. So our choice of the weight $\boldsymbol{w}$ and the compactification in (1.4) is essential to prove Theorem 1.2.

It is well-known [**4, 3**] that the moduli space $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})$ should be an affine extension of the cotangent bundle of $P_{-1}(\mathbf{t})$; once we have choosen a projective chart $P_{-1}^{\boldsymbol{w}}(\mathbf{t}) \simeq \mathbb{P}^{n-3}_{\mathbf{b}}$, the restricted affine bundle $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0$ must be either the cotangent bundle $T^*\mathbb{P}^{n-3}_{\mathbf{b}}$, or the unique non trivial affine extension of $T^*\mathbb{P}^{n-3}_{\mathbf{b}}$. Here, we prove that we are in the latter case if, and only if, $\rho \neq 0$. But the nice fact is that apparent map provides a natural trivialization for the compactification of this affine bundle.

The projective space $|\mathcal{O}_X(n-3)| \simeq \mathbb{P}^{n-3}_{\mathbf{a}}$ may be considered as the space of polynomial equations $\sum_k a_k z^k = 0$. We can consider the $(n-3)!$-fold cover $(\mathbb{P}^1_{\mathbb{C}})^{n-3} \to (\mathbb{P}^1_{\mathbb{C}})^{(n-3)} := (\mathbb{P}^1_{\mathbb{C}})^{n-3}/\mathfrak{S}_{n-3} = \mathbb{P}^{n-3}_{\mathbf{a}}$ parametrized by ordered roots $(q_1 : \cdots : q_{n-3})$. Since we have a morphism App : $M(\mathbf{t},\boldsymbol{\nu}) \dashrightarrow \mathbb{P}^{n-3}_{\mathbf{a}}$, by the fibered product, we get a $(n-3)!$-fold cover $\widetilde{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0} \longrightarrow M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0$. This latter one (or some natural partial compactification $\widetilde{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})}$) has been described in many papers [**18, 8, 22, 19**]. The space $\widetilde{M(\mathbf{t},\boldsymbol{\nu})}$ can be covered by affine charts $\mathbb{C}^{2n-6}$ with Darboux coordinates (p_k, q_k) for the symplectic structure: $\omega = \sum_k dp_k \wedge dq_k$. Our parameters can be expressed in terms of symmetric functions of p_k's and q_k's, what we do explicitely in the five pole case $n = 5$ at the end of the paper. From this point of view, S. Oblezin constructs in [**17**] a natural birational map $M(\mathbf{t},\boldsymbol{\nu}) \dashrightarrow$

$(K_n)^{(n-3)}$ where K_n is an open subset of the total space of $\Omega^1_X(D)$ blown up at $2n$ points. Precisely, at each fiber $z = t_i$, we have the residual map

$$\Omega^1_X(D)_{|t_i} \stackrel{\mathrm{Res}_{t_i}}{\longrightarrow} \mathbb{C}$$

and we blow-up the two points corresponding to $\nu_i^+, \nu_i^- \in \mathbb{C}$ in the fiber; then we delete the strict transforms of fibers $z = t_i$ to obtain the open set K_n. So far, no natural system of coordinates was provided on $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$ for $n \geq 5$.

In [**1**], Arinkin investigated the geometric Langlands problem related to the case of $n = 4$ by using the natural morphism of coarse modui spaces $\mathrm{Bun} : M(\mathbf{t}, \boldsymbol{\nu}) \longrightarrow P_{-1}(\mathbf{t})$ (see also [**2**]). Though we cannot directly extend his methods to the case of $n > 4$, we may expect that our main Theorem 1.1 and 1.2 may give some approach to obtain a similar result.

In the last part of the paper, we investigate with many details the case $n = 5$. We provide a precise description of the non-separated coarse moduli space $P_{-1}(\mathbf{t})$ of undecomposable parabolic bundles, which turns to be closely related to the geometry of degree 4 Del Pezzo surfaces. Precisely, there is a natural embedding $X \hookrightarrow V := \mathbb{P}^2_{\mathbf{b}}$ as a conic. We then consider the blow-up $\phi : \hat{V} \to V$ of the images of the 5 points $t_1, \ldots, t_5 \in X$: this is the Del Pezzo surface associated to our problem. If we denote by $\Pi_i \subset \hat{V}$ the exceptional divisor over t_i, and by $\Pi_{i,j} \in \hat{V}$ the strict transform of the line in V passing through t_i and t_j for any i, j, then these $\Pi, \Pi_i, \Pi_{i,j}$ are the well-known 16 rational curves with self-intersection (-1) in $\hat{V}$. For any $i = 1, \ldots, 5$, by contracting all five (-1)-curves intersecting Π_i, we get a new morphism $\phi_i : \hat{V} \to V_i \simeq \mathbb{P}^2_{\mathbb{C}}$.

THEOREM 1.3. *The coarse moduli space $P_{-1}(\mathbf{t})$ is given by:*

$$P_{-1}(\mathbf{t}) = \hat{V} \cup V \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$$

where $\cup$ means that we are patching these projective manifolds by means of the birational maps ϕ, ϕ_i and $\phi_i \circ \phi_j^{-1}$ along the maximal open subsets where they are one-to-one.

For instance, $\phi : \hat{V} \to V$ induces an isomorphism

$$\hat{V} \setminus (\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4 \cup \Pi_5) \stackrel{\sim}{\longrightarrow} V \setminus \{t_1, t_2, t_3, t_4, t_5\}$$

and we patch V to $\hat{V}$ along these open subsets by means of this isomorphism. Moreover, all these projective charts $\hat{V}, V, V_i$ are realized as coarse moduli spaces of stable parabolic bundles $P^{\boldsymbol{w}}_{-1}(\mathbf{t})$ for convenient choices of weights $\boldsymbol{w}$ and, in the patching, we just identify all isomorphism classes of bundles that are shared by any two of these projective charts. Finally, we explain how to recover the total moduli space $M(\mathbf{t}, \boldsymbol{\nu})$ by blowing-up $|\mathcal{O}_X(n-3)| \times |\mathcal{O}_X(n-3)|^*$ along some lifting of these curves inside the incidence variety Σ.

Acknowledgement: The authors would like to warmly thank Michele Bolognesi for useful discussions about moduli spaces of n-points configurations on the projective line.

2. Moduli space of connections

In this section, we will recall some results in [**16, 15, 21, 14, 10, 9**].

2.1. Definition of the moduli space (as geometric quotient). Let us fix a set of n-distinct points $\mathbf{t} = \{t_1, \cdots, t_n\}$ on the Riemann sphere $X := \mathbb{P}^1_{\mathbb{C}}$ and define the divisor $D = t_1 + \cdots + t_n$. In this paper, a logarithmic connection of rank 2 on X with singularities (or poles) at D is a pair (E, ∇) consisting of an algebraic (or holomorphic) vector bundle E on $\mathbb{P}^1_{\mathbb{C}}$ of rank 2 and a linear algebraic connection $\nabla : E \longrightarrow E \otimes \Omega^1_X(D)$. We can define the residue homomorphism $\mathsf{res}_{t_i}(\nabla) \in \mathrm{End}(E_{|t_i}) \simeq M_2(\mathbb{C})$ and then let ν_i^+, ν_i^- be the eigenvalues of $\mathsf{res}_{t_i}(\nabla)$, that we call *local exponents*. Fuchs relation says that $\sum_i (\nu_i^+ + \nu_i^-) = -\deg E = -d$. So we define the set of local exponents of degree d

$$
(2.1)\quad \mathcal{N}_n(d) := \left\{ \boldsymbol{\nu} = (\nu_i^\pm)_{1\le i\le n} \in \mathbb{C}^{2n} \;\middle|\; d + \sum_{1\le i\le n} (\nu_i^+ + \nu_i^-) = 0 \right\} \simeq \mathbb{C}^{2n-1}
$$

DEFINITION 2.1. *Fix $\boldsymbol{\nu} \in \mathcal{N}_n(d)$. A $\boldsymbol{\nu}$-parabolic connection on $(X, D) = (\mathbb{P}^1_{\mathbb{C}}, \mathbf{t})$ is a collection $(E, \nabla, \boldsymbol{l} = \{l_i\})$ consisting of the following data:*

1. *a logarithmic connection (E, ∇) on (X, D) of rank 2 with spectral data $\boldsymbol{\nu}$,*
2. *a one dimensional subspace $l_i \subset E_{|t_i}$ on which $\mathsf{res}_{t_i}(\nabla)$ acts by multiplication by ν_i^+.*

For generic $\boldsymbol{\nu}$, the parabolic direction l_i is nothing but the eigenspace for $\mathsf{res}_{t_i}(\nabla)$ with respect to ν_i^+ so that the parabolic data is uniquely defined by the connection (E, ∇) itself. However, when $\nu_i^+ = \nu_i^-$ and $\mathsf{res}_{t_i}(\nabla)$ is scalar (i.e. diagonal), then the parabolic l_i add a non trivial data and this allow to avoid singularities of the moduli space.

In order to obtain a good moduli space, we have to introduce a stability condition for parabolic connections. For this, fix weights $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0, 1]^n$. Then for any line subbundle $F \subset E$, define the $\boldsymbol{w}$-stability index of F to be the real number

$$
(2.2)\qquad \mathrm{Stab}(F) := \deg(E) - 2\deg(F) + \sum_{l_i \neq F_{|t_i}} w_i - \sum_{l_i = F_{|t_i}} w_i.
$$

DEFINITION 2.2. *A $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \boldsymbol{l})$ will be called* $\boldsymbol{w}$-stable *(resp.* $\boldsymbol{w}$-semistable*) if for any rank one ∇-invariant subbundle $F \subset E$,*

$$
\nabla(F) \subset F \otimes \Omega^1_X(D)
$$

the following inequality holds

$$
(2.3)\qquad \mathrm{Stab}(F) > 0 \quad (\textit{resp.}\ \geq 0).
$$

A rank 2 parabolic bundle $(E, \boldsymbol{l})$ is called *$\boldsymbol{w}$-stable* (resp. *$\boldsymbol{w}$-semistable*) if inequality (2.3) holds for any rank one subbundle $F \subset E$. In particular, a $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \boldsymbol{l})$ may be stable while the underlying parabolic bundle (E, ∇) is not. For instance, if $(E, \nabla, \boldsymbol{l})$ is irreducible, there is no strict ∇-invariant subbundle and condition (2.3) is just empty.

REMARK 2.3. To fit into usual notations (see [**15, 21, 14, 9, 10**]), one should rather consider the flag $\{l_0^{(i)} \supset l_1^{(i)} \supset l_2^{(i)}\} := \{E_{|t_i} \supset l_i \supset \{0\}\}$ and ask that $(\mathsf{res}_{t_i}(\nabla) - \nu_j^{(i)} Id)(l_j^{(i)}) = l_{j+1}^{(i)}$ for each $j = 0, 1$, where $(\nu_0^{(i)}, \nu_1^{(i)}) := (\nu_i^-, \nu_i^+)$. In the rank 2 case, this is equivalent to Definition 2.1. Then, weights rather look like:

$$
\boldsymbol{\alpha} = (\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_1^{(n)}, \alpha_2^{(n)}) \in \mathbb{R}^{2n}
$$

satisfying $\alpha_1^{(i)} \leq \alpha_2^{(i)} \leq \alpha_1^{(i)} + 1$. Then, for any nonzero ∇-invariant subbundle $F \subset E$, we define integers

$$\mathrm{length}(F)_j^{(i)} = \dim(F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}),$$

and stability is defined by the inequality

$$\deg F + \sum_{i=1}^{n} \sum_{j=1}^{2} \alpha_j^{(i)} \mathrm{length}(F)_j^{(i)} < \frac{\deg E + \sum_{i=1}^{n} \sum_{j=1}^{2} \alpha_j^{(i)} \mathrm{length}(E)_j^{(i)}}{2}.$$

However, it is straightforward to check that this condition is equivalent to (2.3) after setting $w_i = \alpha_2^{(i)} - \alpha_1^{(i)}$. Sometimes, a parabolic degree is defined by

$$\deg^{\mathrm{par}} F := \deg F + \sum_{i=1}^{n} \sum_{j=1}^{2} \alpha_j^{(i)} \mathrm{length}(F)_j^{(i)} = \deg(F) + \sum_{i=1}^{n} \alpha_{\epsilon_i}^{(i)}$$

(including the case $F = E$) and in this case, the stability index for a line bundle is given by $\mathrm{Stab}(F) := \deg^{\mathrm{par}} E - 2 \deg^{\mathrm{par}} F$. We also note that, in [**10**], it was assumed that $0 < \alpha_2^{(i)} < \alpha_1^{(i)} < 1$ and genericity conditions to obtain the smooth moduli space for all $\boldsymbol{\nu} \in \mathcal{N}_n(d)$ simultaneously. In this paper, we will vary the weights $\boldsymbol{\alpha}$ and may consider the case of the equality $\alpha_1^{(i)} = \alpha_2^{(i)}$ for some i. Note also that in [**13**], we use the minus sign in front of the weight $\alpha_j^{(i)}$.

Consider the line bundle $L := \mathcal{O}_X(d)$ and denote by $\nabla_L : L \longrightarrow L \otimes \Omega_X^1(D)$ the unique logarithmic connection having residual eigenvalue $\nu_i^+ + \nu_i^-$ at each pole t_i. For any $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \boldsymbol{l})$ like above, there exists an isomorphism $\varphi : \wedge^2 E \longrightarrow L$; it is unique up to a scalar and automatically conjugates the trace connection $\mathrm{tr}(\nabla)$ with ∇_L. We must add a choice of such isomorphism in the data $(E, \nabla, \varphi, \boldsymbol{l})$ in order to kill-out automorphisms; this is needed in the construction of the moduli space. We omit this data φ in the sequel for simplicity.

Define the moduli space $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$ of isomorphism classes of $\boldsymbol{w}$-stable $\boldsymbol{\nu}$-parabolic connections $(E, \nabla, \boldsymbol{l})$. Set $T_n = \{\mathbf{t} = (t_1, \cdots, t_n) \in X^n \ ; \ t_i \neq t_j \text{ for } i \neq j\}$. Considering the relative setting of moduli space over $T_n \times \mathcal{N}_n(d)$, we obtain a family of moduli space

$$\pi_n : M^{\boldsymbol{w}} \longrightarrow T_n \times \mathcal{N}_n(d),$$

such that $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu}) = \pi_n^{-1}(\mathbf{t}, \boldsymbol{\nu})$.

Theorem 2.4. ([Theorem 2.1, Proposition 6.1, [**10**]]). *Assume that $n > 3$. For a generic weight $\boldsymbol{w}$, we can construct a relative fine moduli space*

$$\pi_n : M^{\boldsymbol{w}} \longrightarrow T_n \times \mathcal{N}_n(d),$$

which is a smooth, quasi-projective *morphism of relative dimension $2n-6$ with irreducible closed fibers. Therefore, the moduli space $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$ is a* smooth, irreducible *quasi-projective algebraic variety of dimension $2n-6$ for all $(\mathbf{t}, \boldsymbol{\nu}) \in T_n \times \mathcal{N}_n(d)$. Moreover the moduli space $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$ admits a natural holomorphic symplectic structure.*

2.2. Isomorphisms between moduli spaces: twist and elementary transformations. Given a connection (F,∇_F) of rank 1,

$$\nabla_F : F \to F \otimes \Omega^1_X(D)$$

with local exponents $(\mu_1,\ldots,\mu_n)$, we can define the *twisting map*

$$\otimes(F,\nabla_F) : \begin{cases} M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu}) & \to & M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu}') \\ (E,\nabla,\boldsymbol{l}) & \mapsto & (E\otimes F,\nabla\otimes\nabla_F,\boldsymbol{l}) \end{cases}$$

where $\boldsymbol{\nu}' = (\nu_i^\pm + \mu_i)$. It is an isomorphism. It follows that our moduli space only depend on differences $\nu_i^+ - \nu_i^-$. On the other hand, this allow to rather freely modify the trace connection. Precisely, depending on the parity of the degree d, we can go into one of the following two cases

- in the even case, $(L,\nabla_L) = (\mathcal{O}_X, d)$ and $(\nu_i^+,\nu_i^-) = (\frac{\kappa_i}{2}, -\frac{\kappa_i}{2})$;
- in the odd case, $(L,\nabla_L) = (\mathcal{O}_X(-t_n), d + \frac{dz}{z-t_n})$ and
  $(\nu_i^+,\nu_i^-) = (\frac{\kappa_i}{2}, -\frac{\kappa_i}{2})$ except $(\nu_n^+,\nu_n^-) = (\frac{\kappa_n}{2}+\frac{1}{2}, -\frac{\kappa_n}{2}+\frac{1}{2})$;

where (L,∇_L) is the fixed trace connection as above.

For each $i=1,\ldots,n$, we can define the *elementary transformation*

$$\mathrm{Elm}^-_{t_i} : \begin{cases} M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu}) & \to & M^{\boldsymbol{w}'}(\mathbf{t},\boldsymbol{\nu}') \\ (E,\nabla,\boldsymbol{l}) & \mapsto & (E',\nabla',\boldsymbol{l}') \end{cases}$$

The vector bundle E' is defined by the exact sequence of sheaves

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E/l_i \longrightarrow 0$$

where l_i is viewed here as a sky-scrapper sheaf. The parabolic direction l_i' is therefore defined as the kernel of the natural morphism $E' \longrightarrow E$. The new connection ∇' is deduced from the action of ∇ on the subsheaf $E' \subset E$ and, over t_i, eigenvalues are changed by

$$(\nu_i^+,\nu_i^-)' = (\nu_i^- + 1, \nu_i^+) \quad \text{(and other } \nu_j^\pm \text{ are left unchanged for } j\neq i).$$

If a line bundle $F\subset E$ contains the parabolic l_i, it is left unchanged and we get $F\subset E'$; on the other hand, when $l_i \neq F_{|t_i}$ then we get $F'\subset E'$ with $F' = F\otimes\mathcal{O}_X(-t_i)$. It follows that stability condition is preserved for $\boldsymbol{w}'$ defined by

$$w_i' = 1 - w_i \quad \text{(and other } w_j \text{ are left unchanged for } j\neq i).$$

The composition $\mathrm{Elm}^-_{t_i}\circ\mathrm{Elm}^-_{t_i}$ is just the twisting map by $(F,\nabla_F) = (\mathcal{O}_X(-t_i), d + \frac{dz}{z-t_n})$. We may also define $\mathrm{Elm}^+_{t_i}$ as the inverse of $\mathrm{Elm}^-_{t_i}$:

$$\mathrm{Elm}^+_{t_i} = \mathrm{Elm}^-_{t_i} \otimes \left(\mathcal{O}_X(t_i), d - \frac{dz}{z-t_i}\right).$$

Although we are mainly interested in the degree $d=-1$ case where the two Lagrangian fibrations naturally occur, we will also consider the degree 0 case of $\mathfrak{sl}_2$-connections to compare with [**4, 3, 17**]. Explicit computations will be made by means of

$$\mathrm{Elm}^-_{t_n} : M(\mathbf{t},\boldsymbol{\nu}) \to M(\mathbf{t},\boldsymbol{\nu}')$$

going from degree 0 to degree -1 case.

3. The coarse moduli of undecomposable quasi-parabolic bundles

Here we describe the coarse moduli space $P_d(\mathbf{t})$ of undecomposable quasi-parabolic bundles $(E, \boldsymbol{l})$ of rank 2 and of degree d over $(X, D) = (\mathbb{P}^1_{\mathbb{C}}, \mathbf{t})$. When $\boldsymbol{\nu}$ is generic, $P_d(\mathbf{t})$ can be also identified with the coarse moduli space of $\boldsymbol{\nu}$-flat quasi-paraboli bundles of rank 2 of degree d. Here a quasi-parabolic bundle $(E, \boldsymbol{l})$ is called *$\boldsymbol{\nu}$-flat* if it admit a connection ∇ with given local exponents $\boldsymbol{\nu}$.

It is a non separated scheme constructed by patching together moduli spaces $P_d^{\boldsymbol{w}}(\mathbf{t})$ of (semi)stable parabolic bundles for different choices of weights $\boldsymbol{w}$; those charts are smooth projective manifolds. A similar description has been done in [**4**]. In the degree $d = -1$ case, using Higgs fields and apparent map for a convenient cyclic vector, we define a natural "birational" map $P_{-1}(\mathbf{t}) \dashrightarrow |\mathcal{O}_X(n-3)|^* \simeq \mathbb{P}^{n-3}_{\mathbb{C}}$, which turn to be an isomorphism in restriction to one of the projective charts $P_{-1}^{\boldsymbol{w}}(\mathbf{t})$, for a convenient choice of weights. Under the condition that $\boldsymbol{\nu}$ is generic, this will be used in the next section to compute and describe the forgetful map

$$M(\mathbf{t}, \boldsymbol{\nu}) \to P_{-1}(\mathbf{t}) \ ; \ (E, \nabla, \boldsymbol{l}) \mapsto (E, \boldsymbol{l}). \tag{3.1}$$

3.1. The flatness of undecomposable quasi-parabolic bundles. In order to define the forgetful map (3.1), we would like to characterize $\boldsymbol{\nu}$-flat quasi-parabolic bundles $(E, \boldsymbol{l})$ of rank 2, which are, by definition, quasi-parabolic bundles $(E, \boldsymbol{l})$ of rank 2 and of degree d on $(\mathbb{P}^1_{\mathbb{C}}, \mathbf{t})$ arising in our moduli spaces of parabolic connections $M(\mathbf{t}, \boldsymbol{\nu})$, i.e. admitting a connection ∇ with prescribed poles, parabolics and eigenvalues. Under the condition that $\boldsymbol{\nu}$ is generic, this is given by the parabolic version of Weil criterium, see for instance in [**4**, Proposition 3].

PROPOSITION 3.1. *Assume $\nu_1^{\epsilon_1} + \cdots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$ for any $\epsilon_i \in \{+, -\}$. Given a quasi-parabolic bundle $(E, \boldsymbol{l})$, the following condition are equivalent:*

1. *$(E, \boldsymbol{l})$ is $\boldsymbol{\nu}$-flat, that is, $(E, \boldsymbol{l})$ admits a parabolic connection ∇ with eigenvalues $\boldsymbol{\nu}$,*
2. *$(E, \boldsymbol{l})$ is simple: the only automorphisms of E preserving parabolics are scalar,*
3. *$(E, \boldsymbol{l})$ is undecomposable: there does not exist decomposition $E = L_1 \oplus L_2$ such that each parabolic direction l_i is contained either in L_1 or in L_2.*

REMARK 3.2. When $\nu_1^{\epsilon_1} + \cdots + \nu_n^{\epsilon_n} \in \mathbb{Z}$ for some $\epsilon_i \in \{+, -\}$, we still have [simple $\Leftrightarrow$ undecomposable $\Rightarrow$ flat] but some decomposable parabolic bundles also admit connections compatible with $\boldsymbol{\nu}$.

We promptly deduce the following obstruction on E.

COROLLARY 3.3. *Write $E = \mathcal{O}_X(d_1) \oplus \mathcal{O}_X(d_2)$ with $d_1 \leq d_2$, $\deg(E) = d_1 + d_2 = d$. Then E admits an undecomposable quasi-parabolic structure if, and only if*

$$d_2 - d_1 \leq n - 2 \quad \textit{(except } n = 2 \textit{ and } d_1 = d_2 \textit{ which is decomposable).}$$

PROOF. When $d_1 = d_2$, any decomposition of $E = \mathcal{O}_X(d_1) \oplus \mathcal{O}_X(d_1)$ is given by two distinct embeddings $\mathcal{O}_X(d_1) \hookrightarrow E$; one such embedding is determined once you ask it to contain one parabolic direction. Then for $n = 2$ (or less) we can decompose the parabolic data, while for $n \geq 3$, we can easily construct a non decomposable parabolic structure.

When $d_1 < d_2$, any decomposition of E is given by the destabilizing bundle $L_2 = \mathcal{O}_X(d_2)$ and any embeddings $\mathcal{O}_X(d_1) \simeq L_1 \hookrightarrow E$. Latter ones form a family of dimension $n' = d_2 - d_1 + 1$: more precisely, n' parabolics lying outside of $\mathcal{O}_X(d_2)$ are always contained in such subbundle and determine it. If $n \leq d_2 - d_1 + 1$ then any quasi-parabolic structure is thus decomposable; if $n > d_2 - d_1 + 1$, it suffices to choose all l_i's outside $\mathcal{O}_X(d_2)$, and l_n outside of the $\mathcal{O}_X(d_1)$ defined by the $d_2 - d_1 + 1$ first parabolics. □

One of the difficulty to define the forgetful map $(E, \nabla, \boldsymbol{l}) \mapsto (E, \boldsymbol{l})$ is that, although the moduli space of connections is smooth and separated (for generic $\boldsymbol{\nu}$), the image is never separated. The reason is that, although the former moduli spaces can be constructed as geometrical quotient of stable objects, the latter one always contain unstable ones and will be not a good scheme.

However, we have

PROPOSITION 3.4. *A quasi-parabolic bundle $(E, \boldsymbol{l})$ is undecomposable if, and only if, it is stable for a convenient choice of weights $\boldsymbol{w}$. Hence, if we assume that $\boldsymbol{\nu}$ is generic, that is, $\nu_1^{\epsilon_1} + \cdots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$ for any $\epsilon_i \in \{+, -\}$, a quasi-parabolic bundle $(E, \boldsymbol{l})$ is $\boldsymbol{\nu}$-flat if and only if it is stable for a convenient choice of weights $\boldsymbol{w}$.*

This will allow us to construct our coarse moduli space of undecomposable quasi-parabolic bundles $P_d(\mathbf{t})$ by patching together moduli spaces $P_d^{\boldsymbol{w}}(\mathbf{t})$ where $\boldsymbol{w}$ runs over a finite family of convenient weights.

PROOF. Let $(E, \boldsymbol{l})$ be a quasi-parabolic bundle and let us write $E = L_1 \oplus L_2$ as above, $L_i = \mathcal{O}_X(d_i)$, $d_1 \leq d_2$, $d_1 + d_2 = d$.

In the decomposable case, all parabolics are contained in the union $L_1 \sqcup L_2$. In this case, it is easy to check that $\mathrm{Stab}(L_1) + \mathrm{Stab}(L_2) = 0$, whatever the weights are, so that one of the two must be ≤ 0: $(E, \boldsymbol{l})$ is not stable for any choice of weights. Note that it is however (strictly) semi-stable for a convenient choice of weights.

In the undecomposable case, we may choose L_1 passing through a maximum number of parabolics, i.e. at least $d_2 - d_1 + 1$. Choose $d_2 - d_1$ of them and apply a negative elementary transformation at each of those directions. We get a new undecomposable quasi-parabolic bundle $(E', \boldsymbol{l}')$ with $E' \simeq L_1 \oplus L_1$. In particular, there are 3 parabolics, say l'_1, l'_2 and l'_3, lying on 3 distincts embeddings $L_1 \subset E'$. It is easy to check that this bundle is stable for weights $\boldsymbol{w}'$

$$0 < w'_1 = w'_2 = w'_3 < \frac{2}{3} \quad \text{and other} \quad w'_i = 0.$$

Thereore, the original quasi-parabolic bundle $(E, \boldsymbol{l})$ is stable for weights $\boldsymbol{w}$ defined by

- $w'_j = 1 - w_j$ if l_j is one of the directions where we made elementary transformation,
- $w'_j = w_j$ for other parabolics.

□

3.2. GIT moduli spaces of stable parabolic bundles. Given weights, the moduli space of semistable points $P_d^{\boldsymbol{w}}(\mathbf{t})$ is a (separated but may be singular) projective variety; moreover, stable points are smooth (cf. [**15**], [**5**], [**7**], [**14**], [**6**] and [**23**]).

Let $\mathcal{W} := [0, 1]^n$ be the set of weights. Given a parabolic bundle $(E, \boldsymbol{l})$ of degree d and a line subbundle $L \hookrightarrow E$ of degree k, denote by I_1 the set of indices of those parabolic directions contained in L, and $I_2 = \{1,\ldots,n\} \setminus I_1$, so that $\{1,\ldots,n\} = I_1 \sqcup I_2$. Then, the (parabolic) stability index of L is zero if, and only if, the weights lie along the hyperplane

$$H_d(k, I_1) := \{\boldsymbol{w} \ ; \ d - 2k - \sum_{i\in I_1} w_i + \sum_{i\in I_2} w_i = 0\}.$$

This equality cuts out the set of weights $[0,1]^n$ into two open sets (possibly one is empty, e.g. for large k). Note that the wall $H_d(k, I_1) = H_d(d-k, I_2)$ also bound the stability locus of those $L' = \mathcal{O}_X(d-k) \hookrightarrow E'$ passing through parabolics I_2 (with $\deg(E') = d$). On one side those $L \hookrightarrow E$ are destabilizing, on the other side those $L' \hookrightarrow E'$ are destabilizing. Along the wall, decomposable parabolic bundles $\mathcal{O}_X(k) \oplus \mathcal{O}_X(d-k)$ with parabolics distributed on the two factors as $\{1,\ldots,n\} = I_1 \sqcup I_2$ may occur as strictly semi-stable points in $P_d^{\boldsymbol{w}}(\mathbf{t})$.

When we cut out $[0,1]^n$ by all possible walls $H_d(k, I_1)$ (only finitely many intersect) we get in the complement many chambers (connected components) along which the moduli space only consists of stable parabolic bundles (semi-stable $\Rightarrow$ stable). Thus $P_d^{\boldsymbol{w}}(\mathbf{t})$ is smooth and locally constant along each chamber.

Mind that $P_d^{\boldsymbol{w}}(\mathbf{t})$ may be empty over some chambers like it so happens for k odd and $\boldsymbol{w} = (0,\ldots,0)$: the bundle is unstable since it is in the usual sense (parabolics are not taken into account for vanishing weights).

A parabolic bundle $(E, \boldsymbol{l})$ is said to be *generic* if

- $E = \mathcal{O}_X(d_1) \oplus \mathcal{O}_X(d_2)$ with $0 \leq d_2 - d_1 \leq 1$,
- a line bundle $L \subset E$ cannot contain more that $m+1$ parabolics, where $m = \deg(E) - 2\deg(L)$.

Note that $m+1$ is the dimension of deformation for such a subbundle $L \subset E$. It is easy to see that the set $P_d(\mathbf{t})^0 \subset P_d(\mathbf{t})$ of generic bundles $(E, \boldsymbol{l})$ over $(X, D) = (\mathbb{P}^1_{\mathbb{C}}, \mathbf{t})$ is a variety of dimension $n-3$.

Proposition 3.5. *A generic parabolic bundle* $(E, \boldsymbol{l}) \in P_d(\mathbf{t})^0$ *is stable for weights* $\boldsymbol{w}$ *if, and only if, the weights* $\boldsymbol{w}$ *satisfy all inequalities*

$$m - \sum_{i\in I_1} w_i + \sum_{i\in I_2} w_i > 0$$

with $\#I_1 = m+1$ *and* $m \geq 0$ *integer,* $m \equiv d \bmod 2$*. The moduli space* $P_d^{\boldsymbol{w}}(\mathbf{t})$ *has therefore expected dimension* $n-3$ *and contains* $P_d(\mathbf{t})^0$ *as an open subset.*

A weight $\boldsymbol{w}$ is said to be *admissible* if it lies outside the walls and moreover satisfies all above inequalities; a chamber is admissible if it consists of admissible weights. It follows that the intersection of all $P_d^{\boldsymbol{w}}(\mathbf{t})$ over admissible weights have a common open subset of dimension $n-3$, namely the moduli space of generic bundles $P_d(\mathbf{t})^0$ (that does not depend on choice of generic weights).

Let us denote by $\mathcal{W}_{adm}$ the set of admissible weights and decompose it into the finite number of connected components $\mathcal{W}_{adm} = \cup_{i,finite}\mathcal{W}_{adm,i}$ separated by walls. Then the moduli space $P_d^{\boldsymbol{w}}(\mathbf{t})$ is constant over the connected components $\boldsymbol{w} \in \mathcal{W}_{adm,i}$. So we may choose a representative $\boldsymbol{w}_i \in \mathcal{W}_{adm,i}$ for each i, and for different i_1, i_2, we have a big nonempty open set $U_{i_1,i_2} \supset P_d(\mathbf{t})^0$ such that

$$P_d^{\boldsymbol{w}_{i_1}}(\mathbf{t}) \supset U_{i_1,i_2} \subset P_d^{\boldsymbol{w}_{i_2}}(\mathbf{t}).$$

We can patch all of $P_d^{\boldsymbol{w}_i}(\mathbf{t})$ over these open subets U_{i_1,i_2} and obtain the coarse moduli space of undecomposable parabolic bundles of degree d. Thus we have the following

PROPOSITION 3.6. *The coarse moduli space $P_d(\mathbf{t})$ of undecomposable parabolic bundles of degree d can be obtained as*

$$P_d(\mathbf{t}) = \cup_{i,finite} P_d^{\boldsymbol{w}_i}(\mathbf{t}). \tag{3.2}$$

Recall (cf. Proposition 3.1) that undecomposable $\Leftrightarrow$ flat for generic $\boldsymbol{\nu}$. However, for special $\boldsymbol{\nu}$, only $\Rightarrow$ holds and there are decomposable flat bundles. The coarse moduli space of flat bundles might more difficult to describe then.

3.3. Wall-crossing and non separated phenomena. Let us compare the moduli spaces $P_d^{\boldsymbol{w}}(\mathbf{t})$ corresponding to admissible chambers, say W^1 and W^2, separated by a wall $H_d(k, I_1)$. Applying elementary transformations if necessary, we can assume $d = k = 0$. Those bundles with parabolics I_1 (resp. I_2) on the same $\mathcal{O}_X \hookrightarrow E$ are excluded on W^2 (resp. on W^1). Along the wall $H_0(0, I_1) = H_0(0, I_2)$, both are allowed as strictly semi-stable bundles; they identify, in the moduli space, with the decomposable bundle $E = L_1 \oplus L_2$ having parabolics I_i on L_i. The special two kinds of bundles previously described yield non separated points in the quotient space. Indeed, they are defined, on the trivial bundle ($L_1 = L_2 = \mathcal{O}_X$), by

- $\boldsymbol{l}^1$ spanned by $\begin{pmatrix}1\\0\end{pmatrix}$ for $i \in I_1$ and $\begin{pmatrix}u_i\\1\end{pmatrix}$ for $i \in I_2$,
- $\boldsymbol{l}^2$ spanned by $\begin{pmatrix}1\\v_i\end{pmatrix}$ for $i \in I_1$ and $\begin{pmatrix}0\\1\end{pmatrix}$ for $i \in I_2$.

If u_i (resp. v_i) are generic enough in $\mathbb{C}$, then $\boldsymbol{l}^i$ defines a stable parabolic bundle on W^i but is no more semi-stable for the other chamber W^j, $\{i,j\} = \{1,2\}$. Now, consider the one-parameter family of parabolic structures defined by

- $\boldsymbol{l}^\varepsilon$ spanned by $\begin{pmatrix}1\\\varepsilon v_i\end{pmatrix}$ for $i \in I_1$ and $\begin{pmatrix}\varepsilon u_i\\1\end{pmatrix}$ for $i \in I_2$.

When $\varepsilon \sim 0$, this parabolic structure is stable on both W^i's and when $\varepsilon \to 0$, it tends to either $\boldsymbol{l}^1$, or $\boldsymbol{l}^2$, depending on the chamber.

We can easily deduce any other non separating phenomenon applying back elementary transformations. For instance, the wall $H_0(1, \emptyset)$ (i.e. $I_1 = \emptyset$) is separating the locus of stability of

- those parabolic structure on the non trivial bundle $E = \mathcal{O}_X(-1) \oplus \mathcal{O}_X(1)$;
- those parabolic structure on the trivial bundle $E = \mathcal{O}_X \oplus \mathcal{O}_X$ where all parabolics lie along the same $\mathcal{O}_X(-1) \hookrightarrow E$.

This provides a non separated phenomenon: former parabolic bundles are arbitrary close to latter ones and *vice-versa*. Indeed, after applying elementary transformation to, say, l_1 and l_2, we are back to a special case of the above discussion.

3.4. The coarse moduli space of undecomposable quasi-parabolic structures on the trivial bundle. Now we describe (following and completing [**4**, Section 2.3]) the coarse moduli space P_0 of undecomposable quasi-parabolic bundles $(E, \boldsymbol{l})$ on $(\mathbb{P}^1_{\mathbb{C}}, \mathbf{t})$ of rank 2 and of degree 0. The cases of all even degrees are similar after twisting by a convenient line bundle.

As suggested by the proof of Proposition 3.4, the coarse moduli space $P_0(\mathbf{t})$ is covered by open charts of the following type.

For $\{i,j,k\} \subset \{1,\ldots,n\}$, consider the moduli space of stable parabolic bundles of degree 0 with respect to weights $\boldsymbol{w}$ defined by

$$0 < w_i = w_j = w_k < \frac{2}{3} \quad \text{and other} \quad w_l = 0.$$

Such parabolic bundles are exactly given by those parabolic structures on the trivial bundle $E = \mathcal{O}_X \oplus \mathcal{O}_X$ such that l_i, l_j and l_k are pairwise distinct (through the trivialization of E). Indeed, it cannot be $E = \mathcal{O}_X(-1) \oplus \mathcal{O}_X(1)$ for instance, since in this case $\mathcal{O}_X(1)$ is destabilizing (taking weights into account). Also, on $E = \mathcal{O}_X \oplus \mathcal{O}_X$, $l_i \neq l_j$ otherwise the trivial line bundle $\mathcal{O}_X \hookrightarrow E$ that contains these directions would be destabilizing. Here we get a fine moduli space that can be described as follows. Choose a trivialization $\mathbb{C}^2$ of E such that $l_i = (1:0)$, $l_j = (1:1)$ and $l_k = (0:1)$. Then our moduli space identifies with

$$U_{i,j,k} = \left\{ \boldsymbol{l} \;;\; \begin{matrix} l_i = (1:0), \\ l_j = (1:1), \\ l_k = (0:1) \end{matrix} \quad \text{and other } l_l \in \mathbb{P}^1_{\mathbb{C}} \text{ arbitrary} \right\} \simeq \left(\mathbb{P}^1_{\mathbb{C}}\right)^{n-3}.$$

Let $P_{0,0}(\mathbf{t})$ denote the moduli subspace of $P_0(\mathbf{t})$ of $(E,\boldsymbol{l})$ with the trivial vector bundle $E \simeq \mathcal{O}_X \oplus \mathcal{O}_X$. To get all points of $P_{0,0}(\mathbf{t})$, we have to patch all these projective smooth charts $U_{i,j,k}$ together: any two of them intersect on a non empty open subset. We already obtain a non separated scheme. For $n = 4$, we obtain $\mathbb{P}^1_{\mathbb{C}}$ with 3 double points at 0, 1 and ∞, or equivalently, two copies of $\mathbb{P}^1_{\mathbb{C}}$ glued outside 0, 1 and ∞. For $n = 5$, gluing maps are birational and non separated phenomena increase: there are rational curves arbitrary close to points. For $n = 6$, there are rational curves arbitrary close to each other through a flop.

Usually, the GIT compactification $\overline{M(0,n)}$ of the moduli of n-punctured sphere is constructed by setting all $w_i = 1/n$. When n is even, this weight is along the walls $H(0, I_1)$ with $\# I_1 = \frac{n}{2}$ and there are strictly semi-stable (and decomposable) bundles. On the other hand, when n is odd, the weight is inside a chamber. For instance, for $n = 5$, we get the 3 blow-up of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ at the points $(0,0)$, $(1,1)$ and (∞,∞). Although this latter moduli space does not embed in any chart $U_{i,j,k} \simeq \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ considered above, it embeds in the total coarse moduli space $P_0(\mathbf{t})$ as an open subset.

3.5. The coarse moduli space of undecomposable quasi-parabolic structures on degree d bundles. All charts $U_{i,j,k}$ are not enough to cover all quasi-parabolic bundles $(E,\boldsymbol{l})$ of degree 0: we only get those for which E is the trivial bundle. We have to add other charts that can be deduced from previous ones by making any even number of elementary transformations (and twisting by the convenient line bundle). All in all, it is enough to consider the following set of weights

$$W := \left\{ \boldsymbol{w} \;;\; \begin{matrix} 3 \text{ of } w_i\text{'s are } \frac{1}{2} \\ \text{other } w_i\text{'s are 0 or 1} \end{matrix} \right\} \tag{3.3}$$

and the corresponding moduli spaces, all isomorphic to $\left(\mathbb{P}^1_{\mathbb{C}}\right)^{n-3}$. In fact, those $\boldsymbol{w} \in W$ for which 1 does not occur are exactly those charts $U_{i,j,k}$ above; other ones are deduced by even numbers of elementary transformations. Recall that stability and flatness are invariant under elementary transformations.

Let us set

$$P_{d,k}(\mathbf{t}) = \{(E,\boldsymbol{l}) \in P_d(\mathbf{t}) \mid E \simeq \mathcal{O}_X(k) \oplus \mathcal{O}_X(d-k)\}$$

Then we have a stratification $P_0(\mathbf{t}) = P_{0,0}(\mathbf{t}) \sqcup P_{0,1}(\mathbf{t}) \sqcup \cdots \sqcup P_{0,m}(\mathbf{t})$, where m is positive integer maximal such that $m \leq \frac{n-2}{2}$. All $P_{d,k}(\mathbf{t})$ with $k > 0$ are on the non separated locus of $P_0(\mathbf{t})$. The open separated locus $P_0(\mathbf{t})^0$ (generic bundles) is, inside $P_{0,0}(\mathbf{t})$, the complement of those parabolic structures for which a subbundle $L \hookrightarrow \mathcal{O}_X \oplus \mathcal{O}_X$ passes through an exceeding number of parabolics.

From the consideration as above, we can see that in the patching (3.2) $P_0(\mathbf{t}) = \cup_i P_0^{\boldsymbol{w}_i}(\mathbf{t})$, the charts $P_0^{\boldsymbol{w}_i}(\mathbf{t})$ with $\boldsymbol{w}_i \in W$ given by (3.3) are enough to cover the whole coarse moduli space.

We can promtly deduce the coarse moduli space $P_{-1}(\mathbf{t})$ of quasi-parabolic bundles $(E, \boldsymbol{l})$ of degree -1 from the previous discussion by applying a single elementary transformation, say at t_n

$$\mathrm{Elm}_{t_n}^- : P_0(\mathbf{t}) \xrightarrow{\sim} P_{-1}(\mathbf{t}).$$

We get a stratification $P_{-1}(\mathbf{t}) = P_{-1,0}(\mathbf{t}) \sqcup P_{-1,1}(\mathbf{t}) \sqcup \cdots \sqcup P_{-1,m}(\mathbf{t})$, where m is maximal such that $m \leq \frac{n-3}{2}$.

3.6. A natural projective chart for coarse moduli space of degree -1 bundles. A natural projective chart V is given by those undecomposable parabolic structures on $E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ where no parabolic lie on $\mathcal{O}_X$.

PROPOSITION 3.7. *Assume $n \geq 3$. For "democratic" weights $w_i = w$, $i = 1, \ldots, n$, with $\frac{1}{n} < w < \frac{1}{n-2}$, a degree -1 parabolic bundle $(E, \boldsymbol{l})$ is (semi-)stable if, and only if*

- *$E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$,*
- *no parabolic l_i lie on $\mathcal{O}_X$,*
- *not all l_i lie on the same $\mathcal{O}_X(-1)$ (flatness).*

symmetry Moreover, for these weights, the moduli space $V := P_{-1}^{\boldsymbol{w}}(\mathbf{t})$ is naturally isomorphic to $\mathbb{P}H^0(X, L \otimes \Omega_X^1(D))^* \simeq \mathbb{P}_{\mathbb{C}}^{n-3}$, where $L = \mathcal{O}_X(-1)$.

PROOF. That $\mathcal{O}_X$ free of parabolics does not destabilize the parabolic bundle is equivalent to $\frac{1}{n} < w$. On the other hand, for $w < \frac{1}{n-2}$, a $\mathcal{O}_X(-1)$ passing through $n-1$ parabolics does not destabilize, but the parabolic bundle becomes unstable whenever one parabolic lie on $\mathcal{O}_X$. Finally, to eliminate parabolic structures on degree -1 vector bundles $E \neq \mathcal{O}_X \oplus \mathcal{O}_X(-1)$, we just need $w < \frac{3}{n}$ which is already implied by the above inequalities provided that $n \geq 3$.

Parabolic bundles of the chart V are precisely non trivial *extensions*

$$0 \to (\mathcal{O}_X, \emptyset) \to (E, \boldsymbol{l}) \to (L, D) \to 0$$

which means that the pair is defined by gluing local models $U_i \times \mathbb{C}^2$ (for an open analytic covering (U_k) of X) by transition matrices

$$M_{kl} = \begin{pmatrix} 1 & b_{kl} \\ 0 & a_{kl} \end{pmatrix}$$

with b_{kl} vanishing on D. Here, on each chart U_k, the vector e_1 generates the trivial subbundle $\mathcal{O}_X \hookrightarrow E$ and e_2 gives the parabolic direction over each point of D.

The multiplicative cocycle $(a_{kl})_{kl} \in H^1(X, \mathcal{O}_X^*)$ defines the line bundle L. Let $a_{kl} = \frac{a_k}{a_l}$ be a meromorphic resolution: a_i is meromorphic on U_k with $\mathrm{div}(a_k) = \mathrm{div}(L)$. The obstruction to split the extension is measured by an element of

$$H^1(X, \mathrm{Hom}(L(D), \mathcal{O}_X)) = H^1(X, L^{-1}(-D)) = H^0(X, L \otimes \Omega_X^1(D))^*$$

(by Serre duality) which is explicitely given by $(b_{kl}a_l)_{kl} \in H^1(X, L^{-1}(-D))$. Any two non trivial extensions define isomorphic parabolic bundles if and only if the corresponding cocycles are proportional: the moduli space of extensions is parametrized by $\mathbb{P}H^0(X, L \otimes \Omega^1_X(D))^*$. $\square$

3.7. Case $n = 4$ detailled. For degree 0 and undecomposable parabolic bundles, we have the following possibilities:

- E is the trivial bundle and at most two of the l_i's coincide;
- $E = \mathcal{O}_X(-1) \oplus \mathcal{O}_X(1)$ and in this case, there is a unique undecomposable quasi-parabolic structure up to automorphism, say $l_1, l_2, l_3 \in \mathcal{O}_X(-1)$ and l_4 outside of the two factors.

On the space $[0,1]^4$ of weights, the walls are defined by equations of the type

$$\epsilon_1 w_1 + \cdots + \epsilon_4 w_4 \in 2\mathbb{Z}$$

where $\epsilon_i = \pm$ and we get the following possibilities (other ones do not cut out $[0,1]^4$ into two non empty pieces)

$$\begin{array}{rcl} w_1 + w_2 + w_3 + w_4 & = & 2 \\ w_i + w_j + w_k - w_l & = & 0 \text{ or } 2 \\ w_i + w_j - w_k - w_l & = & 0 \end{array}$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$, which gives $1 + 2 \cdot 4 + 3 = 12$ walls. It is easy to check that the moduli space of (semi-)stable parabolic bundles is non empty if, and only if, we have the following inequalities

$$0 \leq w_i + w_j + w_k - w_l \leq 2.$$

For instance, when $w_1 + w_2 + w_3 < w_4$, then the line bundle $\mathcal{O}_X \hookrightarrow E$ passing through l_4 destabilizes the bundle.

Now, under above inequalities, the remaining 4 walls cut out the remaining space of weights into 16 chambers. For $w_1 + w_2 + w_3 + w_4 < 2$, the moduli space $P_0^{\boldsymbol{w}}(\mathbf{t})$ consists only of parabolic structures on the trivial bundle: $\mathcal{O}_X(1) \hookrightarrow \mathcal{O}_X(-1) \oplus \mathcal{O}_X(1)$ is destabilizing in this case. This half-space splits into 8 admissible chambers, but only 4 are enough to cover all quasi-parabolic structures on $\mathcal{O}_X \oplus \mathcal{O}_X$, namely those containing $\boldsymbol{w}_4 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$, and its permutations $\boldsymbol{w}_i$ (the ith weight is zero). For $\boldsymbol{w}_4$, we get the following chart

$$U_{1,2,3} := \{\boldsymbol{l} = (0, 1, \infty, u) \ ; \ u \in \mathbb{P}^1_{\mathbb{C}}\}.$$

The classical moduli space $M(0,4)$ is given by the open set $u \neq 0, 1, \infty$ and this is the locus $P_0(\mathbf{t})^0$ of generic parabolic bundles. The chart given by $\boldsymbol{w}_1$ can be for instance described as

$$U_{4,2,3} := \{\boldsymbol{l} = (v, 1, \infty, 0) \ ; \ v \in \mathbb{P}^1_{\mathbb{C}}\}.$$

The intersection is given, in $U_{1,2,3}$, by the complement of $l_4 = l_2$ and $l_4 = l_3$, i.e. by $u \neq 1, \infty$. The two projective charts glue along the latter open subset through the fractional linear transformation $U_{1,2,3} \to U_{4,2,3}; u \mapsto v = \frac{u}{u-1}$. We have already added two non separated points, namely at $u = 1$ and $u = \infty$.

After patching all 4 charts together, we get a non separated scheme over $\mathbb{P}^1_{\mathbb{C}} \ni u$ with double points over $u = 0$, 1 and ∞; they correspond to pairs of special parabolic structures respectively defined by

$$\{l_1 = l_4 \text{ or } l_2 = l_3\}, \quad \{l_2 = l_4 \text{ or } l_1 = l_3\} \quad \text{and} \quad \{l_3 = l_4 \text{ or } l_1 = l_2\}.$$

Finally, one has to add the unique undecomposable quasi-parabolic structure on the non trivial bundle $\mathcal{O}_X(1)\oplus\mathcal{O}_X(-1)$. This adds a 4th non separated point over $u=t$ where cross-ratio$(0,1,\infty,t)$ = cross-ratio(t_1,t_2,t_3,t_4). Indeed, it is infinitesimally closed to the (unique) quasi-parabolic structure lying on an embedding $\mathcal{O}_X(-1)\hookrightarrow \mathcal{O}_X\oplus\mathcal{O}_X$.

To end with the degree 0 case, we note that, although the coarse moduli space is constructed *a posteriori* by gluing two copies of $\mathbb{P}^1_{\mathbb{C}}$ along the complement of t_1,t_2,t_3,t_4. However, this identification strongly depend on our choice of the initial chart $U_{1,2,3}\in u$. Starting from another chart will give another identification; this is up to the 4-group that preserves the cross-ratio.

In case of degree -1 bundles, we necessarily have $E=\mathcal{O}_X\oplus\mathcal{O}_X(-1)$ by undecomposability. Let us choose weights $w_1=w_2=w_3=w_4=:w$. The moduli space $P^{\boldsymbol{w}}_{-1}(\mathbf{t})$ is non empty for $\frac{1}{4}\le w\le\frac{3}{4}$. At $w=\frac{1}{2}$ only, strictly semistable bundles occur. There are two chambers, namely

- $\frac{1}{4}<w<\frac{1}{2}$ where no parabolic l_i is contained in $\mathcal{O}_X$;
- $\frac{1}{2}<w<\frac{3}{4}$ where not 3 of the l_i's is contained in the same $\mathcal{O}_X(-1)$.

By this way, the coarse moduli space is constructed by only two open projective charts, and the four double points are given by those pairs

$$\{l_i \text{ is contained in } \mathcal{O}_X\} \quad \text{and} \quad \{l_j,l_k,l_l \text{ are contained in the same } \mathcal{O}_X(-1)\}$$

which naturally identify with t_i. Here, we get a natural identification with two copies of $\mathbb{P}^1$ glued along the complement of t_1,t_2,t_3,t_4.

4. The two Lagrangian fibrations

4.1. Moduli of generic connections. All along this section, we fix "democratic" weights

$$\boldsymbol{w}=(w,\dots,w) \quad\text{with}\quad \frac{1}{n}<w<\frac{1}{n-2} \tag{4.1}$$

like in Proposition 3.7 and we consider the moduli space $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})$ of $\boldsymbol{w}$-stable $\boldsymbol{\nu}$-parabolic connections $(E,\nabla,\boldsymbol{l})$ where $\boldsymbol{\nu}=(\nu_i^{\pm})$ with $\sum_i(\nu_i^++\nu_i^-)=1$ (see Section 2). Denote by $L=\mathcal{O}_X(-1)$ the determinant line bundle. By Proposition 3.7, for the weights $\boldsymbol{w}=(w,\dots,w)$ in (4.1), the coarse moduli space $V=P^{\boldsymbol{w}}_{-1}(\mathbf{t})$ of $\boldsymbol{w}$-stable parabolc bundles of degree -1 is isomorphic to $PH^0(X,L\otimes\Omega^1_X(D))^*\simeq\mathbb{P}^{n-3}_{\mathbb{C}}$ and consists of $(E,\boldsymbol{l})$ satisfying the conditions:

- $E=\mathcal{O}_X\oplus\mathcal{O}_X(-1)$,
- $l_i\not\subset\mathcal{O}_X$ for $i=1,\dots,n$,
- not all l_i lie in the same $\mathcal{O}_X(-1)\hookrightarrow E$.

Now we introduce the following open subset of the moduli space $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})$

Definition 4.1. For the weight $\boldsymbol{w}$ in (4.1), let us define the open subset

$$M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0=\{(E,\nabla,\boldsymbol{l})\in M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu}) \mid (E,\boldsymbol{l})\in P^{\boldsymbol{w}}_{-1}(\mathbf{t})\} \tag{4.2}$$

of $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})$, which we call the moduli space of *generic $\boldsymbol{\nu}$-parabolic connections.*

We can define two natural Lagrangian maps on $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0$. The first one

$$\mathrm{App}: M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0\to\mathbb{P}H^0(X,L\otimes\Omega^1_X(D))\simeq|\mathcal{O}_X(n-3)|\simeq\mathbb{P}^{n-3}_{\mathbf{a}} \tag{4.3}$$

is obtained by taking the apparent singular points with respect to the cyclic vector of the global section of $\mathcal{O}_X$. Precisely, each connection ∇ on $E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ defines a $\mathcal{O}_X$-linear map

$$\mathcal{O}_X \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega^1_X(D) \to (E/\mathcal{O}_X) \otimes \Omega^1_X(D) \simeq L \otimes \Omega^1_X(D)$$

(where the last arrow is the quotient by the subbundle defined by $\mathcal{O}_X \hookrightarrow E$) i.e. a map

$$\varphi_\nabla : \mathcal{O}_X \to L \otimes \Omega^1_X(D).$$

Its zero divisor is an element of the linear system $\mathbb{P}H^0(X, L\otimes\Omega^1_X(D)) \simeq |\mathcal{O}_X(n-3)|$. This map extends as a rational map

$$\mathrm{App} : M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu}) \dashrightarrow |\mathcal{O}_X(n-3)|$$

on the whole moduli space with some indeterminacy points (See [**19**]).

The second Lagrangian map

$$\mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0 \to P^{\boldsymbol{w}}_{-1}(\mathbf{t}) \simeq \mathbb{P}H^0(X, L \otimes \Omega^1_X(D))^* \simeq (\mathbb{P}^{n-3}_{\mathbf{a}})^* \simeq \mathbb{P}^{n-3}_{\mathbf{b}}. \tag{4.4}$$

comes from the forgetfull map towards the coarse moduli space of undecomposable parabolic bundles

$$\mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu}) \to P_{-1}(\mathbf{t}) \ ; \ (E, \nabla, \boldsymbol{l}) \mapsto (E, \boldsymbol{l})$$

that we restrict to the open projective chart $V := P^{\boldsymbol{w}}_{-1}(\mathbf{t})$ of Section 3.6.

One of main results of this section is the following

THEOREM 4.2. *Under the assumption that* $\sum_i \nu_i^- \neq 0$ *(*$\Leftrightarrow \sum_i \nu_i^+ \neq 1$*), the morphism*

$$\mathrm{App} \times \mathrm{Bun} \ : \ M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0 \ \to \ |\mathcal{O}_X(n-3)| \times |\mathcal{O}_X(n-3)|^* \simeq \mathbb{P}^{n-3}_{\mathbf{a}} \times \mathbb{P}^{n-3}_{\mathbf{b}} \tag{4.5}$$

is an embedding. Precisely, the image is the complement of the incidence variety Σ *for the above duality.*

4.2. Compactification of the moduli space. In order to prove Theorem 4.2, we introduce a nice compactification $\overline{M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0}$ of the moduli space $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0$ of generic connections and will show that the extended map App$\times$Bun to $\overline{M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0}$ is in fact an isomorphism.

In [Definition 2, [**1**]], the moduli stack $\overline{\mathcal{M}(\mathbf{t}, \boldsymbol{\nu})}$ of λ-$\boldsymbol{\nu}$-parabolic connections $(E, \nabla, \varphi, \lambda \in \mathbb{C}, \boldsymbol{l})$ over $X = \mathbb{P}^1$ are introduced. (Note that in [**1**], λ-$\boldsymbol{\nu}$-parabolic connections are called as ϵ-bundles.) Then under the conditions that (E, ∇) is irreducible, Arinkin ([Theorem 1 in [**1**]]) showed that the moduli stack $\overline{\mathcal{M}(\mathbf{t}, \boldsymbol{\nu})}$ is a complete smooth Deligne-Mumford stack. Moreover he also showed that the $\lambda = 0$ locus $\overline{\mathcal{M}}(\mathbf{t}, \boldsymbol{\nu})_H \subset \overline{\mathcal{M}(\mathbf{t}, \boldsymbol{\nu})}$, which is the moduli stack of parabolic Higgs bundles, is also a smooth algebraic stack. On the other hand, as remarked in the proof of [Proposition 7, [**1**]], the coarse moduli space $\overline{M(\mathbf{t}, \boldsymbol{\nu})}$ corresponding to $\overline{\mathcal{M}(\mathbf{t}, \boldsymbol{\nu})}$is not smooth: it has quotient singularities. (As for the possible smooth compactification by ϕ-parabolic connections, one may refer [**10**] and [**11**] (cf. Remark 4.4).)

Our main strategy is to consider the coarse moduli space of $\boldsymbol{w}$-stable λ-$\boldsymbol{\nu}$-parabolic connections for the democratic weight $\boldsymbol{w}$. Define the coarse moduli space

$$\overline{M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0} := \left\{ \begin{array}{l} (E, \nabla, \lambda \in \mathbb{C}, \boldsymbol{l}) \\ \lambda\text{-}\boldsymbol{\nu}\text{-parabolic connection} \end{array} \ \middle| \ (E, \boldsymbol{l}) \in P^{\boldsymbol{w}}_{-1}(\mathbf{t}) \right\} / \simeq . \tag{4.6}$$

Note that if $(E,\boldsymbol{l}) \in P^{\boldsymbol{w}}_{-1}(\mathbf{t})$, λ-parabolic connections $(E,\nabla,\lambda\in\mathbb{C},\boldsymbol{l})$ are always $\boldsymbol{w}$-stable, and there exists a natural embedding $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0 \subset \overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0}$ such that

$$M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0_H := \overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0} \setminus M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0 \tag{4.7}$$

is the coarse moduli space of parabolic Higgs bundles $(E,\nabla,0\in\mathbb{C},\boldsymbol{l})$ such that $(E,\boldsymbol{l}) \in P^{\boldsymbol{w}}_{-1}(\mathbf{t})$.

We can describe the moduli space $\overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0}$ naively as follows. Thinking of $\mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0 \to P^{\boldsymbol{w}}_{-1}(\mathbf{t})$ as an affine $\mathbb{A}^{n-3}$-bundle over the projective chart $P^{\boldsymbol{w}}_{-1}(\mathbf{t})$. On each parabolic bundle $(E,\boldsymbol{l}) \in P^{\boldsymbol{w}}_{-1}(\mathbf{t})$, any two connections ∇_0,∇_1 compatible with $\boldsymbol{l}$ differ to each other by a parabolic Higgs field

$$\nabla_1 - \nabla_0 = \Theta \in H^0(\mathrm{End}(E,\boldsymbol{l})\otimes\Omega^1_X(D))$$

(residues of Θ are nilpotent on each fiber $E_{|t_i}$ fixing the parabolic direction l_i). The moduli space of connections identifies with the $(n-3)$-dimensional affine space $\nabla_0 + H^0(\mathrm{End}(E,\boldsymbol{l})\otimes\Omega^1_X(D))$ (recall that $(E,\boldsymbol{l})$ is simple). Let us consider the fiber $\mathrm{Bun}^{-1}(E,\boldsymbol{l})$ of the map $\mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0 \to P^{\boldsymbol{w}}_{-1}(\mathbf{t})$ in (4.4) over $(E,\boldsymbol{l})$. A natural compactification of the fiber $\mathrm{Bun}^{-1}(E,\boldsymbol{l})$ is given by

$$\overline{\mathrm{Bun}^{-1}(E,\boldsymbol{l})} := \mathbb{P}\left(\mathbb{C}\cdot\nabla_0 \oplus H^0(\mathrm{End}(E,\boldsymbol{l})\otimes\Omega^1_X(D))\right).$$

An element $\nabla := \lambda\cdot\nabla_0 + \Theta$ is a λ-connection; if $\lambda\neq 0$, it is homothetic equivalent to a unique connection, namely $\frac{1}{\lambda}\nabla$; if $\lambda = 0$, it is a parabolic Higgs field. By this way, we compactify the fiber $\mathrm{Bun}^{-1}(E,\boldsymbol{l})$ by adding $\mathbb{P}H^0(\mathrm{End}(E,\boldsymbol{l})\otimes\Omega^1_X(D))$. Varying $(E,\boldsymbol{l}) \in P^{\boldsymbol{w}}_{-1}(\mathbf{t})$ and choose a local section ∇_0 over local open sets of $P^{\boldsymbol{w}}_{-1}(\mathbf{t})$, we can construct a $\mathbb{P}^{n-3}$-bundle

$$\mathrm{Bun} : \overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0} \longrightarrow V = P^{\boldsymbol{w}}_{-1}(\mathbf{t}) \tag{4.8}$$

and its restriction to the boundary

$$\mathrm{Bun}_H : M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0_H := (\overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0} \setminus M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0) \longrightarrow V \tag{4.9}$$

naturally identifies with the total space of the projectivized cotangent bundle $\mathbb{P}T^*V \to V$.

4.3. Main Theorem. The apparent map naturally extends on the compactification since φ_∇ can be defined in the same way for λ-connections (an Higgs fields).

Our main result, which will give a proof of Theorem 4.2, now reads

THEOREM 4.3. *We fix the democratic weight $\boldsymbol{w} = (w,\ldots,w)$ with $\frac{1}{n} < w < \frac{1}{n-2}$ and consider the moduli space $\overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0}$ as in (4.6). If $\sum_i \nu_i^- \neq 0$, the moduli space $\overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0}$ is a smooth projective variety and the map* App $\times$ Bun *induces an isomorphism*

$$\mathrm{App}\times\mathrm{Bun} : \overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0} \xrightarrow{\sim} \mathbb{P}H^0(X, L\otimes\Omega^1_X(D)) \times \mathbb{P}H^0(X, L\otimes\Omega^1_X(D))^*. \tag{4.10}$$

Moreover, by restriction, we also obtain the isomorphism

$$\mathrm{App}\times\mathrm{Bun}_{|M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0_H} : M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0_H \xrightarrow{\sim} \Sigma \tag{4.11}$$

where Σ is the incidence variety for the duality.

PROOF. It is enough to show that the natural morphism App $\times$ Bun (4.10) induces a regular isomorphism between algebraic varieties. Like in the proof of Proposition 3.7, we consider a parabolic bundle $(E,\boldsymbol{l}) \in P_{-1}^{\boldsymbol{w}}(\mathbf{t})$ defined as an extension class, i.e. by a matrix cocycle

$$M_{kl} = \begin{pmatrix} 1 & b_{kl} \\ 0 & a_{kl} \end{pmatrix}$$

where the multiplicative cocycle (a_{kl}) defines the line bundle L and the extension is equivalently defined by the cocycle $(b_{kl}a_l)_{kl} \in H^1(X, L^{-1}(-D)) \simeq H^0(X, L \otimes \Omega^1_X(D))^*$ where $a_{kl} = \frac{a_k}{a_l}$ is a meromorphic resolution ($\mathrm{div}(a_k) = \mathrm{div}(L)$). Let us fix also a non zero element $\gamma \in H^0(X, L \otimes \Omega^1_X(D)) \setminus \{0\}$. We want to show that there is a unique $\lambda \in \mathbb{C}$ and a unique λ-connection $\nabla : E \to E \otimes \Omega^1_X(D)$ (compatible with $\boldsymbol{\nu}$ and $\boldsymbol{l}$) realizing γ as the apparent map.

Such a λ-connection ∇ is given in charts U_k by $\nabla = \lambda d + A_k$

$$A_k = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \in \mathrm{GL}_2(\Omega^1_{U_k}(D)) \tag{4.12}$$

with compatibility condition

$$\lambda \cdot dM_{kl} + A_k M_{kl} = M_{kl} A_l \tag{4.13}$$

on each intersection $U_k \cap U_l$. For each pole $z = t_i$, the residue of A_k takes the form

$$\mathrm{Res}_{t_i}(A_k) = \begin{pmatrix} \lambda\nu_i^- & 0 \\ * & \lambda\nu_i^+ \end{pmatrix}. \tag{4.14}$$

The trace connection ζ is defined on U_k by $d + \omega_k$ with compatibility conditions $\frac{da_{kl}}{a_{kl}} + \omega_k - \omega_k = 0$ on $U_k \cap U_l$. We must have

$$\alpha_k + \delta_k = \lambda\omega_k \tag{4.15}$$

on U_k. We note that $\frac{da_{kl}}{a_{kl}} = \frac{da_k}{a_k} - \frac{da_l}{a_l}$ so that $(\omega_k + \frac{da_k}{a_k})_k$ defines a global 1-form, say $\omega \in H^0(X, L \otimes \Omega^1_X(D))$, and thus $\omega_k = \omega - \frac{da_k}{a_k}$.

Now, compatibility conditions 4.13 expand as

$$\begin{cases} \frac{\gamma_k}{a_k} - \frac{\gamma_l}{a_l} & = & 0 & (\to \gamma := \frac{\gamma_k}{a_k} = \frac{\gamma_l}{a_l}) \\ \alpha_k - \alpha_l & = & (b_{kl}a_l)\gamma & \\ \delta_k - \delta_l & = & -(b_{kl}a_l)\gamma - \lambda\frac{da_{kl}}{a_{kl}} & \\ a_k\beta_k - a_l\beta_l & = & -(\lambda a_l db_{kl} + (b_{kl}a_l)(\alpha_k - \delta_l)) & \end{cases} \tag{4.16}$$

The first condition says that all $(\frac{\gamma_k}{a_k})_k$ glue together to form a global section $\gamma \in H^0(X, L \otimes \Omega^1_X(D))$. It defines the image of the apparent map.

Our problem is now: given $(b_{kl}a_l)_{kl} \in H^1(X, L^{-1}(-D))$ defining the parabolic bundle and $\gamma \in H^0(X, L \otimes \Omega^1_X(D)) \setminus \{0\}$ defining the apparent data, prove that the matrix connections $\lambda d + A_k$ can be completed in a unique way, with a unique λ.

Step1: finding γ_k**.** Given γ, we obviously define $\gamma_k := a_k\gamma \in H^0(U_k, \Omega^1_X(D))$.

Step2: finding α_k**.** Fix α_k^0 sections of $\Omega^1_X(D)$ on each U_k realizing the residual data $\mathrm{Res}_{t_i}(\alpha_k^0) = \nu_i^-$. The cocycle $(\alpha_k^0 - \alpha_l^0)$ defines an element of $H^1(X, \Omega^1_X)$ which is non zero: indeed, if we were able to solve the cocycle by $\alpha_k^0 - \alpha_l^0 = \tilde{\alpha}_k - \tilde{\alpha}_l$ for some holomorphic 1-forms $\tilde{\alpha}_k$, then $(\alpha_k^0 - \tilde{\alpha}_k)_k$ would define a global meromorphic 1-form whose sum of residue $\sum_{i=1,\dots,n} \nu_i^- \neq 0$ contradicts Residue Theorem. Now, we want to find α_k of the form $\lambda\alpha_k^0 + \tilde{\alpha}_k$ with $\tilde{\alpha}_k$ holomorphic. This means that we have to solve

$$\tilde{\alpha}_k - \tilde{\alpha}_l = (b_{kl}a_l)\gamma - \lambda(\alpha_k^0 - \alpha_l^0)$$

in $H^1(X, \Omega_X^1)$; since this cohomology group is one dimensional, $(\alpha_k^0 - \alpha_l^0)$ is a generator (being non zero) and there is a unique λ such that the right-hand-side is zero in cohomology, providing solutions $(\tilde{\alpha}_k)$. We have now fixed λ, and the solution $\alpha_k = \lambda\alpha_k^0 + \tilde{\alpha}_k$ is unique (there is no global 1-form on $X = \mathbb{P}^1_{\mathbb{C}}$).

Step 3: finding δ_k. Since the trace connection must be ζ, we have to set $\delta_k := \omega_k - \alpha_k$. It is straighforward that it satisfies the 3rd equation of 4.16 and the correct residual term of 4.14. Actually, the sum of 2nd and 3rd equations of 4.16 exactly give the compatibility condition of $(\omega_k = \alpha_k + \delta_k)_k$ forming ζ.

Step 4: finding β_k : The right-hand-side of 4th equation of 4.16 defines an element of $H^1(X, L^{-1}\otimes\Omega_X^1) = \{0\}$. We can solve $\lambda a_l db_{kl} + (b_{kl}a_l)(\alpha_k - \delta_l) = \tilde{\beta}_k - \tilde{\beta}_l$ with $\tilde{\beta}_k$ belonging to $L^{-1}\otimes\Omega_X^1$, so that $\beta_k := \frac{\tilde{\beta}_k}{a_k}$ are sections of Ω_X^1, thus satisfying the residual condition 4.14.

We have constructed a unique λ-connection from data γ and $(b_{kl}a_l)_{kl}$.

Locus of Higgs fields. By Serre Duality, we have a perfect pairing

$$H^0(X, L\otimes\Omega_X^1(D)) \times H^1(X, L^{-1}(-D)) \longrightarrow H^1(X, \Omega_X^1) \xrightarrow{\sim} \mathbb{C}.$$

More precisely, in our construction, to the data (γ , $(b_{kl}a_l)_{kl}$), we associate the cocycle $(b_{kl}a_l)\gamma \in H^1(X, \Omega_X^1)$ which admits the meromorphic resolution $(\alpha_k - \alpha_l)_{kl}$. The principal (polar) part of $(\alpha_k)_k$ is well-defined; for instance, $\mathrm{Res}_{t_i}(\alpha_k)_k = \nu_i^-$ does not depend on the chart U_k. The last arrow is given by the sum of residues: it measures the obstruction to realize the principal part by a global meromorphic 1-form. Concretely, the image is

$$\sum_{i=1,\dots,n} \mathrm{Res}_{t_i}(\alpha_k)_k = \lambda \cdot \sum_{i=1,\dots,n} \nu_i^- .$$

We get a Higgs field precisely when $\lambda = 0$, i.e. when the image is zero. Finally, the locus of Higgs fields in our theorem is given by the incidence variety for the above Serre Duality. □

REMARK 4.4. Note that without the condition of $(E, \boldsymbol{l}) \in P_{-1}^{\boldsymbol{w}}$ for our choice of the weights $\boldsymbol{w}$ or the conditions in Proposition 3.7, the coarse moduli space $\overline{M(\mathbf{t}, \boldsymbol{\nu})}$ of λ-$\boldsymbol{\nu}$-connections have singularities. We will explain about this in the case of $n = 4$. (See Figure 1). In this case, the coarse moduli space $\widehat{\overline{M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})}}$ of stable parabolic $\boldsymbol{\nu}$-ϕ-connections gives a smooth compactification of the moduli space $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$ of $\boldsymbol{w}$-stable $\boldsymbol{\nu}$-parabolic connections and it gives an Okamoto-Painlevé pair for Painlevé VI equations ([**20, 10, 11**]). In fact, $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$ is the complement of $\Sigma + \sum_i F_i$ in $\widehat{\overline{M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})}}$. This moduli space $\widehat{\overline{M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})}}$ is isomorphic to the blown-up of 8-points of $\mathbb{F}_2$ as in Figure 1. Note that in this case, ϕ is the endomorphism of $E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$. For simplicity, we assume that $\boldsymbol{\nu}$ is generic and all connections are $\boldsymbol{w}$-stable. Here the exceptional curves $E_i^+ \setminus F_i \cap E_i^+$ (resp. $E_i^- \setminus F_i \cap E_i^-$) is the locus of the parabolic connections such that the apparent coordinate $q = t_i$ and $l_i \subset \mathcal{O}_X$ (resp. $l_i \not\subset \mathcal{O}_X$) and $F_i \setminus (F_i \cap \Sigma)$ is the locus of ϕ-connections with $\operatorname{rank}\phi = 1$. Moreover Σ is the locus of $\phi = 0$, that is, the locus of Higgs bundles. In order to obtain the moduli space $\overline{M(\mathbf{t}, \boldsymbol{w})}$ of λ-connections ([**1**]), we just contract F_i's which are (-2)-rational curves. Hence $\overline{M(\mathbf{t}, \boldsymbol{w})}$ has four A_1-singular points.

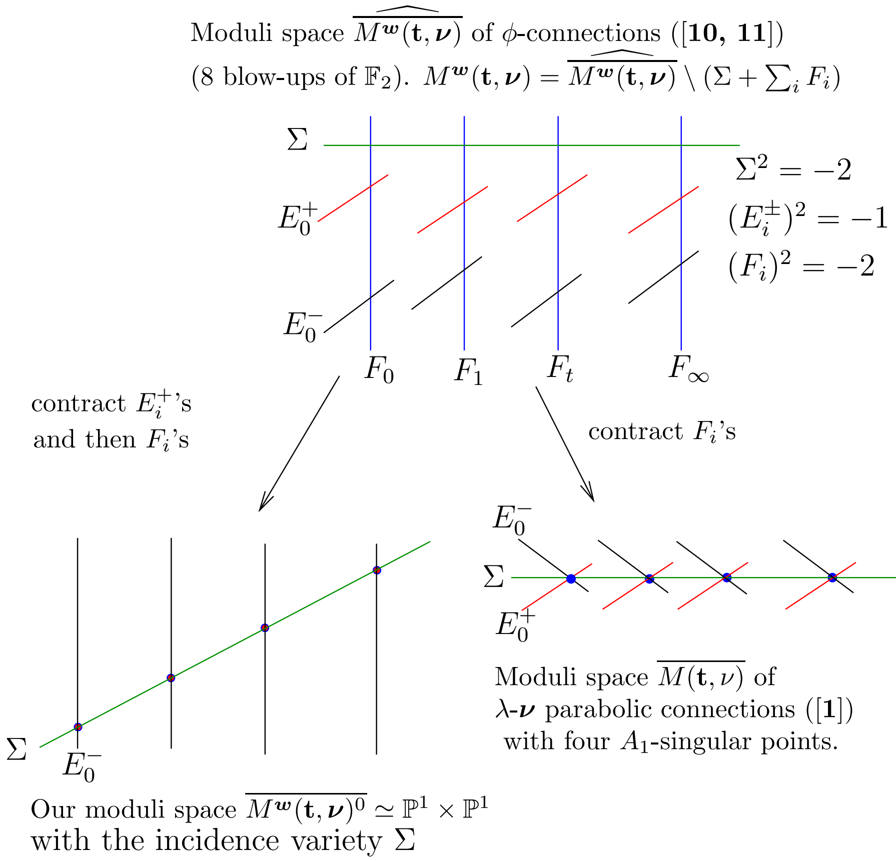

FIGURE 1. Different moduli spaces and their relations in case $n = 4$.

On the other hand, our moduli space $\overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0}$ can be obtained by contracting E_i^+'s and F_i's and $\overline{M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0}$ is smooth and isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$ with the diagonal incidence variety $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$ as in Figure 1.

Another interpretation of our main theorem is that the image of the apparent map for Higgs fields characterizes the bundle. Precisely, given $(E, \boldsymbol{l}) \in P_{-1}^{\boldsymbol{w}}(\mathbf{t})$, let us consider the fiber $\mathrm{Bun_H}^{-1}((E,\boldsymbol{l})) \subset$ of $\mathrm{Bun_H}$ in (4.9) and one can look at the restriction

$$\mathrm{App} : \mathrm{Bun_H}^{-1}((E,\boldsymbol{l})) \simeq \mathbb{P}H^0(\mathrm{End}(E,\boldsymbol{l}) \otimes \Omega_X(D)) \longrightarrow |\mathcal{O}_X(n-3)| \tag{4.17}$$

to the boundary at infinity of connections $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0$. Our main results says first that the image of (4.17) is non degenerate, i.e. defines an hyperplane in $|\mathcal{O}_X(n-3)|$, thus defining an element of the dual $|\mathcal{O}_X(n-3)|^*$; moreover, this hyperplane determines the parabolic structure $\boldsymbol{l}$.

COROLLARY 4.5. *The map*

$$P_{-1}^{\boldsymbol{w}}(\mathbf{t}) \longrightarrow |\mathcal{O}_X(n-3)|^* \ ; \ (E,\boldsymbol{l}) \mapsto \mathrm{image}(\mathrm{App}(\mathbb{P}H^0(\mathrm{End}(E,\boldsymbol{l}) \otimes \Omega_X(D))))$$

is well-defined and is an isomorphism.

In fact, it is not difficult to deduce our main result from this corollary; for instance, that the above map is well-defined shows the injectivity of App $\times$ Bun in restriction to each fiber $\mathrm{Bun}^{-1}(E,\boldsymbol{l})$. We will provide an alternate proof of this Corollary by direct computation in the next section.

4.4. The degenerate case. When $\sum_i \nu_i^- = 0$ ($\Leftrightarrow \sum_i \nu_i^+ = 1$), we get

Proposition 4.6. *If $\sum_i \nu_i^- = 0$, then $M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0$ identifies with the total space of the cotangent bundle T^*V, and the map $\mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0 \to V$, with the natural projection $T^*V \to V$. Here, the section $\nabla_0 : V \to M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0$ corresponding to the zero section of $T^*V \to V$ is given by those reducible connections preserving the destabilizing subbundle $\mathcal{O}_X$. Moreover, the map $\mathrm{App}\times\mathrm{Bun}$ is the natural map between total spaces*

$$\begin{tikzcd} M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0 \simeq T^*V \arrow[rr, dashed, "\mathrm{App}\times\mathrm{Bun}"] \arrow[dr, "\mathrm{Bun}"'] & & \mathbb{P}T^*V \simeq \Sigma \arrow[dl, "\mathrm{Bun}"] \\ & V & \end{tikzcd}$$

with indeterminacy locus ∇_0.

Here, the restriction $(\nabla_0)_{|\mathcal{O}_X}$ has eigenvalues ν_i^- and Fuchs relation is just $\sum_i \nu_i^- = 0$. Fibers of $\mathrm{App}\times\mathrm{Bun}$ are one-dimensional in this case.

Proof. Going back to the proof of Theorem 4.3, we see that, under assumption $\gamma = 0$ (reducibility condition) and setting $\lambda = 1$ we get a unique connection ∇_0 for each given parabolic bundle $(E,\boldsymbol{l}) \in P_{-1}^{\boldsymbol{w}}(\mathbf{t})$. This section ∇_0 allows to reduce the group structure of the affine bundle $\mathrm{Bun} : M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0 \to V$: in this case, $M(\mathbf{t},\boldsymbol{\nu})^0$ is the trivial affine extension of the cotangent bundle T^*V, i.e. the total space of the cotangent bundle itself, and Bun is the natural projection $T^*V \to V$.

The unique λ-connection with vanishing apparent map φ_∇ (or γ if we follow again the proof of Theorem 4.3) is $\lambda\cdot\nabla_0$. The apparent map of the general λ-connection $\nabla = \lambda\cdot\nabla_0 + \Theta$ is thus given by $\varphi_\nabla \equiv \varphi_\Theta$ by linearity. On each fiber $\mathrm{Bun}^{-1}(E,\boldsymbol{l})$, the apparent map is thus the projection on the image of Higgs bundles, namely Σ. □

5. Some computations

Here we provide some explicit formulae for the two fibrations.

5.1. Higgs fields and connections. By fractional linear transformation, set $(t_{n-2}, t_{n-1}, t_n) = (0, 1, \infty)$ for simplicity. In order to describe the generic Higgs bundle or connection in matrix form, we use the following isomorphism

$$\mathrm{Elm}_{t_n}^- : P_0(\mathbf{t}) \to P_{-1}(\mathbf{t}); (E,\boldsymbol{l}) \mapsto (E',\boldsymbol{l}').$$

It induces a birational map $U \dashrightarrow V$ between the projective charts $U := U_{n-2,n-1,n}$ introduced in Section 3.4 and $V := P_{-1}^{\boldsymbol{w}}(\mathbf{t})$, in Section 3.6. Precisely, denote by e_1 and e_2 a basis of $E = \mathcal{O}_X \oplus \mathcal{O}_X$ with $l_n = \mathbb{C}\cdot e_1$, $l_{n-2} = \mathbb{C}\cdot e_2$ and $l_{n-1} = \mathbb{C}\cdot(e_1+e_2)$; then, choose the basis (e_1', e_2') for $(E',\boldsymbol{l}') := \mathrm{Elm}_{t_n}^-(E,\boldsymbol{l})$ given by $e_i' := e_i$ outside of t_n: e_2' has a pole at t_n and generates $\mathcal{O}_X(-1)$. Note that $e_1 = e_1'$ is the cyclic vector for the apparent map. The rational map $U \dashrightarrow V$ is therefore given by

$$(u_1, \ldots, u_{n-3}, 0, 1, \infty) \mapsto (v_1, \ldots, v_n) = (u_1, \ldots, u_{n-3}, 0, 1, 0)$$

where parabolic structures $\boldsymbol{l}$ and $\boldsymbol{l}'$ are respectively generated by $u_ie_1 + e_2$ and $v_ie_1' + e_2'$. We interpret this as a map

$$\begin{array}{ccc} U \simeq (\mathbb{P}^1_{\mathbb{C}})^{n-3} & \dashrightarrow & V \simeq \mathbb{P}^{n-3}_{\mathbb{C}} \\ \mathbf{u} = (u_1, \ldots, u_{n-3}) & \mapsto & \mathbf{v} = (u_1 : \ldots : u_{n-3} : 1) \end{array}$$

Assume, for computations, that $(u_1, \ldots, u_{n-3}) \in \mathbb{C}^{n-3}$. We also rename spectral data as follows:

$$\begin{array}{cc} E = \mathcal{O}_X \oplus \mathcal{O}_X & E' = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \\ \begin{pmatrix} t_1 & \cdots & t_{n-3} & 0 & 1 & \infty \\ \nu_{t_1}^+ & \cdots & \nu_{t_{n-3}}^+ & \nu_0^+ & \nu_1^+ & \nu_\infty^- \\ \nu_{t_1}^- & \cdots & \nu_{t_{n-3}}^- & \nu_0^- & \nu_1^- & \nu_\infty^+ - 1 \end{pmatrix} & \begin{pmatrix} t_1 & \cdots & t_{n-3} & 0 & 1 & \infty \\ \nu_{t_1}^+ & \cdots & \nu_{t_{n-3}}^+ & \nu_0^+ & \nu_1^+ & \nu_\infty^+ \\ \nu_{t_1}^- & \cdots & \nu_{t_{n-3}}^- & \nu_0^- & \nu_1^- & \nu_\infty^- \end{pmatrix} \end{array}$$

Then, the general connection on $(E, \boldsymbol{l})$ or $(E', \boldsymbol{l}')$ writes

$$\nabla = \nabla_0 + c_1\Theta_1 + \cdots + c_{n-3}\Theta_{n-3}, \quad (c_i) \in \mathbb{C}^{n-3}$$

where

$$\nabla_0 := d + \begin{pmatrix} \nu_0^- & 0 \\ \rho & \nu_0^+ \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} \nu_1^- - \rho & \nu_1^+ - \nu_1^- + \rho \\ -\rho & \nu_1^+ + \rho \end{pmatrix} \frac{dz}{z-1} \tag{5.1}$$

$$+ \sum_{i=1}^{n-3} \begin{pmatrix} \nu_{t_i}^- & (\nu_{t_i}^+ - \nu_{t_i}^-)u_i \\ 0 & \nu_{t_i}^+ \end{pmatrix} \frac{dz}{z - t_i}, \quad \text{with } \rho = \nu_0^- + \nu_1^- + \nu_\infty^- + \sum_{i=1}^{n-3} \nu_{t_i}^-,$$

and

$$\Theta_i := \begin{pmatrix} 0 & 0 \\ 1 - u_i & 0 \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} u_i & -u_i \\ u_i & -u_i \end{pmatrix} \frac{dz}{z-1} + \begin{pmatrix} -u_i & u_i^2 \\ -1 & u_i \end{pmatrix} \frac{dz}{z - t_i}. \tag{5.2}$$

The connection ∇_0 is the unique connection (compatible with the given parabolic structure) such that the divisor of the apparent map $\mathrm{App}(\nabla_0)$ takes the form $\mathrm{div}(\varphi_{\nabla_0}) = t_1 + \cdots + t_{n-3}$: in this case, e_1 is the $\nu_{t_i}^-$-eigendirection for $i = 1, \ldots, n-3$. We note that

$$\rho = 0 \Leftrightarrow \nu_0^- + \nu_1^- + \nu_\infty^- + \sum_{i=1}^{n-3} \nu_{t_i}^- = 0 \Leftrightarrow \nu_0^+ + \nu_1^+ + \nu_\infty^+ + \sum_{i=1}^{n-3} \nu_{t_i}^+ = 1$$

in which case ∇_0 is the reducible connection (see Proposition 4.6): the subbundle $\mathcal{O}_X \subset E$ (resp. E') generated by the cyclic vector e_1 (resp. e_1') is ∇_0-invariant.

The parabolic Higgs fields Θ_i, $i = 1, \ldots, n-3$, are independent over $\mathbb{C}$ (they do not share the same poles) and any other one is a linear combination of these Θ_i's. These generators have been choosen so that the apparent map has divisor $\mathrm{div}(\varphi_{\Theta_i}) = \mu_i + \sum_{j \neq i} t_j$ where $\mu_i = \frac{t_i(u_i - 1)}{u_i - ti}$. Moreover, the moduli space of parabolic Higgs bundles is naturally isomorphic to the total space of the cotangent bundle T^*U (for those parabolic Higgs bundles $(E, \boldsymbol{l}, \Theta)$ with $(E, \boldsymbol{l}) \in U$). Under this identification, Θ_i corresponds to the differential form du_i.

Denoting $\nabla = d + Adz$, the apparent map is given by the coefficient $\varphi_\nabla := A(2,1)$ and we get

$$\varphi_\nabla = -\frac{\rho}{z(z-1)} + \sum_{i=1}^{n-3} c_i \frac{(u_i - t_i)z + (1 - u_i)t_i}{z(z-1)(z-t_i)} = \frac{\tilde{\varphi}_\nabla(z)}{z(z-1)\prod_j (z - t_j)}, \tag{5.3}$$

where $\tilde{\varphi}_\nabla(z)$ is a polynomial of z degree $n-3$. The roots $z = q_1, \ldots, q_{n-3}$ of φ_∇ (or of $\tilde{\varphi}_\nabla(z)$) are the apparent singular points with respect to the cyclic vector e_1 (resp. e_1'). For such a variable q, we define the dual variable as

$$p := A(1,1)|_{z=q} - \frac{\nu_0^-}{q} - \frac{\nu_1^-}{q-1} - \sum_{i=1}^{n-3} \frac{\nu_{t_i}^-}{q-t_i}$$

i.e.

$$p = -\frac{\rho}{q-1} + \sum_{i=1}^{n-3} c_i u_i \left(\frac{1}{q-1} - \frac{1}{q-t_i} \right). \tag{5.4}$$

The natural symplectic structure on the moduli space $M(\mathbf{t}, \boldsymbol{\nu})$ is defined by

$$\omega = \sum_{i=1}^{n-3} dp_i \wedge dq_i \tag{5.5}$$

and the two maps App and Bun are Lagrangian with respect to ω. Here recall that (q_i, p_i) are not the coordinates for the moduli space $M(\mathbf{t}, \boldsymbol{\nu})$, but the coordinates for some $(n-3)!$-covering of $M(\mathbf{t}, \boldsymbol{\nu})$. However the symplectic form ω in (5.5) is invariant under the changing the order of roots q_i, so it descends to a symplectic form on $M(\mathbf{t}, \boldsymbol{\nu})$.

Under these explicit notation, we can give the following

Alternate proof of Corollary 4.5. We want first to show that the map φ_Θ is not identically zero for any Higgs bundle $(E, \boldsymbol{l}, \Theta)$ with $(E, \boldsymbol{l}) \in V$. A Higgs field writes in a matrix form as

$$\Theta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and the map φ_Θ is given by the coefficient γ which is a holomorphic section of $\mathcal{O}_X(n-3)$. We want to check that $\gamma \equiv 0$ implies that either $\Theta \equiv 0$, or one of the parabolics $l_i \in \mathcal{O}_X$. If $\gamma \equiv 0$, then α and δ have to vanish over each $z = t_i$, $i = 1, \ldots, n$, since Θ has to be nilpotent over these points. But α and δ are sections of $\mathcal{O}_X(n-2)$ and have thus to be identically zero. Finally, if $\beta \not\equiv 0$, then, as a section of $\mathcal{O}_X(n-1)$, it cannot vanish at all $z = t_i$: for some t_i the matrix is not zero and take the form

$$\Theta|_{z=t_i} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the corresponding parabolic l_i lies on $\mathcal{O}_X$. The map $\Theta \mapsto \varphi_\Theta$ defines an homomorphism

$$H^0(\mathrm{End}(E, \boldsymbol{l}) \otimes \Omega_X(D)) \to H^0(X, L \otimes \Omega_X^1(D))$$

and we have just proved that it has zero kernel: it is injective. Therefore, its image image$(\mathrm{App}(\mathbb{P}H^0(\mathrm{End}(E, \boldsymbol{l}) \otimes \Omega_X(D))))$ defines an hyperplane of $H^0(X, L \otimes \Omega_X^1(D))$, i.e. an element of the dual $\mathcal{O}_X(n-3)|^*$, which depends only on $(E, \boldsymbol{l}) \in V$.

We have thus proved that the map $V \to |\mathcal{O}_X(n-3)|^*$ is a well-defined morphism and may be viewed as an endomorphism of $\mathbb{P}^{n-3}_{\mathbb{C}}$ (after fixing isomorphisms with $\mathbb{P}^{n-3}_{\mathbb{C}}$). In order to prove that it is an isomorphism, we just have to check that it is birational. For this, it is enough to prove that the composition

$$U \overset{\mathrm{Elm}^-_{t_n}}{\dashrightarrow} V \xrightarrow{\mathrm{Bun}} |\mathcal{O}_X(n-3)|^*$$

is birational (since the left-hand-side is). We compute this latter one in the affine chart $(u_1,\ldots,u_{n-3})\in\mathbb{C}^{n-3}\subset U$. For $\Theta=\Theta_i$, the map φ_{Θ_i} is the multiplication by

$$\gamma=\frac{P_i(z)\cdot dz}{z(z-1)\prod_j(z-t_j)}\quad\text{where}\quad P_i(z)=[(u_i-t_i)z+(1-u_i)t_i]\prod_{j\neq i}(z-t_j).$$

The zero of φ_{Θ_i} is thus given by $P_i(z)=0$. Now, consider the line Δ_i defined in $|\mathcal{O}_X(n-3)|$ by those polynomials vanishing on all t_j, $j\neq i$; these lines all intersect at the single point defined by the very special polynomial $\prod_i(z-t_i)$ and span $\mathbb{P}^{n-3}_{\mathbb{C}}$ ($t_i\neq t_j$ for any $i\neq j$). For generic u_i's, the hyperplane image $H\subset\mathbb{P}^{n-3}_{\mathbb{C}}$ of App cuts out all Δ_i's outside of their common intersection point. Conversely, a generic hyperplane H cuts out each Δ_i at a single point defined by say $(z-\mu_i)\prod_{j\neq i}(z-t_j)$; solving $\mu_i=\frac{t_i(u_i-1)}{u_i-t_i}$ gives the parabolic structure $(u_1,\ldots,u_{n-3})$. □

Let us start with $|\mathcal{O}_X(n-3)|\simeq\mathbb{P}^{n-3}_{\mathbb{C}}$ equipped with the following projective coordinates : $\mathbf{a}=(a_0:a_1:\cdots:a_{n-3})$ stands for the polynomial equation $a_{n-3}z^{n-3}+\cdots+a_1z+a_0=0$. It can be interesting to view also this space as $\mathrm{Sym}^{n-3}X$ with $X=\mathbb{P}^1_{\mathbb{C}}$ our initial base curve, and we have a natural map

$$\mathrm{Sym}:X^{n-3}\to\mathrm{Sym}^{n-3}X\ ;\ (q_1,\ldots,q_{n-3})\mapsto(z-q_1)\cdots(z-q_{n-3}).$$

The dual $|\mathcal{O}_X(n-3)|^*$ is the set of hyperplanes $a_0b_0+a_1b_1+\cdots+a_{n-3}b_{n-3}=0$ and has thus natural projective coordinates $\mathbf{b}=(b_0:b_1:\cdots:b_{n-3})$. Let us explicitely compute the relation ship between usual Darboux coordinates (p_i,q_i), our basic coordinates (u_i,c_i) and the new coordinates $(\mathbf{a},\mathbf{b})$ from our main Theorem 4.3. We do this for the Painlevé case $n=4$ and the first Garnier case $n=5$.

5.2. Case $n=4$**.** Our starting variables are $u\in\mathbb{C}\subset U$ and $c\in\mathbb{C}$. From (5.3) and (5.4), we get Darboux variables:

$$p=-\frac{(t-u)(\rho+c(t-u))}{t(t-1)}\quad\text{and}\quad q=t\frac{\rho+c(1-u)}{\rho+c(t-u)};$$

reversing, we get:

$$u=t\frac{\rho+p(q-1)}{\rho+p(q-t)}\quad\text{and}\quad c=-\frac{(q-t)(\rho+c(q-t))}{t(t-1)}.$$

The apparent map for Higgs bundle (set $c=\infty$ in above formula) vanishes at

$$\mu=t\frac{1-u}{t-u}=q+\frac{\rho}{p}$$

and we get

$$(a_1:a_0)=(1:-q)\quad\text{and}\quad(b_1:b_0)=(\mu:1).$$

The symplectic structure is given by

$$dp\wedge dq=dc\wedge du=\rho\cdot d\left(\frac{a_0db_0+a_1db_1}{a_0b_0+a_1b_1}\right).$$

Our μ-variable is exactly the Q-variable involved in Section 8 of [**13**] and it was observed, there, that Okamoto symmetry is just given by the involution $(q,\mu)\mapsto(\mu,q)$ (i.e. $(a_1:a_0)\leftrightarrow(b_0:-b_1)$) permuting the two fibrations. We will see in the next section that there is no such symmetric (global on $M(\mathbf{t},\boldsymbol{\nu})$) permuting the two fibrations for $n=5$.

6. Computations for the case $n = 5$

A straightforward computation shows that the map

$$\mathrm{Bun}\circ\mathrm{Elm}^-_{t_n} : \begin{cases} U & \to \quad V = |\mathcal{O}_X(n-3)|^* \\ (u_1,u_2) & \mapsto \quad (b_0 : b_1 : b_2) \end{cases}$$

is given by

$$\begin{cases} b_2 = & t_1t_2(t_1(t_2-1)u_1 - (t_1-1)t_2u_2 + (t_1-t_2)) \\ b_1 = & t_1t_2((t_2-1)u_1 - (t_1-1)u_2 + (t_1-t_2)) \\ b_0 = & t_2(t_2-1)u_1 - t_1(t_1-1)u_2 + t_1t_2(t_1-t_2) \end{cases}$$

The (u_1,u_2) affine chart may thus be seen as an affine chart of V, or equivalently, V as an alternate compactification of the (u_1,u_2)-chart. The inverse map is given by

$$\begin{cases} u_1 = & t_1\frac{b_2-(t_2+1)b_1+t_2b_0}{b_2-(t_1+t_2)b_1+t_1t_2b_0} \\ u_2 = & t_2\frac{b_2-(t_1+1)b_1+t_1b_0}{b_2-(t_1+t_2)b_1+t_1t_2b_0} \end{cases}$$

Apparent singular points are the roots of the polynomial

$$P(z) = -\rho(z-t_1)(z-t_2)$$

$$+c_1\left[(u_1-t_1)z + (1-u_1)t_1\right](z-t_2) + c_2(z-t_1)\left[(u_2-t_2)z + (1-u_2)t_2\right]$$

$$= \left[c_1(u_1-t_1) + c_2(u_2-t_2) - \rho\right]z^2$$

$$+\left[\rho(t_1+t_2) + c_1(t_1(t_2+1) - u_1(t_1+t_2)) + c_2((t_1+1)t_2 - u_2(t_1+t_2))\right]z$$

$$+t_1t_2\left[c_1(u_1-1) + c_2(u_2-1) - \rho\right]$$

We can re-write

$$P(z) = \rho[b_2 - (t_1+t_2)b_1 + t_1t_2b_0](z-t_1)(z-t_2)$$

$$+c_1t_1(t_1-1)(b_2 - (z+t_2)b_1 + zt_2b_0)(z-t_1)$$

$$+c_2t_2(t_2-1)(b_2 - (z+t_1)b_1 + zt_1b_0)(z-t_2).$$

Denoting $z = q_1$ and $z = q_2$ the two apparent singular points, we get

$$c_1 = \rho\,\frac{(q_1-t_1)(q_2-t_1)}{t_1(t_1-1)(t_1-t_2)}\,\frac{b_2-(t_1+t_2)b_1+t_1t_2b_0}{b_2-(q_1+q_2)b_1+q_1q_2b_0}$$

and

$$c_2 = \rho\,\frac{(q_1-t_2)(q_2-t_2)}{t_2(t_2-1)(t_2-t_1)}\,\frac{b_2-(t_1+t_2)b_1+t_1t_2b_0}{b_2-(q_1+q_2)b_1+q_1q_2b_0}$$

and we already see strong transversality between parabolic and apparent fibrations. In a more symmetric way, we can introduce the equation $a_2q^2+a_1q+a_0=0$ of the two apparent singular points and we get the following formula

$$c_1 = \rho\,\frac{a_2t_1^2+a_1t_1+a_0}{t_1(t_1-1)(t_1-t_2)}\,\frac{b_2-(t_1+t_2)b_1+t_1t_2b_0}{a_2b_2+a_1b_1+a_0b_0}$$

and

$$c_2 = \rho\,\frac{a_2t_2^2+a_1t_2+a_0}{t_2(t_2-1)(t_2-t_1)}\,\frac{b_2-(t_1+t_2)b_1+t_1t_2b_0}{a_2b_2+a_1b_1+a_0b_0}.$$

As expected by our choice of coordinates, the locus of Higgs bundles, where c_1 and/or c_2 goes to the infinity, is given by the incidence variety $a_2b_2+a_1b_1+a_0b_0 = 0$. For each root $z = q_i$, the dual variable is expressed by

$$p_i = -\frac{\rho}{q_i - 1} + c_1u_1\left(\frac{1}{q_i-1} - \frac{1}{q_i-t_1}\right) + c_2u_2\left(\frac{1}{q_i-1} - \frac{1}{q_i-t_2}\right)$$

and we get

$$p_1 = \rho\frac{b_1 - q_2b_0}{b_2 - (q_1+q_2)b_1 + q_1q_2b_0} \quad \text{and} \quad p_2 = \rho\frac{b_1 - q_1b_0}{b_2 - (q_1+q_2)b_1 + q_1q_2b_0}.$$

We find that

$$\eta := p_1dq_1 + p_2dq_2 = \rho\frac{a_2db_2 + a_1db_1 + a_0db_0}{a_2b_2 + a_1b_1 + a_0b_0} - \rho\frac{d(a_2b_2 + a_1b_1 + a_0b_0)}{a_2b_2 + a_1b_1 + a_0b_0} + \rho\frac{da_2}{a_2}$$

where $(a_2 : a_1 : a_0) \sim (1 : -q_1 - q_2 : q_1q_2)$. The differential

$$\omega := d\eta = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 = \rho \cdot d\left(\frac{a_2db_2 + a_1db_1 + a_0db_0}{a_2b_2 + a_1b_1 + a_0b_0}\right)$$

is anti-invariant under the involution $(a_2 : a_1 : a_0) \leftrightarrow (b_2 : b_1 : b_0)$ that exchanges the two sets of projective coordinates.

A straightforward computation shows that, pulling-back the symplectic form ω to our initial parameters c_i and u_i, we obtain

$$dp_1 \wedge dq_1 + dp_2 \wedge dq_2 = dc_1 \wedge du_1 + dc_2 \wedge du_2$$

which is the Liouville form on moduli space of Higgs bundles.

We have also the following formula comparing classical coordinates to our parabolic ones:

$$(b_2 : b_1 : b_0) = (p_1q_1^2 - p_2q_2^2 + \rho(q_1 - q_2) \ : \ p_1q_1 - p_2q_2 \ : \ p_1 - p_2).$$

6.1. The coarse moduli space of parabolic bundles and Del Pezzo geometry. Here, we want to explicitely describe the full coarse moduli space $P_{-1}(\mathbf{t})$ of all undecomposable parabolic bundles as a finite union of projective charts patched together by birational maps between open subsets (see end of Section 3.2). We already get our main projective chart $V \subset P_{-1}(\mathbf{t})$ (defined in Section 3.6) that contains almost all undecomposable parabolic bundles: $V := P_{-1}^{\boldsymbol{w}}(\mathbf{t}) \simeq \mathbb{P}^2_{\mathbf{b}}$ where $\boldsymbol{w} = (w, w, w, w, w)$ with $\frac{1}{5} < w < \frac{1}{3}$. To get the full coarse moduli space $P_{-1}(\mathbf{t})$, we have to add those undecomposable parabolic structures on $\mathcal{O}_X \oplus \mathcal{O}_X(-1)$ with 1 or 2 parabolics lying on $\mathcal{O}_X$, and the unique undecomposable parabolic structure on $\mathcal{O}_X(1) \oplus \mathcal{O}_X(-2)$. They will occur in $P_{-1}(\mathbf{t})$ as points infinitesimally close to special points of $V = \mathbb{P}^2_{\mathbf{b}}$, namely those non generic bundles (see Section 3.2). Let us list them.

Defining Π_i and D_i. The locus Π_i of those undecomposable parabolic structures on $E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ having exactly $l_i \subset \mathcal{O}_X$ (other parabolics outside $\mathcal{O}_X$) is naturally isomorphic to X: it is the moduli space of undecomposable parabolic structures on E over the 4 other points, none of them lying on $\mathcal{O}_X$ (see Section 3.7). Each of these parabolic bundles is infinitesimally close to the unique undecomposable parabolic structures on $\mathcal{O}_X \oplus \mathcal{O}_X(-1)$ with all l_j, $j \neq i$, lying on the same $\mathcal{O}_X(-1) \hookrightarrow E$ (see Section 3.3). There is only one undecomposable bundle with this latter property and it defines a single point $D_i \in V$. As we shall see, Π_i will occur, when we pass to another projective chart for $P_{-1}(\mathbf{t})$, as the exceptional divisor after blowing-up the point D_i (wall-crossing phenomenon).

	V	$P_{-1}(\mathbf{t}) \setminus V$
5	$D_i : \begin{cases} E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \\ l_j, l_k, l_m, l_n \subset \mathcal{O}_X(-1) \end{cases}$	$\Pi_i : \begin{cases} E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \\ l_i \subset \mathcal{O}_X \end{cases}$
10	$\Pi_{i,j} : \begin{cases} E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \\ l_k, l_m, l_n \subset \mathcal{O}_X(-1) \end{cases}$	$D_{i,j} : \begin{cases} E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \\ l_i, l_j \subset \mathcal{O}_X \end{cases}$
1	$\Pi : \begin{cases} E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \\ l_i, l_j, l_k, l_m, l_n \subset \mathcal{O}_X(-2) \end{cases}$	$D : \begin{cases} E = \mathcal{O}_X(1) \oplus \mathcal{O}_X(-2) \\ - \end{cases}$

TABLE 1. Non generic bundles (here, $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$).

Defining $\Pi_{i,j}$ and $D_{i,j}$. There is a unique undecomposable parabolic structure $\boldsymbol{l}$ on $E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ having exactly $l_i, l_j \in \mathcal{O}_X$ (other parabolics outside $\mathcal{O}_X$). It is infinitesimally close to the one-parameter family of undecomposable parabolic structures on $\mathcal{O}_X \oplus \mathcal{O}_X(-1)$ with all l_k, $k \neq i, j$, lying on the same $\mathcal{O}_X(-1) \hookrightarrow E$; this latter family form a rational curve $\Pi_{i,j} \subset V$ which is also naturally parametrized by X. Indeed, there is also a $\mathcal{O}_X(-1) \hookrightarrow E$ passing through l_i and l_j, and these two embeddings intersect over a point $z \in X$. The locus of $\boldsymbol{l}$ is given by a single point $D_{i,j} \notin V$ in the coarse moduli space $P_{-1}(\mathbf{t})$ which is infinitesimally close to any point of $\Pi_{i,j}$. When switching to some other projective charts of $P_{-1}(\mathbf{t})$, by moving weights, the rational curve $D_{i,j}$ is eventually contracted, replaced by the single point $D_{i,j}$.

Defining Π and D. Finally, the unique undecomposable parabolic structure on $\mathcal{O}_X(1) \oplus \mathcal{O}_X(-2)$ is infinitesimally close to the one-parameter family of parabolic structures on $E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ with all parabolics lying on the same $\mathcal{O}_X(-2) \hookrightarrow E$. The latter family is again a rational curve $\Pi \subset V$ naturally parametrized by X: the subbundles $\mathcal{O}_X(-2)$ and $\mathcal{O}_X$ coincide over a unique point of X. The undecomposable parabolic bundle $\mathcal{O}_X(1) \oplus \mathcal{O}_X(-2)$ is thus represented by a single point $D \in P_{-1}(\mathbf{t}) \setminus V$ infinitesimally closed to Π.

Computations in $V = \mathbb{P}^2_{\mathbf{b}}$. We have summarized the list of non generic undecomposable parabolic bundles in the table above. All $\Pi, \Pi_i, \Pi_{i,j}$ are one-parameter families naturally parametrized by X; they form rational curves in $P_{-1}(\mathbf{t})$. All $D, D_i, D_{i,j}$ are just points. On each line, bundles from the two columns are infinitesimally closed; bundles from the left side are contained in the main chart V while those on the right side are outside. There are 16 one parameter families of special bundles infinitesimally closed to 16 special bundles. A straightforward computation shows that:

- Π is the conic with equation $b_1^2 - b_0 b_2$ parametrized by
$$X \to \Pi \subset \mathbb{P}^2_{\mathbf{b}} \ ; \ z \mapsto (1 : z : z^2).$$
- D_i is the image of $z = t_i$ through the previous mapping.
- $\Pi_{i,j}$ is the line passing through D_i and D_j.

Moving democratic weights Let us consider the moduli space $P_{-1}^{\boldsymbol{w}}(\mathbf{t})$ for "democratic" weights $w_i = w$, $i = 1, \ldots, n$, and see how $P_{-1}^{\boldsymbol{w}}(\mathbf{t})$ is varrying while w goes from 0 to 1. At the beginning, when $w < \frac{1}{5}$, the weight is not admissible (see Section 3.2) and $P_{-1}^{\boldsymbol{w}}(\mathbf{t}) = \emptyset$.

For $\frac{1}{5} < w < \frac{1}{3}$, $P_{-1}^{\boldsymbol{w}}(\mathbf{t}) = V$ is the main projective chart discussed above, the projective plane $\mathbb{P}^2_{\mathbb{C}}$ with coordinates $(b_0 : b_1 : b_2)$.

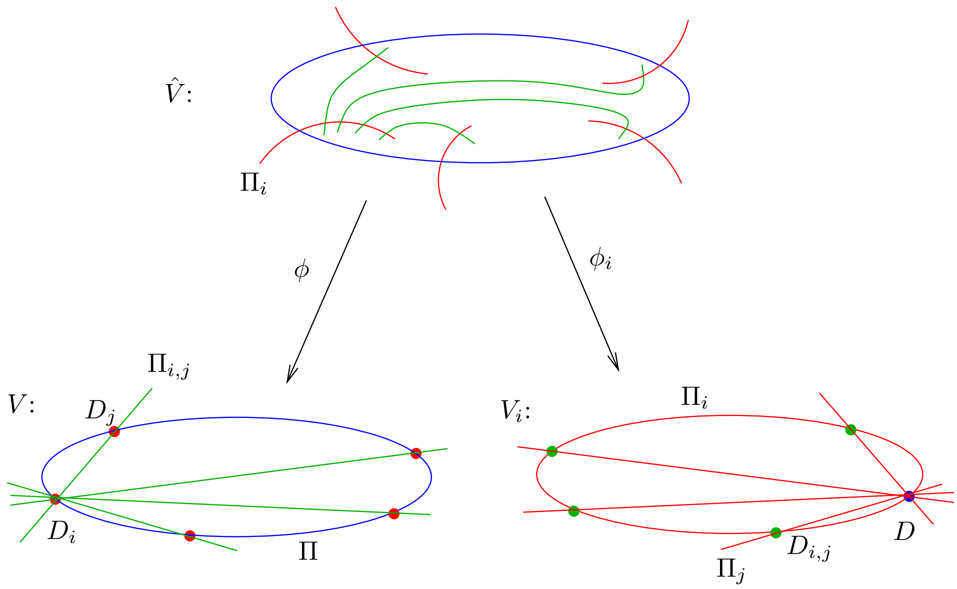

FIGURE 2. Projective charts V, $\hat{V}$ and V_i's and Del Pezzo geometry

When we pass to the next admissible chamber $\frac{1}{3} < w < \frac{3}{5}$, all five bundles D_i's become unstable while all five families Π_i's now consist in stable bundles. The new moduli space of stable bundles $P_{-1}^{\boldsymbol{w}}(\mathbf{t})$ is obtained from the previous one by blowing-up the five points D_i's, that are replaced by the corresponding Π_i's in the moduli space. We thus get a 5-points blow-up of $\mathbb{P}^2_{\mathbf{b}}$, a Del Pezzo surface of degree 4. Let us denote by $\hat{V}$ the Del Pezzo surface and $\phi : \hat{V} \to V \simeq \mathbb{P}^2_{\mathbf{b}}$ the blowing-up of five points D_i's. There are exactly 16 rational curves having -1 self-intersection on it, namely the 5 exceptional divisors Π_i's (arising from blowing-up the D_i's) and the strict transforms of the conic Π and of the 10 lines $\Pi_{i,j}$: they precisely correspond to our 16 families of special bundles (see Figure 2).

Finally, when we pass to the last chamber $\frac{3}{5} < w < 1$, bundles of the family Π become unstable and the parabolic structure D on the special bundle $E = \mathcal{O}_X(1) \oplus \mathcal{O}_X(-2)$ becomes stable. The corresponding moduli space $P_{-1}^{\boldsymbol{w}}(\mathbf{t})$ is thus obtained by contracting the (-1)-curve $\Pi \in \hat{V}$ onto the single point D. This may be viewed as a 4-points blow-up of $\mathbb{P}^2_{\mathbb{C}}$ (the degree 5 Del Pezzo surface).

Patching two charts Let us now focus on the two projective charts V and $\hat{V}$. Restricting the blowing-up $\phi : \hat{V} \to V$ to the complement of Π_i, we have a natural isomorphism

$$\phi^0 : \hat{V} \setminus (\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4 \cup \Pi_5) \xrightarrow{\sim} V \setminus \{D_1, D_2, D_3, D_4, D_5\}$$

that identify equivalence classes of bundles that both occur in $\hat{V}$ and V. Indeed, when w crosses the special value $\frac{1}{3}$, sign of stability index changes only for those bundles D_i's and P_i's. The map ϕ obviously contracts each Π_i to D_i, it is the blow-up morphism. Now, the non reduced scheme obtained by patching together V and $\hat{V}$ by ϕ^0 is still an open subset of the moduli space $P_{-1}(\mathbf{t})$ that contains both V and the non separated locus Π_i:

$$V \hookrightarrow V \cup_{\phi^0} \hat{V} \hookrightarrow P_{-1}(\mathbf{t}).$$

In other words, by patching $\hat{V}$ to V like above, we have added to V the non separated points of Π_i's. Mind that not only generic bundles are identified by ϕ^0, also generic points of Π, for example, occur in both charts and are identified by ϕ^0.

Geometry of Del Pezzo We want now to cover the full coarse moduli space $P_{-1}(\mathbf{t})$ by a finite number of smooth projective charts and patch them together like we have just done. There are many projective charts $V' \subset P_{-1}(\mathbf{t})$ that can be defined as moduli space of stable bundles $P_{-1}^{\boldsymbol{w}}(\mathbf{t})$ (many chambers in the space of weights) but we do not need all of them to cover the coarse moduli space $P_{-1}(\mathbf{t})$. We will use the classical geometry of the Del Pezzo surface $\hat{V}$ in order to select a few number of them. First of all, note that $\hat{V}$ is dominating all other projective charts in the following sense: if V' is another chart, the natural birational map $\phi' : \hat{V} \to V$ is actually a morphism. Indeed, V' only differ from $\hat{V}$ by the fact that some of the one-parameter families Π' are contracted to points D'.

The Del Pezzo surface $\hat{V}$ contains 16 rational (-1)-curves that correspond, in our modular setting, to the 16 families of special bundles. Each of these "lines" intersects 5 other ones with cross-ratio determined by the t_i's. Recall that each of these curves are naturally parametrized by $X \supset \{t_1, t_2, t_3, t_4, t_5\}$ in our modular interpretation, and each poles t_i corresponds to intersecting lines.

Appart from special symmetric values of the t_i's, the automorphism group of this Del Pezzo surface has order 16 and it acts 1-transitively on -1 rational curves: given 2 of these lines $\Pi, \Pi' \subset \hat{V}$, there is exactly one automorphism $\phi : \hat{V} \to \hat{V}$ sending Π to Π'. This group has also modular interpretation. If we apply two elementary transformations (see Section 2.2), we get an automorphism

$$\mathrm{Elm}_{t_i}^- \circ \mathrm{Elm}_{t_j}^+ : P_{-1}(\mathbf{t}) \xrightarrow{\sim} P_{-1}(\mathbf{t}).$$

This automorphism must permute special points $D, D_i, D_{i,j}$'s and permute special families $\Pi, \Pi_i, \Pi_{i,j}$ and induces, in particular, an automorphism

$$(\mathrm{Elm}_{t_i}^- \circ \mathrm{Elm}_{t_j}^+)|_{\hat{V}} : \hat{V} \xrightarrow{\sim} \hat{V}$$

of the projective chart $\hat{V}$. It is straightforward to check that the group generated by these automorphisms has order 16 and acts 1-transitively on the 16 lines. Precisely, we have

$$(\mathrm{Elm}_{t_i}^- \circ \mathrm{Elm}_{t_j}^+) : \Pi \xrightarrow{\sim} \Pi_{i,j}$$

for all i, j and

$$(\mathrm{Elm}_{t_j}^- \circ \mathrm{Elm}_{t_k}^+) \circ (\mathrm{Elm}_{t_m}^- \circ \mathrm{Elm}_{t_n}^+) : \Pi \xrightarrow{\sim} \Pi_i$$

where $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$.

There are 16 ways to go back to $\mathbb{P}^2_{\mathbb{C}}$ by blowing-down 5 curves in $\hat{V}$: we have to choose 5 lines intersecting a given one Π'. All these $\mathbb{P}^2_{\mathbb{C}}$ correspond to moduli spaces for different choices of weights. Indeed, after an even number of elementary tranformations, we can assume $\Pi' = \Pi$. So the chart $V' \simeq \mathbb{P}^2_{\mathbb{C}}$ obtained by contracting those 5 lines intersecting Π' in $\hat{V}$ is given as moduli space $V' = P_{-1}^{\boldsymbol{w}}(\mathbf{t})$ with weights of the form $w_i \in \{w, 1-w\}$, $\frac{1}{5} < w < \frac{1}{3}$, with an even number of occurences $w_i = 1-w$; note that there are 16 such possibilities and we denote them by $V, V_i, V_{i,j}$ in the obvious way.

The whole moduli space $P_{-1}(\mathbf{t})$ Consider, like before, the projective charts $V_i \simeq \mathbb{P}^2_{\mathbb{C}}$ obtained by contracting in $\hat{V}$ the 5 lines intersecting Π_i. The automorphism

$$(\mathrm{Elm}^-_{t_j} \circ \mathrm{Elm}^+_{t_k}) \circ (\mathrm{Elm}^-_{t_m} \circ \mathrm{Elm}^+_{t_n}) : P_{-1}(\mathbf{t}) \xrightarrow{\sim} P_{-1}(\mathbf{t})$$

where $\{i,j,k,m,n\} = \{1,2,3,4,5\}$, permutes the lines Π and Π_i, and thus the charts V and V_i; it follows that $V_i = P^{\boldsymbol{w}}_{-1}(\mathbf{t})$ for weights of the form $w_i = w$ and $w_j = w_k = w_m = w_n = 1 - w$ with $\frac{1}{5} < w < \frac{1}{3}$. It is then easy to check that these charts are enough to cover the whole coarse moduli space. Precisely, we have:

- $D \in V_i$ for all i,
- $D_i \in V$,
- $D_{i,j} \in V_i$ and V_j,
- $\Pi \subset V, \hat{V}$,
- $\Pi_i \subset \hat{V}$ and all V_j,
- $\Pi_{i,j} \subset V$, $\hat{V}$ and all $V_k \neq V_i, V_j$.

We finally obtain the following description:

$$P_{-1}(\mathbf{t}) = \hat{V} \cup V \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \tag{6.1}$$

where $\cup$ means that we identify all isomorphism classes of bundles that are shared by any two of these projective charts by means of the natural birational isomorphisms. More precisely, we have the explicit blowing-down morphisms $\phi : \hat{V} \to V \simeq \mathbb{P}^2_{\mathbf{b}}$ and $\phi_i : \hat{V} \to V_i \simeq \mathbb{P}^2_{\mathbb{C}}$ introduced above and the patching (6.1) are induced by ϕ and ϕ_i on the maximal open subsets where it is one-to-one.

The union of Proposition 1.3 is not sharp: we can delete one of the V_i's, remaining 6 charts are enough to cover the whole moduli. However, we stress that we cannot delete $\hat{V}$. Indeed, so far, we have not been very rigorous with those special t_i points occuring along our one-parameter families $X \xrightarrow{\sim} \Pi, \Pi_i, \Pi_{i,j}$, namely those intersection points between two such families. Let us explain on an example. The family Π of those parabolic structures $\boldsymbol{l}$ on $E = \mathcal{O}_X \oplus \mathcal{O}(-1)$ such that all parabolics l_i's are contained in the same subbundle $\mathcal{O}_X(-2) \hookrightarrow E$. Such a subbundle is determined, up to automorphism of E, by its intersection locus with the special subbundle $\mathcal{O}_X$: there is a unique point $z \in X$ such that these two subbundles coincide over this point. Since the parabolic structure is determined by $\mathcal{O}_X(-2)$, it is also determined by the corresponding point z and we get a natural parametrization $X \xrightarrow{\sim} \Pi$. There are 5 special points corresponding to the case where the two subbundles $\mathcal{O}_X(-2)$ and $\mathcal{O}_X$ coincide over t_i: then $l_i \subset \mathcal{O}_X$ and it is the intersection point with the family Π_i. This special parabolic bundle in Π occurs in $\hat{V}$, but not in our main chart V. Indeed, stability assumption for V excludes the possibility of $l_i \subset \mathcal{O}_X$, and this special bundle is replaced by $D_i \in V$.

6.2. The duality picture. We now go back to the moduli space of connections $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})$. An open subset $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0$ is given by those connection $(E, \nabla, \boldsymbol{l})$ whose underlying parabolic bundle belongs to our main chart $(E, \boldsymbol{l}) \in V$. Recall that a natural compactification is obtained by adding projective Higgs bundles

$$\mathrm{App} \times \mathrm{Bun} : \overline{M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0} \xrightarrow{\sim} \mathbb{P}^2_{\mathbf{a}} \times \mathbb{P}^2_{\mathbf{b}}$$

and the boundary of $M^{\boldsymbol{w}}(\mathbf{t}, \boldsymbol{\nu})^0$ corresponds to the incidence variety $\Sigma : \{a_0 b_0 + a_1 b_1 + a_2 b_2 = 0\}$. We would like to add to this picture those missing connections, i.e. the connections on missing parabolic bundles.

In order to do this, let us denote by C the image of the diagonal through the map

$$\mathrm{Sym} : X \times X \to \mathrm{Sym}^2 X = \mathbb{P}^2_{\mathbf{a}} \; ; \; (q_1, q_2) \mapsto (z - q_1)(z - q_2),$$

namely the conic $C : \{a_1^2 - 4a_0a_2 = 0\}$, which is the locus of double roots $q_1 = q_2$: it is naturally parametrized by our initial base curve

$$X \to C \; ; \; q \mapsto (q^2 : -2q : 1).$$

Those lines $a_0b_0 + a_1b_1 + a_2b_2 = 0$ tangent to the conic are defined by the dual conic $C^* : \{b_1^2 - b_0b_2 = 0\}$ (denoted by Π in the previous section) which is also naturally parametrized by our initial base curve

$$X \to C^* \; ; \; z \mapsto (1 : z : z^2).$$

The locus $q = t_i$ of poles provide 5 special points on the conic C, namely $(a_0 : a_1 : a_2) = (t_i^2 : -2t_i : 1)$, and we will denote by $\Delta_i : \{t_i^2 a_2 + t_i a_1 + a_0 = 0\}$ the line tangent to C at this point. Any two of those lines intersect at a point $\Delta_i \cap \Delta_j = \{P_{i,j}\}$ (outside of C); we get 10 special points with coordinates $(t_it_j : -t_i - t_j : 1)$. Passing to the dual picture, we get 5 points $D_i := \Delta_i^*$ on the dual conic C^* defined by $(b_0 : b_1 : b_2) = (1 : t_i : t_i^2)$ and 10 lines, $\Pi_{i,j} := P_{i,j}^*$ passing through both D_i and D_j with equation $t_it_jb_0 - (t_i + t_j)b_1 + b_2 = 0$ (see Figure 3).

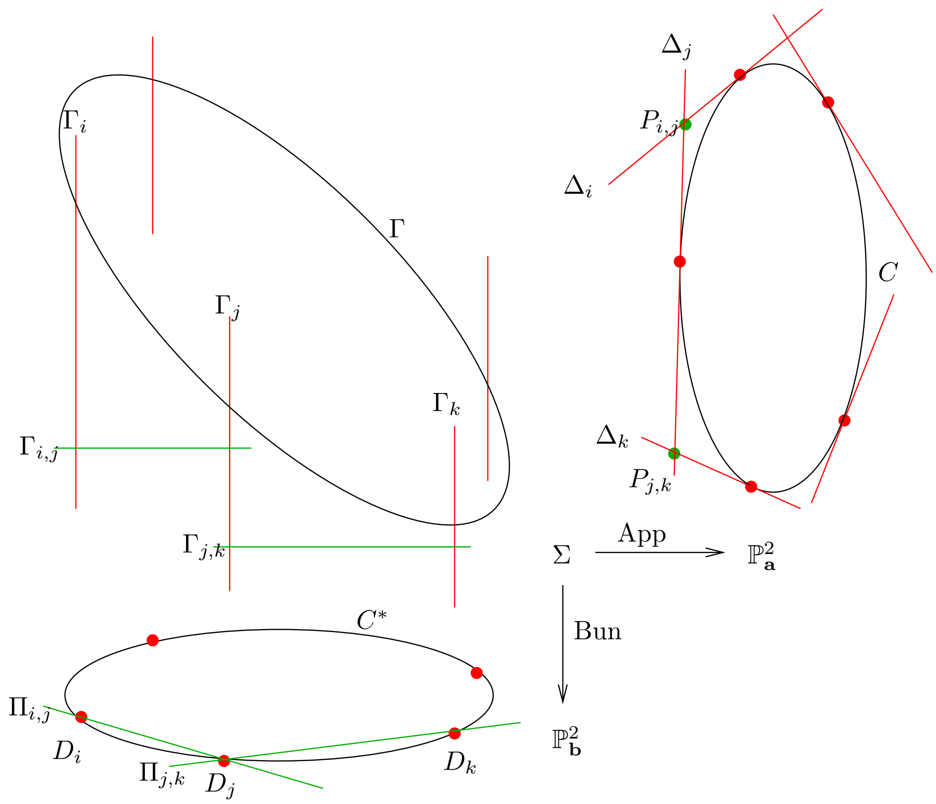

FIGURE 3. Duality picture in the case $n = 5$

Denote by $\Sigma \subset \mathbb{P}^2_{\mathbf{a}} \times \mathbb{P}^2_{\mathbf{b}}$ the incidence variety defined by $a_0b_0 + a_1b_1 + a_2b_2 = 0$; recall that $\Sigma = M^{\boldsymbol{w}}(\mathbf{t},\boldsymbol{\nu})^0_H$ (see equation (4.9)). The conic $C \subset \mathbb{P}^2_{\mathbf{a}}$ lifts-up as a rational curve $\Gamma \subset \Sigma$ parametrized by

$$\mathbb{P}^1 \to \Gamma \ ; \ q \mapsto \big(\ (1 : -2q : q^2) \ , \ (q^2 : q : 1) \ \big)$$

(which projects down to both C and C^*). It is defined by equations

$$a_1^2 = 4a_0a_2, \quad a_0b_0 = a_2b_2 \quad \text{and} \quad 2a_2b_2 + a_1b_1 = 0.$$

Inside Σ, we also get 5 lines

$$\Gamma_i := \Delta_i \times \{D_i\}$$

and 10 more lines

$$\Gamma_{i,j} := \{P_{i,j}\} \times \Pi_{i,j}.$$

All these 16 curves intersect like the corresponding 16 special rational curves in the Del Pezzo surface discussed in Section 6 (the blow-up of $\mathbb{P}^2_{\mathbf{b}}$ at the 5 points D_i, see picture 3); they moreover intersect transversally.

As we shall see, the locus of those connections that we have forgotten so far is given by points infinitesimally closed to some points of Σ, namely

- to Γ_i for those connections on a bundle having the parabolic $l_i \in \mathcal{O}_X$,
- to Γ for those connections on $\mathcal{O}_X(1) \oplus \mathcal{O}_X(-2)$.

To recover the full moduli space, we will have (at least) to blow-up these curves.

6.3. Those connections on $\mathcal{O}_X \oplus \mathcal{O}_X(-1)$ having a parabolic $l_i \in \mathcal{O}_X$. To simplify formulae, set

$$\kappa_i := \nu_i^+ - \nu_i^- \quad \text{for } i = 0, 1, t_1, t_2, \infty.$$

In order to recover such connections in our moduli space, we would like to construct, for each such connection $(E, \boldsymbol{l}^0, \nabla^0)$, a deformation $(E, \boldsymbol{l}^t, \nabla^t)$ on the fixed bundle $E = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ such that it belongs to our main chart $M(\mathbf{t},\boldsymbol{\nu})^0$ for $t \neq 0$ (no parabolic l_i^t is contained in $\mathcal{O}_X$). We will do this for a connection ∇^0 having $l_{t_1}^0 \in \mathcal{O}_X$ and other p_i's being generic (mainly $l_\infty \notin \mathcal{O}_X$). After applying the elementary transformation Elm^+_∞, we get a connection with parabolic structure in the chart U with $u_1 = \infty$. A deformation like above can be given by setting

$$c_1^t = -t\kappa_{t_1} + t^2 \cdot c_1, \quad c_2^t = c_2, \quad u_1^t = \frac{1}{t} \quad \text{and} \quad u_2^t = u_2$$

(note that $\nabla_0 - \frac{\kappa_{t_1}}{u_1}\Theta_1$ and $\frac{1}{u_1^2}\Theta_i$ have limit when $u_1 \to \infty$). By the way, we will get all connections for a generic parabolic structure $\boldsymbol{l}$ having $l_{t_1} \in \mathcal{O}_X$. Going back with Elm^-_∞, we get a curve in $\mathbb{P}^2_{\mathbf{a}} \times \mathbb{P}^2_{\mathbf{b}}$ that tends to Σ when $t \to 0$. The limit point is given by

$$(a_2 : a_1 : a_0) \sim (1 : -t_1 - q : t_1q) \quad \text{and} \quad (b_2 : b_1 : b_0) \sim (t_1^2 : t_1 : 0)$$

(i.e. we tend to a point of the special line Γ_{t_1}) with apparent points given by

$$q_1 = t_1 \quad \text{and} \quad q_2 = \frac{t_2(c_2(u_2-1) - \rho - \kappa_{t_1}}{c_2(u_2-t_2) - \rho - \kappa_{t_1}}.$$

In order to distinguish between all connections having the same limit point (so far, c_1 does not appear for instance) we have to blow-up Γ_1 and compute the limit point on the exceptional divisor F_{t_1}. This latter one is parametrized by q_2 and the restriction of the projective coordinates

$$(u : v : w) \sim (\ b_2(t_1^2a_2 + t_1a_1 + a_0) \ : \ a_2(b_2 - t_1b_1) \ : \ a_2(b_2 - t_1^2b_0)\ ).$$

One easily check that, when $t \to 0$, the three entries above tend to 0 but the triple projectively tends to

$$(u:v:w) \to \Big(\ \frac{\kappa_{t_1} t_1^2 t_2(t_2-1)}{c_2(u_2-t_2)-\rho-\kappa_{t_1}} \ : \ t_2(u_2-1) \ : \ (t_1+t_2)u_2-(t_1+1)t_2 \ \Big)$$

From the discussion of Section 3.3 and more particularly Section 6, it is not surprising that u_2, and thus the parabolic structure, is determined by the ratio $\frac{v}{w} = \frac{b_2-t_1b_1}{b_2-t_1^2b_0}$ which is also the coordinate for the blow-up of the point $D_{t_1} \in \mathbb{P}^2_{\mathbf{b}}$. For u_2 fixed, we see that the parameter c_2 is determined by q_2, i.e. by the apparent map. We still not see the parameter c_1 and cannot determine yet the limit connection. We have to blow-up once again.

Precisely, we have now to blow-up the surface defined in F_{t_1} by

$$(\rho+\kappa_{t_1})u+\kappa_{t_1}t_1(t_1+q_2)v-\kappa_{t_1}t_1q_2w=0.$$

One can check that the locus of those connections $p_{t_i} \in \mathcal{O}_X$ is parametrized by an open subset of the latter exceptional divisor F'_{t_1}.

6.4. Those connections on $\mathcal{O}_X(1) \oplus \mathcal{O}_X(-2)$. After an elementary transformation at the 5 parabolics of the form

$$\mathrm{Elm}_0^+ \circ \mathrm{Elm}_1^+ \circ \mathrm{Elm}_\infty^+ \circ \mathrm{Elm}_{t_1}^- \circ \mathrm{Elm}_{t_1}^-,$$

such a connection $(E = \mathcal{O}_X(1) \oplus \mathcal{O}_X(-2), \boldsymbol{l}, \nabla)$ can be transformed into a trace-free connection $(E' = \mathcal{O}_X \oplus \mathcal{O}_X, \boldsymbol{l}', \nabla')$ on the trivial bundle with the property that all parabolics l'_i now lie along the diagonal section $\mathcal{O}_X(-1) \hookrightarrow \mathcal{O}_X \oplus \mathcal{O}_X$. We can now work in the chart U of Section 5.1 and parametrize a small deformation, say ∇'_t, on the trivial bundle whose parabolics become generic (not lying anymore on a same $\mathcal{O}_X(-1)$) for $t \neq 0$ and $\nabla'_0 = \nabla'$. After coming back with the same 5 elementary transformations (but opposite signs), we get a deformation ∇_t of connections on the main bundle $E_t = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$ for $t \neq 0$ that tends to the initial connection $\nabla_0 = \nabla$ on the special bundle $E_0 = \mathcal{O}_X(1) \oplus \mathcal{O}_X(-2)$. We thus get a curve in our moduli space $\Sigma \subset \mathbb{P}^2_{\mathbf{a}} \times \mathbb{P}^2_{\mathbf{b}}$ that tends to Σ when $t \to 0$. After Maple computations, we get the following.

First of all, the corresponding point in $\mathbb{P}^2_{\mathbf{a}} \times \mathbb{P}^2_{\mathbf{b}}$ tends to the "conic" Γ, the limit point depending on the first variation of the parabolics l'_i at $t=0$: if we normalize so that the parabolic structure $\boldsymbol{l}'$ of ∇' is

$$(1:0), \quad (1:1), \quad (1:u'_1), \quad (1:u'_2) \quad \text{and} \quad (0:1)$$

(like notations of Section 5.1) then the limit point on Γ depends on the slope $\lambda = \frac{u'_2-t_2}{u'_1-t_1}$ when $(u'_1,u'_2) \to (t_1,t_2)$. Precisely, the limit point is

$$\big(\ (1:-2q:q^2)\ ,\ (q^2:q:1)\ \big) \in \Gamma \quad \text{where} \quad q = \frac{t_1t_2((t_1-1)\lambda-(t_2-1))}{t_1(t_1-1)\lambda-t_2(t_2-1)}.$$

We fix this point from now on with genericity condition $q \neq 0, 1, t_1, t_2, \infty$.

At the neighborhood of q, the curve Γ is given as complete intersection of

$$a_1^2 = 4a_0a_2, \quad a_0b_0 = a_2b_2 \quad \text{and} \quad 2a_2b_2 + a_1b_1 = 0.$$

Denote by F the exceptional divisor obtained after blowing-up the curve Γ. One can reduce our discussion to the hyperplane $2qa_2 + a_1 = 0$ which is transversal to

Γ. Affine coordinates on F are given by restricting the two rational functions

$$U = \frac{a_2}{b_2}\,\frac{b_2 - qb_1}{q^2a_2 - a_0} \quad \text{and} \quad V = \frac{a_2}{b_2}\,\frac{b_2 - q^2b_0}{q^2a_2 - a_0}.$$

Here, the strict transform of Σ is given by $q^2(V-2U)+1=0$. The limit of these two rational functions along a deformation (E_t,∇_t) like above is

$$U \to \frac{1}{2q}\left(-\frac{2\rho+\kappa_0+5}{q} + \frac{\kappa_1-1}{q-1} + \frac{\kappa_{t_1}-1}{q-t_1} + \frac{\kappa_{t_2}-1}{q-t_2}\right)$$

and

$$V \to \frac{1}{q}\left(-\frac{\rho+\kappa_0+2}{q} + \frac{\kappa_1-1}{q-1} + \frac{\kappa_{t_1}-1}{q-t_1} + \frac{\kappa_{t_2}-1}{q-t_2}\right)$$

In particular, we can check that $q^2(V-2U)+1 \to \rho+4 \neq 0$ for generic parameters κ_i. This defines a curve $\Gamma' \subset F$ parametrized by $(1:-2q)=(a_2:a_1)$ on F (or a single point since q is fixed) that we have to blow-up once again; let us call F' the new exceptional divisor and still denote by F the strict transform by abuse of notation. We then check by a direct computation that $F' \setminus (F\cap F')$ is the locus of those parabolic connections the bundle $E = \mathcal{O}_X(1)\oplus\mathcal{O}_X(-2)$.

If we switch to Darboux coordinates, we can check that, along the above limit process, we get

$$q_1, q_2 \to q \quad \text{and} \quad p_1, p_2 \to \infty$$

with the constraints

$$p_1 + p_2 \to \frac{\rho}{\rho+4}\left(\frac{\kappa_0-1}{q} + \frac{\kappa_1-1}{q-1} + \frac{\kappa_{t_1}-1}{q-t_1} + \frac{\kappa_{t_2}-1}{q-t_2}\right)$$

and

$$\frac{p_1q_1+p_2q_2}{q} \to \frac{\rho}{\rho+4}\left(\frac{\kappa_0-3}{q} + \frac{\kappa_1-1}{q-1} + \frac{\kappa_{t_1}-1}{q-t_1} + \frac{\kappa_{t_2}-1}{q-t_2}\right).$$

References

[1] D. Arinkin. Orthogonality of natural sheaves on moduli stacks of SL(2)-bundles with connections on $\mathbb{P}^1$ minus 4 points. *Selecta Math. (N.S.)*, 7(2):213–239, 2001.

[2] D. Arinkin and R. Fedorov. An example of the Langlands correspondence for irregular rank two connections on $\mathbb{P}^1$. *Adv. Math.*, 230(3):1078–1123, 2012.

[3] D. Arinkin and S. Lysenko. Isomorphisms between moduli spaces of SL(2)-bundles with connections on $\mathbb{P}^1 \setminus \{x_1,\cdots,x_4\}$. *Math. Res. Lett.*, 4(2-3):181–190, 1997.

[4] D. Arinkin and S. Lysenko. On the moduli of SL(2)-bundles with connections on $\mathbb{P}^1 \setminus \{x_1,\cdots,x_4\}$. *Internat. Math. Res. Notices*, 19:983–999, 1997.

[5] U. N. Bhosle. Parabolic vector bundles on curves. *Ark. Mat.*, 27(1):15–22, 1989.

[6] I. Biswas, Y. I. Holla, and C. Kumar. On moduli spaces of parabolic vector bundles of rank 2 over $\mathbb{CP}^1$. *Michigan Math. J.*, 59(2):467–475, 2010.

[7] H. U. Boden and Y. Hu. Variations of moduli of parabolic bundles. *Math. Ann.*, 301(3):539–559, 1995.

[8] B. Dubrovin and M. Mazzocco. Canonical structure and symmetries of the Schlesinger equations. *Comm. Math. Phys.*, 271(2):289–373, 2007.

[9] M. Inaba. Moduli of parabolic connections on a curve and Riemann-Hilbert correspondence. *to appear in J. Algebraic Geom.,* (`http://arXiv:math/0602004v2`), 2006.

[10] M. Inaba, K. Iwasaki, and M.-H. Saito. Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. I. *Publ. Res. Inst. Math. Sci.*, 42(4):987–1089, 2006.

[11] M. Inaba, K. Iwasaki, and M.-H. Saito. Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. II. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 387–432. Math. Soc. Japan, Tokyo, 2006.

[12] M. Jimbo and T. Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. *Phys. D*, 2(3):407–448, 1981.

[13] F. Loray, M.-H. Saito, and C. Simpson. Foliations on the moduli space of rank two connections on the projective line minus four points. In *Geometric and differential Galois theories*, volume 27 of *Sémin. Congr.*, pages 115–168. Soc. Math. France, Paris, 2012.

[14] M. Maruyama and K. Yokogawa. Moduli of parabolic stable sheaves. *Math. Ann.*, 293(1):77–99, 1992.

[15] V. B. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Math. Ann.*, 248(3):205–239, 1980.

[16] N. Nitsure. Moduli of semistable logarithmic connections. *J. Amer. Math. Soc.*, 6(3):597–609, 1993.

[17] S. Oblezin. Isomonodromic deformations of $\mathfrak{sl}(2)$ Fuchsian systems on the Riemann sphere. *Mosc. Math. J.*, 5(2):415–441, 494–495, 2005.

[18] K. Okamoto. Isomonodromic deformation and Painlevé equations, and the Garnier system. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 33(3):575–618, 1986.

[19] M.-H. Saito and S. Szabó. Apparent singularities of logarithmic connections and canonical coordinates for their moduli spaces. *In preparation.*

[20] M.-H. Saito, T. Takebe, and H. Terajima. Deformation of Okamoto-Painlevé pairs and Painlevé equations. *J. Algebraic Geom.*, 11(2):311–362, 2002.

[21] C. S. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Bull. Amer. Math. Soc.*, 83(1):124–126, 1977.

[22] M. Suzuki. Spaces of initial conditions of Garnier system and its degenerate systems in two variables. *J. Math. Soc. Japan*, 58(4):1079–1117, 2006.

[23] K. Yokogawa. Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves. *J. Math. Kyoto Univ.*, 33(2):451–504, 1993.

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