Poissonian ensembles of loops of one-dimensional diffusions

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Abstract. We study the analogue of Poissonian ensembles of Markov loops ("loop soups") in the setting of one-dimensional diffusions. First we give a detailed description of the corresponding intensity measure. Then we identify the law of the occupation field of the Poissonian ensembles of loops. Finally we explain how to sample this Poissonian ensembles using two-dimensional Markov processes. We also state and prove a Vervaat-like relation between bridges conditioned by the value of their minimum and excursion that holds for all the diffusion we consider and not just for the Brownian motion.

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Introduction

Lawler and Werner introduced in [14] the notion of Poissonian ensemble of Markov loops ("loop soup") for two-dimensional conformal diffusions. In [21] it was used by Sheffield and Werner to construct the Conformal Loops Ensemble (CLE). Le Jan studied in [10] the analogue of the Poissonian ensembles of Markov loops in the setting of a symmetric Markov jump process on a finite graph. In both cases one defines an infinite measure μ^* on time-parametrizes unrooted loops (i.e. loops parametrized by a circle where it is not specified when the cut between the beginning and the end occurs) and considers the Poisson point ensemble of intensity $\alpha\mu^*$, $\alpha > 0$, denoted here $\mathcal{L}_\alpha$. In both cases the ensemble $\mathcal{L}_1$ (where $\alpha = 1$) is related to the loops erased during the loop-erasure procedure applied to Markovian sample path. In [10] Le Jan also studied the occupation field of $\mathcal{L}_\alpha$, that is the sum of the occupation times in a given vertex of the graph of individual loops. In case $\alpha = \frac{1}{2}$ he found that it the square of a Gaussian Free Field and related it to the Dynkin's Isomorphism ([6]).

The analogue of the measure μ^* can be defined for a much larger class of Markovian processes. The aim of this paper is to study the measure μ^* and the Poissonian ensembles of Markov loops in case of one-dimensional, not necessarily conservative, diffusion processes. The diffusion processes we consider evolve on a subinterval I of $\mathbb{R}$, are always killed et hitting a boundary point of I, and may be killed by a killing measure on the interior of I. One can transform a diffusion process into an other applying a change of scale, a random change of time, a restriction to a subinterval, an increase of the killing measure or an h-transform. The measure μ^* is covariant with all this transformations on Markovian processes. In other words the map diffusion to measure on loops is a covariant functor. Moreover we will show that μ^* is invariant by h-transform on underlying diffusions. We will also enlarge the scope of our study by associating a measure on loops to "generators" which contain a creation of mass term: If $L = L^{(0)} + \nu$ where $L^{(0)}$ is a second order differential operator on I and ν is a signed measure, and if one sets zero Dirichlet boundary conditions for L, one can define in a consistent way a measure on loops related to L even in case the semi-group $(e^{tL})_{t\geq 0}$ does not make sense. This extended definition of μ^* will be particularly handy for computing the exponential moments of the Poissonian ensemble of Markov loops.

The layout of this paper is the following: In section 1 we will recall some facts on one-dimensional diffusions and set important notation. We will further consider "generators" with creation of mass term and characterize a class of such operators which up to an h-transform are equivalent to the generators of diffusions. In section 2 we will define the measure μ^* and point out different covariance and invariance properties. Further we will make a connection between the Brownian measure on loops and the Levy-Itô measure on Brownian excursion using the Vervaat's bridge-to-excursion transformation. This in turn will lead us to a conditioned version of Vervaat's transformation that holds for every one-dimensional diffusions, that is an absolute continuity relation between the bridge conditioned to have a given minimum and an excursion of the same duration above this minimum. The Vervaat's transformation is deeply related to the measure on loops μ^*: The loops are unrooted, so one can freely chose a moment to separate the end from the start. If one chooses this moment uniformly over the life-time of

the loop, then the loop under the measure μ^* looks in some sense like a bridge. If one chooses this moment when the loop hits its minimum, then it looks like an excursion. In section 3 we will study the occupation field of the Poissonian ensemble of Markov loops. Each loops is endowed with a family of local times. The occupation field is the sum of local times over the loops. We will identify its law as an non-homogeneous continuous state branching process with immigration parametrised by the position points in I. In case $\alpha = \frac{1}{2}$ we will identify it as the square of a Gaussian Free Field and show how it is possible to derive particular versions of the Dynkin's Isomorphism using this fact and Palm's identity for Poissonian ensembles. In section 4 we will explain how to sample the Poissonian ensembles of Markov loops using sample paths of two-dimensional Markov processes. For the particular case $\alpha = 1$, one can sample $\mathcal{L}_1$ by slicing the sample path of an one-dimensional diffusion. This is the analogue in the setting of one-dimensional diffusions of the relation between $\mathcal{L}_1$ and the loop-erasure procedure observed in the setting of the two-dimensional Brownian motion or of the symmetric Markov jump processes on graphs. Results of sections 2 and 3 lead to an interpretation, in terms of Poissonian ensembles of Markov loops, of the Ray-Knight theorem on the law of the local times of a Brownian motion stopped at a first hitting time.

This paper contains definitions, propositions, lemmas, corollaries and properties. Properties are non-proved statements that are either obvious or already known. Propositions, lemmas, corollaries and properties have common counters which are restarted at the beginning of each of four sections. Definitions are numbered separately.

1. Preliminaries on generators and semi-groups

1.1. **A second order differential equation.** Let I be an open interval of $\mathbb{R}$ and ν a signed measure on I. By signed measure we mean that the total variation $|\nu|$ is a positive Radon measure, but not necessarily finite, and $\nu(dx) = \epsilon(x)|\nu|(dx)$ where ϵ takes values in $\{\pm 1\}$. We look for the solutions of the linear second order differential equation on I:

$$d\frac{du}{dx} + ud\nu = 0 \tag{1.1}$$

In case ν is a negative non-zero measure, the equation (1.1) commonly appears when studying the Brownian motion with a Killing measure. In this case the two-dimensional linear space of solutions is spanned by two positive solutions $u_\uparrow$ and $u_\downarrow$, $u_\uparrow$ being non-decreasing and $u_\downarrow$ non-increasing. Given $x_0 \in I$, we can construct $u_\uparrow$ as the limit when $x_1 \to \inf I$ of the unique solution which equals 0 in x_1 and 1 in x_0. For $u_\downarrow$ we take the limit as $x_1 \to \sup I$. See [3], section 16.11, or [19], Appendix 8, for more details. Here we are mostly interested in the less common case of a signed measure $|\nu|$. For a solution u to (1.1) we will write $\frac{du}{dx}(x^+)$ and $\frac{du}{dx}(x^-)$ for the right-hand side respectively left-hand side derivative of u at x. The two are related by $\frac{du}{dx}(x^+) - \frac{du}{dx}(x^-) = -u(x)\nu(\{x\})$. Next we give a Cauchy-Lipschitz principle for (1.1):

Proposition 1.1. *If $x_0 \in I$ and $u_0, v_0 \in \mathbb{R}$, there is a unique solution u to (1.1), continuous on I, satisfying $u(x_0) = u_0$ and $\frac{du}{dx}(x_0^+) = v_0$.*

Proof. Let $x_1 \in I$, $x_1 > x_0$. Let $\mathcal{J}$ be the map from $\mathbb{L}^\infty([x_0, x_1])$ to itself defined by

$$(\mathcal{J}V)(x) = v_0 - u_0\nu((x_0, x]) - \int_{x_0}^{x} \nu((y, x])V(y)dy$$

A continuous function u on $[x_0, x_1]$ is solution to (1.1) on $[x_0, x_1]$ with given initial values at x_0 if and only if $u(x_0) = u_0$, $\frac{du}{dx} \in \mathbb{L}^\infty([x_0, x_1])$ and $\frac{du}{dx}$ is a fixed point of $\mathcal{J}$. Conversely if V is a fixed point of $\mathcal{J}$, then $u(x) = u_0 + \int_{x_0}^{x} V(y)dy$ is solution to (1.1).

For $V_1, V_2 \in \mathbb{L}^\infty([x_0, x_1])$, $\mathcal{J}^n(V_2) - \mathcal{J}^n(V_1)$ equals

$$(-1)^n \int_{x_0<y_1<\ldots<y_n<x} \nu((y_1, x])...\nu((y_n, x])(V_2(y_n) - V_1(y_n))\, dy_1...dy_n$$

The Lipschitz norm of $\mathcal{J}^n$ is smaller or equal to $\frac{|\nu|([x_0, x_1])^n (x_1 - x_0)^n}{n!}$. So for n large enough $\mathcal{J}^n$ is contracting and thus $\mathcal{J}$ has a unique fixed point in $\mathbb{L}^\infty([x_0, x_1])$. This implies existence and uniqueness of a solution to (1.1) on $[x_0, x_1]$. The same is true if we take $x_1 < x_0$. By gluing together solutions on different compact subintervals we get a solution on I. □

Let $W(u_1, u_2)(x)$ be the wronskian of two functions u_1, u_2:

$$W(u_1, u_2)(x) := u_1(x)\frac{du_2}{dx}(x^+) - u_2(x)\frac{du_1}{dx}(x^+)$$

If u_1, u_2 are both solutions to (1.1), $W(u_1, u_2)$ is constant on I. Using this fact we get a results which is similar to Sturm's separation theorem for the case of a measure ν with a continuous density with respect to the Lebesgue measure (see theorem 7, section 2.6 in [2]):

Property 1.2. *Let $x_0 < x_1$ be two points in I.*

- *(i) Let u_1 be a solution to (1.1) satisfying $u_1(x_0) = 0$, $\frac{du_1}{dx}(x_0^+) > 0$, and u_2 a solution such that $u_2(x_0) > 0$. Assume that $u_2 \geq 0$ on $[x_0, x_1]$. Then $u_1 > 0$ on $(x_0, x_1]$.*
- *(ii) Let u_1, u_2 be two solutions such that $u_1(x_0) = u_2(x_0) > 0$ and $\frac{du_1}{dx}(x_0^+) > \frac{du_2}{dx}(x_0^+)$. Assume that $u_2 \geq 0$ on $[x_0, x_1]$. Then $u_1 > u_2$ on $(x_0, x_1]$.*
- *(iii) If there is a solution u to (1.1) positive on (x_0, x_1) and zero at x_0 and x_1 then any other linearly independent solutions to (1.1) has exactly one zero in (x_0, x_1).*

Next we prove a lemma that will be useful in the section 1.3.

Lemma 1.3. *Let ν_+ be the positive part of ν. Let $x_0 < x_1 \in I$. Let f be a continuous positive function on $[x_0, x_1]$ such that $\min_{[x_0,x_1]} f > \nu_+([x_0, x_1])^2$. Then the equation*

$$d\frac{du}{dx} + udv - uf\, dx = 0 \tag{1.2}$$

has a positive solution that is non-decreasing on $[x_0, x_1]$.

Proof. Set $\lambda := \min_{[x_0,x_1]} f$. Let u be the solution to (1.2) with the initial values $u(x_0) = 1$, $\frac{du}{dx}(x_0^+) = \sqrt{\lambda}$. We will show that u is non-decreasing on $[x_0, x_1]$. Assume that this is not the case. This means that $\frac{du}{dx}(x^+)$ takes negative values somewhere in $[x_0, x_1]$. Let

$$x_2 := \inf\{x \in [x_0, x_1] | \frac{du}{dx}(x^+) \leq 0\}$$

Since $\frac{du}{dx}(x^+)$ is right-continuous, $\frac{du}{dx}(x_2^+) \leq 0$. Let $r(x) := \frac{1}{u(x)}\frac{du}{dx}(x^+)$. u is positive on $[x_0, x_2]$ hence r is defined $[x_0, x_2]$. $r(x_0) = \sqrt{\lambda}$. r is cadlag and satisfies the equation

$$dr = (f - r^2)\, dx - d\nu$$

Let $x_3 := \sup\{x \in [x_0, x_2] | r(x) \geq \sqrt{\lambda}\}$. We have

$$r(x_2) = r(x_3^-) + \int_{x_3}^{x_2} (f(x) - r^2(x))dx - \nu([x_3, x_2])$$

By construction $r(x_3^-) \geq \sqrt{\lambda}$. By definition $f - r^2 \geq 0$ on $(x_3, x_2]$.
Thus $r(x_2) \geq \sqrt{\lambda} - \nu([x_3, x_2]) > 0$. It follows that $r(x_2) > 0$, which is absurd. □

1.2. **One-dimensional diffusions.** In this subsection we will describe the king of linear diffusion we are interested in, recall some facts and set notations to be used subsequently. For a detailed presentation of one-dimensional diffusions see [9] and [3], chapter 16.

Let I be an open interval of $\mathbb{R}$, m and w continuous positive functions on I. We consider a diffusion $(X_t)_{0 \leq t < \zeta^{(0)}}$ on I with generator

$$L^{(0)} := \frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{w(x)}\frac{d}{dx}\right)$$

and killed as it hits the boundary of I. In case I is unbounded, we also allow for X to blow up to infinity in finite time. $\zeta^{(0)}$ is the first time X either hits the boundary or blows up. For the sake of simplicity we will assume that $\frac{dw}{dx} \in \mathbb{L}^\infty_{loc}(I)$, the space of functions on I that are bounded on compact subintervals, although this condition is unimportant. Given such a diffusion, the speed measure $m(x)\, dx$ and the scale measure $w(x)\, dx$ are defined up to a positive multiplicative constant, but the product mw is uniquely defined. A primitive S of w is a natural scale function of X. Consider the random time change $d\tilde{t} = \frac{1}{m(X_t)}\, dt$. Then $(\frac{1}{2}S(X_{\tilde{t}}))_{0 \leq \tilde{t} < \tilde{\zeta}^{(0)}}$ is a standard Brownian motion on $S(I)$ killed when it first hits the boundary of $S(I)$. For all f, g smooth, compactly supported in I, $\int_I (L^{(0)}f)(x)g(x)m(x)\, dx = \int_I f(x)(L^{(0)}g)(x)m(x)\, dx$. The diffusion X has a family of local times $(\ell_t^x(X))_{x \in I, t \geq 0}$ with respect to the measure $m(x)\, dx$ such that $(x, t) \mapsto \ell_t^x(X)$ is continuous. We can further consider diffusions with killing measures. Let k be a non-negative Radon measure on I. We kill X as soon as $\int_I \ell_t^x(X)m(x)\, dk(x)$ hits an independent exponential time with parameter 1. The corresponding generator is

$$L = \frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{w(x)}\frac{d}{dx}\right) - k \tag{1.3}$$

Let $(X_t)_{0 \leq t < \zeta}$ be the diffusion of generator (1.3), which is killed either by hitting ∂I, or through blowing up, or by the killing measure k. For $x \in I$ let $\eta_{exc}^{>x}$ and $\eta_{exc}^{<x}$ be the excursion measures of X above and below the level x up to the last time X visits x. The behaviour of X from the first to the last time it visits x is a Poissonian point process with intensity $\eta_{exc}^{>x} + \eta_{exc}^{<x}$, parametrized by the local time at x up to the value $\ell_t^\zeta(X)$. $\eta_{exc}^{>x}$ and $\eta_{exc}^{<x}$ are obtained from the Levy-Itô measure on Brownian excursions through scale change, time change and multiplication by a density function accounting for the killing. See [20] for details on excursion measures in case of recurrent diffusions.

If X is transient the Green function of L,

$$G(x,y) := \mathbb{E}_x[\ell_t^\zeta(X)]$$

is finite, continuous and symmetric. For $x \leq y$ it can be written

$$G(x,y) = u_\uparrow(x)u_\downarrow(y)$$

where $u_\uparrow(x)$ and $u_\downarrow(y)$ are positive , respectively non-decreasing and non-increasing solutions to the equation $Lu = 0$, which through a change of scale reduces to an equation of form (1.1). If S is bounded from below, $u_\uparrow(\inf I^+) = 0$. If S is bounded from above, $u_\downarrow(\sup I^-) = 0$. $u_\uparrow(x)$ and $u_\downarrow(y)$ are each determined up to a multiplication by a positive constant, but when entering the expression of G, the two constants are related. See [9] or [3], chapter 16 for details. Let $W(u_\downarrow, u_\uparrow)$ be the Wronskian of $u_\downarrow$ and $u_\uparrow$:

$$W(u_\downarrow, u_\uparrow)(x) := u_\downarrow(x)\frac{du_\uparrow}{dx}(x^+) - u_\uparrow(x)\frac{du_\downarrow}{dx}(x^+)$$

This Wronskian is actually the density of the scale measure: $W(u_\downarrow, u_\uparrow) \equiv w$.

If the killing measure k is non zero, then the probability that X, starting from x, gets killed by k before reaching a boundary of I or blowing up to infinity equals $\int_I G(x,y)m(y)k(dy)$. Conditionally on this event, the distribution of X_{ζ^-} is:

$$\frac{G(x,z)m(z)k(dz)}{\int_I G(x,y)m(y)k(dy)}$$

The semi-group of L has positive transition densities $p_t(x,y)$ with respect to the speed measure $m(y)\,dy$ and $(t,x,y) \mapsto p_t(x,y)$ is continuous on $(0,+\infty) \times I \times I$. McKean gives a proof of this in [15] in case when the killing measure k has a continuous density with respect to the Lebesgue measure. If this is not the case, we can take u a positive continuous solution to $Lu = 0$ and consider the h-transform of L by u: $u^{-1}Lu$. The latter is the generator of a diffusion without killing measure and by [15] this diffusion has continuous transition densities $\tilde{p}_t(x,y)$ with respect to $m(y)\,dy$. Then $u(x)\tilde{p}_t(x,y)\frac{1}{u(y)}$ are the transition densities of the semi-group of L. Transition densities with respect to the speed measure are symmetric: $p_t(x,y) = p_t(y,x)$. For all $x, y \in I$ and $t \geq 0$ the following equality holds:

$$\mathbb{E}_x[\ell_y^{t\wedge\zeta}(X)] = \int_0^t p_s(x,y)ds \tag{1.4}$$

Newt we deal with bridge probability measures.

Proposition 1.4. *The bridge probability measures $\mathbb{P}^t_{x,y}(\cdot)$ (bridge of X from x to y in time t conditioned neither to die nor to explode in the interval) satisfy: for all $x \in I$ the map $(x,y,t) \mapsto \mathbb{P}^t_{x,y}(\cdot)$ is continuous for the weak topology on probability measures on continuous paths.*

Proof. Our proof mainly relies on absolute continuity arguments of [16] and [4], and the time reversal argument of [16]. [4] gives a proof of weak continuity of bridges for conservative Feller cadlag processes on second countable locally compact spaces. But since the proof contains an error and we do not restrict to conservative diffusions, we give here accurate arguments for the weak continuity.

First we can restrict to the case $k = 0$. Otherwise consider u a solution to $Lu = 0$, positive on I. The generator of the h-transform of L by u is

$$\frac{1}{u(x)^2 m(x)} \frac{d}{dx}\left(\frac{u(x)^2}{w(x)}\frac{d}{dx}\right)$$

and does not contain any killing measure. The h-transform preserves the bridge measures and changes the density functions with respect to $m(x)\,dx$ to $\dfrac{1}{u(x)} p_t(x,y)u(y)$, and thus preserves their continuity.

Then we normalise the length of bridges: if $(X_s^{(x,y,t)})_{0\le s\le t}$ is a path under the law $\mathbb{P}^t_{x,y}(\cdot)$, let $\tilde{\mathbb{P}}^t_{x,y}(\cdot)$ be the law of $(X_{rt}^{(x,y,t)})_{0\le r\le 1}$. It is sufficient to prove that $(x,y,t) \mapsto \tilde{\mathbb{P}}^t_{x,y}(\cdot)$ is continuous. For $v \in [0,1]$, let $\tilde{\mathbb{P}}^{t,v}_{x,y}(\cdot)$ be the law of $(X_{rt}^{(x,y,t)})_{0\le r\le v}$. Let $\tilde{\mathbb{P}}^{t,v}_{x}(\cdot)$ be the law of the Markovian path $(X_{rt})_{0\le r\le v}$ starting from x. For $v \in [0,1)$ we have the following absolute continuity relationship:

$$d\tilde{\mathbb{P}}^{t,v}_{x,y} = 1_{vt<\zeta}\frac{p_{(1-v)t}(X_{vt},y)}{p_t(x,y)}\, d\tilde{\mathbb{P}}^{t,v}_{x} \tag{1.5}$$

Let $(J_n)_{n\ge 0}$ be an increasing sequence of compact subintervals of I such that $I = \bigcup_{n\ge 0} J_n$. Let T_n be the first exit time from J_n. Let f_n be continuous compactly supported function on I such that $0 \le f_n \le 1$ and $f_{n|J_n} \equiv 1$. We can further assume that the sequence $(f_n)_{n\ge 0}$ is non-decreasing. The map

$$(x,y,t) \mapsto f_n(\sup_{[0,vt]} X) f_n(\inf_{[0,vt]} X) d\tilde{\mathbb{P}}^{t,v}_{x}$$

is weakly continuous. Let $(x_j,y_j,t_j)_{j\ge 0}$ be a sequence converging to (x,y,t). Let F be a continuous bounded functional on $\mathcal{C}([0,v])$. Then applying (1.5) we get:

$$\tilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(f_n(\sup_{[0,v]}\gamma) f_n(\inf_{[0,v]}\gamma)F(\gamma)) - \tilde{\mathbb{P}}^{t,v}_{x,y}(f_n(\sup_{[0,v]}\gamma) f_n(\inf_{[0,v]}\gamma)F(\gamma)) = \tag{1.6}$$

$$\tilde{\mathbb{P}}^{t_j,v}_{x_j}\left(\frac{p_{(1-v)t}(\gamma(v),y)}{p_t(x,y)} f_n(\sup_{[0,v]}\gamma) f_n(\inf_{[0,v]}\gamma)F(\gamma)\right) \tag{1.7}$$

$$-\tilde{\mathbb{P}}^{t,v}_{x}\left(\frac{p_{(1-v)t}(\gamma(v),y)}{p_t(x,y)} f_n(\sup_{[0,v]}\gamma) f_n(\inf_{[0,v]}\gamma)F(\gamma)\right) \tag{1.8}$$

$$+\tilde{\mathbb{P}}^{t_j,v}_{x_j}\left(\frac{p_{(1-v)t_j}(\gamma(v),y_j)}{p_{t_j}(x_j,y_j)} f_n(\sup_{[0,v]}\gamma) f_n(\inf_{[0,v]}\gamma)F(\gamma)\right) \tag{1.9}$$

$$-\tilde{\mathbb{P}}^{t_j,v}_{x_j}\left(\frac{p_{(1-v)t}(\gamma(v),y)}{p_t(x,y)} f_n(\sup_{[0,v]}\gamma) f_n(\inf_{[0,v]}\gamma)F(\gamma)\right) \tag{1.10}$$

Since $\dfrac{p_{(1-v)t}(\cdot,y)}{p_t(x,y)}$ is continuous and bounded on J_n, (1.7)$-$(1.8) converges to 0. Moreover for j large enough, $\dfrac{p_{(1-v)t_j}(\cdot,y_j)}{p_{t_j}(x_j,y_j)}$ is uniformly close on J_n to $\dfrac{p_{(1-v)t}(\cdot,y)}{p_t(x,y)}$. Thus (1.9)$-$(1.10)

converges to 0 and finally (1.6) converges to 0. Let $n_0 \in \mathbb{N}$ and $n \geq n_0$. Then

$$\tilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(1 - f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma)) = 1 - \tilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma)) \leq$$

$$1 - \tilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(f_{n_0}(\sup_{[0,v]}\gamma)f_{n_0}(\inf_{[0,v]}\gamma)) \to 1 - \tilde{\mathbb{P}}^{t,v}_{x,y}(f_{n_0}(\sup_{[0,v]}\gamma)f_{n_0}(\inf_{[0,v]}\gamma)) \tag{1.11}$$

From (1.11) we deduce that

$$\lim_{n\to+\infty} \limsup_{j\to+\infty} \tilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(1 - f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma)) = 0$$

It follows that

$$\lim_{j\to+\infty} \tilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(F(\gamma)) = \tilde{\mathbb{P}}^{t,v}_{x,y}(F(\gamma))$$

From this we get that the law of any finite-dimensional family of marginals of $\tilde{\mathbb{P}}^{t}_{x,y}(\cdot)$ depends continuously on (x, y, t). To conclude we need a tightness result for $(x, y, t) \mapsto \tilde{\mathbb{P}}^{t}_{x,y}(\cdot)$. We have already tightness for $(x, y, t) \mapsto \tilde{\mathbb{P}}^{t,v}_{x,y}(\cdot)$. The image of $\tilde{\mathbb{P}}^{t}_{x,y}(\cdot)$ through time reversal is $\tilde{\mathbb{P}}^{t}_{y,x}(\cdot)$. So we also have tightness on intervals $[1 - v', 1]$ where $v' \in (0, 1)$. But if $v + v' > 1$, tightness on $[0, v]$ and on $[1 - v', 1]$ implies tightness on $[0, 1]$. This concludes. The article [4] contains an error in the proof of the tightness of bridge measures in the neighbourhood of the ending point. □

1.3. **"Generators" with creation of mass.** In this section we consider more general operators

$$L = \frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{w(x)}\frac{d}{dx}\right) + \nu \tag{1.12}$$

with zero Dirichlet boundary conditions on ∂I, where ν is a signed measure on I which is no longer assumed to be negative. We set

$$L^{(0)} := L - \nu$$

In the sequel we may call L "generator" even in case the semi-group $(e^{tL})_{t\geq 0}$ does not make sense. Our main goal in this subsection is to characterize through a positivity condition the subclass of operators of form (1.12) that are equivalent up to an h-transform to the generator of a diffusion of form (1.3).

We will consider several kinds of transformations on operators of the form (1.12). First, the h-transform: Let h be a positive continuous function on I such that $\frac{d^2h}{dx^2}$ is a signed measure. We call $Conj(h, L)$ the operator

$$Conj(h, L) = \frac{1}{h(x)^2 m(x)}\frac{d}{dx}\left(\frac{h(x)^2}{w(x)}\frac{d}{dx}\right) + \nu + \frac{1}{h}L^{(0)}h$$

If f is smooth function compactly supported in I then

$$Conj(h, L)f = h^{-1}L(hf)$$

We will call $Conj(h, L)$ the h-transform of L by h even though h may not be harmonic ($Lh = 0$) or sub-harmonic ($Lh \leq 0$) and L is not necessarily the generator of a diffusion.

Second, the change of scale: If A is a $\mathcal{C}^1$ function on I such that $\frac{dA}{dx} > 0$ and $\frac{d^2A}{dx^2} \in \mathbb{L}^\infty_{loc}(I)$ and $(\gamma(t))_{0\leq t\leq T}$ a continuous path in I, then we will set $Scale_A(\gamma)$ to be the continuous path $(A(\gamma(s)))_{0\leq t\leq T}$ in $A(I)$. Let $Scale^\dagger_A(L)$ be the operator on functions on $A(I)$ with zero Dirichlet boundary conditions induced by this change of scale:

$$Scale^\dagger_A(L) = \frac{1}{m\circ A^{-1}(a)}\frac{d}{da}\left(\frac{1}{w\circ A^{-1}(a)}\frac{d}{da}\right) + A_*\nu$$

where $A_*\nu$ is the push-forward of the measure ν by A.

Third, the change of time: If V is positive continuous on I then we can consider the change of time $ds = V(\gamma(t))\,dt$. Let $Speed_V$ be the corresponding transformation on paths. The corresponding "generator" is $\frac{1}{V}L$.

Finally, the restriction: if $\tilde{I}$ is an open subinterval of I then set $L_{|I}$ be the operator L acting on functions supported in $\tilde{I}$ and with zero Dirichlet conditions on $\partial\tilde{I}$.

For the analysis of L we will use a bit of spectral theory: If $[x_0, x_1]$ is a compact interval of $\mathbb{R}$ and $\tilde{m}$, $\tilde{w}$ are positive continuous functions on $[x_0, x_1]$, then the operator $\frac{1}{\tilde{m}(x)}\frac{d}{dx}\left(\frac{1}{\tilde{w}(x)}\frac{d}{dx}\right)$ with zero Dirichlet boundary conditions has a discrete spectrum of negative eigenvalues. Let $-\tilde{\lambda}_1$ be the first eigenvalue. It is simple. According to Sturm-Liouville theory (see for instance [23], section 5.5) we have the following picture:

Property 1.5. *Let $\lambda > 0$ and u a solution to*

$$\frac{1}{\tilde{m}}\frac{d}{dx}\left(\frac{1}{\tilde{w}}\frac{d}{dx}\right) + \lambda u = 0$$

with initial conditions $u(x_0) = 0, \frac{du}{dx}(x_0) > 0$.

- *(i) If u is positive on (x_0, x_1) and $u(x_1) = 0$ then $\lambda = \tilde{\lambda}_1$ and u is the fundamental eigenfunction.*
- *(ii) If u is positive on $(x_0, x_1]$ then $\lambda < \tilde{\lambda}_1$*
- *(iii) If u changes sign on (x_0, x_1) then $\lambda > \tilde{\lambda}_1$*

Next we state and prove the main result of this section.

Proposition 1.6. *The following two conditions are equivalent:*

- *(i) There is a positive continuous function u on I satisfying $Lu = 0$.*
- *(ii) For any f smooth compactly supported in I*

$$\int_I (L^{(0)}f)(x)f(x)m(x)\,dx + \int_I f(x)^2 m(x)\nu(dx) \leq 0 \tag{1.13}$$

Proof. (i) implies (ii): First observe that the equation $Lu = 0$ reduces through a change of scale to an equation of the form (1.1). Let u be given by condition (i). Let $\tilde{L} := Conj(u, L)$. Since $Lu = 0$, $\tilde{L}$ is a generator of a diffusion without killing measure. Let $\tilde{m}(x) := u^2(x)m(x)$. Then for all g smooth compactly supported in I, $\int_I (\tilde{L}g)g\,\tilde{m}\,dx \leq 0$. But

$$\int_I (\tilde{L}g)g\,\tilde{m}\,dx = \int_I (L^{(0)}(ug))(ug)\,m\,dx + \int_I (ug)^2\,m\nu(dx)$$

Thus (1.13) holds for all f positive compactly supported in I such that $u^{-1}f$ is smooth. By density arguments, this holds for general smooth f.

(ii) implies (i): First we will show that for every compact subinterval J of I there is a positive continuous function u_J on $\mathring{J}$ satisfying $Lu_J = 0$ on $\mathring{J}$. Let J be such an interval. By lemma 1.3 there is $\lambda > 0$ and u_λ positive continuous on J satisfying $Lu_\lambda - \lambda u_\lambda = 0$ on J. Let $L_\lambda := Conj(u_\lambda, L_{|\mathring{J}})$. Then

$$L_\lambda = \frac{1}{u^2 m}\frac{d}{dx}\left(\frac{u^2}{w}\frac{d}{dx}\right) + \lambda$$

Let $L_\lambda^{(0)} := L_\lambda - \lambda$. $L^{(0)}$ is the generator of a diffusion on $\mathring{J}$. We can apply the standard spectral theorem to $L_\lambda^{(0)}$. Let $-\lambda_1$ be its fundamental eigenvalue. $L_\lambda^{(0)} + \lambda = L_\lambda$ is a non-positive operator because it is an h-transform of $L_{|J}$ which satisfies condition (ii). This implies that $\lambda \leq \lambda_1$. Let $\tilde{u}$ be a solution to $L_\lambda^{(0)}\tilde{u} + \lambda\tilde{u} = 0$ with initial conditions $\tilde{u}(\min J) = 0$ and $\frac{d\tilde{u}}{dx}(\min J) > 0$. Since $\lambda \leq \lambda_1$, according to property 1.5, $\tilde{u}$ is positive on $\mathring{J}$. We set $u_J := u_\lambda \tilde{u}$. Then u_J is positive continuous on $\mathring{J}$ and satisfies $Lu_J = 0$. This finishes the proof of the first step.

Now consider a fixed point x_0 in I and $(J_n)_{n\geq 0}$ an increasing sequence of compact subintervals of I such that $x_0 \in \mathring{J}_0$ and $\bigcup_{n\geq 0} J_n = I$. Let u_{J_n} be a positive L-harmonic function on $\mathring{J}_n$. We may assume that $u_{J_n}(x_0) = 1$. The sequence $\left(\frac{du_{J_n}}{dx}(x_0^+)\right)_{n\geq 0}$ is bounded from below. Otherwise some of the u_{J_n} would change sign on $I \cap (x_0, +\infty)$. Similarly, since none of the u_{J_n} changes sign on $I \cap (-\infty, x_0)$, $\frac{du_{J_n}}{dx}(x_0^+)$ is bounded from above. Let v be an accumulation value of the sequence $\left(\frac{du_{J_n}}{dx}(x_0^+)\right)_{n\geq 0}$. Then the L-harmonic function satisfying the initial conditions $u(x_0) = 1$ and $\frac{du}{dx}(x_0^+) = v$ is positive on I. □

We will divide the operators of the form (1.12) in two sets: $\mathfrak{D}^{0,-}$ for those that satisfies either of the constraints of the proposition 1.6 and $\mathfrak{D}^+$ for those that don't. $\mathfrak{D}^{0,-}$ is made exactly of operators that are equivalent up to an h-transform to the generator of a diffusion. We will subdivide the set $\mathfrak{D}^{0,-}$ in two: $\mathfrak{D}^-$ for the operators that are an h-transform of the generator of a transient diffusion and $\mathfrak{D}^0$ for those that are an h-transform of the generator of a recurrent diffusion. These two subclasses are well defined since a transient diffusion can not be an h-transform of a recurrent one. Observe that each of $L \in \mathfrak{D}^-$, $\mathfrak{D}^0$ and $\mathfrak{D}^+$ is stable under h-transforms, changes of scale and of speed. Operators in $\mathfrak{D}^-$ and $\mathfrak{D}^0$ do not need to be generators of transient or recurrent diffusions themselves. For instance consider on $\mathbb{R}$

$$L = \frac{1}{2}\frac{d^2}{dx^2} + a_+\delta_1 - a_-\delta_{-1}$$

where $a_+, a_- > 0$. If $3a_+ - a_- > 0$ then $L \in \mathfrak{D}^+$, if $3a_+ - a_- = 0$ then $L \in \mathfrak{D}^0$, if $3a_+ - a_- < 0$ then $L \in \mathfrak{D}^-$.

If $L \in \mathfrak{D}^{0,-}$, the semi-group $(e^{tL})_{t\geq 0}$ is well defined. Indeed, let X be the diffusion on I of generator $L^{(0)}$ and ζ the first time it hits the boundary of I or blows up to infinity. Let u be a positive L-harmonic function and $\tilde{L} := Conj(u, L)$. $\tilde{L}$ is the generator of a diffusion

$\tilde{X}$ on I without killing measure. Let $\tilde{\zeta}$ be the first time $\tilde{X}$ hits the boundary of I or blows up to infinity. Then for any f positive bounded compactly supported in I, $x \in I$ and $t > 0$, $\mathbb{E}_x\left[1_{t<\zeta}\exp(\int_I l_t^y m(y)\nu(dy))f(X_t)\right] < +\infty$ and we have the equality:

$$\mathbb{E}_x\left[1_{t<\zeta}\exp(\int_I \ell_t^y(X)m(y)\nu(dy))\,f(X_t)\right] = \frac{1}{u(x)}\mathbb{E}_x\left[1_{t<\tilde{\zeta}}u(\tilde{X}_t)f(\tilde{X}_t)\right] \tag{1.14}$$

Identity (1.14) can be proved using Girsanov's theorem. If $L \in \mathfrak{D}^-$, let $(\tilde{G}(x,y))_{x,y\in I}$ be the Green function of $\tilde{L}$ relatively to the measure $u(x)^2 m(x)\,dx$. Then L has a Green function $(G(x,y))_{x,y\in I}$ that equals

$$G(x,y) = \mathbb{E}_x\left[\exp(\int_I \ell^z_{t\wedge\zeta}(X)m(z)\nu(dz))\,\ell^y_{t\wedge\zeta}(X)\right] = u(x)u(y)\tilde{G}(x,y)$$

For $x \le y \in I$, $\tilde{G}(x,y) = \tilde{u}_\uparrow(x)\tilde{u}_\downarrow(y)$ where $\tilde{u}_\uparrow$ and $\tilde{u}_\downarrow$ are $\tilde{L}$-harmonic. Then we set $u_\uparrow := u\tilde{u}_\uparrow$ and $u_\downarrow := u\tilde{u}_\downarrow$. $u_\uparrow$ and $u_\downarrow$ are L-harmonic and for $x \le y \in I$, $G(x,y) = u_\uparrow(x)u_\downarrow(y)$. But contrary to $\tilde{u}_\uparrow$ respectively $\tilde{u}_\downarrow$, $u_\uparrow$ respectively $u_\downarrow$ is not necessarily non-decreasing respectively non-increasing.

The discrete analogue of the sets $\mathfrak{D}^-$, $\mathfrak{D}^0$ and $\mathfrak{D}^+$ are symmetric matrices with non-negative off-diagonal coefficients inducing a connected transition graph, with the highest eigenvalue that is respectively negative, zero and positive. However in continuous case the sets $L \in \mathfrak{D}^-$, $\mathfrak{D}^0$ and $\mathfrak{D}^+$ can not be defined spectrally because for operators from $L \in \mathfrak{D}^-$ and $\mathfrak{D}^+$ the maximum of the spectrum can also equal zero. However the next result shows that the sets $\mathfrak{D}^-$ and $\mathfrak{D}^+$ are stable under small perturbations of the measure ν and that $\mathfrak{D}^0$ is not.

Proposition 1.7.
- *(i) If $L \in \mathfrak{D}^0$ and k is a non-zero positive Radon measure on I then $L - k \in \mathfrak{D}^-$ and $L + k \in \mathfrak{D}^+$.*
- *(ii) If $L \in \mathfrak{D}^-$ and J is a compact subinterval of I then there is $K > 0$ such that for any positive measure k supported in J satisfying $k(J) < K$ we have $L + k \in \mathfrak{D}^-$.*
- *(iii) If $L \in \mathfrak{D}^+$ then there is $K > 0$ such that for any positive finite measure k satisfying $k(I) < K$ we have $L - k \in \mathfrak{D}^+$.*
- *(iv) If $L \in \mathfrak{D}^+$, there is a positive Radon measure k on I such that $L - k \in \mathfrak{D}^0$.*
- *(v) Let $L \in \mathfrak{D}^+$ and $x_0 < x_1 \in I$. Then $L_{|(x_0,x_1)} \in \mathfrak{D}^0$ if and only if there is an L-harmonic function u positive on (x_0,x_1) and zero in x_0 and x_1.*

Proof. (i): Consider h positive continuous on I such that $Conj(h,L)$ is the generator of a recurrent diffusion. Since $Conj(h,L-k) = Conj(h,L) - k$, $Conj(h,L-k)$ is the generator of a diffusion killed at rate k and thus $L - k \in \mathfrak{D}^-$. Similarly we can not have $L + k \in \mathfrak{D}^{0,-}$ because this would mean $L = (L+k) - k \in \mathfrak{D}^-$.

(ii): Without loss of generality we may assume that L is the generator of a transient diffusion and that it is at natural scale, that is $L = \dfrac{1}{m(x)}\dfrac{d^2}{dx^2}$. Since the diffusion is transient, $I \neq \mathbb{R}$. We may assume that $x_0 := \inf I > -\infty$. Write $J = [x_1, x_2]$. Let k be a positive measure supported in $[x_1,x_2]$. Let u be the solution to $Lu + u\,dk = 0$ with the initial conditions $u(x_0) = 0, \dfrac{du}{dx}(x_0^+) = 1$. u is affine on $[x_0,x_1]$ and on $[x_2, \sup I)$. On $[x_1,x_2]$ u is bounded

from above by $x_2 - x_0$. Thus, if $k([x_1, x_2]) \leq \dfrac{\min_{[x_1,x_2]} m}{(x_2 - x_0)}$ then u is non-decreasing on I and hence positive. This implies that $L + k \in \mathfrak{D}^{0,-}$. By the point (i) of current proposition, if $k([x_1, x_2]) < \dfrac{\min_{[x_1,x_2]} m}{(x_2 - x_0)}$ then $L + k \in \mathfrak{D}^{-}$.

(iii): By definition there is f smooth compactly supported in I such that (1.13) does not hold for f. Let U be the value of the left-hand side in (1.13). $U > 0$. If k is a positive finite measure on I satisfying $k(I) < \dfrac{U}{\|f\|_\infty^2 \max_{Suppf} m}$ then if we replace ν by $\nu - k$ in (1.13), keeping the same function f, we still get something positive. Thus $L - k \in \mathfrak{D}^{+}$.

(iv): Let f be a smooth function compactly supported in I such that (1.13) does not hold for f. Let J be a compact subinterval of I containing the support of f. The set

$$\{s \in [0,1] | L - \nu_+ + s\, 1_J\, \nu_+ \in \mathfrak{D}^{-}\}$$

is not empty because it contains 0, and open by proposition 1.7 (ii). Let s_{max} by its supremum. Then $s_{max} < 1$ and $L - \nu_+ + s_{max}\, 1_J\, \nu_+ \in \mathfrak{D}^0$. Then $k := 1_{I \setminus J}\, \nu_+ + (1 - s_{max}) 1_J\, \nu_+$ is appropriate.

(v): First assume that there is such a function u. Then by definition $L_{|(x_0,x_1)} \in \mathfrak{D}^{0,-}$. $Conj(u, L_{|(x_0,x_1)})$ does not have any killing measure and the derivative of its natural scale function is $\dfrac{w}{u^2}$. It is not integrable in the neighbourhood of x_0 or x_1. Thus the corresponding diffusion never hits x_0 or x_1. This means that it is recurrent. Conversely, assume that $L_{|(x_0,x_2)} \in \mathfrak{D}^0$. Let u be a solution to $Lu = 0$ satisfying $u(x_0) = 0$ and $\dfrac{du}{dx}(x_0^+) > 0$. If u changed its sign on (x_0, x_1) then according to the preceding we would have $L_{|(x_0,x_1)} \in \mathfrak{D}^{+}$. If u were positive on an interval larger that (x_0, x_1) we would have $L_{|(x_0,x_1)} \in \mathfrak{D}^{-}$. The only possibility is that u is positive on (x_0, x_1) and zero in x_1. □

2. Measure on loops, invariance, covariance and disintegration properties

2.1. **Spaces of loops.** In this subsection we introduce the spaces of paths and loops on witch will be defined the measures we will consider throughout the paper. First we will consider continuous, time parametrized, paths on $\mathbb{R}$, $(\gamma(t))_{0 \leq t \leq T(\gamma)}$, with finite life-time $T(\gamma) \in (0, +\infty)$. Given two such paths $(\gamma(t))_{0 \leq t \leq T(\gamma)}$ and $(\gamma'(t))_{0 \leq t \leq T(\gamma')}$, a natural distance between them is

$$\delta_{paths}(\gamma, \gamma) := |\log(T(\gamma)) - \log(T(\gamma'))| + \max_{v \in [0,1]} |\gamma(vT(\gamma)) - \gamma'(vT(\gamma'))|$$

A rooted loop in $\mathbb{R}$ will be a continuous finite life-time path $(\gamma(t))_{0 \leq t \leq T(\gamma)}$ such that $\gamma(T(\gamma)) = \gamma(0)$ and we will write $\mathfrak{L}$ for the space of such loops. $\mathfrak{L}$ endowed with the metric δ_{paths} is a Polish space. In the sequel we will use the corresponding Borel σ-algebra, $\mathcal{B}_{\mathfrak{L}}$, for the definition of measures on $\mathfrak{L}$. For $v \in [0,1]$ we define a parametrisation shift transformation $shift_v$ on $\mathfrak{L}$: $shift_v(\gamma) = \tilde{\gamma}$ where $T(\tilde{\gamma}) = T(\gamma)$ and

$$\tilde{\gamma}(t) = \begin{cases} \gamma(vT(\gamma) + t) & if\ t \leq (1-v)T(\gamma) \\ \gamma(t - (1-v)T(\gamma)) & if\ t \geq (1-v)T(\gamma) \end{cases}$$

We introduce an equivalence relation on $\mathfrak{L}$: $\gamma \sim \gamma$ if $T(\gamma') = T(\gamma)$ and there is $v \in [0,1]$ such that $\gamma' = shift_v(\gamma)$. We call the quotient space $\mathfrak{L}/_\sim$ the space of unrooted loops, or just loops,

and write it $\mathfrak{L}^*$. Let π be the projection $\pi : \mathfrak{L} \to \mathfrak{L}^*$. There is a natural metric $\delta_{\mathfrak{L}^*}$ on $\mathfrak{L}^*$:

$$\delta_{\mathfrak{L}^*}(\pi(\gamma), \pi(\gamma')) := \min_{v\in[0,1]} \delta_{paths}(shift_v(\gamma), \gamma')$$

$(\mathfrak{L}^*, \delta_{\mathfrak{L}^*})$ is a Polish space and π is continuous. For defining measures on $\mathfrak{L}^*$ we will use its Borel σ-algebra, $\mathcal{B}_{\mathfrak{L}^*}$. $\pi^{-1}(\mathcal{B}_{\mathfrak{L}^*})$, the inverse image of $\mathcal{B}_{\mathfrak{L}^*}$ by π, is a sub-algebra of $\mathcal{B}_{\mathfrak{L}}$.

In the sequel we will consider paths and loops that have a continuous family of local times $(\ell^x_t(\gamma))_{x\in\mathbb{R},0\le t\le T(\gamma)}$ relatively to a measure $m(x)\,dx$ such that for any positive measurable function f on $\mathbb{R}$ and any $t \in [0, T(\gamma)]$.

$$\int_0^t f(\gamma(s))ds = \int_I \ell^x_t(\gamma)m(x)\,dx$$

We will simply write $\ell^x(\gamma)$ for $\ell^x_{T(\gamma)}(\gamma)$.

In the sequel we will also consider transformations on paths and loops and the images of different measures by these transformation. We will use everywhere the following notation: If $\mathcal{E}$ and $\mathcal{E}'$ are two measurable spaces, $\varphi : \mathcal{E} \mapsto \mathcal{E}'$ a measurable map and η a positive measure on $\mathcal{E}$, $\varphi_*\eta$ will be the measure on $\mathcal{E}'$ obtained as the image of η trough φ.

2.2. **Measures $\mu^{x,y}$ on finite life-time paths.** First we recall the framework that Le Jan used in [10]: $\mathbb{G} = (V, E)$ is a finite connected graph. $L_{\mathbb{G}}$ is the generator of a symmetric Markov jump process with killing on $\mathbb{G}$. $m_{\mathbb{G}}$ is the duality measure for $L_{\mathbb{G}}$. $(p^{\mathbb{G}}_t(x, y))_{x,y\in V,t\ge0}$ is the family of transition densities of the jump process and $(\mathbb{P}^{\mathbb{G},t}_{x,y}(\cdot))_{x,y\in V,t\ge0}$ the family of bridge probability measures. The measure on rooted loops associated with $L_{\mathbb{G}}$ is

$$\mu_{L_{\mathbb{G}}}(\cdot) = \int_{t>0} \sum_{x\in V} \mathbb{P}^{\mathbb{G},t}_{x,x}(\cdot)p^{\mathbb{G}}_t(x, x)m_{\mathbb{G}}(x)\,\frac{dt}{t} \tag{2.1}$$

$\mu^*_{L_{\mathbb{G}}}$ is the image of $\mu_{L_{\mathbb{G}}}$ by the projection on unrooted loops. The definition of $\mu^*_{L_{\mathbb{G}}}$ is the exact formal analogue of the definition used in [14] for the loops of the two-dimensional Brownian motion. In [10] also appear variable life-time bridge measures $(\mu^{x,y}_{L_{\mathbb{G}}})_{x,y\in V}$ which are related to $\mu^*_{L_{\mathbb{G}}}$:

$$\mu^{x,y}_{L_{\mathbb{G}}}(\cdot) = \int_0^{+\infty} \mathbb{P}^{\mathbb{G},t}_{x,y}(\cdot)p^{\mathbb{G}}_t(x, y)\,dt \tag{2.2}$$

In this subsection we will define and give the important properties of the formal analogue of the measures $\mu^{x,y}_{L_{\mathbb{G}}}$ in case of one-dimensional diffusions. In the next subsection 2.3 we will do the same with the measure on loops $\mu^*_{L_{\mathbb{G}}}$.

I is an open interval of $\mathbb{R}$. $(X_t)_{0\le t<\zeta}$ is a diffusion on I with a generator L of the form (1.3). We use the notations of the section 1.2. Let $x, y \in I$. Following the pattern of (2.2) we define:

Definition 1.

$$\mu^{x,y}_L(\cdot) := \int_0^{+\infty} \mathbb{P}^t_{x,y}(\cdot)p_t(x, y)dt$$

We will write $\mu^{x,y}$ instead of $\mu^{x,y}_L$ whenever there is no ambiguity on L. The definition of $\mu^{x,y}$ depends on the choice of m, but $m(y)\mu^{x,y}$ does not. Measures $\mu^{x,y}$ were first introduced by Dynkin in [5] and enter the expression of Dynkin's isomorphism between the Gaussian Free Field and the local times of random paths. Pitman and Yor studied this measures in [17] in the

setting of one-dimensional diffusions without killing measure ($k = 0$). Next we give a handy representation of $\mu^{x,y}$ in the setting of one-dimensional diffusions. It was observed and proved by Pitman and Yor in case $k = 0$. We consider the general case.

Proposition 2.1. *Let F be a non-negative measurable functional on the space of variable life-time paths starting from x. Then*

$$\mu^{x,y}(F(\gamma)) = \mathbb{E}_x \left[\int_0^{\zeta} F((X_s)_{0\le s\le t}) d_t \ell_t^y(X) \right] \tag{2.3}$$

Equivalently

$$\mu^{x,y}(F(\gamma)) = \mathbb{E}_x \left[\int_0^{\ell_\zeta^y(X)} F((X_s)_{0\le s\le \tau_l^y})\, dl \right]$$

where

$$\tau_l^y := \inf\{t \ge 0 | \ell_t^z(X) > l\}$$

Proof. It is enough to prove this for F non-negative continuous bounded functional witch takes value 0 if either the life-time of the paths exceeds some value $t_{max} < +\infty$ or of it is inferior to some value t_{min} or if the end point of the path lies out of a compact subinterval $[z_1, z_2]$ of I (with $y \in (z_1, z_2)$). For $j \le n \in \mathbb{N}$, set $t_{j,n} := t_{min} + \frac{j(t_{max}-t_{min})}{n}$ and $\Delta t_n := \frac{t_{max}-t_{min}}{n}$. Almost surely $\int_0^{\zeta} F((X_s)_{0\le s\le t}) d_t l_t^y$ is a limit as $n \to +\infty$ of

$$\sum_{j=0}^{n-1} F((X_s)_{0\le s\le t_{j,n}})(\ell^y_{t_{j+1,n}\wedge\zeta}(X) - \ell^y_{t_{j,n}\wedge\zeta}(X)) \tag{2.4}$$

Moreover (2.4) is dominated by $\|F\|_\infty l^y_{t_{max}\wedge\zeta}$. It follows that the expectations converge too. Using the Markov property and (1.4), we get that the expectation of (2.4) equals

$$\sum_{j=0}^{n-1} \int_{z\in I} \int_0^{\Delta t_n} \mathbb{P}^{t_{j,n}}_{x,z} \left(F((X_s)_{0\le s\le t_{j,n}}) \right) p_{t_{j,n}}(x,z) p_r(z,y)\, dr\, m(z)\, dz \tag{2.5}$$

Using the fact that $p_r(\cdot,\cdot)$ is symmetric, we can rewrite (2.5) as

$$\int_{z_1}^{z_2} \left(\sum_{j=0}^{n-1} \Delta t_n\, \mathbb{P}^{t_{j,n}}_{x,z} \left(F((X_s)_{0\le s\le t_{j,n}}) \right) p_{t_{j,n}}(x,z) \right) \frac{1}{\Delta t_n} \int_0^{\Delta t_n} p_r(y,z)\, dr\, m(z)\, dz \tag{2.6}$$

As $n \to +\infty$ the measure $\frac{1}{\Delta t_n} \int_0^{\Delta t_n} p_r(y,z)\, dr\, m(z)\, dz$ converges weakly to δ_y. Using the weak continuity of bridge probabilities (proposition 1.4) we get that (2.6) converges to

$$\int_{t_{min}}^{t_{max}} \mathbb{P}^t_{x,y} \left(F((X_s)_{0\le s\le t}) \right) p_t(x,y) dt$$

□

Proposition 2.1 also holds in case of a Markov jump processes on a graph, where the local time is replaced by the occupation time in a vertex dived by its weight. Proposition 2.1 shows that we can consider $\mu^{x,y}$ as a measure on paths $(\gamma(t))_{0\le t\le T(\gamma)}$ endowed with continuous occupation density functions $(\ell_t^z(\gamma))_{z\in I, 0\le t\le T(\gamma)}$. Next we state several properties that follow almost immediately either from the definition 1 or proposition 2.1:

Property 2.2.

- *(i) The total mass of the measure $\mu^{x,y}$ is finite if and only if X is transient and then it equals $G(x,y)$.*
- *(ii) The measure $\mu^{y,x}$ is image of the measure $\mu^{x,y}$ by time reversal.*
- *(iii) If $\tilde{I}$ is an open subinterval of I then*

$$\mu^{x,y}_{L_{|\tilde{I}}}(d\gamma) = 1_{\gamma \, contained \, in \, \tilde{I}} \, \mu^{x,y}_L(d\gamma)$$

- *(iv) If $\tilde{k}$ is a positive Radon measure on I then*

$$\mu^{x,y}_{L-\tilde{k}}(d\gamma) = \exp\left(-\int_I \ell^z(\gamma) m(z) \tilde{k}(dz)\right) \mu^{x,y}_L(d\gamma)$$

- *(v) If A is a change of scale function then*

$$\mu^{A(x),A(y)}_{Scale_A^\dagger L} = Scale_{A*} \mu^{x,y}_L$$

- *(vi) If V is a positive continuous function on I then for the time changed diffusion of generator $\frac{1}{V}L$:*

$$\mu^{x,y}_{\frac{1}{V}L} = Speed_{V*} \mu^{x,y}_L$$

- *(vii) If h is a positive continuous function on I such that $\dfrac{d^2h}{dx^2}$ is a signed measure and Lu is a negative measure then*

$$\mu^{x,y}_{Conj(h,L)} = \frac{1}{h(x)h(y)} \mu^{x,y}_L$$

Previous equalities depend on a particular choice of the speed measure for the modified generator. For (iv) we keep the measure $m(y)\,dy$. For (iii) we restrict $m(y)\,dy$ to $\tilde{I}$. For (v) we choose $\left(\dfrac{dA}{dx} \circ A^{-1}\right)^{-1} m \circ A^{-1}\,da$. For (vi) we choose $\dfrac{1}{V(y)} m(y)\,dy$. For (vii) we choose $h(y)^2 m(y)\,dy$. Property (ii) follows from that $p_t(x,y) = p_t(y,x)$ and $\mathbb{P}^t_{y,x}(\cdot)$ is the image of $\mathbb{P}^t_{x,y}(\cdot)$ by time reversal. Property (vi) is not immediate from definition 1 because fixed times are transformed by time change in random times, but follows from proposition 2.1. Property (vii) follows from that an h-transform does not change bridge probability measures and changes the semi-group $(p_t(x,y)m(y)\,dy)_{t\geq 0, x\in I}$ to $(\frac{1}{u(x)} p_t(x,y) u(y) m(y)\,dy)_{t\geq 0, x\in I}$.

Next we state different representations for the measures $\mu^{x,y}$

Property 2.3.

- *(i) Assume $k \neq 0$. Let $\mathbb{P}_x(\cdot)$ be the law of $(X_t)_{0\leq t<\zeta}$ where $X(0) = 0$. Then*

$$\int_{y\in I} \mu^{x,y}(\cdot) m(y) k(dy) = 1_{X \, killed \, by \, k} \, \mathbb{P}_x(\cdot)$$

- *(ii) Assume that X is transient. Then $\frac{1}{G(x,x)}\mu^{x,x}$ is the law of X, starting from $X(0) = x$, up to the last time it visits x. $\frac{1}{G(x,y)}\mu^{x,y}$ is the law of X, starting from $X(0) = x$, conditioned to visit y before ζ, up to the last time it visits y.*
- *(iii) Let X and $\tilde{X}$ be two independent Markovian paths of generator L starting from $X(0) = x$ and $\tilde{X}(0) = y$. For $a \leq x \wedge y$, we introduce T_a and $\tilde{T}_a$ the first time X respectively $\tilde{X}$ hits a. Let $\mathbb{P}^{T_a}_x$ be the law of X up to time T_a, conditioned by the event $T_a < \zeta$. Let $\tilde{\mathbb{P}}^{\tilde{T}_a}_y$ be the analogue for $\tilde{X}$. Let $\tilde{\mathbb{P}}^{\tilde{T}_a\wedge}_y$ be the image of $\tilde{\mathbb{P}}^{\tilde{T}_a}_y$ through time*

reversal and $\mathbb{P}_x^{T_a} \lhd \tilde{\mathbb{P}}_y^{\tilde{T}_a\wedge}$ *the image of* $\mathbb{P}_x^{T_a} \otimes \tilde{\mathbb{P}}_y^{\tilde{T}_a\wedge}$ *through concatenation at* a *of two paths, one ending and the other starting in* a*. Then*

$$\mu^{x,y}(\cdot) = \int_{a\in I, a\leq x\wedge y} \mathbb{P}_x(T_a < \zeta)\mathbb{P}_y(\tilde{T}_a < \tilde{\zeta}) \left(\mathbb{P}_x^{T_a} \lhd \tilde{\mathbb{P}}_y^{\tilde{T}_a\wedge}\right)(\cdot) w(a)\, da$$

Property 2.3 (i) was noticed by Dynkin in [5]. Properties 2.3 (ii) and (iii) were proved by Pitman and Yor in case $k = 0$. See [17]. The case $k \neq 0$ can be obtained through h-transforms. Indeed, and h-transform does not change the law of a diffusion from the first to the last time it visits a point x, and does not change the measures $\mathbb{P}_x^{T_a}(\cdot)$.

Next we study the continuity of $(x, y) \mapsto \mu^{x,y}$.

Lemma 2.4. *Let* J *be a compact subinterval of* I*. Then the family of local times of* X *satisfies: for every* $\varepsilon > 0$

$$\lim_{t\to 0^+} \sup_{x\in J} \mathbb{P}_x \left(\sup_{y\in I} \ell^y_{t\wedge\zeta}(X) > \varepsilon \right) = 0 \tag{2.7}$$

Proof. It is enough to prove it in case the killing measure k is zero because adding a killing measure only lowers $\ell^y_{t\wedge\zeta}(X)$. Without loss of generality we may also assume that the diffusion is on its natural scale, that is to say $w \equiv 2$. Then X is just a time changed Brownian motion on some open subinterval of $\mathbb{R}$. For a Brownian motion $(B_t)_{t\geq 0}$ (2.7) is clear. In this case $\mathbb{P}_x \left(\sup_{y\in\mathbb{R}} \ell^y_{t\wedge\zeta}(B) > \varepsilon\right)$ does not depend on x and for a given x

$$\lim_{t\to 0^+} \mathbb{P}_x \left(\sup_{y\in\mathbb{R}} \ell^y_{t\wedge\zeta}(B) > \varepsilon \right) = 0$$

Otherwise let

$$\mathcal{I}_t := \int_0^t m(X_s)\, ds$$

Then given the time change that transforms X into a Brownian motion B, we have

$$\ell^y_t(X) = \ell^y_{\mathcal{I}_t}(B)$$

Let $J = [x_0, x_1]$. Let $x_{min} \in I$, $x_{min} < x_0$ and $x_{max} \in I$, $x_{max} > x_1$. Let $T_{x_{min},x_{max}}$ the first time X hits either x_{min} or x_{max}. Let $s > 0$, $\varepsilon > 0$ and $x \in J$. If $t \leq \dfrac{s}{\max_{[x_{min},x_{max}]} m}$ then on the event $T_{x_{min},x_{max}} \geq t$, $\mathcal{I}_t$ is less or equal to s. So for t small enough

$$\mathbb{P}_x \left(\sup_{y\in I} \ell^y_{t\wedge\zeta}(X) > \varepsilon \right) \leq \mathbb{P}_x \left(\sup_{y\in\mathbb{R}} \ell^y_s(B) > \varepsilon \right) + \mathbb{P}_x \left(T_{x_{min},x_{max}} < t \right)$$

But

$$\mathbb{P}_x \left(T_{x_{min},x_{max}} < t \right) = \mathbb{P}_{x_0} \left(T_{x_{min},x_{max}} < t \right) + \mathbb{P}_{x_1} \left(T_{x_{min},x_{max}} < t \right)$$

and

$$\lim_{t\to 0^+} \sup_{x\in J} \mathbb{P}_x \left(T_{x_{min},x_{max}} < t \right) = 0$$

Thus

$$\limsup_{t\to 0^+} \sup_{x\in J} \mathbb{P}_x \left(\sup_{y\in I} \ell^y_{t\wedge\zeta}(X) > \varepsilon \right) \leq \mathbb{P} \left(\sup_{y\in\mathbb{R}} \ell^y_s(B) > \varepsilon \right)$$

Letting s go to 0 we get (2.7). □

Proposition 2.5. *Let $t_{max} > 0$. Let F be a bounded functional on finite life-time paths endowed with continuous local times that depends continuously on the path $(\gamma_t)_{0\leq t\leq T(\gamma)}$ and on $(l^x_{T(\gamma)}(\gamma))_{x\in I}$ where we take the topology of uniform convergence for the occupation densities on I. On top of that we assume that F is zero if $T(\gamma) > t_{max}$. Then the function $(x, y) \mapsto \mu^{x,y}(F(\gamma))$ is continuous on $I \times I$.*

Proof. If we had assumed that F does only depend on the path regardless to its occupation field then the continuity of $(x, y) \mapsto \mu^{x,y}(F(\gamma))$ would just be a consequence of the continuity of transition densities and of the weak continuity of bridge probability measures. For our proof we further assume that L does not contain any killing measure. If this is not the case, then we can consider a continuous positive L-harmonic function u. Then $Conj(u, L)$ does not contain any killing measure and up to a continuous factor $u(x)u(y)$ gives the same measure $\mu^{x,y}$ (property 2.2 (vii)). We will mainly rely on the representation given by proposition 2.1.

Let $x, y \in I$ and $(x_j, y_j)_{j\geq 0}$ a sequence in $I\times I$ converging to (x, y). Without loss of generality we assume that $(x_j)_{j\geq 0}$ is increasing. We consider sample paths $(X_t)_{0\leq t<\zeta}$ and $(X^{(j)}_t)_{0\leq t<\zeta_j}$ of the diffusion of generator L starting from x and each of x_j, coupled on a same probability space in the following way: First we sample X starting from x. Then we sample $X^{(0)}$ starting from x_0. It starts independently from X until the first time $X^{(0)}_t = X_t$. After that time $X^{(0)}$ sticks to X. This two paths may never meet if one of them dies to early. If X, $X^{(0)}$,..., $X^{(j)}$ are already sampled, we start $X^{(j+1)}$ from x_{j+1} independently from the preceding sample paths until it meets one of them. After that time $X^{(j+1)}$ sticks to the path it has met. Let

$$T^{(j)} := \inf\{t \geq 0 | X^{(j)}_t = X_t\}$$

If $X^{(j)}$ does not meet X, we set $T^{(j)} = +\infty$. By construction, $(T^{(j)})_{j\geq 0}$ is a non-increasing sequence. Here we use that there is no killing measure. $T^{(j)}$ is equal in law to the first time two independent sample paths of the diffusion, one starting from x and the other from x_j, meet. Thus the sequence $(T^{(j)})_{j\geq 0}$ converges to 0 in probability. Since it is decreasing, it converges almost surely to 0.

We use reduction to absurdity. The sequence $(\mu^{x_j,y_j}(F(\gamma)))_{j\geq 0}$ is bounded because F is bounded and zero on paths with life-time greater then t_{max}. Assume that it does not converge to $\mu^{x,y}(F(\gamma))$. Then there is a subsequence that converges to a value other than $\mu^{x,y}(F(\gamma))$. We may as well assume that the whole sequence $(\mu^{x_j,y_j}(F(\gamma)))_{j\geq 0}$ converges to a value $v \neq \mu^{x,y}(F(\gamma))$. According to lemma 2.4, the sequence of occupation density functions $((\ell^z_{T^{(j)}}(X^{(j)}))_{z\in I})_{j\geq 0}$ converges in probability to the null function. Thus there is an extracted subsequence $((\ell^z_{T^{(j_n)}}(X^{(j_n)}))_{z\in I})_{n\geq 0}$ that converges almost surely uniformly to the null function. We will show that $(\mu^{x_{j_n},y_{j_n}}(F(\gamma)))_{n\geq 0}$ converges to $\mu^{x,y}(F(\gamma))$ and obtain a contradiction.

For $z \in I$ and $l > 0$ let

$$\tau^z_l := \inf\{t \geq 0 | \ell^z_t(X) > l\}$$

and

$$\tau^z_{j,l} := \inf\{t \geq 0 | \ell^z_t(X^{(j)}) > l\}$$

Then according to proposition 2.1:

$$\mu^{x,y}(F(\gamma)) = \mathbb{E}\left[\int_0^{\ell^y_{t_{max}\wedge\zeta}(X)} F((X_s)_{0\le s\le \tau^y_l})\,dl\right]$$

$$\mu^{x_j,y_j}(F(\gamma)) = \mathbb{E}\left[\int_0^{\ell^{y_j}_{t_{max}\wedge\zeta_j}(X^{(j)})} F((X^{(j)}_s)_{0\le s\le \tau^{y_j}_{j,l}})\,dl\right]$$

For any $z\in I$, if $\tau^z_{j,l}\in[T^{(j)},\zeta_j)$ then

$$\tau^z_{j,l} = \tau^z_{l'}$$

where

$$l' = l + \ell^z_{T^{(j)}}(X) - \ell^z_{T^{(j)}}(X^{(j)})$$

Along the subset of indices $(j_n)_{n\ge0}$, $\tau^{y_{j_n}}_{j_n,l}$ converges to τ^y_l for every $l\in(0,l^y_\zeta(X))$ except possibly the countable set of values of l where $l\mapsto\tau^y_{j,l}$ jumps. For any l such that $\tau^{y_{j_n}}_{j_n,l}$ converges to τ^y_l, the path $(X^{(j)}_s)_{0\le s\le\tau^{y_{j_n}}_{j_n,l}}$ converges to the path $(X_s)_{0\le s\le\tau^y_l}$. Moreover for such l the occupation densities $(l^z_{\tau^{y_{j_n}}_{j_n,l}}(X^{(j_n)}))_{z\in I}$ converge uniformly to $(l^z_{\tau^y_l}(X))_{z\in I}$. Indeed

$$\ell^z_{\tau^{y_{j_n}}_{j_n,l}}(X^{(j_n)}) = \ell^z_{\tau^{y_{j_n}}_{j_n,l}}(X) - \ell^z_{T^{(j)}}(X) + \ell^z_{T^{(j)}}(X^{(j_n)})$$

Thus for all $l\in(0,\ell^y_\zeta(X))$, except possibly countably many,

$$\lim_{n\to+\infty} F((X^{(j_n)}_s)_{0\le s\le\tau^{y_{j_n}}_{j_n,l}}) = F((X_s)_{0\le s\le\tau^y_l})$$

For n large enough, $\zeta_j=\zeta$ and $\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})$ converges to $\ell^y_{t_{max}\wedge\zeta}(X)$. It follows that the following almost sure convergence holds

$$\lim_{n\to+\infty}\int_0^{\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})} F((X^{(j_n)}_s)_{0\le s\le\tau^{y_{j_n}}_{j_n,l}})\,dl = \int_0^{l^y_{t_{max}\wedge\zeta}(X)} F((X_s)_{0\le s\le\tau^y_l})\,dl \tag{2.8}$$

The absolute value of the left-hand side of (2.8) is bounded by $\|F\|_{+\infty}\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})$. In order to conclude that the almost sure convergence (2.8) is also an $\mathbb{L}^1$ convergence we need only to show that

$$\mathbb{E}\left[|\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)}) - \ell^y_{t_{max}\wedge\zeta}(X)|\right] = 0 \tag{2.9}$$

We already know that $\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})$ converges almost surely to $\ell^y_{t_{max}\wedge\zeta}(X)$. Moreover

$$\mathbb{E}\left[\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})\right] = \int_0^{t_{max}} p_t(x_{j_n},y_{j_n})$$

and

$$\mathbb{E}\left[\ell^y_{t_{max}\wedge\zeta}(X)\right] = \int_0^{t_{max}} p_t(x,y)$$

It follows that the expectations converge. By Scheffe's lemma, the $\mathbb{L}^1$ convergence (2.9) holds.

Finally there is a subsequence $(\mu^{x_{j_n},y_{j_n}}(F(\gamma)))_{n\ge0}$ that converges to $\mu^{x,y}(F(\gamma))$ which contradict the convergence of $(\mu^{x_j,y_j}(F(\gamma)))_{j\ge0}$ to a different value. □

2.3. **The measure μ^* on unrooted loops.** The measure $\mu^{x,x}$ can be seen as a measure on the space of rooted loops $\mathfrak{L}$. Next we define a natural measure μ_L^* on $\mathfrak{L}^*$ following the pattern (2.1):

Definition 2. *Let μ_L be the following measure on $\mathfrak{L}$:*

$$\mu_L(d\gamma) := \int_{t>0}\int_{x\in I} \mathbb{P}^t_{x,x}(d\gamma)p_t(x,x)m(x)\,dx\,\frac{dt}{t} = \frac{1}{T(\gamma)}\int_{x\in I}\mu_L^{x,x}(d\gamma)m(x)\,dx$$

$\mu_L^* := \pi_*\mu_L$ is a measure on $\mathfrak{L}^*$.

We will drop the subscript L whenever there is no ambiguity on L. The definition 2 does not depend on the choice of the speed measure $m(x)\,dx$. The measures μ and μ^* are σ-finite but not finite. They satisfy the following elementary properties:

Property 2.6.

- *(i) μ is invariant by time reversal.*
- *(ii) If $\tilde{I}$ is an open subinterval of I then*

$$\mu_{L_{|\tilde{I}}}(d\gamma) = 1_{\gamma\, contained\, in\, \tilde{I}}\,\mu_L(d\gamma)$$

- *(iii) If $\tilde{k}$ is a positive Radon measure on I then*

$$\mu_{L-\tilde{k}}(d\gamma) = \exp\left(-\int_I \ell^z(\gamma)m(z)\tilde{k}(dz)\right)\mu_L(d\gamma)$$

- *(iv) If A is a change of scale function then*

$$\mu_{Scale_A^\dagger L} = Scale_{A*}\mu_L$$

- *(v) If h is a positive continuous function on I such that $\dfrac{d^2h}{dx^2}$ is a signed measure and Lu is a negative measure then*

$$\mu_{Conj(h,L)} = \mu_L$$

Same properties hold for μ^*.

The measures μ and μ^* contain some information on the diffusion X but the invariance by h-transforms (property 2.6 (v)) shows that they do not capture its asymptotic behaviour. In the subsection 2.4 we will prove a converse to the property property 2.6 (v). In our setting, most important examples of h-transforms are:

- The Bessel 3 process on $(0,+\infty)$ is an h-transform of the Brownian motion on $(0,+\infty)$, killed when hitting 0, through the function $x\mapsto x$.
- The Brownian motion on $\mathbb{R}$ killed with uniform rate $k\,dx$ (i.e. k constant) is an h-transform of the drifted Brownian motion on $\mathbb{R}$ with constant drift $\sqrt{2k}$, through the function $x\mapsto e^{-\sqrt{2k}x}$.

In the sequel we will be interested mostly in μ^* and not μ. As it will be clear from the next propositions, the measure μ^* has some nice features that μ does not.

Proposition 2.7. *Let $v \in [0,1]$. Then*

$$shift_{v*}\mu = \mu$$

In particular

$$\mu = \int_{v\in[0,1]} dv\, shift_{v*}\mu \tag{2.10}$$

Proof. For a rooted loop γ of life-time $T(\gamma)$ we will introduce γ_1 the path restricted to time interval $[0, vT(\gamma)]$ and γ_2 the path restricted to $[vT(\gamma), T(\gamma)]$. By bridge decomposition property, the measure $\mu(d\gamma_1, d\gamma_2)$ equals

$$\int_{t>0}\int_I\int_I \mathbb{P}^{vt}_{x,y}(d\gamma_1)\mathbb{P}^{(1-v)t}_{y,x}(d\gamma_2)p_{vt}(x,y)p_{(1-v)t}(y,x)m(y)\,dy\,m(x)\,dx\,\frac{dt}{t} \tag{2.11}$$

In (2.11) γ_1 and γ_2 play symmetric roles, so changing the order of γ_1 and γ_2 does not change the measure μ. □

Formula (2.10) shows that we can get back to the measure μ from the measure μ^* by cutting the circle parametrizing a loop in $\mathfrak{L}^*$ in a point chosen uniformly on this circle, in order to separate the start from the end.

Corollary 2.8. *Let F be a positive measurable functional on $\mathfrak{L}$. Then $\gamma \mapsto \int_0^1 F(shift_v(\gamma))\,dv$ is $\pi^{-1}(\mathcal{B}_{\mathfrak{L}^*})$-measurable and*

$$\frac{d(F(\gamma)\mu)}{d\mu}_{|\pi^{-1}(\mathcal{B}_{\mathfrak{L}^*})} = \int_0^1 F(shift_v(\gamma))\,dv$$

Proof. We need only to show that for every F' measurable functional on $\mathfrak{L}^*$:

$$\int_{\mathfrak{L}} F(\gamma)F'(\pi(\gamma))\mu(d\gamma) = \int_0^1\int_{\mathfrak{L}} F(shift_v(\gamma))F'(\pi(\gamma))\mu(d\gamma)\,dv \tag{2.12}$$

From proposition 2.7 follows that for every $v \in [0,1]$:

$$\int_{\mathfrak{L}} F(\gamma)F'(\pi(\gamma))\mu(d\gamma) = \int_{\mathfrak{L}} F(shift_v(\gamma))F'(\pi(\gamma))\mu(d\gamma) \tag{2.13}$$

Integrating (2.13) on $[0,1]$ leads to (2.12). □

The next identity appears in [10] in the setting of Markov jump processes on graphs. We will give a proof for that suits our framework.

Corollary 2.9. *Let $x \in I$. Then*

$$\ell^x(\gamma)\mu^*(d\gamma) = \pi_*\mu^{x,x}(d\gamma) \tag{2.14}$$

For $l > 0$, let $\mathbb{P}_x^{\tau_l^x}(\cdot)$ be the law of the sample paths of a diffusion X of generator L, started from x, until the time τ_l^x when $\ell_t^x(X)$ hits l, conditioned by $\tau_l^x < \zeta$. Then

$$1_{\gamma\ visits\ x}\mu^*(d\gamma) = \int_0^{+\infty} \pi_*\mathbb{P}_x^{\tau_l^x}(d\gamma)e^{-\frac{l}{G(x,x)}}\,\frac{dl}{l} \tag{2.15}$$

Conventionally we set $G(x,x) = +\infty$ if X is recurrent.

Proof. Let $\varepsilon > 0$ such that $[x-\varepsilon, x+\varepsilon] \subseteq I$. Let $T_{[x-\varepsilon,x+\varepsilon]}(\gamma)$ be the time a loop γ spends in $[x-\varepsilon, x+\varepsilon]$. From the identity (2.10) follows that

$$\frac{T_{[x-\varepsilon,x+\varepsilon]}(\gamma)}{T(\gamma)}\mu^*(d\gamma) = \frac{1}{T(\gamma)}\int_{x-\varepsilon}^{x+\varepsilon} \pi_*\mu^{z,z}(d\gamma)m(z)\,dz$$

and simplifying $T(\gamma)$:

$$T_{[x-\varepsilon,x+\varepsilon]}(\gamma)\mu^*(d\gamma) = \int_{x-\varepsilon}^{x+\varepsilon} \pi_*\mu^{z,z}(d\gamma)m(z)\,dz \tag{2.16}$$

Using local times we rewrite (2.16) as

$$\frac{\int_{x-\varepsilon}^{x+\varepsilon} \ell^z(\gamma)m(z)\,dz}{\int_{x-\varepsilon}^{x+\varepsilon} m(z)\,dz}\,\mu^*(d\gamma) = \frac{1}{\int_{x-\varepsilon}^{x+\varepsilon} m(z)\,dz}\int_{x-\varepsilon}^{x+\varepsilon} \pi_*\mu^{z,z}(d\gamma)m(z)\,dz \tag{2.17}$$

Let $\varepsilon_0 > 0$ such that $[x-\varepsilon_0, x+\varepsilon_0] \subseteq I$. Let F be a continuous bounded functional on loops endowed with continuous local times such that F is zero if the life-time of the loop exceeds $t_{max} > 0$ and if $\sup_{z\in[x-\varepsilon_0,x+\varepsilon_0]} l^z(\gamma)$ exceeds l_{max}. According to the proposition 2.5, the right-hand side of (2.17) applied to F converges as $\varepsilon \to 0$ to $(\pi_*\mu^{x,x})(F(\gamma))$. By dominated convergence it follows that the left-hand side of (2.17) applied to F converges as $\varepsilon \to 0$ to

$$\int_{\mathfrak{L}^*} \ell^x(\gamma)F(\gamma)\mu^*(d\gamma)$$

Thus we have the equality

$$\int_{\mathfrak{L}^*} \ell^x(\gamma)F(\gamma)\mu^*(d\gamma) = (\pi_*\mu^{x,x})(F(\gamma)) \tag{2.18}$$

The set of test functionals F that satisfy (2.18) is large enough to deduce the equality (2.14) between measures.

From proposition 2.1 follows that

$$\mu^{x,x}(\cdot) = \int_0^{+\infty} \mathbb{P}_x^{\tau_l^x}(\cdot)e^{-\frac{l}{G(x,x)}}\,dl$$

Applying (2.14) to the above disintegration, we get (2.15). □

Corollary 2.10. *Let V be a positive continuous function on I. We consider a time change with speed V: $ds = V(x)dt$. Then*

$$\mu^*_{\frac{1}{V}L} = Speed_{V*}\mu^*_L \tag{2.19}$$

Proof. By definition 2 and property 2.2 (vi):

$$\mu_{\frac{1}{V}L}(d\gamma) = \frac{1}{T(\gamma)}\int_0^{T(\gamma)} \frac{V(\gamma(0))}{V(\gamma(s))}\,ds\ Speed_{V*}(\mu_L(d\gamma))$$

Applying corollary 2.8 we obtain:

$$\frac{dSpeed_{V*}\mu_L}{d\mu_{\frac{1}{V}L}}_{|\pi^{-1}(\mathcal{B}_{\mathfrak{L}^*})} = \frac{\int_0^1 V^{-1}(\gamma(vT(\gamma)))\,dv}{\frac{1}{T(\gamma)}\int_0^{T(\gamma)} V^{-1}(\gamma(s))\,ds} = 1$$

This concludes. □

In dimension two, the time change covariance of the measure μ^* on loops plays a key role for the construction of the Conformal Loop Ensembles (CLE) using loop soups as in [21]: Let D be an open domain of the complex plane, $(B_t)_{0\leq t<\zeta}$ the two-dimensional standard Brownian motion in D killed when hitting ∂D and μ^* the corresponding measure on loops. If $f : D \to D$ is a conformal map, then $(f(B_t))_{0\leq t<\zeta}$ is a time changed Brownian motion. If we consider μ^* not as a measure on loops parametrized by time but a measure on the geometrical drawings of loops, then μ^* is invariant by the transformation $(\gamma(t))_{0\leq t\leq T(\gamma)} \mapsto (f(\gamma(t)))_{0\leq t\leq T(\gamma)}$. This is proved in [14].

Given that μ^* is invariant through h-transforms and covariant with the change of scale and change of time, if X is a recurrent diffusion, then up to a change of scale and time, μ^* is the same as for the Brownian motion on $\mathbb{R}$, and if X is a transient diffusion, even if the killing measure k is non-zero, then up to a change of scale and time, μ^* is the same as for the Brownian motion on a bounded interval, killed when it hits the boundary.

2.4. **Multiple local times.** In this subsection we define the multiple local time functional on loops. Corollary 2.9 gives a link between the measure μ^* and the measures $(\mu^{x,x})_{x\in I}$. Using multiple local time we will get a further relation between μ^* and $(\mu^{x,y})_{x,y\in I}$. This will allow us to prove a converse to the property 2.6 (v): two diffusions that have the same measure on unrooted loops are related trough an h-transform.

Definition 3. *If $(\gamma(t))_{0\leq t\leq T(\gamma)}$ is a continuous path in I having a family of local times $(l^x_t(\gamma))_{x\in I,0\leq t\leq T(\gamma)}$ relatively to the measure $m(x)\,dx$, we introduce multiple local times $\ell^{x_1,x_2,...,x_n}(\gamma)$ for $x_1, x_2, ..., x_n \in I$:*

$$\ell^{x_1,x_2,...,x_n}(\gamma) := \int_{0\leq t_1\leq t_2\leq ...\leq t_n\leq T(\gamma)} d_{t_1}\ell^{x_1}_{t_1}(\gamma)\, d_{t_2}\ell^{x_2}_{t_2}(\gamma)...d_{t_n}\ell^{x_n}_{t_n}(\gamma)$$

If $\gamma \in \mathfrak{L}$ and has local times, we introduce circular local times for γ:

$$\ell^{*x_1,x_2,...,x_n}(\gamma) := \sum_{\substack{c\, circular\\ permutation\\ of\, \{1,2,...,n\}}} \ell^{x_{c(1)},x_{c(2)},...,x_{c(n)}}(\gamma)$$

$\ell^{*x_1,x_2,...,x_n}$ being invariant under the transformations $(shift_v)_{v\in[0,1]}$, we see it as a functional defined on $\mathfrak{L}^*$.

Multiple local times of the form $l^{x,x,...,x}(\gamma)$, called self intersection local times, were studied by Dynkin in [7]. Circular local times were introduced by Le Jan in [10].

Let $n \in \mathbb{N}^*$ and $p \in \{1, ..., n\}$. Let $Shuffle_{p,n}$ be the set of permutations σ of $\{1, ..., n\}$ such that for all $i \leq j \in \{1, ..., p\}$, $\sigma(i) \leq \sigma(j)$ and for all $i \leq j \in \{p+1, ..., n\}$, $\sigma(i) \leq \sigma(j)$. Permutations in $Shuffle_{p,n}$ are obtained by shuffling two card decks $\{1, ..., p\}$ and $\{p+1, ..., n\}$. Let $Shuffle'_{p,n}$ be the permutations of $\{1, ..., n\}$ of the form $\sigma \circ c$ where c is a circular permutation of $\{p+1, ..., n\}$ and $\sigma \in Shuffle_{p,n}$ satisfies $\sigma(1) = 1$. One can check that

Property 2.11. *For all $x_1, ..., x_p, x_{p+1}, ..., x_n \in I$:*

- *(i)*

$$\ell^{x_1,...,x_p}(\gamma)\ell^{x_{p+1},...,x_n}(\gamma) = \sum_{\sigma\in Shuffle_{p,n}} \ell^{x_{\sigma(1)},...,x_{\sigma(p)},x_{\sigma(p+1)},...,x_{\sigma(n)}}(\gamma)$$

- *(ii)*

$$\ell^{*x_1,...,x_p}(\gamma)\ell^{*x_{p+1},...,x_n}(\gamma) = \sum_{\sigma'\in Shuffle'_{p,n}} \ell^{x_{\sigma'(1)},...,x_{\sigma'(p)},x_{\sigma'(p+1)},...,x_{\sigma'(n)}}(\gamma)$$

The equality 2.11 (ii) appears in [10]. It is also shown in [10] that for transient Markov jump processes:

$$\int \ell^{*x_1,x_2,...,x_n}(\gamma)\mu(d\gamma) = G(x_1,x_2)\times ...\times G(x_{n-1},x_n)\times G(x_n,x_1) \tag{2.20}$$

It turns out that we have more: We consider L a generator of a diffusion on I of form (1.3). If γ_i for $i\in\{1,2,...,n-1\}$ is a continuous path from x_i to x_{i+1}, then we can concatenate $\gamma_1,\gamma_2,...,\gamma_{n-1}$ to obtain a continuous path $\gamma_1\lhd\gamma_2\lhd ...\lhd\gamma_{n-1}$ from x_1 to x_n. Let $\mu^{x_1,x_2}\lhd ...\lhd\mu^{x_{n-1},x_n}$ be the image measure of $\mu^{x_1,x_2}\otimes ...\otimes\mu^{x_{n-1},x_n}$ by this concatenation procedure.

Proposition 2.12. *The following absolute continuity relationships hold:*

- *(i)* $(\mu^{x_1,x_2}\lhd ...\lhd\mu^{x_{n-1},x_n})(d\gamma) = \ell^{x_2,...,x_{n-1}}(\gamma)\mu^{x_1,x_n}(d\gamma)$
- *(ii)* $\pi_*(\mu^{x_1,x_2}\lhd ...\lhd\mu^{x_{n-1},x_n}\lhd\mu^{x_n,x_1})(d\gamma) = \ell^{*x_1,x_2,...,x_n}(\gamma)\mu^*(d\gamma)$

Proof. (i): Let $((X_t^{(i)})_{0\le t<\zeta_i})_{0\le i\le n-1}$ be $n-1$ independent diffusions of generator L, with $X_0^{(i)} = x_i$. For $l\ge 0$, let

$$\tau_{i,l} := \inf\left\{t_i\ge 0 | \ell_{t_i}^{x_{i+1}}(X^{(i)}) > l\right\}$$

According to proposition 2.1, $(\mu^{x_1,x_2}\lhd ...\lhd\mu^{x_{n-1},x_n})(F(\gamma))$ equals:

$$\mathbb{E}\left[\int 1_{l_i<l_{\zeta_i}^{x_{i+1}},1\le i\le n-1} F\left((X_t^{(1)})_{0\le t\le\tau_{1,l_1}}\lhd ...\lhd (X_t^{(n-1)})_{0\le t\le\tau_{n-1,l_{n-1}}}\right) dl_1...dl_{n-1}\right] \tag{2.21}$$

Let $(X_t)_{0\le t<\zeta}$ be an other diffusion of generator L. Let

$$\tau_{l_1} := \inf\{t\ge 0 | l_t^{x_2}(X) > l_1\}$$

and recursively defined:

$$\tau_{l_1,...l_{i-1},l_i} := \inf\{t\ge\tau_{l_1,...l_{i-1}} | \ell_t^{x_{i+1}}(X) > l_i\}$$

Then by strong Markov property, (2.21) equals

$$\mathbb{E}\left[\int 1_{\tau_{l_1,...,l_{n-1}}\le\zeta} F\left((X_t)_{0\le t\le\tau_{l_1,...,l_{n-1}}}\right) dl_1...dl_{n-1}\right] \tag{2.22}$$

(2.22) in turn equals

$$\mathbb{E}\left[\int 1_{\forall i,t_i<\zeta} F\left((X_t)_{0\le t\le t_{n-1}}\right) d\ell_{t_1}^{x_2}(X)...d\ell_{t_{n-1}}^{x_n}(X)\right] \tag{2.23}$$

By proposition 2.1, (2.23) equals $\int \ell^{x_1,...,x_{n-1}}(\gamma)F(\gamma)\mu^{x_1,x_n}(d\gamma)$.

(ii): According to the identity (i) and corollary 2.8, we have

$$\pi_*(\mu^{x_1,x_2} \lhd ... \lhd \mu^{x_{n-1},x_n} \lhd \mu^{x_n,x_1})(d\gamma) = \left(\int_0^1 \ell^{x_2,...,x_n}(shift_v(\gamma))\, dv\right) \pi_*\mu^{x_1,x_1}(d\gamma) \tag{2.24}$$

According to corollary 2.9:

$$\left(\int_0^1 \ell^{x_2,...,x_n}(shift_v(\gamma))\, dv\right) \pi_*\mu^{x_1,x_1}(d\gamma) = \ell^{x_1}(\gamma) \left(\int_0^1 \ell^{x_2,...,x_n}(shift_v(\gamma))\, dv\right) \mu^*(d\gamma)$$

But

$$\ell^{x_1}(\gamma) \int_0^1 \ell^{x_2,...,x_n}(shift_v(\gamma))\, dv = \ell^{*x_1,x_2,...,x_n}(\gamma)$$

which ends the proof. □

The proposition 2.12 (ii) implies (2.20).

Proposition 2.13. *If L and L' are two generators of diffusions on I of the form* (1.3) *such that $\mu_L^* = \mu_{L'}^*$, then there is a positive continuous function h on I such that $\frac{d^2h}{dx^2}$ is a signed measure, Lh a negative measure and $L' = Conj(h, L)$. If the diffusion of generator L is recurrent then $L' = L$.*

Proof. Let $m(x)\,dx$ be a speed measure for L and $m'(x)\,dx$ be a speed measure for L'. First let's assume that both L and L' are generators of transient diffusions. Let $(G(x,y))_{x,y\in I}$ be the Green's function of L relatively to the measure $m(x)\,dx$ and $(G'(x,y))_{x,y\in I}$ be the Green's function of L' relatively to the measure $m'(x)\,dx$. Applying the identity (2.20) to $\int_{\mathfrak{L}^*} \ell^{*x,y}(\gamma)\mu^*(d\gamma)$ we get that for all $x, y \in I$:

$$G'(x,y)G'(y,x)m'(x)m'(y) = G(x,y)G(y,x)m(x)m(y) \tag{2.25}$$

and for all $x, y, z \in I$:

$$G'(x,y)G'(y,z)G'(z,x)m'(x)m'(y)m'(z) = G(x,y)G(y,z)G(z,x)m(x)m(y)m(z) \tag{2.26}$$

Fix $x_0 \in I$. Let h be

$$h(x) := \frac{G'(x_0,x)m'(x)}{G(x_0,x)m(x)}$$

h is positive and continuous. $\frac{1}{h(x)}G(x,y)h(y)m(y)$ equals:

$$\frac{G(x_0,x)G(x,y)G(y,x_0)m(x_0)m(x)m(y)}{G'(x_0,x)G'(x,y)G'(y,x_0)m'(x_0)m'(x)m'(y)}$$

$$\times \frac{G'(x_0,y)G'(y,x_0)m'(x_0)m'(y)}{G(x_0,y)G(y,x_0)m(x_0)m(y)} \times G'(x,y)m'(y) \tag{2.27}$$

Applying (2.25) and (2.26) to (2.27) we get that

$$\frac{1}{h(x)}G(x,y)h(y)m(y) = G'(x,y)m'(y) \tag{2.28}$$

Applying (2.28) once to (x,y) and once do (x,x) we get that

$$h(y) = h(x)\frac{G'(x,y)}{G(x,y)}\frac{G(y,y)}{G'(y,y)} \tag{2.29}$$

From (2.29) we deduce that $\frac{d^2h}{dx^2}$ is a signed measure. From (2.28) we deduce that $L' = Conj(h, L)$. $-Lh$ is the killing measure of L' and is positive.

If we no longer assume that L and L' generate transient diffusions then consider $\lambda > 0$. Then $\mu^*_{L-\lambda} = \mu^*_{L'-\lambda}$. According to the above, there is h positive continuous function on I such that $\frac{d^2}{dx^2}$ is a signed measure and

$$L' - \lambda = Conj(h, L - \lambda) = Conj(h, L) - \lambda$$

Then $L' = Conj(h, L)$ and necessarily Lh is a negative measure.

The class of recurrent diffusions is preserved by h-transforms. So if L is the generator of a recurrent diffusion then so is L', and thus h is bound to satisfy $Lh = 0$. But since the diffusion of L is recurrent, the only solutions to $Lh = 0$ are constant functions. Thus $L' = L$. □

2.5. **A disintegration of μ^* induced by the Vervaat's transformation.** By conditioning the measure μ by the life-time of loops we get a sum of bridge measures. Vervaat in [24] shows a relation between Brownian bridges and Brownian excursions. Using the Vervaat's transformation we will disintegrate the measure μ^* as a measure on the minimal value of the loop and its behaviour above this value. We will obtain a sum of excursion measures $\eta^{>x}$.

Vervaat's Transformation. *Let $(\gamma(s))_{0\le s\le t}$ be a random path following the Brownian bridge probability measure $\mathbb{P}^{0,0}_{t,BM}(\cdot)$. Let $s_{min} := argmin\,\gamma$. Then the path*

$$s \mapsto -\min\gamma + (shift_{\frac{s_{min}}{t}}\gamma)(s)$$

has the law of a positive Brownian excursion of life-time t.

In the sequel if η is a measure on paths and $x \in \mathbb{R}$, we will write $(x+\eta)$ for the image of η by $\gamma \mapsto x + \gamma$. $\eta^{>0}_{BM}$ will be the Levy-Itô measure on positive Brownian excursions and $\eta^{>0}_{t,BM}$ the probability measure on positive Brownian excursions of duration t. Given a continuous loop $(\gamma_t)_{0\le t\le T(\gamma)}$ and t_0 the first time γ hits $\min\gamma$, let $\mathcal{V}(\gamma)$ be the transformation $shift_{\frac{t_0}{T(\gamma)}}$. $\mathcal{V}$ is $\mathcal{B}_{\mathfrak{L}}$-measurable.

Proposition 2.14. *Let μ^*_{BM} be the measure on loops associated to the Brownian motion on $\mathbb{R}$. Then:*

$$\mu^*_{BM}(d\gamma) = 2\int_{a\in\mathbb{R}} \pi_*(a + \eta^{>0}_{BM})(d\gamma)\,da \tag{2.30}$$

The measure on $(\min\gamma, \max\gamma)$ induced by μ^*_{BM} is $1_{a<b}(b-a)^{-2}\,dx\,dy$. Let $a < b \in \mathbb{R}$ and $\rho, \tilde\rho$ two independent Bessel 3 processes starting from 0. Let T_{b-a} and $\tilde T_{b-a}$ be the first times ρ respectively $\tilde\rho$ hit $b-a$. Let $(\beta_t)_{0\le t\le T_{b-a}+\tilde T_{b-a}}$ be the path

$$\beta_t := \begin{cases} a + \rho_t & if\ t \le T_{b-a} \\ a + \tilde\rho_{T_{b-a}+\tilde T_{b-a}-t} & if\ t \ge T_{b-a} \end{cases}$$

Then the law of $(\beta_t)_{0\le t\le T_{b-a}+\tilde T_{b-a}}$ is the probability measure obtained by conditioning the measure μ^*_{BM} by $(\min\gamma, \max\gamma) = (a, b)$.

Proof. For the Brownian motion on $\mathbb{R}$, μ_{BM} writes

$$\mu_{BM}(\cdot) = \int_{x\in\mathbb{R}} \int_{t>0} (x + \mathbb{P}^{0,0}_{t,BM})(\cdot) \frac{dt}{\sqrt{2\pi t^3}}\, dx$$

Let $\chi(a)\, da$ be the law of the minimum of the bridge under $\mathbb{P}^{0,0}_{t,BM}$. Applying the Vervaat's transformation, we get that

$$\mathcal{V}_*\mu_{BM}(\cdot) = \int_{a\in\mathbb{R}} \int_{t>0} \left(\int_{x>a} \chi(x-a)\, dx \right) (a + \eta^{>0}_{t,BM})(\cdot) \frac{dt}{\sqrt{2\pi t^3}}\, da \tag{2.31}$$

Since $\int_{x>a} \chi(x-a)\, dx = 1$, the right-hand side of (2.31) equals

$$\int_{a\in\mathbb{R}} \int_{t>0} (a + \eta^{>0}_{t,BM})(\cdot) \frac{dt}{\sqrt{2\pi t^3}}\, da$$

But

$$\int_{t>0} (a + \eta^{>0}_{t,BM})(\cdot) \frac{dt}{\sqrt{2\pi t^3}} = 2(a + \eta^{>0}_{BM})(\cdot)$$

The equality (2.30) follows. The rest of the proposition 2.14 is a consequence of the William's representation of Brownian excursions. □

Corollary 2.15. *Let I be an open interval of $\mathbb{R}$ and $\lambda \geq 0$. Let L be the generator $\frac{1}{2}\frac{d^2}{dx^2} - \lambda$ on I with zero Dirichlet boundary conditions and μ^* the associated measure on loops. Given a loop $(\gamma(t))_{0\leq t\leq T(\gamma)}$, let $R(\gamma)$ be the loop $(\max\gamma + \min\gamma - \gamma(t))_{0\leq t\leq T(\gamma)}$, that is the image of γ through reflection relatively to $\frac{\max\gamma + \min\gamma}{2}$. Then*

$$R_*\mu^* = \mu^*$$

Proof. It is enough to prove this in case $\lambda = 0$ and $I = \mathbb{R}$. Otherwise we multiply the measure μ^*_{BM} by a density function that is left invariant by R. Then we use the description of the measure μ^*_{BM} conditioned by the value of $(\min\gamma, \max\gamma)$ and the fact that if $a > 0$, $(\rho_t)_{t\geq}$ is a Bessel 3 process starting from 0 and T_b is the first time it hits b, then $(y - \rho_{T_b - t})_{0\leq t\leq T_b}$ has the same law as $(\rho_t)_{0\leq t\leq T_b}$ (see [19], chapter VII, §4). □

Now we consider that L is a generator of a diffusion on I of form (1.3). Given a point $x_0 \in I$, u^{+,x_0} and u^{-,x_0} will be the L-harmonic functions satisfying $u^{+,x_0}(x_0) = u^{-,x_0}(x_0) = 0$, $\frac{du^{+,x_0}}{dx}(x_0^+) = 1$ and $\frac{du^{-,x_0}}{dx}(x_0^-) = -1$. If $x \leq y \in I$ then

$$w(y)u^{-,y}(x) = w(x)u^{+,x}(y) \tag{2.32}$$

Indeed, the Wronskian $W(u^{-,y}, u^{+,x})$ takes in x the value $u^{-,y}(x)$ and in y the value $u^{+,x}(y)$, and the ratio $\frac{1}{w(z)} W(u^{-,y}, u^{+,x})(z)$ is constant. If $\nu = 0$, then the both sides of (2.32) equal $\int_x^y w(z)\, dz$. u^{+,x_0} is positive on $I \cap (x_0, +\infty)$ and u^{-,x_0} is positive on $I \cap (-\infty, x_0)$. Let L^{+,x_0} be $Conj(u^{+,x_0}, L)$ restricted to $I \cap (x_0, +\infty)$ and L^{-,x_0} be $Conj(u^{-,x_0}, L)$ restricted to $I \cap (-\infty, x_0)$. L^{+,x_0} and L^{-,x_0} are generators of transient diffusions without killing measures. If L is the generator of the Brownian motion on $\mathbb{R}$, then $L^{+,0}$ is just the generator of a Bessel 3 process. In general case, x_0 is an entrance boundary for L^{+,x_0} and L^{-,x_0}, that is to say a diffusion started from $x \neq x_0$ will never reach the boundary at x_0, and we can also start this diffusions at the boundary point x_0, in which case it will be immediately repelled away

from x_0. Let $x \in I$ and $(\rho_t^{+,x})_{0\le t<\zeta^{+,x}}$ be a diffusion of generator $L^{+,x}$ starting from x. Let $y \in I$, $y > x$. Let $T_y^{+,x}$ be the first time $\rho^{+,x}$ hits y and $\hat{T}_y^{+,x}$ the last time it visits y. Then $(\rho^{+,x}_{\hat{T}_y^{+,x}+t})_{0\le t<\zeta^{+,x}-\hat{T}_y^{+,x}}$ is a diffusion of generator $L^{+,y}$ starting from y. Let $(\rho_t^{-,y})_{0\le t<\zeta^{-,y}}$ be a diffusion of generator $L^{-,y}$ starting from y and $T_x^{-,y}$ the first time it hits x. Then $(\rho_t^{+,x})_{0\le t\le T_y^{+,x}}$ and $(\rho^{-,y}_{T_x^{-,y}-t})_{0\le t\le T_x^{-,y}}$ are equal in law: Indeed let C be the constant

$$C = \frac{w(z)}{W(u^{-,y}, u^{+,x})(z)}$$

The Green's operator of $\rho^{+,x}$ killed in y is

$$((-L^{+,x}_{|(x,y)})^{-1}f)(x') = C\int_x^y u^{+,x}(x'\wedge y')u^{-,y}(x'\vee y')\frac{u^{+,x}(y')}{u^{+,x}(x')}m(y')\,dy'$$

and the Green's operator of $\rho^{-,y}$ killed in x is

$$((-L^{-,y}_{|(x,y)})^{-1}f)(x') = C\int_x^y u^{+,x}(x'\wedge y')u^{-,y}(x'\vee y')\frac{u^{-,y}(y')}{u^{-,y}(x')}m(y')\,dy'$$

The potential measure of $(\rho_t^{+,x})_{0\le t\le T_y^{+,x}}$ starting from x is

$$U(x')\,dx' = Cu^{+,x}(x')u^{-,y}(x')m(x')\,dx'$$

and for any f, g bounded functions on (x, y)

$$\int_x^y ((-L^{+,x}_{|(x,y)})^{-1}f)(x')g(x')U(x')\,dx' = \int_x^y f(x')((-L^{-,y}_{|(x,y)})^{-1}g)(x')U(x')\,dx' \tag{2.33}$$

The time reversal property for $(\rho_t^{+,x})_{0\le t\le T_y^{+,x}}$ follows from the duality relation (2.33). See [19], chapter VII, §4 for details on time reversal.

Corollary 2.16. *If L is a generator of a diffusion on I of form* (1.3)*, then*

$$\mu^*(\cdot) = \int_{a\in I}\pi_*\eta^{>a}(\cdot)\,w(a)\,da \tag{2.34}$$

The measure on $(\min\gamma, \max\gamma)$ induced by μ^* is $1_{a<b\in I}\dfrac{da\,db}{u^{+,a}(b)u^{-,b}(a)}$. Let $a < b \in I$. Let $(\rho_t^{+,a})_{0\le t<\zeta^{+,a}}$ and $(\rho_t^{-,b})_{0\le t<\zeta^{-,b}}$ be two independent diffusion, the first of generator $L^{+,a}$ starting from a and the second of generator $L^{-,b}$ starting from b. Let $T_b^{+,a}$ be the first time $\rho^{+,a}$ hits b and $T_a^{-,b}$ the first time $\rho^{-,b}$ hits a. Let $(\beta_t)_{0\le t\le T_b^{+,a}+T_a^{-,b}}$ be the path

$$\beta_t := \begin{cases} \rho_t^{+,a} & if\ t \le T_b^{+,a} \\ \rho^{-,b}_{t-T_b^{+,a}} & if\ t \ge T_b^{+,a} \end{cases}$$

Then the law of $(\beta_t)_{0\le t\le T_b^{+,a}+T_a^{-,b}}$ is the probability measure obtained by conditioning the measure μ^* by $(\min\gamma, \max\gamma) = (a, b)$.

Proof. Both sides of (2.34) are covariant by scale and time change. Moreover both sides satisfy the property 2.6 (ii) for the restriction to a subinterval and the property 2.6 (iii) when adding a killing measure.

Regarding the description of the measure on $(\min\gamma, \max\gamma)$ and the probabilities obtained after conditioning by $(\min\gamma, \max\gamma) = (a, b)$, if L is a generator without killing measure ($k = 0$),

then the result follows through a change of scale and time from the analogous description in proposition 2.14. If $k \neq 0$, then we can take u a positive L-harmonic function and deduce the result for L from the result for $Conj(u, L)$ using the fact that $\mu_L^* = \mu_{Conj(u,L)}^*$. □

The relation between the measure on loops and the excursions measures in dimension 1 (identity (2.34)) is analogous to the relation between the measure on Brownian loops and the so called bubble measures observed by Lawler and Werner in dimension 2. See propositions 7 and 8 in [14].

2.6. **A generalization of the Vervaat's transformation.** In this subsection we will show a conditioned version of the Vervaat's transformation that holds for any one-dimensional diffusion of form (1.3) and not just for the Brownian motion. L will be a generator of a diffusion on I of form (1.3). From corollary 2.9 and identity (2.34) follows that for every $x \in I$:

$$\int_{t>0} \mathcal{V}_* \mathbb{P}^t_{x,x}(d\gamma) p_t(x,x) dt = \int_{a \in I, a<x} \ell^x(\gamma) \eta^{>a}(d\gamma) w(a)\, da \tag{2.35}$$

Let $\mathbb{P}^t_{x,x}(d\gamma | \min \gamma = a)$ be the bridge probability measure condition by the value of the minimum to equal a. Further we will show that there is a version that depends continuously on (a,t). Let $\eta_t^{>a}$ the probability measure obtained from $\eta^{>a}$ by conditioning the excursion to have a life-time t. The identity (2.35) suggests the following:

Proposition 2.17. *For every $a < x \in I$ and $t > 0$*

$$\mathcal{V}_* \mathbb{P}^t_{x,x}(d\gamma | \min \gamma = a) = \frac{\ell_t^x(\gamma) \eta_t^{>a}(d\gamma)}{\eta_t^{>a}(l_t^x(\gamma))} \tag{2.36}$$

The distribution of $\min \gamma$ under $\mathbb{P}^t_{x,x}$ equals

$$w(a) \eta_t^{>a}(\ell_t^x(\gamma)) \frac{1}{p_t(x,x)} \frac{\eta^{>a}(T(\gamma) \in (t, t+dt))}{dt} \quad da \tag{2.37}$$

where $\dfrac{\eta^{>a}(T(\gamma) \in (t, t+dt))}{dt}$ is the density of the measure on the life-time of the excursion induced by $\eta^{>a}$. Given an excursion γ following the law $\dfrac{\ell_t^x(\gamma) \eta_t^{>a}(d\gamma)}{\eta_t^{>a}(\ell_t^x(\gamma))}$, the local time in x is a measure on $\{s \in [0,t] | \gamma(s) = x\}$. The transformation $\mathcal{V}$ sends the starting point of the bridge to a point $s \in [0,t]$ distributed conditionally on the excursion γ according the measure $\dfrac{d_s \ell_s^x(\gamma)}{\ell_t^x(\gamma)}$.

Identities (2.36) and (2.37) can be viewed as a conditioned analogue of the Vervaat's relation between the Brownian bridge and the Brownian excursion. The latter can be deduced from (2.36) and (2.37) using the translation invariance of the Brownian motion. From (2.35) we can only deduce that (2.36) and (2.37) hold for Lebesgue almost all t and a. We need to show the weak continuity in (a,t) of conditioned bridge probabilities and biased conditioned excursion probabilities to conclude. It is enough to prove the proposition 2.17 for L not containing any killing measure and such that for all $a < x \in I$, a diffusion starting from x reaches a almost surely. Indeed, for a general generator, $Conj(u_\downarrow, L)$ does satisfy the above constraints and if the proposition 2.17 is true for $Conj(u_\downarrow, L)$ then it is also for L. From now on we assume

that L satisfies the above constraints. Next we give a more "constructive" description of the conditioned bridges and biased conditioned excursions. We start with bridges.

Property 2.3 (iii) shows that the measure $\mathbb{P}_x^{T_a} \lhd \tilde{\mathbb{P}}_x^{\tilde{T}_a\wedge}$ conditioned on $T_a + \tilde{T}_a = t$ is a version of $\mathbb{P}^t_{x,x}(d\gamma|\min\gamma = a)$. Let $p_t^{(a)}(x,y)$ be the transition density on $I \cap (a,+\infty)$ relatively to $m(y)\,dy$ of the semi-group generated by $L_{|I\cap(a,+\infty)}$. Then $p_t^{(a)}(x,a^+) = 0$. According to [15], for all $t > 0$, $y \mapsto p_t^{(a)}(x,y)$ is $\mathcal{C}^1$. Let $\partial_2 p_t^{(a)}(x,y)$ be the derivative relatively to y. It has a positive limit $\partial_2 p_t^{(a)}(x,a^+)$ as $y \to a^+$. Extended in this way, the map $(t,x,y) \mapsto \partial_2 p_t^{(a)}(x,y)$ is continuous on $(0+\infty) \times I \cap (a,+\infty) \times I \cap [a,+\infty)$. The distribution of T_a under $\mathbb{P}_x$ is (see [7]):

$$\frac{1}{w(a)}\partial_2 p_t^{(a)}(x,a^+)\,dt$$

Let $\mathbb{P}^{(a),t}_{x,y}$ be the bridge probability measures of $L_{|I\cap(a,+\infty)}$. It has a weak limit $\mathbb{P}^{(a),t}_{x,a^+}$ as $y \to a^+$. Let $\mathcal{F}_s$ be the sigma-algebra generated by the restriction of a continuous path to the time interval $[0,s]$. Let $\mathbb{P}_a^{+,a}$ be the law of $\rho^{+,a}$ starting from a. For all $s \in (0,t)$ we have the following absolute continuity relations:

$$\frac{d\mathbb{P}^{(a),t}_{x,a^+}}{d\mathbb{P}_x}_{|\mathcal{F}_s} = 1_{s<T_a}\frac{\partial_2 p^{(a)}_{t-s}(X_s,a^+)}{\partial_2 p_t^{(a)}(x,a^+)} \tag{2.38}$$

and for the time reversed bridge

$$\frac{d\mathbb{P}^{(a),t\wedge}_{x,a^+}}{d\mathbb{P}_a^{+,a}}_{|\mathcal{F}_s} = \frac{p^{(a)}_{t-s}(\rho_s^{+,a},x)}{\partial_2 p_t^{(a)}(x,a^+)} \tag{2.39}$$

Using the absolute continuity relation (2.38) and (2.39) one can prove in a similar way as in proposition 1.4 that the map $(t,y) \mapsto \mathbb{P}^{(a),t}_{x,a^+}$ is continuous for the weak topology. The measure $\mathbb{P}_x^{T_a}$ disintegrates as follows

$$\mathbb{P}_x^{T_a}(\cdot) = \frac{1}{w(a)}\int_{t>0} \mathbb{P}^{(a),t}_{x,a^+}(\cdot)\partial_2 p_t^{(a)}(x,a^+)\,dt \tag{2.40}$$

From the property 2.3 (iii) and (2.40) we get that

Property 2.18. *The distribution of* $\min\gamma$ *under* $P^t_{x,x}$ *is*

$$\frac{da}{w(a)p_t(x,x)}\int_0^t \partial_2 p_s^{(a)}(x,a^+)\partial_2 p^{(a)}_{t-s}(x,a^+)\,ds \tag{2.41}$$

There is a version of $\mathbb{P}^t_{x,x}(d\gamma|\min\gamma = a)$ *that disintegrates as*

$$\frac{\int_0^t \left(\mathbb{P}^{(a),s}_{x,a^+} \lhd \mathbb{P}^{(a),t-s\wedge}_{x,a^+}\right)(d\gamma)\partial_2 p_s^{(a)}(x,a^+)\partial_2 p^{(a)}_{t-s}(x,a^+)\,ds}{\int_0^t \partial_2 p_s^{(a)}(x,a^+)\partial_2 p^{(a)}_{t-s}(x,a^+)\,ds} \tag{2.42}$$

Next we show that the probability measure given by (2.42) depends continuously on (a,t).

Lemma 2.19. *The functions* $(x,a,t) \mapsto p_t^{(a)}(x,a^+)$ *and* $(x,a,t) \mapsto \partial_2 p_t^{(a)}(x,a^+)$ *are continuous on* $\{(x,a)|x > a \in I\} \times (0,+\infty)$.

Proof. As in [15], we can use the eigendifferential expansion of L to express $p_t^{(a)}(x,a^+)$ and $\partial_2 p_t^{(a)}(x,a^+)$. Let x_0. For $\lambda\in\mathbb{R}$ consider $e_1(\cdot,\lambda)$ and $e_2(\cdot,\lambda)$ two solutions to $Lu+\lambda u=0$ with initial conditions

$$e_1(x_0,\lambda)=1\quad \frac{\partial e_1}{\partial x}(x_0,\lambda)=0\quad e_2(x_0,\lambda)=0\quad \frac{\partial e_2}{\partial x}(x_0,\lambda)=1$$

Let $\mathfrak{e}(x,\lambda)$ be the 2-vector whose entries are $e_1(x,\lambda)$ and $e_2(x,\lambda)$. According to theorems 3.2 and 4.3 in [15], for all $a\in I$ there is a Radon measure $\mathfrak{f}^{(a)}$ on $(-\infty,0]$ with values in the space of 2×2 symmetric positive semi-definite matrices such that for all $x\in I\cap(a,+\infty)$

$$p_t^{(a)}(x,a^+)=\int_{-\infty}^0 e^{t\lambda}\mathfrak{e}(x,\lambda)\mathfrak{f}^{(a)}(d\lambda)\mathfrak{e}(a,\lambda)$$

$$\partial_2 p_t^{(a)}(x,a^+)=\int_{-\infty}^0 e^{t\lambda}\mathfrak{e}(x,\lambda)\mathfrak{f}^{(a)}(d\lambda)\frac{\partial\mathfrak{e}}{\partial x}(a,\lambda)$$

Let $x>a\in I$. Consider a two sequences $(x_n)_{n\geq 0}$ and $(a_n)_{n\geq 0}$ in $I\cap(-\infty,x)$ converging to x respectively a such that for all $n\geq 0$, $x_n>a_n$. Let $(b_j)_{j\geq 0}$ be an increasing sequence in $I\cap(x,\sup I)$ converging to $\sup I$. Let $\mathfrak{f}_{n,j}$ be the 2×2-matrix valued measure on $(-\infty,0]$ corresponding to the eigendifferential expansion of L restricted to (a_n,b_j). $\mathfrak{f}_{n,j}$ charges only a discrete set of atoms. As shown in the proof of theorem 3.2 in [15], the total mass of the measures $1\wedge|\lambda|^{-2}\|\mathfrak{f}_{n,j}\|(d\lambda)$, $1\wedge|\lambda|^{-2}\|\mathfrak{f}^{(a_n)}\|(d\lambda)$ and $1\wedge|\lambda|^{-2}\|\mathfrak{f}^{(a)}\|(d\lambda)$ is uniformly bounded. Moreover for a fixed n, as $j\to+\infty$, $1\wedge|\lambda|^{-2}\mathfrak{f}_{n,j}(d\lambda)$ converges vaguely, that is against continuous functions vanishing at infinity, to the measure $1\wedge|\lambda|^{-2}\mathfrak{f}^{(a_n)}(d\lambda)$. Moreover, for any increasing integer-valued sequence $(j_n)_{n\geq 0}$ converging to $+\infty$, $1\wedge|\lambda|^{-2}\mathfrak{f}_{n,j_n}(d\lambda)$ converges vaguely as $n\to+\infty$ to $1\wedge|\lambda|^{-2}\mathfrak{f}^{(a)}(d\lambda)$. Since the sequence $(j_n)_{n\geq 0}$ is arbitrary, this implies that $1\wedge|\lambda|^{-2}\mathfrak{f}^{(a_n)}(d\lambda)$ converges vaguely as $n\to+\infty$ to $1\wedge|\lambda|^{-2}\mathfrak{f}^{(a)}(d\lambda)$.

There are constants $C,c'>0$ such that for all $\lambda\leq 0$ and $n\geq 0$

$$\|\mathfrak{e}(x_n,\lambda)\|\leq Ce^{c'\sqrt{|\lambda|}}\quad \|\mathfrak{e}(a_n,\lambda)\|\leq Ce^{c'\sqrt{|\lambda|}}\quad \|\frac{\partial\mathfrak{e}}{\partial x}(a_n,\lambda)\|\leq Ce^{c'\sqrt{|\lambda|}} \tag{2.43}$$

Let $t>0$ and $(t_n)_{n\geq 0}$ a sequence of times converging to t. From (2.43) follows that

$$\lim_{\lambda\to-\infty}\sup_{n\geq 0}|\lambda|^2e^{t_n\lambda}\|\mathfrak{e}(x_n,\lambda)\|\times\|\mathfrak{e}(a_n,\lambda)\|=0$$

$\lambda\mapsto 1\vee|\lambda|^2e^{t_n\lambda}\left(\mathfrak{e}(x_n,\lambda),\partial\mathfrak{e}(a_n,\lambda)\right)$ vanishes at infinity an converges uniformly on $(-\infty,0]$ to $\lambda\mapsto 1\vee|\lambda|^2e^{t\lambda}\left(\mathfrak{e}(x,\lambda),\mathfrak{e}(a,\lambda)\right)$. The vague convergence of measures implies that

$$\lim_{n\to+\infty}\int_{-\infty}^0 e^{t_n\lambda}\mathfrak{e}(x_n,\lambda)\mathfrak{f}^{(a_n)}(d\lambda)\mathfrak{e}(a_n,\lambda)=\int_{-\infty}^0 e^{t\lambda}\mathfrak{e}(x,\lambda)\mathfrak{f}^{(a)}(d\lambda)\mathfrak{e}(a,\lambda)$$

Similarly $\partial_2 p_{t_n}^{(a_n)}(x_n,a_n^+)$ converges to $\partial_2 p_t^{(a)}(x,a^+)$. □

Lemma 2.20. *The map $a\mapsto\mathbb{P}_a^{+,a}$ is weakly continuous.*

Proof. Let $a_0\in I$. Consider the process $(\rho_t^{+,a_0})_{t\geq 0}$ following the law $\mathbb{P}_{a_0}^{+,a_0}$. For $a\in I\cap(a_0,+\infty)$, let $\hat{T}_a$ be the last time ρ^{+,a_0} visits a. Then $(\rho_{\hat{T}_a+t}^{+,a_0})_{t\geq 0}$ follows the law $\mathbb{P}_a^{+,a}$. The process valued map $a\mapsto(\rho_{\hat{T}_a+t}^{+,a_0})_{t\geq 0}$ is almost surely continuous on $I\cap(a_0,+\infty)$ and thus the laws depend weakly continuously on a. □

Proposition 2.21. *The version of* $\mathbb{P}^t_{x,x}(d\gamma|\min\gamma = a)$ *given by* (2.42) *is weakly continuous in* (a,t).

Proof. From the absolute continuity relations (2.38) for the bridge $\mathbb{P}^{(a),t}_{x,a^+}$ and (2.39) for its time reversal, together with the continuity of the densities which follows from lemma 2.19, and the weak continuity of $a \mapsto \mathbb{P}^{+,a}_a$, we can deduce in a very similar way as in proposition 1.4 that the map $(a,t) \mapsto \mathbb{P}^{(a),t}_{x,a^+}$ is weakly continuous on $(0,+\infty)\times I \cap (-\infty,x)$ and hence $(a,s,t) \mapsto \mathbb{P}^{(a),s}_{x,a^+} \lhd \mathbb{P}^{(a),t-s\wedge}_{x,a^+}$ is weakly continuous. Finally the densities that appear in expression (2.42) are continuous with respect to (a,s,t). □

Next we will describe the measure $\eta^{>a}$. $\partial_2 p^{(a)}_t(x,a^+)$ is $\mathcal{C}^1$ relatively to x and the derivative $\partial_{1,2}p^{(a)}_t(x,a^+)$ has a positive limit $\partial_{1,2}p^{(a)}_t(a^+,a^+)$ as $y \to a^+$. Moreover $t \mapsto \partial_{1,2}p^{(a)}_t(a^+,a^+)$ is continuous. The measure on the life-time of the excursion induced by $\eta^{>a}$ is (see [20]):

$$\frac{1}{w(a)^2}\partial_{1,2}p^{(a)}_t(a^+,a^+)\,dt$$

Let $s \in [0,t]$. The measure $\eta^{>a}_t(\cdot)$ disintegrates as (see [20]):

$$\int_{x\in I, x>a}\left(\mathbb{P}^{(a),s\wedge}_{x,a^+} \lhd \mathbb{P}^{(a),t-s}_{x,a^+}\right)(\cdot)\frac{\partial_2 p^{(a)}_s(x,a^+)\partial_2 p^{(a)}_{t-s}(x,a^+)m(y)}{\partial_{1,2}p^{(a)}_t(a^+,a^+)}\,dy \tag{2.44}$$

For every $s_1 < s_2 \in [0,s]$, under the bridge measure $\mathbb{P}^{(a),s}_{y,z}$:

$$\mathbb{P}^{(a),t}_{y,z}(\ell^x_{s_2}(\gamma) - \ell^x_{s_1}(\gamma)) = \int_{s_1}^{s_2}\frac{p^{(a)}_r(y,x)p^{(a)}_{s-r}(x,z)}{p^{(a)}_s(y,z)}\,dr \tag{2.45}$$

and under the bridge measure $\mathbb{P}^{(a),s}_{y,a^+}$:

$$\mathbb{P}^{(a),t}_{y,a^+}(\ell^x_{s_2}(\gamma) - \ell^x_{s_1}(\gamma)) = \int_{s_1}^{s_2}\frac{p^{(a)}_r(y,x)\partial_2 p^{(a)}_{s-r}(x,a^+)}{\partial_2 p^{(a)}_s(y,a^+)}\,dr \tag{2.46}$$

Combining (2.44) and (2.46) we get that for every $s_1 < s_2 \in [0,s]$:

$$\eta^{>a}_t(\ell^x_{s_2}(\gamma) - \ell^x_{s_1}(\gamma)) = \int_{s_1}^{s_2}\frac{\partial_2 p^{(a)}_s(x,a^+)\partial_2 p^{(a)}_{t-s}(x,a^+)}{\partial_{1,2}p^{(a)}_t(a^+,a^+)}\,ds \tag{2.47}$$

Proposition 2.22. *Let* F_1 *and* F_2 *be two non-negative measurable functional on the paths with variable life-time. Then*

$$\eta^{>a}_t\left(\int_0^t F_1((\gamma(r))_{0\le r\le s})F_2((\gamma(s+r))_{0\le r\le t-s})\,d_s\ell^x_s(\gamma)\right) =$$

$$\int_0^t \mathbb{P}^{(a),s\wedge}_{x,a^+}(F_1)\mathbb{P}^{(a),t-s}_{x,a^+}(F_2)\frac{\partial_2 p^{(a)}_s(x,a^+)\partial_2 p^{(a)}_{t-s}(x,a^+)}{\partial_{1,2}p^{(a)}_t(a^+,a^+)}\,ds \tag{2.48}$$

In particular

$$\ell^x_t(\gamma)\eta^{>a}_t(d\gamma) = \int_0^t\left(\mathbb{P}^{(a),s\wedge}_{x,a^+} \lhd \mathbb{P}^{(a),t-s}_{x,a^+}\right)(d\gamma)\frac{\partial_2 p^{(a)}_s(x,a^+)\partial_2 p^{(a)}_{t-s}(x,a^+)}{\partial_{1,2}p^{(a)}_t(a^+,a^+)}\,ds \tag{2.49}$$

Proof. It is enough to prove the result in case F_1 and F_2 are non-negative, continuous and bounded. On top of that we may assume that there are $s_{min} < s_{max} \in (0,t)$ such that F_1 respectively F_2 takes value 0 if the life-time of a path is smaller than s_{min} respectively $t - s_{max}$, and that there is $C \in I$, $C > a$, such that F_1 and F_2 take value 0 if $\max \gamma > C$. For $j \leq n \in \mathbb{N}$ set $\Delta s_n := \frac{1}{n}(s_{max} - s_{min})$ and $s_{j,n} := s_{min} + j\Delta s_n$. Then almost surely

$$\int_0^t F_1((\gamma(r))_{0\leq r\leq s}) F_2((\gamma(s+r))_{0\leq r\leq t-s})\, d_s\ell_s^x(\gamma) =$$

$$\lim_{n\to+\infty} \sum_{j=0}^{n-1} F_1((\gamma(r))_{0\leq r\leq s_{j,n}})(\ell^x_{s_{j+1,n}}(\gamma) - \ell^x_{s_{j,n}}(\gamma)) F_2((\gamma(s_{j+1,n}+r))_{0\leq r\leq t-s_{j+1,n}}) \tag{2.50}$$

Moreover the right-hand side of (2.50) is dominated by $l_t^x(\gamma)\|F_1\|_\infty\|F_2\|_\infty$. Thus the $\eta_t^{>a}$-expectation converges too. Applying (2.44) and (2.45) we get

$$\eta_t^{>a}\left(F_1((\gamma(r))_{0\leq r\leq s_{j,n}})(\ell^x_{s_{j+1,n}}(\gamma) - \ell^x_{s_{j,n}}(\gamma)) F_2((\gamma(s_{j+1,n}+r))_{0\leq r\leq t-s_{j+1,n}})\right) =$$

$$\int_0^{\Delta s_n} \int_{(a,C)^2} \mathbb{P}^{(a),s_{j,n}\wedge}_{y,a^+}(F_1)\mathbb{P}^{(a),t-s_{j+1,n}}_{z,a^+}(F_2) q_n(r,y,z)\, m(y)dy\, m(z)dz\, dr$$

where

$$q_n(r,y,z) = \frac{\partial_2 p^{(a)}_{s_{j,n}}(y,a^+)\partial_2 p^{(a)}_{t-s_{j+1,n}}(z,a^+)}{\partial_{1,2} p^{(a)}_t(a^+,a^+)} p^{(a)}_r(y,x) p^{(a)}_{\Delta s_n - r}(x,z)$$

The measure $1_{y,z>a\in I,}\dfrac{1}{\Delta s_n}\int_0^{\Delta s_n} q_n(r,y,z)\, dr\, dy\, dz$ converges weakly as $n \to +\infty$ to $\delta_{(x,x)}$. The maps $(s,y) \mapsto \partial_2 p^{(a)}_s(x,a^+)$ and $(s,y) \mapsto \mathbb{P}^{(a),y,a^+}_s(\cdot)$ are continuous. Moreover $\partial_2 p^{(a)}_{s_{j,n}}(y,a^+)\partial_2 p^{(a)}_{t-s_{j+1,n}}(z,a^+)$ is uniformly bounded for $j \leq n \in \mathbb{N}$ and $y,z \in (a,C]$. All this ensures that the $\eta_t^{>a}$-expectation of the right-hand side of (2.50) converges as $n \to +\infty$ to the right-hand side of (2.48). □

Now we need only to match the preceding descriptions to prove proposition 2.17. (2.42) and (2.49) imply (2.36). (2.41) and (2.47) imply (2.37). The fact that the point where the excursion is split is distributed according to $\dfrac{d_s\ell_s^x(\gamma)}{\ell_t^x(\gamma)}$ follows from (2.48).

2.7. **Restricting loops to a discrete subset.** Let L be the generator of a diffusion on I of form (1.3) and $(X_t)_{0\leq t<\zeta}$ be the corresponding diffusion. Let $\mathbb{J}$ be a countable discrete subset of I. A Markov jump process to the nearest neighbours on $\mathbb{J}$ is naturally embedded in the diffusion X. In this subsection we will show that, given any $x,y \in \mathbb{J}$, the image of the measure $\mu_L^{x,y}$ through the restriction application that sends a sample paths of the diffusion $(X_t)_{0\leq t<\zeta}$ to a sample path of a Markov jump process on $\mathbb{J}$ is a measure on $\mathbb{J}$-valued paths that follows the pattern (2.2). From this we will deduce that the image of the measure μ_L^* through the restriction to $\mathbb{J}$ is a measure on $\mathbb{J}$-valued loops following the pattern (2.1) and which was studied in [10]. This property will be used in section 3.2 to express the law of finite-dimensional marginals of the occupation field of a Possonian ensemble of intensity $\alpha\mu_L^*$.

For a continuous path $(\gamma(t))_{0\leq t\leq T(\gamma)}$ in I, endowed with continuous local times, let

$$\mathcal{I}_t^{\mathbb{J}}(\gamma) := \sum_{x\in\mathbb{J}} \ell_t^x(\gamma)m(x)$$

For $s \geq 0$, we introduce the stopping time

$$\tau_s^{\mathbb{J}}(\gamma) := \inf\{t \geq 0 | \mathcal{I}_t^{\mathbb{J}}(\gamma) \geq s\}$$

We write $\gamma^{\mathbb{J}}$ for the path $(\gamma(\tau_s^{\mathbb{J}}))_{0\leq s\leq \mathcal{I}^{\mathbb{J}}_{T(\gamma)}(\gamma)}$ on $\mathbb{J}$. Let $m_{\mathbb{J}}$ be the measure

$$m_{\mathbb{J}} := \sum_{x\in\mathbb{J}} m(x)\delta_x$$

The occupation measure of $\gamma^{\mathbb{J}}$ is

$$\sum_{x\in\mathbb{J}} \ell^x(\gamma)m(x)\delta_x$$

and $(l^x(\gamma))_{x\in\mathbb{J}}$ are also occupation densities of the restricted path $\gamma^{\mathbb{J}}$ relatively to $m_{\mathbb{J}}$.

The restricted diffusion $X^{\mathbb{J}}$ is a Markov jump process to nearest neighbours on $\mathbb{J}$, potentially with killing. If $x_0 < x_1$ are two consecutive points in $\mathbb{J}$, the jump rate from x_0 to x_1 is $\frac{1}{m(x_0)w(x_0)}\frac{1}{u^{+,x_0}(x_1)}$ and the jump rate from x_1 to x_0 is $\frac{1}{m(x_1)w(x_1)}\frac{1}{u^{-,x_1}(x_0)}$. If $x_0 < x_1 < x_2$ are three consecutive points in $\mathbb{J}$, then the rate of killing while in x_1 is

$$\frac{1}{m(x_1)w(x_1)}\left(\frac{W(u^{-,x_2},u^{+,x_0})(x_1)}{u^{-,x_2}(x_1)u^{+,x_0}(x_1)} - \frac{1}{u^{-,x_1}(x_0)} - \frac{1}{u^{+,x_1}(x_2)}\right)$$

If $\mathbb{J}$ has a minimum x_0 and x_1 is the second lowest point in $\mathbb{J}$, then the killing rate while in x_0 is

$$\frac{1}{m(x_0)w(x_0)}\left(\frac{W(u^{-,x_1},u_\uparrow)(x_0)}{u^{-,x_1}(x_0)u_\uparrow(x_0)} - \frac{1}{u^{+,x_0}(x_1)}\right)$$

An analogous expression holds for the killing rate while in a possible maximum of $\mathbb{J}$. $X^{\mathbb{J}}$ is transient if and only if X is. Let $L_{\mathbb{J}}$ be the generator of $X^{\mathbb{J}}$. $L_{\mathbb{J}}$ is symmetric relatively to $m_{\mathbb{J}}$. Its Green's function relatively to $m_{\mathbb{J}}$ is $(G(x,y))_{x,y\in I}$, that is the restriction of the Green's function of L to $\mathbb{J}\times\mathbb{J}$. $X^{\mathbb{J}}$ may not be conservative even if the diffusion X is. In case if $\mathbb{J}$ is not finite, $X^{\mathbb{J}}$ may blow up performing an infinite number of jumps in finite time. Measures $(\mu_L^{x,y})_{x,y\in I}$, μ_L and μ_L^* have discrete space analogues $(\mu_{L_{\mathbb{J}}}^{x,y})_{x,y\in\mathbb{J}}$, $\mu_{L_{\mathbb{J}}}$ and $\mu_{L_{\mathbb{J}}}^*$ as defined in [10], that follow the patterns (2.2) and (2.1).

Proposition 2.23. *Let $x, y \in \mathbb{J}$. Then $\gamma \mapsto \gamma^{\mathbb{J}}$ transforms $\mu_L^{x,y}$ in $\mu_{L_{\mathbb{J}}}^{x,y}$ and μ_L^* in $\mu_{L_{\mathbb{J}}}^*$.*

Proof. The representation (2.3) also holds for $\mu_{L_{\mathbb{J}}}^{x,y}$. For $l > 0$, let

$$\tau_l^y := \inf\{t \geq 0 | \ell_t^y(X) > l\}$$

and

$$\tau_l^{y,\mathbb{J}} := \inf\{s \geq 0 | \ell_s^y(X^{\mathbb{J}}) > l\}$$

Then for any non-negative measurable functional F

$$\mu_{L_{\mathbb{J}}}^{x,y}(F(\gamma)) = \int_0^{+\infty} dl\, \mathbb{E}_x\left[1_{\tau_l^{y,\mathbb{J}} < \mathcal{I}_\zeta^{\mathbb{J}}} F((X_s^{\mathbb{J}})_{0\leq s\leq \tau_l^{y,\mathbb{J}}})\right]$$

But $(X_s^{\mathbb{J}})_{0\le s\le \tau_\lambda^{y,\mathbb{J}}}$ is the image of $(X_t)_{0\le t\le \tau_\lambda^y}$ by the map $\gamma\mapsto\gamma^{\mathbb{J}}$ and $\tau_l^{y,\mathbb{J}} < \mathcal{I}_\zeta^{\mathbb{J}}$ if and only if $\tau_l^y < \zeta$. Thus $\mu_{L_{\mathbb{J}}}^{x,y}$ is the image of $\mu_L^{x,y}$ through the restriction on path to $\mathbb{J}$. The second part of the proposition can be deduced from that for any $x\in\mathbb{J}$

$$\ell^x(\gamma)\mu_L^*(d\gamma) = \pi_*\mu_L^{x,x}(d\gamma)$$

and as noticed in [10]

$$\ell^x(\gamma)\mu_{L_{\mathbb{J}}}^*(d\gamma^{\mathbb{J}}) = \pi_*\mu_{L_{\mathbb{J}}}^{x,x}(d\gamma^{\mathbb{J}})$$

□

2.8. **Measure on loops associated to a "generator" with creation of mass.** We can further extend the definition of the measures $\mu^{x,y}$ on paths and μ and μ^* on loops to the case of L being a "generator" on I containing a creation of mass term as in (1.12). Doing so will enable us to emphasize further the h-transform invariance of the measure on loops and will be useful in section 3.2 to compute the exponential moments of the occupation field of Poissonian ensembles of Markov loops.

Let ν be signed measure on I. Let $L^{(0)} := \dfrac{1}{m(x)}\dfrac{d}{dx}\left(\dfrac{1}{w(x)}\dfrac{d}{dx}\right)$ and $L := L^{(0)} + \nu$.

Definition 4.
- $\mu_L^{x,y}(d\gamma) := \exp\left(\int_I l^x(\gamma)m(x)\,\nu(dx)\right)\mu_{L^{(0)}}^{x,y}(d\gamma)$
- $\mu_L(d\gamma) := \exp\left(\int_I l^x(\gamma)m(x)\,\nu(dx)\right)\mu_{L^{(0)}}(d\gamma)$
- $\mu_L^* := \pi_*\mu_L$

Definition 4 is consistent with properties 2.2 (iv) and 2.6 (iii). If $\tilde\nu$ is any other signed measure on I, then

$$\mu_{L+\tilde\nu}^{x,y}(d\gamma) := \exp\left(\int_I l^x(\gamma)m(x)\,\tilde\nu(dx)\right)\mu_L^{x,y}(d\gamma) \tag{2.51}$$

Same holds for μ and μ^*. Under the extended definition, the measures $\mu^{x,y}$ still satisfy properties 2.2 (ii), (iii), (v) and (vi). Proposition 2.5 remains true. μ still satisfies properties 2.6 (i), (ii) and (iv). Proposition 2.7 and corollary 2.8 still hold. The identities (2.14) and (2.19) remain true for μ^*. Concerning the h-transforms, we have:

Proposition 2.24. *Let h be a continuous positive function on I such that $\dfrac{d^2h}{dx^2}$ is a signed measure. $h^2m\,dx$ is a speed measure for $Conj(h,L)$. Then for all $x,y\in I$, $\mu_{Conj(h,L)}^{x,y} = \dfrac{1}{h(x)h(y)}\mu_L^{x,y}$, and $\mu_{Conj(h,L)} = \mu_L$. Conversely, if L and L' are two "generators" with or without creation of mass such that $\mu_L = \mu_{L'}$ then there is a positive continuous function h on I such that $\dfrac{d^2h}{dx^2}$ is a signed measure and $L' = Conj(h,L)$.*

Proof. There is a positive Radon measure $\tilde k$ on I such that both $L-\tilde k$ and $Conj(h,L)-\tilde k$ are generators of (killed) diffusions. But $Conj(h,L)-\tilde k = Conj(h,L-\tilde k)$. It follows that $\mu_{Conj(h,L)-\tilde k}^{x,y} = \dfrac{1}{h(x)h(y)}\mu_{L-\tilde k}^{x,y}$ and $\mu_{Conj(h,L)-\tilde k} = \mu_{L-\tilde k}$. Applying (2.51) we get the result.

If $\mu_L = \mu_{L'}$, we can again consider $\tilde k$ a positive Radon measure on I such that both $L-\tilde k$ and $L'-\tilde k$ are generators of (killed) diffusions. Then according to proposition 2.13, there is a positive

continuous function h on I such that $\frac{d^2h}{dx^2}$ is a signed measure and $L' - \tilde{k} = Conj(h, L - \tilde{k})$. Then $L' = Conj(h, L)$. □

As for generators of diffusions is subsection 2.5, one can consider L-harmonic functions $u^{-,x}$ and $u^{+,x}$ in case of L containing creation of mass. If $L \in \mathfrak{D}^+$, then $u^{-,x}$ respectively $u^{+,x}$ is not necessarily positive on $I \cap (-\infty, x)$ respectively $I \cap (x, +\infty)$. Let

$$M(x) := \sup\{y \in I, y \geq x | \forall z \in (x, y),\, u^{+,x}(z) > 0\} \in I \cup \{\sup I\}$$

If $L \in \mathfrak{D}^{0,-}$ then for all $x \in I$, $M(x) = \sup I$. Let $y \in I$, $y > x$. If $y < M(x)$, then $L_{|(x,y)} \in \mathfrak{D}^-$. If $y = M(x)$, then $L_{|(x,y)} \in \mathfrak{D}^0$. If $y > M(x)$, then $L_{|(x,y)} \in \mathfrak{D}^+$. The diffusion $\rho^{+,x}$ of generator $L^{+,x} = Conj(u^{+,x}, L^{+,x}_{|(x,M(x))})$ is defined on $(x, M(x))$. Similarly for $\rho^{-,y}$. Moreover if If $M(x) \in I$, then $L^{+,x}_{|(x,M(x))} = L^{-,M(x)}_{|(x,M(x))}$.

If $L \in \mathfrak{D}^{0,-}$, the description of the measure on $(\min\gamma, \max\gamma)$ induced by μ^* as well as of the probability measures obtained by conditioning μ^* by the value of $(\min\gamma, \max\gamma)$ is the same as given by corollary 2.16, with the same formal expressions. Next we state what happens if $L \in \mathfrak{D}^+$:

Proposition 2.25. *Let $L \in \mathfrak{D}^+$. The measure on $(\min\gamma, \max\gamma)$ induced by μ^* and restricted to the set $\{a \in I, b \in (a, M(a))\}$ is $1_{a\in I, b\in(a,M(a))} \frac{da\, db}{u^{+,a}(b)u^{-,b}(a)}$. If $a < b < M(a)$, then the probability measure obtained through conditioning by $(\min\gamma, \max\gamma) = (a, b)$ has the same description as in corollary 2.16. Outside the set $\{a \in I, b \in (a, M(a))\}$, the measure on $(\min\gamma, \max\gamma)$ is not locally finite. That is to say that, if $a < b \in I$ and $b \geq M(a)$, then for all $\varepsilon > 0$.*

$$\mu^*(\{\min\gamma \in (a, a+\varepsilon), \max\gamma \in (b-\varepsilon, b)\}) = +\infty \tag{2.52}$$

Proof. For the behaviour on $\{a \in I, b \in (a, M(a))\}$: There is a countable collection $(I_j)_{j\geq 0}$ of open subintervals of I such that

$$\{a \in I, b \in (a, M(a))\} = \bigcup_{j\geq 0} \{x < y \in I_j\}$$

Since for all j, $L_{|I_j} \in \mathfrak{D}^{0,-}$, corollary 2.16 applies to $L_{|I_j}$. Combining the descriptions on different $\{a < b \in I_j\}$, we get the description on $\{a \in I, b \in (a, M(a))\}$.

For the behaviour outside $\{a \in I, b \in (a, M(a))\}$: Let $A < B \in \mathbb{R}$. Then

$$\mu^*_{BM}(\{\min\gamma < A, \max\gamma > B\}) = \int_B^{+\infty} \int_{-\infty}^{A} \frac{da\, db}{(b-a)^2} = +\infty \tag{2.53}$$

If $a < b \in I$ and $M(a) = b$, then $1_{a<\gamma<b}\mu^*$ is the image of μ^*_{BM} through a change of scale and time. In this case (2.52) follows from (2.53). If $b > M(a)$, then $L_{|(a,b)} \in \mathfrak{D}^+$. According to proposition 1.7 (iv), there is a positive measure Radon measure k on (a, b) such that $L_{|(a,b)} - k \in \mathfrak{D}^0$. From what precedes, (2.52) holds for $\mu^*_{L_{|(a,b)}-k}$. Moreover, $\mu^*_{L_{|(a,b)}} \geq \mu^*_{L_{|(a,b)}-k}$. So (2.52) holds for $\mu^*_{L_{|(a,b)}}$. □

3. Occupation fields of the Poissonian ensembles of Markov loops

3.1. **Inhomogeneous continuous state branching processes with immigration.** We will identify the occupation fields of the Poissonian ensembles of Markov loops as inhomogeneous continuous state branching processes with immigration. In this subsection we give the basic properties of such processes.

Let I be an open interval of $\mathbb{R}$. We will consider stochastic processes where $x \in I$ is the evolution variable. We do not call it time because in the sequel it will rather represent a space variable. Let $(\mathbb{B}_x)_{x\in\mathbb{R}}$ be a standard Brownian motion. Consider the following SDE:

$$d\tilde{Z}_x = \sigma(x)\sqrt{\tilde{Z}_x}\, d\mathbb{B}_x + b(x)\tilde{Z}_x\, dx \tag{3.1}$$

$$dZ_x = \sigma(x)\sqrt{Z_x}\, d\mathbb{B}_x + b(x)Z_x\, dx + c(x)\, dx \tag{3.2}$$

For our needs we will assume that σ is positive and continuous on I, that b and c are only locally bounded and that c is non negative. In this case existence and pathwise uniqueness holds for (3.1) and (3.2) (see [19], chapter IX, §3), and $\tilde{Z}$ and Z take values in $\mathbb{R}_+$. 0 is an absorbing state for $\tilde{Z}$.

(3.1) satisfies the branching property: if $\tilde{Z}^{(1)}$ and $\tilde{Z}^{(2)}$ are two independent processes solutions in law to (3.1), defined on $I\cap[x_0,+\infty)$, then $\tilde{Z}^{(1)}+\tilde{Z}^{(2)}$ is a solution in law to (3.1). If $\tilde{Z}$ and Z are two independent processes, $\tilde{Z}$ solution in law to (3.1) and Z solution in law to (3.2), defined on $I\cap[x_0,+\infty)$, then $Z+\tilde{Z}$ is a solution in law to (3.2). Solutions to (3.2) are (inhomogeneous) continuous state branching processes with immigration. The branching mechanism is given by (3.1) and the immigration measure is $c(x)\, dx$. The homogeneous case (σ, b and c constant) was extensively studied. See [12].

The case of inhomogeneous branching without immigration reduces to the homogeneous case as follows: Let $x_0 \in I$ and let

$$C(x) := \exp\left(-\int_{x_0}^{x} b(y)\, dy\right)$$

$$A(x) := \int_{x_0}^{x} \sigma(y)^2 C(y)^2\, dy$$

If $(\tilde{Z}_x)_{x\in I}$ is a solution to (3.1), then $(C(A^{-1}(a))\tilde{Z}_{A^{-1}(a)})_{a\in A(I)}$ is a solution in law to

$$d\tilde{\mathcal{Z}}_a = 2\sqrt{\tilde{\mathcal{Z}}_a}\, d\mathbb{B}_a$$

Let $\tilde{Z}$ be a solution to (3.1) defined on $I\cap[x_0,+\infty)$, starting at x_0 with the initial condition $\tilde{Z}_{x_0} = z_0 \geq 0$. Then, for $\lambda \geq 0$ and $x \in I$, $x \geq x_0$:

$$\mathbb{E}_{\tilde{Z}_{x_0}=z_0}\left[e^{-\lambda\tilde{Z}_x}\right] = e^{-z_0\psi(x_0,x,\lambda)}$$

$\psi(x_0,x,\lambda)$ depends continuously on (x_0,x,λ). If $x = x_0$ then

$$\psi(x_0,x_0,\lambda) = \lambda \tag{3.3}$$

If $x_0 \leq x_1 \leq x_2 \in I$ then

$$\psi(x_0,x_2,\lambda) = \psi(x_0,x_1,\psi(x_1,x_2,\lambda))$$

ψ satisfies the differential equation

$$\frac{\partial \psi}{\partial x_0}(x_0, x, \lambda) = \frac{\sigma(x_0)^2}{2}\psi(x_0, x, \lambda)^2 - b(x_0)\psi(x_0, x, \lambda) \tag{3.4}$$

If b is not continuous, equation (3.4) should be understand in the weak sense. If be is continuous, then (3.4) satisfies the Cauchy-Lipschitz conditions, and ψ is uniquely determined by (3.4) and the initial condition (3.3). This is also the case even if b is not continuous. Indeed, by considering $C(x)\tilde{Z}_x$ rather than $\tilde{Z}_x$, that is to say considering $\dfrac{C(x)}{C(x_0)}\psi(x_0, x, \lambda)$ rather than $\psi(x_0, x, \lambda)$, we get rid of b.

Inhomogeneous branching processes are related to the local times of general one-dimensional diffusions:

Proposition 3.1. *Let $x_0 \in I$ and let $(X_t)_{0\leq t<\zeta}$ be a diffusion on I of generator L of form (1.3) starting from x_0. Let $z_0 > 0$ and*

$$\tau_{z_0}^{x_0} := \inf\{t \geq 0 | \ell_t^{x_0}(X) > z_0\}$$

Then conditionally on $\tau_{z_0}^{x_0} < \zeta$, $(\ell^x_{\tau_{z_0}^{x_0}}(X))_{x\in I, x\geq x_0}$ is a solution in law to the SDE:

$$d\tilde{Z}_x = \sqrt{2w(x)}\sqrt{\tilde{Z}_x}\, d\mathbb{B}_x + 2\frac{d\log u_\downarrow}{dx}(x)\tilde{Z}_x\, dx \tag{3.5}$$

Proof. If X is the Brownian motion on $\mathbb{R}$, then $w \equiv 2$ and $u_\downarrow$ is constant. In this case the assertion is the second Ray-Knight theorem. See [19], chapter XI, §2. The equation (3.5) is then the equation of a square of Bessel 0 process. If $x_{min} < x_0$ and X is the Brownian motion on $(x_{min}, +\infty)$ killed in x_{min} then the law of $(\ell^x_{\tau_{z_0}^{x_0}}(X))_{x\in I, x\geq x_0}$ conditionally on $\tau_{z_0}^{x_0} < \zeta$ does not depend on x_{min} and is the same as in case of the Brownian motion on $\mathbb{R}$. Equation (3.5) is still satisfied.

If X is a diffusion on I that satisfies that for all $x > a \in I$, starting from x, X reaches almost surly a, which is equivalent to $u_\downarrow$ being constant, then through a change of scale and time X is the Brownian motion on some $(x_{min}, +\infty)$ where $x_{min} \in [-\infty, +\infty)$. Time change does not change the local times because we defined them relatively to the speed measure. Only the change of scale matters. If S is a primitive of w, then conditionally on $\tau_{z_0}^{x_0} < \zeta$, $(\ell^{S^{-1}(2y)}_{\tau_{z_0}^{x_0}}(X))_{y\geq \frac{1}{2}S(x_0)}$ is a square of Bessel 0 process. The equation (3.5) follows from the equation of the square of Bessel 0 process by deterministic change of variable $dy := \frac{1}{2}w(x)\, dx$.

Now the general case: let $(\tilde{X}_t)_{0\leq t<\tilde{\zeta}}$ be the diffusion of generator $Conj(u_\downarrow, L)$. $\dfrac{w(x)}{u_\downarrow(x)^2}\, dx$ is the natural scale measure of $\tilde{X}$ and $u_\downarrow(x)^2 m(x)\, dx$ is its speed measure. We assume that both X and $\tilde{X}$ start from x_0. The law of $\tilde{X}$ up to the last time it visits x_0 is the same as for X. Let

$$\tilde{\tau} := \inf\left\{t \geq 0 | \ell_t^{x_0}(\tilde{X}) > \frac{1}{u_\downarrow(x_0)^2} z_0\right\}$$

Then the law of $(\ell^x_{\tau_{z_0}^{x_0}}(X))_{x\in I, x\geq x_0}$ conditionally on $\tau_{z_0}^{x_0} < \zeta$ is the same as the law of $(u_\downarrow(x)^2 \ell^x_{\tilde{\tau}}(\tilde{X}))_{x\in I, x\geq x_0}$ conditionally on $\tilde{\tau} < \tilde{\zeta}$. The factor $u_\downarrow(x)^2$ comes from the fact that

performing an h-transform we change the measure relatively to which the local times are defined. For any $a < x_0 \in I$, $\tilde{X}$ reaches a a.s. Thus $(\ell^x_{\tilde{\tau}}(\tilde{X}))_{x\in I, x\geq x_0}$ satisfies the SDE

$$d\tilde{Z}_x = \frac{\sqrt{2w(x)}}{u_\downarrow(x)}\sqrt{\tilde{Z}_x}\, d\mathbb{B}_x$$

and $(u_\downarrow(x)^2 \ell^x_{\tilde{\tau}^{x_0}_{z_0}}(\tilde{X}))_{x\in I, x\geq x_0}$ satisfies (3.5). □

If there is immigration: Let Z be a solution to (3.2) defined on $I \cap [x_0, +\infty)$, starting at x_0 with the initial condition $Z_{x_0} = z_0 \geq 0$. Then, for $\lambda \geq 0$ and $x \in I$, $x \geq x_0$:

$$\mathbb{E}_{Z_{x_0}=z_0}\left[e^{-\lambda Z_x}\right] = \exp\left(-z_0\psi(x_0, x, \lambda) - \int_{x_0}^x \psi(y, x, \lambda)c(y)\, dy\right) \tag{3.6}$$

3.2. **Occupation field.** Let L be the generator of a diffusion on I of form (1.3). Let $\mathcal{L}_{\alpha,L}$ be a Poissonian ensemble of intensity $\alpha\mu^*_L$. $\mathcal{L}_{\alpha,L}$ is a random infinite countable collection of unrooted loops supported in I. It is sometimes called "loop soup".

Definition 5. *The occupation field of $\mathcal{L}_{\alpha,L}$ is $(\widehat{\mathcal{L}}^x_{\alpha,L})_{x\in I}$ where*

$$\widehat{\mathcal{L}}^x_{\alpha,L} := \sum_{\gamma\in\mathcal{L}_{\alpha,L}} \ell^x(\gamma)$$

We will drop out the subscript L whenever there is no ambiguity on L. In this subsection we will identify the law of $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ as an inhomogeneous continuous state branching process with immigration. If $\mathbb{J}$ is a discrete subset of I, then applying proposition 2.23 we deduce that $(\widehat{\mathcal{L}}^x_\alpha)_{x\in\mathbb{J}}$ is the occupation field of the Poisson ensemble of discrete loops of intensity $\alpha\mu^*_{L_\mathbb{J}}$ as defined in [10], chapter 4. This fact allows us to apply the results of [10] in order to describe the finite-dimensional marginals of the occupation field. If the diffusion is recurrent, then for all $x \in I$, $\widehat{\mathcal{L}}^x_\alpha = +\infty$ a.s. If the diffusion is transient, then for all $x \in I$, $\widehat{\mathcal{L}}^x_\alpha < +\infty$ a.s. Next we state how does the occupation field behave if we apply various transformations on L.

Property 3.2. *Let L be the generator of a transient diffusion.*

- *(i) If A is a change of scale function, then*
$$\widehat{\mathcal{L}}^{A(x)}_{\alpha,Scale^\dagger_A L} = \widehat{\mathcal{L}}^x_{\alpha,L}$$
- *(ii) If V is a positive continuous function on I, then*
$$\widehat{\mathcal{L}}^x_{\alpha,\frac{1}{V}L} = \widehat{\mathcal{L}}^x_{\alpha,L}$$
- *(iii) If h is a positive continuous function on I such that Lh is a negative measure, then*
$$\widehat{\mathcal{L}}^x_{\alpha,Conj(h,L)} = \frac{1}{h(x)^2}\widehat{\mathcal{L}}^x_{\alpha,L}$$

Previous equalities depend on a particular choice of the speed measure for the modification of L. For (i) we choose $\left(\frac{dA}{dx}\circ A^{-1}\right)^{-1} m \circ A^{-1}\, da$. For (ii) we choose $\frac{1}{V(x)}m(x)\, dx$. For (iii) we choose $h(x)^2 m(x)\, dx$. The fact that $\widehat{\mathcal{L}}^x_{\alpha,Conj(h,L)} \neq \widehat{\mathcal{L}}^x_{\alpha,L}$ despite $\mathcal{L}_{\alpha,Conj(h,L)} = \mathcal{L}_{\alpha,L}$ comes from a change of speed measure.

Next we characterize the finite-dimensional marginals of the occupation field by stating the results that appear in [10], chapter 4.

Property 3.3. *The distribution of* $\widehat{\mathcal{L}}_\alpha^x$ *is*

$$\frac{(G(x,x))^\alpha}{\Gamma(\alpha)}\, l^{\alpha-1} \exp\left(-\frac{l}{G(x,x)}\right) 1_{l>0}\, dl$$

Let $x_1, x_2, ..., x_n \in I$ *and* $\lambda_1, \lambda_2, ..., \lambda_n \geq 0$. *Let* $(\tilde{G}(x,y))_{x,y\in I}$ *be the Green's function of* $L - \sum_{i=1}^n \lambda_i \delta_{x_i}$. *Then*

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^n \lambda_i \widehat{\mathcal{L}}_\alpha^{x_i}\right)\right] = \left(\frac{\det(\tilde{G}(x_i,x_j))_{1\leq i,j\leq n}}{\det(G(x_i,x_j))_{1\leq i,j\leq n}}\right)^\alpha \tag{3.7}$$

The moment $\mathbb{E}\left[\widehat{\mathcal{L}}_\alpha^{x_1}\widehat{\mathcal{L}}_\alpha^{x_2}...\widehat{\mathcal{L}}_\alpha^{x_n}\right]$ *is an* α*-permanent:*

$$\mathbb{E}\left[\widehat{\mathcal{L}}_\alpha^{x_1}\widehat{\mathcal{L}}_\alpha^{x_2}...\widehat{\mathcal{L}}_\alpha^{x_n}\right] = \sum_{\sigma\in\mathfrak{S}_n} \alpha^{\sharp\, cycles\, of\, \sigma} \prod_{i=1}^n G(x_i, x_{\sigma(i)})$$

If $\mathbb{J}$ *is a discrete subset of* I, *then* $(\widehat{\mathcal{L}}_\alpha^x)_{x\in\mathbb{J}}$, *viewed as a stochastic process that evolves when* x *increases, is an inhomogeneous continuous state branching process with immigration defined on the discrete set* $\mathbb{J}$. *In particular, for any* $x_1 \leq x_2 \leq ... \leq x_n \in I$ *and* $p \in \{1, 2, ..., n\}$, $\left(\widehat{\mathcal{L}}_\alpha^{x_1}, \widehat{\mathcal{L}}_\alpha^{x_2}, ...\widehat{\mathcal{L}}_\alpha^{x_p}\right)$ *and* $\left(\widehat{\mathcal{L}}_\alpha^{x_p}, \widehat{\mathcal{L}}_\alpha^{x_{p+1}}, ...\widehat{\mathcal{L}}_\alpha^{x_n}\right)$ *are independent conditionally on* $\widehat{\mathcal{L}}_\alpha^{x_p}$.

Next we show that the processes $x \mapsto \widehat{\mathcal{L}}_\alpha^x$ parametrized by $x \in I$, where x is assumed to increase, is an inhomogeneous branching process with immigration of form (3.2). In particular, it has a continuous version and is inhomogeneous Markov.

Proposition 3.4. $(\widehat{\mathcal{L}}_\alpha^x)_{x\in I}$ *has the same finite-dimensional marginals as a solution to the stochastic differential equation*

$$dZ_x = \sqrt{2w(x)}\sqrt{Z_x}\, d\mathbb{B}_x + 2\frac{d\log u_\downarrow}{dx}(x) Z_x\, dx + \alpha w(x)\, dx \tag{3.8}$$

If L *is the generator of a Brownian motion on* $(0, +\infty)$ *killed when it hits* 0, *then* $(\widehat{\mathcal{L}}_\alpha^x)_{x>0}$ *has the same law as the square of a Bessel process of dimension* 2α *starting from* 0 *at* $x = 0$. *If* L *is the generator of a Brownian motion on* $(0, x_{max})$, *killed when hitting the boundary, then* $(\widehat{\mathcal{L}}_\alpha^x)_{0<x<x_{max}}$ *has the same law as the square of a Bessel bridge of dimension* 2α *from* 0 *at* $x = 0$ *to* 0 *at* $x = x_{max}$.

Proof. Let $x_0 < x \in I$ and $\lambda_0, \lambda \geq 0$. Applying the identity (3.7) to the case of two points, we get that

$$\mathbb{E}\left[\exp\left(-\lambda_0\widehat{\mathcal{L}}_\alpha^{x_0} - \lambda\widehat{\mathcal{L}}_\alpha^x\right)\right] = \left((1+\lambda_0 G(x_0,x_0))(1+\lambda G(x,x)) - \lambda_0\lambda(G(x_0,x))^2\right)^{-\alpha} \tag{3.9}$$

Let

$$\Lambda(x_0,\lambda_0) := \mathbb{E}\left[e^{-\lambda_0\widehat{\mathcal{L}}_\alpha^{x_0}}\right] = \left(\frac{G(x_0,x_0)}{G(x_0,x_0)+\lambda_0}\right)^\alpha$$

For $y \leq x$, let

$$\psi(y,x,\lambda) := \frac{G(x,y)G(y,x)\lambda}{G(y,y)(G(y,y)+\lambda\det_{y,x} G)}$$

$$\varphi(y,x,\lambda) := -\log\left(\frac{G(y,y)}{G(y,y)+\lambda \det_{y,x} G}\right)$$

One can check that the right-hand side of (3.9) equals

$$\Lambda(x_0, \lambda_0 + \psi(x_0,x,\lambda)) \exp(-\alpha\varphi(x_0,x,\lambda))$$

In particular for the conditional Laplace transform:

$$\mathbb{E}\left[\exp\left(-\lambda\widehat{\mathcal{L}}^x_\alpha\right)|\widehat{\mathcal{L}}^{x_0}_\alpha\right] = \exp\left(-\widehat{\mathcal{L}}^{x_0}_\alpha\psi(x_0,x,\lambda)\right)\exp(-\alpha\varphi(x_0,x,\lambda)) \ a.s. \tag{3.10}$$

Moreover

$$\begin{aligned}\frac{\partial\psi}{\partial y}(y,x,\lambda) &= W(u_\downarrow,u_\uparrow)(y)\psi(y,x,\lambda)^2 - \frac{2}{u_\downarrow(y)}\frac{du_\downarrow}{dy}(y)\psi(y,x,\lambda)\\ &= w(y)\psi(y,x,\lambda)^2 - 2\frac{d\log u_\downarrow}{dy}(y)\psi(y,x,\lambda)\end{aligned}$$

and

$$\frac{\partial\varphi}{\partial y}(y,x,\lambda) = -W(u_\downarrow,u_\uparrow)(y)\psi(y,x,\lambda) = -w(y)\psi(y,x,\lambda)$$

and we have the initial conditions $\psi(x,x,\lambda)=\lambda$ and $\varphi(x,x,\lambda)=0$. Thus (3.10) has the same form as (3.6) where $c(y)=\alpha w(y)$. Let $(Z_y)_{y\in I, y\geq x_0}$ be a solution to (3.8) with the initial condition Z_{x_0} being a gamma random variable of parameter α with mean $\alpha G(x_0,x_0)$. It follows from what precedes that $(\widehat{\mathcal{L}}^{x_0}_\alpha, \widehat{\mathcal{L}}^x_\alpha)$ has the same law as (Z_{x_0}, Z_x). Using the conditional independence satisfied by the occupation field, we deduce that $(\widehat{\mathcal{L}}^y_\alpha)_{y\in I, y\geq x_0}$ has the same finite-dimensional marginals as $(Z_y)_{y\in I, y\geq x_0}$. Making x_0 converge to $\inf I$ along a countable subset, we get a consistent family of continuous stochastic processes, which induces a continuous stochastic process $(Z_y)_{y\in I}$ defined on whole I. It satisfies (3.8) and has the same finite-dimensional marginals as $(\widehat{\mathcal{L}}^y_\alpha)_{y\in I}$.

In case of a Brownian motion in $(0,+\infty)$ killed in 0, the equation (3.8) becomes

$$dZ_x = 2\sqrt{Z_x}\,\mathbb{B}_x + 2\alpha\,dx$$

which is the SDE satisfied by the square of a Bessel process of dimension 2α. Moreover $(\widehat{\mathcal{L}}^x_\alpha)_{x>0}$ has the same one-dimensional marginals as the latter, more precisely $\widehat{\mathcal{L}}^x_\alpha$ is a gamma r.v. of parameter α with mean $2\alpha x$. This shows the equality in law.

In case of a Brownian motion in $(0, x_{max})$ killed in 0 and x_{max} the equation (3.8) becomes

$$dZ_x = 2\sqrt{Z_x}\,\mathbb{B}_x + \frac{1}{x_{max}-x}Z_x\,dx + 2\alpha\,dx$$

which is the SDE satisfied by the square of a Bessel bridge of dimension 2α from 0 at $x=0$ to 0 at $x=x_{max}$. Moreover the latter process and $(\widehat{\mathcal{L}}^x_\alpha)_{0<x<x_{max}}$ have the same one-dimensional marginals, more precisely gamma r.v. of parameter α with mean $2\alpha(x_{max}-x)\dfrac{x}{x_{max}}$. Thus the two have the same law. □

We showed that $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ has the same finite-dimensional marginals as a continuous stochastic process. We will assume in the sequel and prove in section 4.2 that one can couple the Poissonian ensemble $\mathcal{L}_\alpha$ and a continuous version of its occupation field $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ on the same probability space. This does not follow trivially from the fact that the process $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ has a continuous version. Consider the following counterexample: Let U be an uniform r.v. on $(0,1)$. Let $\mathcal{E}$ be

a countable random set of Brownian excursions defined as follows: conditionally on U $\mathcal{E}$ is a Poissonian ensemble with intensity $\eta_{BM}^{>U}+\eta_{BM}^{<U}$. Let $(\widehat{\mathcal{E}}_x)_{x\in\mathbb{R}}$ be the occupation field of $\mathcal{E}$. Then $\widehat{\mathcal{E}}$ is continuous on $(-\infty,U)$ and $(U,+\infty)$ but not at U. Indeed $\widehat{\mathcal{E}}_U=0$ and

$$\lim_{x\to U^-}\widehat{\mathcal{E}}_x=\lim_{x\to U^-}\widehat{\mathcal{E}}_x=1$$

Let $(\widehat{\mathcal{E}}'_x)_{x\in\mathbb{R}}$ be the field defined by: $\widehat{\mathcal{E}}'_x=\widehat{\mathcal{E}}_x$ if $x\neq U$ and $\widehat{\mathcal{E}}'_U=1$. $(\widehat{\mathcal{E}}'_x)_{x\in\mathbb{R}}$ is continuous and for any fixed $x\in\mathbb{R}$ $\widehat{\mathcal{E}}'_x=\widehat{\mathcal{E}}_x$ a.s. Thus $(\widehat{\mathcal{E}}'_x)_{x\in\mathbb{R}}$ is a continuous version of the process $(\widehat{\mathcal{E}}_x)_{x\in\mathbb{R}}$ but it can not be implemented as a sum of local time across the excursions in $\mathcal{E}$. As we will show in section 4.2, such a difficulty does not arise in case of $\mathcal{L}_\alpha$.

$(\widehat{\mathcal{L}}_\alpha^x)_{x\in I}$ is an inhomogeneous continuous state branching with immigration. The branching mechanism is the same as for the local times of the diffusion X, given by (3.5). The immigration measure is $\alpha w(x)\,dx$. The interpretation is the following: given a loop in $\mathcal{L}_\alpha$, its family of local times performs a branching according to the mechanism (3.5), independently from the other loops. The immigration between x and $x+\Delta x$ comes from the loops whose minima belong to $(x,x+\Delta x)$. It is remarkable that although the immigration measure is absolutely continuous with respect to Lebesgue measure, there is only a countable number of moments at which immigration occurs. These are the positions of the minima of loops in $\mathcal{L}_\alpha$. Moreover the local time of each loop at its minimum is zero. For $x>a\in I$, let

$$\widehat{\mathcal{L}}_\alpha^{(a),x}:=\sum_{\substack{\gamma\,\in\,\mathcal{L}_\alpha\\ \min\gamma\,>\,a}}\ell^x(\gamma)$$

Let $a<b\in I$. For $j\leq n\in\mathbb{N}$, let $\Delta x_n:=\dfrac{1}{n}(b-a)$ and let $x_{j,n}:=a+j\Delta x_n$. Then $\left(\widehat{\mathcal{L}}_\alpha^{(x_{j-1}),x_j}\right)_{1\leq j\leq n}$ is a sequence of independent gamma r.v. of parameter α and the mean of $\widehat{\mathcal{L}}_\alpha^{(x_{j-1}),x_j}$ is $\alpha\left(G(x_j,x_j)-\dfrac{G(x_{j-1},x_j)G(x_j,x_{j-1})}{G(x_{j-1},x_{j-1})}\right)$. For n large

$$G(x_j,x_j)-\frac{G(x_{j-1},x_j)G(x_j,x_{j-1})}{G(x_{j-1},x_{j-1})}=w(x_{j-1})\Delta x_n+o(\Delta x_n)$$

and $o(\Delta x_n)$ is uniform in j. Thus

$$\lim_{n\to+\infty}\mathbb{E}\Big[\sum_{j=1}^n\widehat{\mathcal{L}}_\alpha^{(x_{j-1}),x_j}\Big]=\lim_{n\to+\infty}\alpha\sum_{j=1}^n\left(G(x_j,x_j)-\frac{G(x_{j-1},x_j)G(x_j,x_{j-1})}{G(x_{j-1},x_{j-1})}\right)=\alpha\int_a^b w(x)\,dx$$

and

$$\lim_{n\to+\infty}Var\Big(\sum_{j=1}^n\widehat{\mathcal{L}}_\alpha^{(x_{j-1}),x_j}\Big)=\lim_{n\to+\infty}\alpha\sum_{j=1}^n\left(G(x_j,x_j)-\frac{G(x_{j-1},x_j)G(x_j,x_{j-1})}{G(x_{j-1},x_{j-1})}\right)^2=0$$

It follows that $\sum_{j=1}^n\widehat{\mathcal{L}}_\alpha^{(x_{j-1}),x_j}$ converges in probability to $\alpha\int_a^b w(x)\,dx$. This is consistent with our interpretation of immigration.

Next proposition deals with the zeroes of the occupation field.

Proposition 3.5. *Let $x_0\in I$. If $\int_{\inf I}^{x_0}w(x)\,dx<+\infty$ then*

$$\lim_{x\to\inf I}\widehat{\mathcal{L}}_\alpha^x=0$$

Analogous result holds if $\int_{x_0}^{\sup I} w(x)\,dx < +\infty$.

If $\alpha \geq 1$, *then the continuous process* $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ *stays almost surely positive on* I. *If* $\alpha < 1$ *then* $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ *hits* 0 *infinitely many times on* I.

Proof. If $\int_{\inf I}^{x_0} w(x)\,dx < +\infty$, then $L + k$, where k is the killing measure of L, is also the generator of a transient diffusion. We can couple $(\widehat{\mathcal{L}}^x_{\alpha,L})_{x\in I}$ and $(\widehat{\mathcal{L}}^x_{\alpha,L+k})_{x\in I}$ on the same probability space such that a.s. for all $x \in I$, $\widehat{\mathcal{L}}^x_{\alpha,L} \leq \widehat{\mathcal{L}}^x_{\alpha,L+k}$. But according to property 3.2 (i), $(\widehat{\mathcal{L}}^x_{\alpha,L+k})_{x\in I}$ is just a scale changed square of Bessel process starting from 0 or square of a Bessel bridge from 0 to 0. Thus

$$\lim_{x\to\inf I} \widehat{\mathcal{L}}^x_{\alpha,L} \leq \lim_{x\to\inf I} \widehat{\mathcal{L}}^x_{\alpha,L+k} = 0$$

Regarding the number of zeros of $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ on I, property 3.2 ensures that it remains unchanged if we apply scale, time changes and h-transforms to L. Since any generator of a transient diffusion is equivalent through latter transformation to the generator of a Brownian motion on $(0, +\infty)$ killed in 0, the result on the number of zeros of $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ follows from standard properties of Bessel processes. □

In [21] respectively [11] are studied the clusters of loops induced by a Poisson ensemble of loops in the setting of conformal diffusions respectively Markovian jump processes on graphs. In our setting of one dimensional diffusions the description of such clusters is simple and is related to the zeros of the occupation field. We introduce an equivalence relation on the loops of $\mathcal{L}_\alpha$: γ is in the same class as γ' if there is a chain of loops $\gamma_0, \gamma_1, ..., \gamma_n$ in $\mathcal{L}_\alpha$ such that $\gamma_0 = \gamma$, $\gamma_n = \gamma'$ and for all $i \in \{0, 1, ..., n-1\}$, $\gamma_i([0, T(\gamma_i)]) \cap \gamma_{i+1}([0, T(\gamma_{i+1})]) \neq \emptyset$. A cluster is the union of all $\gamma([0, T(\gamma)])$ where the loops γ belong to the same equivalence class. It is a subinterval of I. By definition clusters corresponding to different equivalence classes are disjoint.

Proposition 3.6. *Let* L *be the generator of a transient diffusion on* I. *If* $\alpha \geq 1$, *the loops in* $\mathcal{L}_\alpha$ *form a single cluster:* I. *If* $\alpha \in (0, 1)$, *there are infinitely many clusters. These are the maximal open intervals on which* $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ *is positive. In case of the Brownian motion on* $(0, +\infty)$ *killed at* 0, *the clusters correspond to the jumps of a stable subordinator with index* $1 - \alpha$. *In case of a general diffusion, by performing a change of scale of derivative* $\frac{1}{2}\frac{w}{u_\downarrow^2}$, *we reduce the problem to the previous case.*

Proof. Assume that $\mathcal{L}_\alpha$ and a continuous version of $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ are defined on the same probability space. Almost surely the following holds

- Given $\gamma \neq \gamma' \in \mathcal{L}_\alpha$, $\min\gamma \neq \max\gamma'$ and $\max\gamma \neq \min\gamma'$.
- For all $\gamma \in \mathcal{L}_\alpha$, $\ell^{\min\gamma}(\gamma) = \ell^{\max\gamma}(\gamma) = 0$ and $\ell^x(\gamma)$ is positive for $x \in (\min\gamma, \max\gamma)$.

Whenever the above two conditions hold it follows deterministically that the clusters are the intervals on which $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ stays positive. We deduce then the number of clusters from proposition 3.5.

If L is the generator of the Brownian motion on $(0,+\infty)$ killed at 0, then $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ is the square of a Bessel process of dimension 2α and its excursions correspond to the jumps of a stable subordinator with index $1-\alpha$. □

The clusters coalesce when α increases and fragment when α decreases. Some information on the coalescence of clusters delimited by the zeroes of Bessel processes is given in [1], section 3. This clusters can be obtained as a limit of clusters of discrete loops on discrete subsets. In case of a symmetric jump process to the nearest neighbours on $\varepsilon\mathbb{N}$, if $\alpha > 1$, there are finitely many clusters, and if $\alpha \in (0,1)$, there are infinitely many clusters and these clusters are given by the holding times of a renewal process, which suitable normalized converges in law as $\varepsilon \to 0^+$ to the inverse of a stable subordinator with index $1-\alpha$. See remark 3.3 in [11].

We can consider the occupation field $(\widehat{\mathcal{L}}^x_{\alpha,L})_{x\in I}$ if L is not the generator of a diffusion but contains creation of mass as in (1.12). In this setting, if h is a positive continuous function on I such that $\dfrac{d^2h}{dx^2}$ is a signed measure, then for all $x \in I$

$$\widehat{\mathcal{L}}^x_{\alpha,Conj(h,L)} = \frac{1}{h(x)^2}\widehat{\mathcal{L}}^x_{\alpha,L}$$

It follows that if $L \in \mathfrak{D}^-$ then for all $x \in I$, $\widehat{\mathcal{L}}^x_{\alpha,L} < +\infty$ a.s. and if $L \in \mathfrak{D}^0$ then for all $x \in I$, $\widehat{\mathcal{L}}^x_{\alpha,L} = +\infty$ a.s. If $L \in \mathfrak{D}^+$, then according to proposition 1.7 (iv), there is a positive Radon measure $\tilde{k}$ such that $L - \tilde{k} \in \mathfrak{D}^0$. Then for all $x \in I$, $\widehat{\mathcal{L}}^x_{\alpha,L} \geq \widehat{\mathcal{L}}^x_{\alpha,L-\tilde{k}} = +\infty$. If $L \in \mathfrak{D}^-$, then properties 3.2 (i) and (ii) are still hold. The description given by the property 3.3 of the finite-dimensional marginals of $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ is still true, although the case of creation of mass wasn't considered in [10]. $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ still satisfies the SDE (3.8).

Proposition 3.7. *Let $L \in \mathfrak{D}^-$ and $\tilde{\nu}$ a finite signed measure with compact support in I. Then there is equivalence between*

- *(i)* $\mathbb{E}\left[\exp\left(\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde{\nu}(dx)\right)\right] < +\infty$
- *(ii)* $L + \tilde{\nu} \in \mathfrak{D}^-$

If $L + \tilde{\nu} \in \mathfrak{D}^-$, let for $s \in [0,1]$ $G_{s\tilde{\nu}}$ be the Green function of $L + s\tilde{\nu}$. Then

$$\mathbb{E}\left[\exp\left(\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde{\nu}(dx)\right)\right] = \exp\left(\alpha\int_0^1\int_I G_{s\tilde{\nu}}(x,x)\,\tilde{\nu}(dx)\,ds\right) \tag{3.11}$$

Proof. First observe that $\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,|\tilde{\nu}|(dx)$ is almost surely finite because $|\tilde{\nu}|$ is finite and has compact support and $(\widehat{\mathcal{L}}^x_{\alpha,L})_{x\in I}$ is continuous. Also observe that $\mathfrak{D}^-$ is convex. So if $L+\tilde{\nu} \in \mathfrak{D}^-$, then for all $s \in [0,1]$, $L + s\tilde{\nu} \in \mathfrak{D}^-$.

(i) implies (ii): Let $\mathbb{P}_{\mathcal{L}_{\alpha,L}}$ be the law of $\mathcal{L}_{\alpha,L}$ and $\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde{\nu}}}$ be the law of $\mathcal{L}_{\alpha,L+\tilde{\nu}}$. There is an absolute continuity relation between the intensity measures:

$$\mu_{L+\tilde{\nu}}(d\gamma) = \exp\left(\int_I \ell^x(\gamma)\right)\mu_L(d\gamma)$$

In case (i) is true $\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde\nu}}$ is absolutely continuous with respect to $\mathbb{P}_{\mathcal{L}_{\alpha,L}}$ and

$$d\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde\nu}} = \frac{\exp\left(\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu(dx)\right)}{\mathbb{E}\left[\exp\left(\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu(dx)\right)\right]}\,d\mathbb{P}_{\mathcal{L}_{\alpha,L}} \tag{3.12}$$

But this can not be if $L+\tilde\nu \notin \mathfrak{D}^-$ because then for any $x\in I$, $\widehat{\mathcal{L}}^x_{\alpha,L} < +\infty$ and $\widehat{\mathcal{L}}^x_{\alpha,L+\tilde\nu} = +\infty$. Thus necessarily $L+\tilde\nu \in \mathfrak{D}^-$.

(ii) implies (i): We first assume that $\tilde\nu$ is a positive measure and $L+\tilde\nu\in\mathfrak{D}^-$. Then $\mathbb{P}_{\mathcal{L}_{\alpha,L}}$ is absolutely continuous with respect to $\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde\nu}}$ and

$$d\mathbb{P}_{\mathcal{L}_{\alpha,L}} = \frac{\exp\left(-\int_I \widehat{\mathcal{L}}^x_{\alpha,L+\tilde\nu}\,\tilde\nu(dx)\right)}{\mathbb{E}\left[\exp\left(-\int_I \widehat{\mathcal{L}}^x_{\alpha,L+\tilde\nu}\,\tilde\nu(dx)\right)\right]}\,d\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde\nu}}$$

Inverting the above absolute continuity relation, we get that

$$\mathbb{E}\left[\exp\left(\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu(dx)\right)\right] = \mathbb{E}\left[\exp\left(-\int_I \widehat{\mathcal{L}}^x_{\alpha,L+\tilde\nu}\,\tilde\nu(dx)\right)\right]^{-1} < +\infty$$

If $\tilde\nu$ is not positive, let $\tilde\nu^+$ and $-\tilde\nu^-$ be its positive respectively negative part. Then

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu(dx)\right)\right] &= \mathbb{E}\left[\exp\left(\int_I \widehat{\mathcal{L}}^x_{\alpha,L-\tilde\nu^-}\,\tilde\nu^+(dx)\right)\right]\mathbb{E}\left[\exp\left(-\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu^-(dx)\right)\right]\\ &= \frac{\mathbb{E}\left[\exp\left(-\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu^-(dx)\right)\right]}{\mathbb{E}\left[\exp\left(-\int_I \widehat{\mathcal{L}}^x_{\alpha,L+\tilde\nu}\,\tilde\nu^+(dx)\right)\right]} < +\infty\end{aligned}$$

For the expression (3.11) of exponential moments:

$$\frac{d}{ds}\mathbb{E}\left[\exp\left(s\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu(dx)\right)\right] = \mathbb{E}\left[\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu(dx)\exp\left(s\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu(dx)\right)\right] \tag{3.13}$$

From the absolute continuity relation (3.12) follows that the right-hand side of (3.13) equals

$$\alpha\int_I G_{s\tilde\nu}(x,x)\tilde\nu(dx)\;\mathbb{E}\left[\exp\left(s\int_I \widehat{\mathcal{L}}^x_{\alpha,L}\,\tilde\nu(dx)\right)\right]$$

This implies (3.11) □

As in discrete space case, the above exponential moments can be expressed using determinants. On the complex Hilbert space $\mathbb{L}^2(d|\tilde\nu|)$ define for $s\in[0,1]$ the operators

$$(\mathfrak{G}_{s\tilde\nu}f)(x) := \int_I G_{s\tilde\nu}(x,y)f(y)\,\tilde\nu(dy)$$

$$(|\mathfrak{G}_{s\tilde\nu}|f)(x) := \int_I G_{s\tilde\nu}(x,y)f(y)\,|\tilde\nu|(dy)$$

The operator $|\mathfrak{G}_{s\tilde\nu}|$ is self-adjoint, positive semi-definite with continuous kernel function, and according to [22], theorem 2.12, it is trace class. Since trace class operators form a two-sided ideal in the algebra of bounded operators, $\mathfrak{G}_{s\tilde\nu}$ is also trace class. Moreover

$$Tr(\mathfrak{G}_{s\tilde\nu}) = \int_I G_{s\tilde\nu}(x,x)\,\tilde\nu(dx) \tag{3.14}$$

The determinant $\det(Id + \mathfrak{G}_{s\tilde{\nu}})$ is well defined as a converging product of eigenvalues (see [22], chapter 3). For any $s \neq s' \in [0,1]$, the following resolvent identity holds

$$\mathfrak{G}_{s\tilde{\nu}}\mathfrak{G}_{s'\tilde{\nu}} = \mathfrak{G}_{s'\tilde{\nu}}\mathfrak{G}_{s\tilde{\nu}} = \frac{1}{s'-s}(\mathfrak{G}_{s'\tilde{\nu}} - \mathfrak{G}_{s\tilde{\nu}}) \tag{3.15}$$

Proposition 3.8.

$$\exp\left(\alpha \int_0^1 \int_I G_{s\tilde{\nu}}(x,x)\, \tilde{\nu}(dx)\, ds\right) = (\det(Id + \mathfrak{G}_{\tilde{\nu}}))^{\alpha}$$

Proof. $\mathfrak{G}_{\tilde{\nu}}$ has only real eigenvalues. Indeed, let λ be such an eigenvalue and f a non zero eigenfunction for λ. Let $sgn(\tilde{\nu})$ be the $\{-1,+1\}$-valued function defined $d|\tilde{\nu}|$ almost everywhere. Then

$$\int_I (sgn(\tilde{\nu})\bar{f})|\mathfrak{G}_{\tilde{\nu}}|(sgn(\tilde{\nu})f)(x)|\tilde{\nu}|(dx) = \lambda \int_I |f|^2(x)\tilde{\nu}(dx) \tag{3.16}$$

The left-hand side of (3.16) is non-negative. If the right-hand side of (3.16) is non-zero, then λ is real. If it is zero, consider $f_\varepsilon := f + \varepsilon\, sgn(\tilde{\nu})f$. Then

$$\lambda = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon}\left(\int_I (sgn(\tilde{\nu})\bar{f}_\varepsilon)|\mathfrak{G}_{\tilde{\nu}}|(sgn(\tilde{\nu})f_\varepsilon)(x)|\tilde{\nu}|(dx)\right)\left(\int_I |f|^2(x)|\tilde{\nu}|(dx)\right)^{-1}$$

and thus λ is real.

The operators $\mathfrak{G}_{s\tilde{\nu}}$ are compact and the characteristic space corresponding to each of their non-zero eigenvalue is of finite dimension. Let $(\lambda_i)_{i\geq 0}$ be the non-increasing sequence of positive eigenvalues of $\mathfrak{G}_{\tilde{\nu}}$. Each eigenvalue λ_i appears as many times as the dimension of its characteristic space $\ker(\mathfrak{G}_{\tilde{\nu}} - \lambda_i Id)^n$ (n large enough). Similarly let $(-\tilde{\lambda}_j)_{j\geq 0}$ be the non-decreasing sequence of the negative eigenvalues of $\mathfrak{G}_{\tilde{\nu}}$. Let $s \in [0,1]$. Since $\mathfrak{G}_{\tilde{\nu}}$ and $\mathfrak{G}_{s\tilde{\nu}}$ commute, these operators have common characteristic spaces. From (3.15) follows that $\left(\frac{\lambda_i}{1+(1-s)\lambda_i}\right)_{i\geq 0}$ is a non-increasing sequence of positive eigenvalues of $\mathfrak{G}_{s\tilde{\nu}}$. If $\frac{-1}{1-s}$ is not an eigenvalue of $\mathfrak{G}_{\tilde{\nu}}$, then $\left(\frac{-\tilde{\lambda}_j}{1-(1-s)\tilde{\lambda}_j}\right)_{j\geq 0}$ is also a sequence of eigenvalues of $\mathfrak{G}_{s\tilde{\nu}}$. But the family of operators $(\mathfrak{G}_{s\tilde{\nu}})_{s\in[0,1]}$ is bounded. Thus none of $\frac{-\tilde{\lambda}_j}{1-(1-s)\tilde{\lambda}_j}$ can blow up when s varies. So it turns out that $\mathfrak{G}_{\tilde{\nu}}$ has no eigenvalues in $(-\infty,-1]$. From (3.14) we get

$$\int_I G_{s\tilde{\nu}}(x,x)\, \tilde{\nu}(dx) = \sum_{i\geq 0} \frac{\lambda_i}{1+(1-s)\lambda_i} - \sum_{j\geq 0} \frac{\tilde{\lambda}_j}{1-(1-s)\tilde{\lambda}_j}$$

The above sum is absolutely convergent, uniformly for $s \in [0,1]$. Integrating over $[0,1]$ yields

$$\int_0^1 \int_I G_{s\tilde{\nu}}(x,x)\, \tilde{\nu}(dx)\, ds = \sum_{i\geq 0} \log(1+\lambda_i) + \sum_{j\geq 0} \log(1-\tilde{\lambda}_j)$$

This concludes the proof. □

3.3. **Dynkin's isomorphism.** In this subsection we recall the equality in law observed in [10] between the occupation field $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in I}$ and the square of a Gaussian Free Field and show how to derive from this particular versions of Dynkin's isomorphism.

Let L be a generator of a transient diffusion on I of form (1.3). Let $(\phi_x)_{x\in I}$ be a centred Gaussian process with variance-covariance function:

$$\mathbb{E}[\phi_x\phi_y] = G(x,y)$$

$(\phi_x)_{x\in I}$ is the Gaussian Free Field associated to L. Let $\tilde{S}$ be a primitive of $\frac{w}{u_\downarrow^2}$. Then $\tilde{S}(\sup I) = +\infty$. Moreover $\tilde{S}(\inf I) > -\infty$ because L is the generator of a transient diffusion. $\left(\frac{1}{u_\downarrow(\tilde{S}^{-1}(a))}\phi_{\tilde{S}^{-1}(a)}\right)_{a\in\tilde{S}(I)}$ is a standard Brownian motion starting from 0 at $\tilde{S}(\inf I)$. In particular $(\phi_x)_{x\in I}$ is inhomogeneous Markov and has continuous sample paths.

It was shown in [10], chapter 5, that when $\alpha = \frac{1}{2}$ $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in I}$ has the same law as $(\frac{1}{2}\phi_x^2)_{x\in I}$. In case of a Brownian motion on $(0,+\infty)$ killed in 0, $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x>0}$ is the square of a standard Brownian motion starting from 0. In case of a Brownian motion on $(0,x_{max})$ killed in 0 and x_{max}, $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{0<x<x_{max}}$ is the square of a standard Brownian bridge on $[0,x_{max}]$ from 0 to 0. In case of a Brownian motion on $\mathbb{R}$ with constant killing rate k, $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in\mathbb{R}}$ is the square of a stationary Ornstein–Uhlenbeck process.

The relation between the occupation field of a Poissonian ensemble of Markov loops and the square of a Gaussian Fee Field extends the Dynkin's isomorphism which we state below (see [5] and [7]):

Dynkin's Isomorphism. *Let $x_1, x_2, ..., x_{2n} \in I$. Then for any non-negative measurable functional F on continuous paths on I,*

$$\mathbb{E}_\phi\left[\prod_{i=1}^{2n}\phi_{x_i}F((\frac{1}{2}\phi_x^2)_{x\in I})\right] = \sum_{pairings}\int \mathbb{E}_\phi\left[F((\frac{1}{2}\phi_x^2 + \sum_{j=1}^{n}\ell^x(\gamma_j))_{x\in I})\right]\prod_{pairs}\mu^{y_j,z_j}(d\gamma_j) \tag{3.17}$$

where $\sum_{pairings}$ means that the n pairs $\{y_j, z_j\}$ are formed with all $2n$ points x_i in all $\frac{(2n)!}{2^n n!}$ possible ways.

Next we will show that in case $x_i = x_{i+n}$, for $i \in \{1,...,n\}$, i.e. $\prod_{i=1}^{2n}\phi_{x_i}$ is a product of squares $\prod_{i=1}^{n}\phi_{x_i}^2$, one can deduce the Dynkin's isomorphism from the relation between the square of the Gaussian Free Field and the occupation field using a somewhat extended version of Palm's identity for Poisson ensembles and the result of proposition 2.12 (ii).

Lemma 3.9. *Let $\mathcal{E}$ be an abstract Polish space. Let $\mathfrak{M}(\mathcal{E})$ be the space of locally finite measures on $\mathcal{E}$ and let $\mathcal{M} \in \mathfrak{M}(\mathcal{E})$. Let Φ be a Poisson random measure of intensity $\mathcal{M}$. Let H be a positive measurable function on $\mathfrak{M}(\mathcal{E}) \times \mathcal{E}^n$. Let $\mathfrak{P}_n$ be the set of partitions of $\{1,...,n\}$. If $\mathcal{P} \in \mathfrak{P}_n$ and $i \in \{1,...,n\}$, then $\mathcal{P}(i)$ will be the equivalence class of i under $\mathcal{P}$. The following identity holds:*

$$\mathbb{E}\left[\int_{\mathcal{E}^n} H(\Phi, q_1, ..., q_n)\prod_{i=1}^{n}\Phi(dq_i)\right] =$$

$$\sum_{\mathcal{P}\in\mathfrak{P}_n}\int_{\mathcal{E}^{\sharp\mathcal{P}}}\mathbb{E}\left[H(\Phi+\sum_{c\in\mathcal{P}}\delta_{q_c},q_{\mathcal{P}(1)},...,q_{\mathcal{P}(n)})\right]\prod_{c\in\mathcal{P}}\mathcal{M}(dq_c) \tag{3.18}$$

Proof. We will make a recurrence over n. If $n=1$, (3.18) is the Palm's identity for Poisson random measures. Assume that $n\geq 2$ and that (3.18) holds for $n-1$. We set

$$\tilde{H}(\Phi,q_1,...,q_{n-1}):=\int_{\mathcal{E}}H(\Phi,q_1,...,q_{n-1},q_n)\Phi(dq_n)$$

Then

$$\mathbb{E}\left[\int_{\mathcal{E}^n}H(\Phi,q_1,...,q_{n-1},q_n)\prod_{i=1}^{n}\Phi(dq_i)\right]=\mathbb{E}\left[\int_{\mathcal{E}^{n-1}}\tilde{H}(\Phi,q_1,...,q_{n-1})\prod_{i=1}^{n-1}\Phi(dq_i)\right]=$$

(3.19)

$$\sum_{\mathcal{P}'\in\mathfrak{P}_{n-1}}\int_{\mathcal{E}^{\sharp\mathcal{P}'}}\mathbb{E}\left[\int_{\mathcal{E}}H(\Phi+\sum_{c'\in\mathcal{P}'}\delta_{q_{c'}},q_{\mathcal{P}'(1)},...,q_{\mathcal{P}'(n-1)},q_n)(\Phi(dq_n)+\sum_{c'\in\mathcal{P}'}\delta_{q_{c'}}(dq_n))\right]\prod_{c'\in\mathcal{P}'}\mathcal{M}(dq_{c'})$$

Given a partition $\mathcal{P}'\in\mathfrak{P}_{n-1}$, one can extend it to a partition of $\{1,...,n-1,n\}$ either by deciding that n is single in its equivalence class or by choosing an equivalence class $c'\in\mathcal{P}'$ and adjoining n to it. In the identity (3.19) the first case corresponds to the integration with respect to $\Phi(dq_n)$, and according to Palm's identity

$$\mathbb{E}\left[\int_{\mathcal{E}}H(\Phi+\sum_{c'\in\mathcal{P}'}\delta_{q_{c'}},q_{\mathcal{P}'(1)},...,q_{\mathcal{P}'(n-1)},q_n)\Phi(dq_n)\right]=$$

$$\int_{\mathcal{E}}\mathbb{E}\left[H(\Phi+\sum_{c'\in\mathcal{P}'}\delta_{q_{c'}},q_{\mathcal{P}'(1)},...,q_{\mathcal{P}'(n-1)},q_n)\right]\mathcal{M}(dq_n)$$

The second case corresponds to the integration with respect to $\delta_{q_{c'}}(dq_n)$. Thus the right-hand side of (3.19) equals the right-hand side of (3.18). □

Next we show how derive a particular case of Dynkin's isomorphism using the above extended Palm's formula. Since $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in I}$ and $(\frac{1}{2}\phi^2_x)_{x\in I}$ are equal in law:

$$\mathbb{E}_\phi\left[\prod_{i=1}^{n}\phi^2_{x_i}F((\frac{1}{2}\phi^2_x)_{x\in I})\right]=2^n\mathbb{E}_{\mathcal{L}_{\frac{1}{2}}}\left[\prod_{i=1}^{n}\widehat{\mathcal{L}}^{x_i}_{\frac{1}{2}}F((\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in I})\right]$$

Applying lemma 3.9 we get that

$$\mathbb{E}_{\mathcal{L}_{\frac{1}{2}}}\left[\prod_{i=1}^{n}\widehat{\mathcal{L}}^{x_i}_{\frac{1}{2}}F((\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in I})\right]=\sum_{\mathcal{P}\in\mathfrak{P}_n}\int\prod_{i=1}^{n}\ell^{x_i}(\gamma_{\mathcal{P}(i)})\mathbb{E}\left[F((\widehat{\mathcal{L}}^x_{\frac{1}{2}}+\sum_{c\in\mathcal{P}}\ell^x(\gamma_c))_{x\in I})\right]\prod_{c\in\mathcal{P}}\frac{1}{2}\mu^*(d\gamma_c)$$

Let $\mathfrak{S}_n(\mathcal{P})$ be all the permutations σ of $\{1,...,n\}$ such that the classes of the partition $\mathcal{P}$ are the supports of the disjoint cycles of σ. Given a class $c\in\mathcal{P}$, let j_c be its smallest element. From property 2.11 (ii) follows that

$$\prod_{i=1}^{n}\ell^{x_i}(\gamma_{\mathcal{P}(i)})=\sum_{\sigma\in\mathfrak{S}_n(\mathcal{P})}\prod_{c\in\mathcal{P}}\ell^{*x_{j_c},x_{\sigma(j_c)},...,x_{\sigma^{|c|}(j_c)}}(\gamma_c)$$

Proposition 2.12 (ii) states that

$$\ell^{*x_{j_c},x_{\sigma(j_c)},...,x_{\sigma^{|c|}(j_c)}}(\gamma_c)\mu^*(d\gamma_c)=$$

$$\pi_*(\mu^{j_c,\sigma(j_c)}(d\tilde{\gamma}_{j_c}) \lhd ... \lhd \mu^{\sigma^{|c|-1}(j_c),\sigma^{|c|}(j_c)}(d\tilde{\gamma}_{\sigma^{|c|-1}(j_c)}) \lhd \mu^{\sigma^{|c|}(j_c),j_c}(d\tilde{\gamma}_{\sigma^{|c|}(j_c)}))$$

and if the loop γ_c is a concatenation of paths $\tilde{\gamma}_{j_c}, ..., \tilde{\gamma}_{\sigma^{|c|-1}(j_c)}, \tilde{\gamma}_{\sigma^{|c|}(j_c)}$ then

$$\ell^x(\gamma_c) = \ell^x(\tilde{\gamma}_{j_c}) + ... + \ell^x(\tilde{\gamma}_{\sigma^{|c|-1}(j_c)}) + \ell^x(\tilde{\gamma}_{\sigma^{|c|}(j_c)})$$

It follows that

(3.20)

$$2^n \mathbb{E}_{\mathcal{L}_{\frac{1}{2}}}\left[\prod_{i=1}^n \widehat{\mathcal{L}}^{x_i}_{\frac{1}{2}} F((\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in I})\right] = \sum_{\sigma\in\mathfrak{S}_n} 2^{n-\sharp cycles\, of\, \sigma} \int \mathbb{E}_{\mathcal{L}_{\frac{1}{2}}}\left[F((\widehat{\mathcal{L}}^x_{\frac{1}{2}} + \sum_{i=1}^n \ell^x(\tilde{\gamma}_i))_{x\in I})\right] \prod_{i=1}^n \mu^{i,\sigma(i)}(d\tilde{\gamma}_i)$$

But the right-hand side of (3.20) is just the same as the right-hand side of (3.17) in the specific case when for all $i \in \{1, ..., n\}$, $x_{i+n} = x_i$. This finishes the derivation of the Dynkin's isomorphism for this case.

4. Poissonian loops rooted at their minimum and ordered by their minimum

4.1. **Glueing together excursions ordered by their minimum.** Let L be the generator of a diffusion on I of form (1.3). A loop of $\mathcal{L}_{\alpha,L}$ rooted at its minimal point is a positive excursion. For a given $x_0 \in I$, we will consider the loops $\gamma \in \mathcal{L}_{\alpha,L}$ such that $\min\gamma \in (\inf I, x_0]$. We will root this loops at their minimum then order the obtained excursions in the decreasing sense of their minima. Then we will glue all this excursions together and obtain a continuous paths $\xi_{\alpha,L}$. The law of this path can be described as a one-dimensional projection of a two-dimensional Markov process. Moreover this paths contains all the information on the ensemble of loops $\mathcal{L}_{\alpha,L} \cap \{\gamma \in \mathfrak{L}^* | \min\gamma < x_0\}$. So this is a way to sample the latter ensemble of loops. In the particular case of $\alpha = 1$, $\xi_{1,L}$ is the sample paths of a one-dimensional diffusion. This is analogue of the link between $\mathcal{L}_1$ and the loop-erasure procedure already observed in [14] and in [10], chapter 8. Moreover this will give an interpretation of a Ray-Knight theorem in terms of Possonian ensemble of Markov loops. In the subsection 4.1 we will consider generalities about glueing together excursions ordered by their minimum and probability laws will be involved. In the subsection 4.2 we will deal with $\xi_{\alpha,L}$ and identify its law. In the subsection 4.3 we will consider the case of L containing a creation of mass term as in (1.12). We will observe that if $L \in \mathcal{D}^+$ then one can no longer construct $\xi_{\alpha,L}$ and show what one can construct instead.

Let $x_0 \in \mathbb{R}$ and let $\mathcal{Q}$ be a countable everywhere dense subset of $(-\infty, x_0)$. We consider a deterministic collection of excursions $(\mathsf{e}_q)_{q\in\mathcal{Q}}$ where $(\mathsf{e}_q(t))_{0\le t\le T(\mathsf{e}_q)}$ is a continuous excursion above 0, $T(\mathsf{e}_q) > 0$ and

$$\mathsf{e}_q(0) = \mathsf{e}_q(T(\mathsf{e}_q)) = 0$$

$$\forall t \in (0, T(\mathsf{e}_q)),\ \mathsf{e}_q(t) > 0$$

We also assume that for all $C > 0$ and $a < x_0$, there are only finitely many $q \in \mathcal{Q} \cap (a, x_0)$ such that $\max \mathsf{e}_q > C$ and that for all $a < x_0$

$$\sum_{q\in\mathcal{Q}\cap(a,x_0)} T(\mathsf{e}_q) < +\infty \tag{4.1}$$

Let $\mathcal{T}(y)$ be the function defined on $[0,+\infty)$ by

$$\mathcal{T}(y) := \sum_{q\in\mathcal{Q}\cap(x_0-y,x_0)} T(\mathtt{e}_q)$$

$\mathcal{T}$ is a non-decreasing function. Since $\mathcal{Q}$ is everywhere dense, $\mathcal{T}$ is increasing. $\mathcal{T}$ is right-continuous and jumps when $x_0 - y \in \mathcal{Q}$. The height of the jump is then $T(\mathtt{e}_{-y})$.

Let $T_{max} := \mathcal{T}(+\infty) \in (0,+\infty]$. For $t \in [0, T_{max})$ we define

$$\theta(t) := x_0 - \sup\{y \in [0,+\infty) | \mathcal{T}(y) > t\}$$

θ is a non-increasing function from $[0, T_{max})$ to $(-\infty, x_0]$. Since $\mathcal{T}$ is increasing, θ is continuous. We define

$$b^-(t) = \inf\{s \in [0, T_{max}) | \theta(s) = \theta(t)\}$$

$$b^+(t) = \sup\{s \in [0, T_{max}) | \theta(s) = \theta(t)\}$$

$b^-(t) < b^+(t)$ if and only if $\theta(t) \in \mathcal{Q}$ and then $b^+(t) - b^-(t) = T(\mathtt{e}_{\theta(t)})$. We introduce the set

$$\mathfrak{b}^- := \{t \in [0, T_{max}) | \theta(t) \in \mathcal{Q},\, b^-(t) = \theta(t)\}$$

$\mathfrak{b}^-$ is in one to one correspondence with $\mathcal{Q}$ by $t \mapsto \theta(t)$.

Finally we define on $[0, T_{max})$ the function ξ:

$$\xi(t) := \begin{cases} \theta(t) & if\ \theta(t) \notin \mathcal{Q} \\ \theta(t) + \mathtt{e}_{\theta(t)}(t - b^-(t)) & if\ \theta(t) \in \mathcal{Q} \end{cases}$$

Intuitively ξ is the function obtained by gluing together the excursions $(q + \mathtt{e}_q)_{q\in\mathcal{Q}}$ ordered in decreasing sense of their minimum. See figure 1 for an example of ξ and θ.

Proposition 4.1. *ξ is continuous. For all $t \in [0, T_{max})$*

$$\theta(t) = \inf_{[0,t]} \xi \tag{4.2}$$

The set $\mathfrak{b}^-$ can be recovered from ξ as follows:

$$\mathfrak{b}^- = \{t \in [0, T_{max}) | \xi(t) = \inf_{[0,t]} \xi \text{ and } \exists \varepsilon > 0, \forall s \in (0,\varepsilon),\, \xi(t+s) > \xi(t)\} \tag{4.3}$$

If $t_0 \in \mathfrak{b}^-$ then

$$b^+(t_0) = \inf\{t \in [t_0, T_{max}] | \xi(t) < \xi(t_0)\} \tag{4.4}$$

Proof. Let $t \in [0, T_{max})$. To prove the continuity of ξ at t we distinguish three case: the first case is when $\theta(t) \in \mathcal{Q}$ and $b^-(t) < t < b^+(t)$, the second case is when $\theta(t) \notin \mathcal{Q}$ and the third case is when $\theta(t) \in \mathcal{Q}$ and either $b^-(t) = t$ or $b^+(t) = t$.

In the first case, for all $s \in (b^-(t), b^+(t))$,

$$\xi(s) = \theta(t) + \mathtt{e}_{\theta(t)}(s - b^-(t))$$

$\mathtt{e}_{\theta(t)}$ being continuous, we get the continuity of ξ at t.

In the second case we consider a sequence $(t_n)_{n\geq 0}$ in $[0, T_{max})$ converging to t. Let $C > 0$. There are only finitely many $q \in \mathcal{Q}$ such that there is $n \geq 0$ such that $\theta(t_n) = q$ and $\max \mathtt{e}_q > C$.

Moreover for any $q \in \mathcal{Q}$, there are only finitely many $n \geq 0$ such that $\theta(t_n) = q$. Thus there are only finitely many $n \geq 0$ such that $\theta(t_n) \in \mathcal{Q}$ and $\max \mathsf{e}_{\theta(t_n)} > C$. So for n large enough

$$\theta(t_n) \leq \xi(t_n) \leq \theta(t_n) + C \tag{4.5}$$

But $\xi(t) = \theta(t)$ and $\theta(t_n)$ converges to $\theta(t)$. Since we may take C arbitrarily small, (4.5) implies that $\xi(t_n)$ converges to $\theta(t)$.

Regarding the third case, assume for instance that $\theta(t) \in \mathcal{Q}$ and $t = b^-(t)$. The right-continuity of ξ at t follows from the same argument as in the first case and left-continuity from the same argument as in the second case.

By definition, for all $t \in [0, T_{max})$, $\theta(t) \leq \xi(t)$. θ being non-increasing, for all $t \in [0, T_{max})$

$$\theta(t) \leq \inf_{[0,t]} \xi$$

For the converse inequality, we have

$$\theta(t) = \xi(b^-(t)) \geq \inf_{[0,t]} \xi$$

Regarding (4.3) and (4.4) we have the following disjunction: if $\theta(t) \in \mathcal{Q}$ and $b^-(t) < t < b^+(t)$ then $\xi(t) > \theta(t)$. If $\theta(t) \in \mathcal{Q}$ and $t = b^-(t)$ then for all $s \in (0, b^+(t) - b^-(t))$, $\xi(t+s) > \xi(t)$. If either $\theta(t) \in \mathcal{Q}$ and $t = b^+(t)$ or $\theta(t) \notin \mathcal{Q}$ then $\xi(t) = \theta(t)$ and there is a positive sequence $(s_n)_{n\geq 0}$ decreasing to 0 such that $\theta(t + s_n) \notin \mathcal{Q}$ and $\xi(t + s_n) = \theta(t + s_n) < \theta(t)$. □

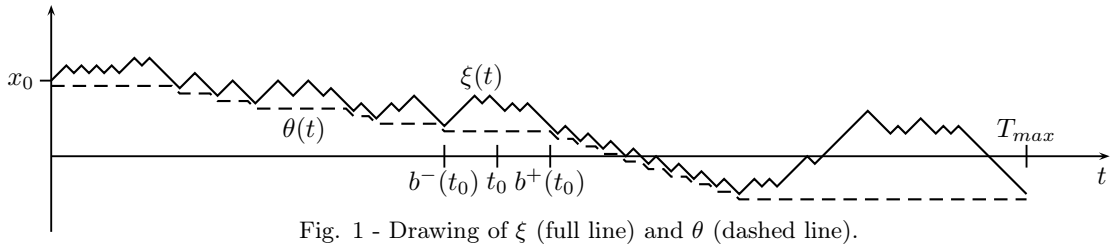

Fig. 1 - Drawing of ξ (full line) and θ (dashed line).

Previous proposition shows that one can reconstruct $\mathcal{Q}$ and the family of excursions $(\mathsf{e}_q)_{q\in\mathcal{Q}}$ only knowing ξ. (4.2) shows how to recover θ from ξ. (4.3) and (4.4) show how to recover the left and the right time boundaries of the excursions of ξ above θ. Also observe that the set defined by the right-hand side of (4.3) is countable whatever the continuous function ξ is, even if it is not obtained by glueing together excursions.

Lemma 4.2. *Let $(\mathcal{Q}_n)_{n\geq 0}$ be an increasing sequence of subsets of $\mathcal{Q}$ such that every $\mathcal{Q}_n$ is everywhere dense in $(-\infty, x_0]$ and such that*

$$\bigcup_{n\geq 0} \mathcal{Q}_n = \mathcal{Q} \tag{4.6}$$

Let

$$T_{n,max} := \sum_{q\in\mathcal{Q}_n} T(\mathsf{e}_q)$$

Let ξ_n be the function on $[0, T_{n,max})$ defined analogously to ξ by gluing together the excursion $(q + \mathsf{e}_q)_{q\in\mathcal{Q}_n}$. Then $T_{n,max}$ converges to T_{max} and ξ_n converges to ξ uniformly on every compact subset of $[0, T_{max})$.

Proof. It is obvious that $T_{n,max}$ converges to T_{max}. For $t \in [0, T_{max})$ let

$$f_n(t) := \int_0^t 1_{\theta(s) \notin \mathcal{Q} \setminus \mathcal{Q}_n} \, ds$$

f_n maps $[0, T_{max})$ to $[0, T_{n,max})$. For $t \in [0, T_{n,max})$ let

$$h_n(t) := \inf\{s \in [0, T_{max}) | f_n(s) > t\}$$

h_n is increasing and right-continuous. Its jumps correspond to $q \in \mathcal{Q} \setminus \mathcal{Q}_n$ an their height is $T(\mathtt{e}_q)$. (4.6) implies that f_n converges uniformly on compact subsets of $[0, T_{max})$ to the identity function and so does its right-continuous inverse h_n. Moreover

$$\xi_n = \xi \circ h_n$$

which implies the convergence of ξ_n to ξ. □

4.2. **Recovering the Poissonian ensembles of loops from Markovian sample paths.** Let $\alpha > 0$ and $\mathcal{L}_{\alpha,BM}$ the Poisson ensemble of loops of intensity $\alpha\mu^*_{BM}$ where μ^*_{BM} is the measure on loops associated to the Brownian motion on $\mathbb{R}$. Let $x_0 \in \mathbb{R}$. We consider the random countable set $\mathcal{Q}$:

$$\mathcal{Q} := \{\min \gamma | \gamma \in \mathcal{L}_{\alpha,BM}\} \cap (-\infty, x_0)$$

Almost surely $\mathcal{Q}$ is everywhere dense in $(-\infty, x_0)$ and for every $q \in \mathcal{Q}$ there is only one $\gamma \in \mathcal{L}_{\alpha,BM}$ such that $\min \gamma = q$. Almost surely $\gamma \in \mathcal{L}_{\alpha,BM}$ reaches its minimum at one single moment. Given $q \in \mathcal{Q}$ and $\gamma \in \mathcal{L}_{\alpha,BM}$ such that $\min \gamma = q$ we consider $\mathtt{e}_q$ to be the excursion above 0 equal to $\gamma - q$ where we root the unrooted loop γ at $argmin\, \gamma$. Then the random set of excursions $(\mathtt{e}_q)_{q \in \mathcal{Q}}$ almost surely satisfies the assumptions of the subsection 4.1. In particular the condition (4.1) follows from the fact that, according to (2.30),

$$\int_{\mathfrak{L}^*} 1 \wedge T(\gamma) 1_{\min \gamma \in (a, x_0)} \, \mu^*_{BM}(d\gamma) = (x_0 - a) \int_0^{+\infty} \frac{t \wedge 1}{\sqrt{2\pi t}} \, dt < +\infty$$

Thus we can consider the random continuous function $(\xi_{\alpha,BM}(t))_{t \geq 0}$ constructed by glueing together the excursions $(q + \mathtt{e}_q)_{q \in \mathcal{Q}}$ in the way described in subsection 4.1. Let

$$\theta_{\alpha,BM}(t) = \inf_{[0,t]} \xi_{\alpha,BM}$$

$$\Xi_{\alpha,BM}(t) := (\xi_{\alpha,BM}(t), \theta_{\alpha,BM}(t))$$

Next we will describe the law of the two-dimensional process $(\Xi_{\alpha,BM}(t))_{t \geq 0}$.

Proposition 4.3. *Let $(\tilde{B}_t)_{t \geq 0}$ be a standard Brownian motion on $\mathbb{R}$ starting from 0. $(\Xi_{\alpha,BM}(t))_{t \geq 0}$ has the same law as*

$$\left(x_0 + |\tilde{B}_t| - \frac{1}{\alpha} \ell^0_t(\tilde{B}),\, x_0 - \frac{1}{\alpha} \ell^0_t(\tilde{B}) \right)_{t \geq 0}$$

In particular for $\alpha = 1$, $(\xi_{1,BM}(t))_{t \leq 0}$ has the same law as a Brownian motion starting from x_0.

Proof. For $a < x_0$ let T_a be the first time $\theta_{\alpha,BM}$ hits a. For $l > 0$ let

$$\tilde{\tau}_l^0 := \inf\{t > 0 | \ell_t^0(\tilde{B}) > l\}$$

According to the disintegration (2.30) of the measure μ_{BM}^* in the proposition 2.14, for all $a < x_0$ the family $(\mathsf{e}_q)_{q\in\mathcal{Q}\cap(a,x_0)}$ of excursions above 0 is a Poissonian point process of intensity $2\alpha\eta_{BM}^{>0}$. This implies the following equality in law

$$(\xi_{\alpha,BM}(t) - \theta_{\alpha,BM}(t))_{0\le t\le T_a} \overset{(law)}{=} (|\tilde{B}_t|)_{0\le t\le\tilde{\tau}^0_{\alpha(x_0-a)}} \tag{4.7}$$

Since (4.7) holds for all $a < x_0$, we have the following equality in law

$$(\xi_{\alpha,BM}(t) - \theta_{\alpha,BM}(t),\, \alpha(x_0 - \theta_{\alpha,BM}(t)))_{t\ge0} \overset{(law)}{=} (|\tilde{B}_t|, \ell_t^0(\tilde{B}))_{t\ge0} \tag{4.8}$$

(4.8) is exactly the equality in law we needed. The fact that for $\alpha = 1$, $(x_0 + |\tilde{B}_t| - \ell_t^0(\tilde{B}))_{t\ge0}$ has the law of a Brownian motion starting from x_0 is well known. See [19], chapter VI, §2. □

Assume that $x_0 > 0$. Let $(B_t)_{t\ge0}$ be a Brownian sample paths. Let T_0 be the first time it hits 0. Then according to the first Ray-Knight theorem, $(\ell^x_{T_0}(B_t))_{0\le x\le x_0}$ is the square of a Bessel 2 process starting from 0 at 0, restricted to the interval $[0, x_0]$ (see theorem 2.2 in [19], chapter XI, §2). But from proposition 4.3 follows that the path $(B_t)_{0\le t\le T_0}$ can be sliced into a Poissonian ensemble of Brownian loops of parameter $\alpha = 1$. The fact that its occupation field on $[0, x_0]$ is the square of a Bessel 2 process starting from 0 at 0 is given by the proposition 3.4.

From proposition 4.3 follows in particular that $(\Xi_{\alpha,BM}(t))_{t\ge0}$ is a sample path of a two-dimensional Feller process. Let

$$D_{\mathbb{R}} := \{(x, a) \in \mathbb{R}^2 | x \ge a\}$$

$$Diag(\mathbb{R}^2) := \{(x, x) | x \in \mathbb{R}\}$$

For $(x_0, a_0) \in D_{\mathbb{R}}$ we define the process

$$(\Xi^{x_0,a_0}_{\alpha,BM}(t))_{t\ge0} = (\xi^{x_0,a_0}_{\alpha,BM}(t), \theta^{x_0,a_0}_{\alpha,BM}(t))_{t\ge0}$$

$$:= \left(a_0 + |x_0 - a_0 + \tilde{B}_t| - \frac{1}{\alpha}\ell_t^{a_0-x_0}(\tilde{B}),\, a_0 - \frac{1}{\alpha}\ell_t^{a_0-x_0}(\tilde{B})\right)_{t\ge0} \tag{4.9}$$

where $(\tilde{B}_t)_{t\ge0}$ is a Brownian motion starting from 0. $\Xi^{x_0,x_0}_{\alpha,BM}$ has the same law as $\Xi_{\alpha,BM}$ starting from x_0. The family of paths $(\Xi^{x_0,a_0}_{\alpha,BM})_{x_0\ge a_0}$ are the sample paths of the same Feller semi-group on $D_{\mathbb{R}}$ starting from all possible positions. Next we describe this semi-group in terms of generator and domain. Let f be a continuous function on $D_{\mathbb{R}}$, $\mathcal{C}^2$ on the interior of $D_{\mathbb{R}}$, such that all its second order derivatives extend continuously to $Diag(\mathbb{R}^2)$. This implies in particular that the first order derivatives also extend continuously to $Diag(\mathbb{R}^2)$. We write $\partial_1 f$, $\partial_2 f$ and $\partial_{1,1} f$ for the first order derivative relatively to the first variable, the second variable and the second order derivative relatively the first variable. Applying Itô-Tanaka's formula we get

$$f(\Xi^{x_0,a_0}_{\alpha,BM}(t)) = f(x_0, a_0) + \int_0^t \partial_1 f(\Xi^{x_0,a_0}_{\alpha,BM}(s)) sgn(x_0 - a_0 + \tilde{B}_s)\, d\tilde{B}_s$$

$$+ \int_0^t \left(\left(1 - \frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right) f(\Xi^{x_0,a_0}_{\alpha,BM}(s))\, d_s\ell_s^{a_0-x_0}(\tilde{B}) + \frac{1}{2}\int_0^t \partial_{1,1} f(\Xi^{x_0,a_0}_{\alpha,BM}(s))\, ds$$

Let $\mathcal{D}_{\alpha,BM}$ be the set of continuous functions f on $D_{\mathbb{R}}$, $\mathcal{C}^2$ on the interior of $D_{\mathbb{R}}$, such that all the second order derivatives extend continuously to $Diag(\mathbb{R}^2)$ and that moreover satisfy the following constraints: f and $\partial_{1,1}f$ are uniformly continuous and bounded (which also implies that $\partial_1 f$ is bounded by the inequality $\|\partial_1 f\|_\infty \leq 2\sqrt{\|f\|_\infty \|\partial_{1,1}f\|_\infty}$) and on $Diag(\mathbb{R}^2)$ the following equality holds:

$$\left(\left(1-\frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right) f(x,x) = 0$$

If $f \in \mathcal{D}_{\alpha,BM}$ then $\frac{1}{t}\big(\mathbb{E}\big[f(\Xi^{x_0,a_0}_{\alpha,BM}(t))\big] - f(x_0,a_0)\big)$ converges as $t \to 0^+$ to $\frac{1}{2}\partial_{1,1}f(x_0,a_0)$ uniformly for $(x_0,a_0) \in D_{\mathbb{R}}$. Moreover $\mathcal{D}_{\alpha,BM}$ is a core for $\frac{1}{2}\partial_{1,1}$ in the space of continuous bounded function on $D_{\mathbb{R}}$.

Next we describe what we obtain if we glue together the loops, seen as excursion, ordered in the decreasing sense of their minimum, where instead of $\mathcal{L}_{\alpha,BM}$ we use the Poissonian ensemble of Markov loops associated to a general generator of form (1.3). Let I be an open interval of $\mathbb{R}$ and $\tilde{L}$ a generator on I of form

$$\tilde{L} = \frac{1}{\tilde{m}(x)}\frac{d}{dx}\left(\frac{1}{\tilde{w}(x)}\frac{d}{dx}\right)$$

with zero Dirichlet boundary conditions which satisfies the assumptions of the section 1.2. Let $\tilde{S}$ be a primitive of $\tilde{w}(x)$. We assume that $\tilde{S}(\sup I) = +\infty$. Let

$$D_I := \{(x,a) \in I^2 | x \geq a\}$$

$$Diag(I^2) := \{(x,x) | x \in I\}$$

Let $\widehat{D}_I$ be the closure of D_I in $(\inf I, \sup I]^2$.

Given any $x_0' \geq a_0' > \frac{1}{2}\tilde{S}(\inf I)$ let $\tilde{\zeta}_\alpha$ be the first time $\Xi^{x_0',a_0'}_{\alpha,BM}$ hits $\frac{1}{2}\tilde{S}(\inf I)$. Let

$$\tilde{I}_t := \int_0^t \frac{1}{\tilde{m}}(\tilde{S}^{-1}(2\,\xi^{x_0',a_0'}_{\alpha,BM}(s)))\,ds$$

Let $(\tilde{I}_t^{-1})_{0\leq t<\tilde{I}_{\tilde{\zeta}_\alpha}}$ be the inverse function of $(\tilde{I}_t)_{0\leq t<\tilde{\zeta}_\alpha}$. It is a family of stopping times for $\Xi^{x_0',a_0'}_{\alpha,BM}$. For $x_0 \geq a_0 \in I$ and $t < \tilde{I}_{\tilde{\zeta}_\alpha}$ let

$$\Xi^{x_0,a_0}_{\alpha,\tilde{L}}(t) = (\xi^{x_0,a_0}_{\alpha,\tilde{L}}(t), \theta^{x_0,a_0}_{\alpha,\tilde{L}}(t)) := \Xi^{\tilde{S}(2x_0),\tilde{S}(2a_0)}_{\alpha,BM}(\tilde{I}_t^{-1})$$

If $\alpha = 1$ then $\xi^{x_0,a_0}_{\alpha,\tilde{L}}$ is just the sample paths starting x_0 of a diffusion of generator $\tilde{L}$. Let $\widehat{\mathcal{D}}_{\alpha,\tilde{L}}$ be the space of continuous functions f on D_I satisfying

- $f\circ\tilde{S}^{-1}$ is $\mathcal{C}^2$ on the interior of D_I and all the second order derivatives extend continuously to $Diag(I^2)$.
- $f(x,a)$ and $\frac{1}{\tilde{m}(x)}\partial_1\left(\frac{1}{\tilde{w}(x)}\partial_1 f(x,a)\right)$ are bounded on D_I and extend continuously to $\widehat{D}_I$.
- $f(x,a)$ and $\frac{1}{\tilde{m}(x)}\partial_1\left(\frac{1}{\tilde{w}(x)}\partial_1 f(x,a)\right)$ converge to 0 as a converges to $\inf I$ uniformly in x.

- On $Diag(I^2)$ the following equality holds:

$$\left(\left(1-\frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right) f(x,x) = 0 \tag{4.10}$$

Lemma 4.4. *$(\Xi^{x_0,a_0}_{\alpha,\tilde{L}})_{x_0\geq a_0\in I}$ is a family of sample path starting from all possible positions of the same Markovian or sub-Markovian semi-group on D_I. The law of the path $\Xi^{x_0,a_0}_{\alpha,\tilde{L}}$ depends weakly continuously on the starting point (x_0,a_0). The domain of the generator of this semi-group contains $\widehat{\mathcal{D}}_{\alpha,\tilde{L}}$, and on this space the generator equals*

$$\frac{1}{\tilde{m}(x)}\partial_1\left(\frac{1}{\tilde{w}(x)}\partial_1\right)$$

Moreover there is only one Markovian or sub-Markovian semi-group with such generator on $\widehat{\mathcal{D}}_{\alpha,\tilde{L}}$.

Proof. Since a change of scale does not alter the validity of the above statement, we can assume that $\tilde{w}\equiv 2$. Then $\sup I = +\infty$. $(\Xi^{x_0,a_0}_{\alpha,\tilde{L}}(t))_{0\leq t\leq \tilde{I}_{\tilde{\zeta}_\alpha}}$ is then obtained from $(\Xi^{x_0,a_0}_{\alpha,BM}(t))_{0\leq t<\tilde{\zeta}_\alpha}$ by a random time change. The Markov property and the continuous dependence on the starting point for $\Xi^{x_0,a_0}_{\alpha,\tilde{L}}$ follows from analogous properties for $\Xi^{x_0,a_0}_{\alpha,BM}$. If $f\in\widehat{\mathcal{D}}_{\alpha,\tilde{L}}$ then

$$\left(f(\Xi^{x_0,a_0}_{\alpha,BM}(\tilde{I}_t^{-1}\wedge\tilde{\zeta}_\alpha)) - \frac{1}{2}\int_0^{\tilde{I}_t^{-1}\wedge\tilde{\zeta}_\alpha}\partial_{1,1}f(\Xi^{x_0,a_0}_{\alpha,BM}(s))\,ds\right)_{t\geq 0}$$

is a local martingale. We can rewrite it as

$$\left(f(\Xi^{x_0,a_0}_{\alpha,\tilde{L}}(t\wedge\tilde{I}_{\tilde{\zeta}_\alpha})) - \int_0^t \frac{1}{2\tilde{m}(\xi^{x_0,a_0}_{\alpha,\tilde{L}}(s))}\partial_{1,1}f(\Xi^{x_0,a_0}_{\alpha,\tilde{L}}(s))\,1_{s<\tilde{I}_{\tilde{\zeta}_\alpha}}\,ds\right)_{t\geq 0}$$

The above local martingale is bounded on all finite time intervals and thus is a true martingale. Since $\frac{1}{2\tilde{m}(x)}\partial_{1,1}f(x,a)$ converges to 0 as a converges to $\inf I$, uniformly in x, it follows that

$$f(\Xi^{x_0,a_0}_{\alpha,\tilde{L}}(t\wedge\tilde{I}_{\tilde{\zeta}_\alpha})) = 1_{t<\tilde{I}_{\tilde{\zeta}_\alpha}} f(\Xi^{x_0,a_0}_{\alpha,\tilde{L}}(t))$$

Thus

$$\lim_{t\to 0^+}\frac{1}{t}\left(\mathbb{E}\left[1_{t<\tilde{I}_{\tilde{\zeta}_\alpha}} f(\Xi^{x_0,a_0}_{\alpha,\tilde{L}}(t))\right] - f(x_0,a_0)\right) = \frac{1}{2\tilde{m}(x_0)}\partial_{1,1}f(x_0,a_0)$$

Moreover the above convergence is uniform in (x_0,a_0) because $\frac{1}{2\tilde{m}(x)}\partial_{1,1}f(x,a)$ extends continuously to $\widehat{D}_I$.

To prove the uniqueness of the semi-group we need to show that there is $\lambda>0$ such that

$$\left(\frac{1}{2\tilde{m}(x)}\partial_{1,1}-\lambda\right)(\widehat{\mathcal{D}}_{\alpha,\tilde{L}})$$

is sufficiently large, for instance that it contains all functions with compact support in D_I. Let g be such a function and $\lambda>0$. Consider the equation

$$\frac{1}{2\tilde{m}(x)}\partial_{1,1}f(x,a) - \lambda f(x,a) = g(x,a) \tag{4.11}$$

Let $\tilde{u}_{\lambda,\downarrow}$ be a positive decreasing solution to

$$\frac{1}{2\tilde{m}(x)}\frac{d^2u}{dx^2}(x) - \lambda u(x) = 0$$

Let

$$f_0(x,a) := \tilde{u}_{\lambda,\downarrow}(x)\int_x^{+\infty}\int_y^{+\infty} 2\tilde{m}(z)g(z,a)\tilde{u}_{\lambda,\downarrow}(z)\,dz\,\frac{dy}{\tilde{u}_{\lambda,\downarrow}(y)^2}$$

Then f_0 is a solution to (4.11) and it is compactly supported in D_I. We look for the solutions to (4.11) of form

$$f(x,a) = f_0(x,a) + C(a)\tilde{u}_{\lambda,\downarrow}(x)$$

f satisfies the constraint (4.10) if and only if C satisfies

$$-\frac{1}{\alpha}\tilde{u}_{\lambda,\downarrow}(a)\frac{dC}{da}(a) + \left(1-\frac{1}{\alpha}\right)\frac{d\tilde{u}_{\lambda,\downarrow}}{dx}(a)C(a) + h(a) = 0$$

where

$$h(a) = \left(\left(1-\frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right) f_0(a,a)$$

h is compactly supported in I. We can set

$$C(a) = \tilde{u}_{\lambda,\downarrow}(a)^{\alpha-1}\int_{\inf I}^{x}\frac{h(y)}{\tilde{u}_{\lambda,\downarrow}(y)^{\alpha}}\,dy$$

C is zero in the neighbourhood of $\inf I$. Moreover $\tilde{u}_{\lambda,\downarrow}$ has a limit at $+\infty$. It follows that $f \in \widehat{\mathcal{D}}_{\alpha,\tilde{L}}$. □

Let L be the generator of a diffusion on I of form (1.3). Let $x_0 \in I$. Consider the loops γ in $\mathcal{L}_{\alpha,L}$ such that $\min\gamma < x_0$, rooted at $argmin\,\gamma$, seen as excursions. Let $(\xi_{\alpha,L}(t))_{0\le t<\zeta_\alpha}$ be the path on I obtained by glueing together this excursions ordered in the decreasing sense of their minimum. Let

$$\theta_{\alpha,L}(t) := \min_{[0,t]}\xi_{\alpha,L}$$

$$\Xi_{\alpha,L} := (\xi_{\alpha,L}, \theta_{\alpha,L})$$

Proposition 4.5. *Let $\tilde{L} := Conj(u_\downarrow, L)$. Then $(\Xi_{\alpha,L}(t))_{0\le t<\zeta_\alpha}$ has the same law as $(\Xi^{x_0,x_0}_{\alpha,\tilde{L}}(t))_{0\le t<\tilde{\zeta}_\alpha}$. So it is a sample path of a two-dimensional Feller process. In particular for $\alpha = 1$, $\xi_{1,L}$ is the sample path of a diffusion of generator $\tilde{L}$. For all $\alpha > 0$*

$$\liminf_{t\to\zeta_\alpha}\xi_{\alpha,L}(t) = \inf I$$

If L is the generator of a recurrent diffusion then

$$\limsup_{t\to\zeta_\alpha}\xi_{\alpha,L}(t) = \sup I$$

Otherwise

$$\limsup_{t\to\zeta_\alpha}\xi_{\alpha,L}(t) = \inf I$$

Proof. First notice that if L is the generator of a recurrent diffusion then $\tilde{L} = L$. Otherwise a diffusion of generator $\tilde{L} = L$ is, put informally, a diffusion of generator L conditioned to converge to $\inf I$ (informally because this may occur with zero probability). From h-transform invariance of the measure on loops follows that $\mathcal{L}_{\alpha,L} = \mathcal{L}_{\alpha,\tilde{L}}$. From property 2.6 (iv) and corollary 2.10 follows that $\Xi_{\alpha,L}$ is obtained from $\Xi_{\alpha,BM}$ by scale and time change in the same way as $\Xi^{x_0,x_0}_{\alpha,\tilde{L}}$ and thus $\Xi_{\alpha,L}$ and $\Xi^{x_0,x_0}_{\alpha,\tilde{L}}$ have the same law. Regarding the limits of $\xi_{\alpha,L}$ at ζ_α, we need just to observe that they hold if L is the generator of the Brownian motion on an interval of form $(a, +\infty)$, $a \in [-\infty, +\infty)$, and by time and scale change they hold in general. □

As explained in the proposition 4.1, the knowledge of the path $(\xi_{\alpha,L}(t))_{0\le t<\zeta_\alpha}$ alone is enough to reconstruct $\mathcal{L}_{\alpha,L} \cap \{\gamma \in \mathfrak{L}^* | \min \gamma < x_0\}$. From this we deduce the following

Corollary 4.6. *If L is the generator of a transient diffusion, it is possible to construct on the same probability space $\mathcal{L}_{\alpha,L}$ and a continuous version of the occupation field $(\widehat{\mathcal{L}}^x_{\alpha,L})_{x\in I}$.*

Proof. By scale and time change covariance and h-transform invariance of the Poisson ensembles of loops, it is enough to prove the proposition in case of a Brownian motion on $(0, +\infty)$ killed in 0. Let $(x_n)_{n\ge 0}$ be an increasing sequence in $(0, +\infty)$ converging to $+\infty$. We consider a sequence of independent paths $(\xi^{x_n,x_n}_{\alpha,BM})_{n\ge 0}$ defined by (4.9). Let

$$T_{n,x_{n-1}} := \inf\{t \ge 0 | \xi^{x_n,x_n}_{\alpha,BM}(t) = x_{n-1}\}$$

where conventionally we set $x_{-1} := 0$. From the restricted path $(\xi^{x_n,x_n}_{\alpha,BM}(t))_{0\le t<T_{n,x_{n-1}}}$ we can reconstruct a family of loops γ such that $\min\gamma \in (x_{n-1}, x_n)$: there is a random countable set $\mathcal{J}_n$ of disjoint compact subintervals $[b^-, b^+]$ of $[0, T_{n,x_{n-1}}]$ such that

$$\{(\xi^{x_n,x_n}_{\alpha,BM}(b^- + t))_{0\le t\le b^+-b^-} | [b^-, b^+] \in \mathcal{J}_n\} = \mathcal{L}_{\alpha,BM} \cap \{\gamma \in \mathfrak{L}^* | \min\gamma \in (x_{n-1}, x_n)\}$$

(see (4.3)). The union of all previous families of loops for $n \ge 0$ is a Poissonian ensemble of loops $\mathcal{L}_{\alpha,BM} \cap \{\gamma \in \mathfrak{L}^* | \min\gamma > 0\}$.

Each of $\xi^{x_n,x_n}_{\alpha,BM}$ is a semi-martingale and its quadratic variation is

$$\langle \xi^{x_n,x_n}_{\alpha,BM}, \xi^{x_n,x_n}_{\alpha,BM}\rangle_t = t$$

Moreover for all $x \in \mathbb{R}$

$$\int_0^t 1_{\xi^{x_n,x_n}_{\alpha,BM}=x}\, d\xi^{x_n,x_n}_{\alpha,BM}(s) = \left(1 - \frac{1}{\alpha}\right)\int_0^t 1_{\ell^0_s(\tilde{B})=\alpha x}\, d_s\ell^0_s(\tilde{B}) = 0$$

From theorems 1.1 and 1.7 in [19], chapter VI, §1, follows that we can construct on the same probability space $\xi^{x_n,x_n}_{\alpha,BM}$ and a space-time continuous version $(\ell^x_t(\xi^{x_n,x_n}_{\alpha,BM}))_{x\in\mathbb{R},t\ge 0}$ of local times of $\xi^{x_n,x_n}_{\alpha,BM}$ relatively to the Lebesgue measure. In particular $x \mapsto \ell^x_{T_{n,x_{n-1}}}(\xi^{x_n,x_n}_{\alpha,BM})$ is continuous. If $[b^-, b^+] \in \mathcal{J}_n$, then $(\ell^x_{b^+}(\xi^{x_n,x_n}_{\alpha,BM}) - \ell^x_{b^-}(\xi^{x_n,x_n}_{\alpha,BM}))_{x>0}$ is the occupation field of the loop corresponding to the time interval $[b^-, b^+]$. We need to check that

$$A.s.\ \forall x > 0,\ \ell^x_{T_{n,x_{n-1}}}(\xi^{x_n,x_n}_{\alpha,BM}) = \sum_{[b^-,b^+]\in\mathcal{J}_n} \ell^x_{b^+}(\xi^{x_n,x_n}_{\alpha,BM}) - \ell^x_{b^-}(\xi^{x_n,x_n}_{\alpha,BM}) \tag{4.12}$$

For $x > 0$, consider the random set of times

$$\{t \in [0, T_{n,x_{n-1}}] | \xi^{x_n,x_n}_{\alpha,BM}(t) = x\} \setminus \bigcup_{[b^-,b^+]\in\mathcal{J}_n} [b^-, b^+] \tag{4.13}$$

If x is a minimum of a loop embedded in $(\xi^{x_n,x_n}_{\alpha,BM}(t))_{0\leq t<T_{n,x_{n-1}}}$ or if $x \notin (x^{n-1}, x_n)$ then the set (4.13) is empty. Otherwise it is reduce to one point: the first hitting time of the level x. Almost surely, for all $x > 0$, the measure $d_t\ell^x_t(\xi^{x_n,x_n}_{\alpha,BM})$ is supported in $\{t \geq 0|\xi^{x_n,x_n}_{\alpha,BM}(t) = x\}$ and has no atoms. This implies (4.12). Finally we can conclude that $(\ell^x_{T_{n,x_{n-1}}}(\xi^{x_n,x_n}_{\alpha,BM}))_{x>0}$ is the occupation field of $\mathcal{L}_{\alpha,BM} \cap \{\gamma \in \mathfrak{L}^*| \min\gamma \in (x_{n-1}, x_n)\}$.

The occupation field of $\mathcal{L}_{\alpha,BM} \cap \{\gamma \in \mathfrak{L}^*| \min\gamma > 0\}$ is

$$\left(\sum_{n\geq 0} \ell^x_{T_{n,x_{n-1}}}(\xi^{x_n,x_n}_{\alpha,BM})\right)_{x>0}$$

The above sum is locally finite and thus varies continuously with x. □

Similarly to ordering the loops of $\mathcal{L}_{\alpha,L} \cap \{\gamma \in \mathfrak{L}^*| \min\gamma < x_0\}$ in the decreasing sense of their minimum, we can order the loops in $\mathcal{L}_{\alpha,L} \cap \{\gamma \in \mathfrak{L}^*| \min\gamma > x_0\}$, rooted in their minimum, in the increasing sense of their minimum. Let $(\hat{\xi}_{\alpha,L}(t))_{0\leq t<\hat{\zeta}_\alpha}$ be the path obtained by glueing this loops-excursion ordered this way. For $t < \hat{\zeta}_\alpha$ let

$$\hat{\theta}_{\alpha,L}(t) := \inf_{[t,\hat{\zeta}_\alpha)} \hat{\xi}_{\alpha,L}$$

The ensemble of loops $\mathcal{L}_{\alpha,L} \cap \{\gamma \in \mathfrak{L}^*| \min\gamma > x_0\}$ is invariant in law through time reversal. Thus, if $x_1 \in I, x_1 > x_0$, then what we obtain by time-reversing the path $\hat{\xi}_{\alpha,L}$, run until the last time it visits x_1, equals in law the path $\xi^{x_1,x_1}_{\alpha,L}$ run until the first time it hits x_0. Both paths are obtained by glueing together the loops in $\mathcal{L}_{\alpha,L} \cap \{\gamma \in \mathfrak{L}^*|x_0 < \min\gamma < x_1\}$ rooted at their minimum. In particular if L is the generator of a Brownian motion on $\mathbb{R}$ then $(\hat{\xi}_{\alpha,BM}(t), \hat{\theta}_{\alpha,BM}(t))_{t\geq 0}$ has the same law as

$$\left(x_0 + |\tilde{B}_t| + \frac{1}{\alpha}\ell^0_t(\tilde{B}),\ x_0 + \frac{1}{\alpha}\ell^0_t(\tilde{B})\right)_{t\geq 0}$$

where $\tilde{B}$ is a standard Brownian motion starting from 0. If $\alpha = 1$ then $(\hat{\xi}_{1,BM}(t) - x_0)_{t\geq 0}$ is a Bessel 3 process starting from 0. In general $(\hat{\xi}_{\alpha,L}, \hat{\theta}_{\alpha,L})$ is the sample path of a two-dimensional Feller Markovian or sub-Markovian process on $D_{\mathbb{R}}$. Stated informally, its generator acts on sufficiently smooth function f on $D_{\mathbb{R}}$ satisfying on $Diag(I^2)$ the constraint

$$\left(\left(1 + \frac{1}{\alpha}\right)\partial_1 + \frac{1}{\alpha}\partial_2\right) f(x, x) = 0$$

Given such a function f, one applies the second order differential operator L^{+,x_0} to the first variable, the second one being fixed. If $\alpha = 1$ then the path $\hat{\xi}_{1,L}$ is Markovian and has the same law as ρ^{+,x_0} started from x_0.

It is significant that for $\alpha = 1$ the Poissonian ensemble of loops $\mathcal{L}_{1,L}$ can be recovered from sample paths of one-dimensional diffusions. A similar property was observed for loops of the two-dimensional Brownian Motion and of Markov jump processes on graphs. In [10], chapter 8, it is shown that by launching consecutively symmetric Markov jump processes from different vertices of a finite graph and applying the Wilson's algorithm ([18]), one can simultaneously construct a uniform spanning tree of the graph with prescribed weights on the edges and an independent Poissonian ensemble of Markov loops of parameter $\alpha = 1$. If $\mathbb{D}$ is a simply-connected open domain of $\mathbb{C}$ other than $\mathbb{C}$, it was shown in [25] that one can couple a Brownian

motion on $\mathbb{D}$, killed when hitting $\partial\mathbb{D}$, and a simple curve with same extremal points such that the latter appears as the loop-erasure of the first. It is conjectured that given this loop-erased Brownian motion and an independent Poissonien ensemble of Brownian loops of parameter 1, by attaching to the simple curve the loops that cross it one reconstructs a Brownian sample path. See [14], conjecture 1, and [13], theorem 7.3.

For one-dimensional diffusions one can partially recover $\mathcal{L}_{1,L}$ from Markovian sample paths otherwise than slicing $\xi_{1,L}$ in excursions. The next result has an analogue for loops of Markov jump processes on graphs. See [10], remark 21.

Proposition 4.7. *Assume that L is the generator of a transient diffusion. Let $x \in I$. Let $(X_t)_{0\leq t<\zeta}$ be the sample path of a diffusion of generator L started from x. Let $\hat{T}_x$ the last time X visits x. For $l \geq 0$ let*

$$\tau_l^x := \{t \geq 0 | \ell_t^x(X) > l\}$$

Let $(Y_j)_{j\in\mathbb{N}}$ be a Poisson-Dirichlet partition of $[0,1]$, independent from X, ordered in an arbitrary way. Let

$$l_j := \ell_\zeta^x(X) \sum_{i\geq j} Y_i$$

The family of bridges $((X_t)_{\tau^x_{l_{j-1}}\leq t\leq\tau^x_{l_j}})_{j\geq 0}$ has, up to unrooting, the same law as the loops in

$$\mathcal{L}_{1,L} \cap \{\gamma \in \mathfrak{L}^* | x \in \gamma([0, T(\gamma)])\}$$

In particular $(X_t)_{0\leq t\leq\hat{T}_{x_0}}$ can be obtained through sticking together all the loops in $\mathcal{L}_{\alpha,L}$ that visit x.

Proof. According to corollary 2.9, $(\ell^x(\gamma))_{\gamma\in\mathcal{L}_{\alpha,L},\,\gamma \text{ visits } x}$ is a Poissonian ensemble of intensity

$$e^{-\frac{l}{G(x,x)}}\,\frac{dl}{l}$$

Thus $\widehat{\mathcal{L}}^x_{\alpha,L}$ is an exponential r.v. with mean $G(x,x)$ and has the same law as $\ell^x_\zeta(X)$. Moreover the Poissonian ensemble $(\ell^x(\gamma))_{\gamma\in\mathcal{L}_{\alpha,L},\,\gamma \text{ visits } x}$ has up to reordering the same law as $(l_j - l_{j-1})_{j\geq 0}$. Almost surely $l \mapsto \tau^x_l$ does not jump at any l_j. Conditionally on $(l_j)_{j\geq 0}$, $((X_t)_{\tau^x_{l_{j-1}}\leq t\leq\tau^x_{l_j}})_{j\geq 0}$ is an independent family of bridges and $(X_t)_{\tau^x_{l_{j-1}}\leq t\leq\tau^x_{l_j}}$ has the same law as $(X_t)_{0\leq t\leq\tau^x_{l_j-l_{j-1}}}$. We conclude using identity (2.15) and the theory of marked Poissonian ensembles. □

Assume that L is the generator of a transient diffusion. Let $x > y \in I$. Let $(X_t^{x,y})_{0\leq t\leq\zeta^{x,y}}$ be the path on I following the law $\frac{1}{G(x,y)}\mu^{x,y}$. Let $T_y^{x,y}$ be the first time $X^{x,y}$ hits y. Then $(X_t^{x,y})_{0\leq t\leq T_y^{x,y}}$ has the same law as $\xi_{1,\tilde{L}}$ run until hitting y, where $\tilde{L} = Conj(u_\downarrow, L)$.
$(X^{x,y}_{T_y^{x,y}+s})_{0\leq s\leq\zeta^{x,y}-T_y^{x,y}}$ has the same law as a diffusion of generator L, starting from y, run until the last time it visits y. By slicing $(X_t^{x,y})_{0\leq t\leq T_y^{x,y}}$ in excursion and $(X^{x,y}_{T_y^{x,y}+s})_{0\leq s\leq\zeta^{x,y}-T_y^{x,y}}$ in bridges from y to y as in proposition 4.7, we obtain the Poissonian ensemble of loops

$$\mathcal{L}_{1,L} \cap \{\gamma \in \mathfrak{L}^* | \min\gamma < x, \max\gamma > y\}$$

4.3. **The case of "generators" with creation of mass.** Now we consider that L is a "generator" on I that contains a creation of mass term as in (1.12). We study the problem of glueing together the loops in $\mathcal{L}_{\alpha,L}$, rooted at their minimum and ordered in the decreasing sense of the minimum. If $L \in \mathfrak{D}^{0,-}$, the situation is the same as for the generators of diffusions: if $\tilde{L} = Conj(u_\downarrow, L)$ and $x_0 \in I$, then $\xi^{x_0,x_0}_{\alpha,\tilde{L}}$ is the continuous path obtained by glueing together the loops $\gamma \in \mathcal{L}_{\alpha,L}$ such that $\min\gamma < x_0$. This can not be done any longer if $L \in \mathfrak{D}^+$. Indeed, according to proposition 2.25, if $x \in I$ is sufficiently low and $y \in I$ sufficiently high, there are infinitely many loops $\gamma \in \mathcal{L}_{\alpha,L}$ such that $\min\gamma < x$ and $\max\gamma > y$. However one can consider a continuous function $H : I \to I$ such that for all $x \in I$, $H(x) > x$ and $L_{|(x,H(x))} \in \mathfrak{D}^-$ (see figure 2). We will show that one can glue together the loops $\gamma \in \mathcal{L}_{\alpha,L}$ such that $\min\gamma < x_0$ and $\max(\gamma) < H(\min\gamma)$, rooted in $argmin\,\gamma$, ordered according to the decreasing sense of their minima, and obtain a continuous path $(\xi_{\alpha,L,H}(t))_{0\le t<\zeta_\alpha}$. We will further identify the law of this path.

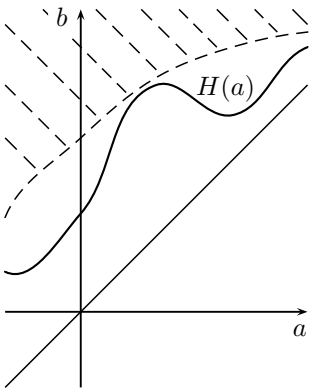

Fig. 2 - The measure on $(\min\gamma, \max\gamma)$ induced by $\mu^*_L(d\gamma)$ is not locally finite in the dashed region.

Lemma 4.8. *For all J compact subinterval of I,*

$$\sum_{\substack{\gamma \in \mathcal{L}_{\alpha,L}, \min\gamma \in J \\ \max\gamma < H(\min\gamma)}} T(\gamma) < +\infty \ a.s. \tag{4.14}$$

Proof. For every $a \in I$, there is $\varepsilon(a) > 0$ and $\tilde{H}(a) \in I$ such that for all $a' \in (a-\varepsilon(a), a+\varepsilon(a))$, $\tilde{H}(a) > H(a')$ and $L_{|(a-\varepsilon(a),\tilde{H}(a))} \in \mathfrak{D}^-$. The set

$$\{(x,a) | a \in J, x \in [a, H(a)]\}$$

is compact and hence there is a finite family $(a_1, ..., a_N) \in I$ such that the preceding set is contained in

$$\bigcup_{i=1}^{N} (a_i - \varepsilon(a_i), \tilde{H}(a_i)) \times (a_i - \varepsilon(a_i), a_i + \varepsilon(a_i))$$

But

$$\sum_{i=1}^{N} \sum_{\substack{\gamma \in \mathcal{L}_{\alpha,L} \\ \min\gamma \in (a_i - \varepsilon(a_i), a_i + \varepsilon(a_i)) \\ \max\gamma < \tilde{H}(a_i)}} T(\gamma) \ < +\infty \ a.s. \tag{4.15}$$

(4.15) implies (4.14). □

Preceding lemma ensures the existence of the continuous path $(\xi_{\alpha,L,H}(t))_{0\leq t<\zeta_\alpha}$. For $t \in [0, \zeta_\alpha)$, let

$$\theta_{\alpha,L,H}(t) := \min_{[0,t]} \xi_{\alpha,L,H}$$

We will show that $(\xi_{\alpha,L,H}, \theta_{\alpha,L,H})$ is the sample path of a two-dimensional Markovian or sub-Markovian process. Next we introduce what will be its domain and generator. Next we will apply the Hille-Yosida's theorem to show that actually there is a Markovian or sub-Markovian process with such a domain and generator. Finally we will show that $(\xi_{\alpha,L,H}, \theta_{\alpha,L,H})$ is indeed its sample path.

Let

$$D_{I,H} := \{(x, a) \in I^2 | a \leq x < H(a)\}$$

First observe the following: Assume that $x_0 \in I$ and h is a $\mathcal{C}^1$ function on I. If there is an L-harmonic function u that is positive in the neighbourhood of x such that $\frac{u^2}{w}\frac{dh}{dx}$ is $\mathcal{C}^1$ in the neighbourhood of x_0, then for any other L-harmonic function $\tilde{u}$ positive in the neighbourhood of x_0, $\frac{\tilde{u}^2}{w}\frac{dh}{dx}$ is $\mathcal{C}^1$ in the neighbourhood of x_0. Indeed the quotient $\frac{\tilde{u}}{u}$ is $\mathcal{C}^1$ on I. We define on $D_{I,H}$ the function $u_H(x, a)$ as follows:

$$u_H(x, a) := \frac{u^{-,H(a)}(x)}{u^{-,H(a)}(a)}$$

u is positive and continuous on $D_{I,H}$ and by definition equals 1 on $Diag(I^2)$. For all $a \in I$, $x \mapsto u_H(x, a)$ is L-harmonic and

$$\lim_{x\to H(a)} u_H(x, a) = 0$$

We will consider functions $f(x, a)$ such that for all $a \in I$, $x \mapsto f(x, a)$ is $\mathcal{C}^1$ and for every $x_0 \geq a \in I$, and u L-harmonic function positive in the neighbourhood of x_0, $x \mapsto \frac{u(x)^2}{w(x)}\partial_1(x, a)$ is $\mathcal{C}^1$ in the neighbourhood of x_0. For such functions we can define the second order partial differential operator

$$L_{H,1}f(x, a) = \frac{1}{u_H(x, a)^2 m(x)}\partial_1\left(\frac{u_H(x, a)^2}{w(x)}\partial_1 f(x, a)\right)$$

For $\hat{a} \in I$, let

$$D_{I,H,\hat{a}} := D_{I,H} \cap (\inf I, \hat{a}]^2$$

$$\widehat{D}_{I,H,\hat{a}} := \{(x, a) \in I^2 | a \leq \hat{a}, a \leq x \leq H(a)\}$$

Let $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$ be the space of continuous bounded functions f on $D_{I,H,\hat{a}}$ that extend continuously to $\widehat{D}_{I,H,\hat{a}}$ and such that $f(x, a)$ converges to 0 as a converges to $\inf I$, uniformly in x. Let $\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ be the subspace of function $f \in \widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$ satisfying the following constraints:

- f is $\mathcal{C}^1$ on the interior of $D_{I,H}$ and all the first order derivatives extend continuously to $Diag(I^2)$.
- $L_{H,1}f$ is well defined and in $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$.
- On $Diag(I^2)$ f satisfies the equation (4.10).

We are interested in Markovian or sub-Markovian processes on $D_{I,H,\hat{a}}$ with domain $\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ and generator $L_{H,1}$. We will show that this $\hat{a}$-parametrized family of semi-groups is consistent in the following way: if $\hat{a}' > \hat{a} \in I$ then any sample path for the semi-group generated by $(L_{H,1}, \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}})$ on $D_{I,H,\hat{a}}$ is also a sample path for the semi-group generated by $(L_{H,1}, \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}'})$ on $D_{I,H,\hat{a}'}$.

Lemma 4.9. *$\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ is dense in $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$ for the uniform topology.*

Proof. Let $\widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$ be the sub-space of functions $f \in \mathcal{D}_{\alpha,L,H}$ satisfying the following additional constraint: $\partial_1 f(x,a)$ and $\dfrac{1}{m(x)w(x)}\partial_1 f(x,a)$ are in $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$. It is enough to show that $\widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$ is dense. $\widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$ is a non-unitary algebra. If $f, g \in \widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$ then

$$L_{H,1}(fg)(x,a) = (L_{H,1}f)g(x,a) + f(L_{H,1}g)(x,a) + \frac{2}{m(x)w(x)}(\partial_1 f \partial_1 g)(x,a)$$

Thus $L_{H,1}(fg) \in \widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$. Let $\widehat{D}_{I,H,\hat{a}} \cup \{\dagger\}$ be the one point compactification of $\widehat{D}_{I,H,\hat{a}}$. Any function in $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$ extends continuously to $\widehat{D}_{I,H,\hat{a}} \cup \{\dagger\}$ and takes value 0 at $\dagger$. The space $\mathbb{R} \oplus \widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$, spanned by the constant function and $\widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$, is a unitary sub-algebra $\mathcal{C}(\widehat{D}_{I,H,\hat{a}} \cup \{\dagger\})$ of continuous functions on $\widehat{D}_{I,H,\hat{a}} \cup \{\dagger\}$. Lets show that this sub-algebra is separating.

The main point here is to show that for every $(x_0, x_0) \in Diag(I^2) \cap \widehat{D}_{I,H,\hat{a}}$ and $\varepsilon > 0$, there is a function $f \in \widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$ such that $f(x_0,x_0) > 0$ and f vanishes outside an ε-neighbourhood of (x_0,x_0). Similar property for other points in $\widehat{D}_{I,H,\hat{a}} \cup \{\dagger\}$ is trivial. Let $\varepsilon > 0$. There is $\varepsilon' \in (0,\varepsilon)$ such that $u^{-,H(x_0)}$ is positive on $(x_0 - \varepsilon', H(x_0))$ and $\inf_{(x_0-\varepsilon', x_0+\varepsilon')} H > x_0 + 2\varepsilon'$. Let h be a smooth non-negative function on I such that $h(x_0) > 0$ and h vanishes outside $(x_0 - \varepsilon', x_0 + \varepsilon')$. Let

$$g(x,a) := h(a) + \frac{1}{\alpha}\frac{dh}{da}(a)\frac{u^{-,H(x_0)}(a)^2}{w(a)}\int_a^x \frac{w(y)}{u^{-,H(x_0)}(y)^2}\,dy$$

If $a < x_0 - \varepsilon'$, we set $g(x,a) = 0$. $L_{H,1}g$ is continuous and g satisfies (4.10). $g(x_0,x_0) = h(x_0)$. Let χ defined on $\mathbb{R}$ be a cut-off function: χ is smooth, equals 1 on $(-\infty, 1]$ and 0 on $[2, +\infty)$. Let

$$f(x,a) := g(x,a)\ \chi\left(2\int_{x_0}^x \frac{w(y)}{u^{-,H(x_0)}(y)^2}\,dy \left(\int_{x_0}^{x_0+2\varepsilon'} \frac{w(y)}{u^{-,H(x_0)}(y)^2}\,dy\right)^{-1}\right)$$

Then $f \in \widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$, f vanishes outside $(x_0 - \varepsilon', x_0 + 2\varepsilon') \times (x_0 - \varepsilon', x_0 + \varepsilon')$, and $f(x_0,x_0) = h(x_0) > 0$.

According to Stone-Weierstrass theorem $\mathbb{R} \oplus \widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$ is dense in $\mathcal{C}(\widehat{D}_{I,H,\hat{a}} \cup \{\dagger\})$. Consequently $\widehat{\mathcal{D}}'_{\alpha,L,H,\hat{a}}$ is dense in $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$. □

Proposition 4.10. *For every $\hat{a} \in I$, there is a continuous positive contraction semi-group on $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$ such that $\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ is a core of the domain of its generator and the generator on $\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ is $L_{H,1}$. Stated otherwise, there is a Feller semi-group on $\mathcal{C}(\widehat{D}_{I,H,\hat{a}} \cup \{\dagger\})$ such that $\mathbb{R} \oplus \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ is a core and the generator values $L_{H,1}$ on $\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ and 0 on constant functions.*

Moreover if $\hat{a}' > \hat{a} \in I$, any sample path for the semi-group generated by $(L_{H,1}, \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}})$ on $D_{I,H,\hat{a}}$ is also a sample path for the semi-group generated by $(L_{H,1}, \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}'})$ on $D_{I,H,\hat{a}'}$.

Proof. Let $\hat{a} \in I$. According to Hille-Yosida theorem (see theorem 2.2, section 4.2 in [8]), we need to check that

- (i) $\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ is dense in $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$.
- (ii) There is $\lambda > 0$ such that $(L_{H,1} - \lambda)(\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}})$ is dense in $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$.
- (iii) Maximum principle: if $f \in \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$, $f(x_0,a_0) = \max f$ and $f(x_0,a_0) \geq 0$ then $L_{H,1}f(x_0,a_0) \leq 0$.

Condition (i) is given by lemma 4.9. Regarding the condition (iii), if $(x_0,a_0) \notin Diag(I^2)$ the maximum principle is obvious. If $x_0 = a_0$, then for ε small enough, $(x_0+\varepsilon, x_0) \in \mathcal{D}_{\alpha,L,H,\hat{a}}$ and necessarily $\partial_1 f(x_0,x_0) \leq 0$. For ε small enough, $(x_0-\varepsilon, x_0-\varepsilon) \in \mathcal{D}_{\alpha,L,H,\hat{a}}$ and necessarily $-\partial_1 f(x_1,x_1) - \partial_2 f(x_1,x_1) \leq 0$. Together with (4.10) this implies that $-\alpha\partial_1 f(x_0,x_0) \leq 0$. Thus $\partial_1 f(x_0,x_0) = 0$. Using again the fact that for ε small enough $f(x_0-\varepsilon,x_0) \leq f(x_0,x_0)$, we get that $L_{H,1}f(x_0,x_0) \leq 0$.

Next we check the condition (ii). Let $\lambda > 0$ and $g \in \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ such that $g(x,a) = 0$ for a close enough to $\inf I$. Consider the equation on $D_{I,H,\hat{a}}$

$$L_{H,1}f - \lambda f = g \tag{4.16}$$

Let $\tilde{u}_{H,\lambda}(x,a)$ be the function on $D_{I,H}$ satisfying the equation $L_{H,1}\tilde{u}_{H,\lambda} - \lambda\tilde{u}_{H,\lambda} = 0$ and the border conditions $\tilde{u}_{H,\lambda}(a,a) = 1$ and $\tilde{u}_{H,\lambda}(H(a),a) = 0$. $\tilde{u}_{H,\lambda}$ is $\mathcal{C}^1$ and positive. Let

$$f_0(x,a) := \tilde{u}_{H,\lambda}(x,a) \int_x^{H(a)} \int_y^{H(a)} m(z)w(y)g(z,a)\tilde{u}_{H,\lambda}(z,a)\frac{u_H(z,a)^2}{u_H(y,a)^2}\,dz\,\frac{dy}{\tilde{u}_{H,\lambda}(y,a)^2} \tag{4.17}$$

The right-hand side of (4.17) is integrable: $u_H(y,a)^{-2}\tilde{u}_{H,\lambda}(y,a)^{-2}$ diverges in the neighbourhood of $H(a)$ like $(H(a)-y)^{-4}$. $\tilde{u}_{H,\lambda}(z,a)u_H(z,a)^2$ is equivalent in the neighbourhood of $H(a)$ to $(H(a)-z)^3$. All other factors are bounded. Moreover

$$\int_x^{H(a)} \int_y^{H(a)} (H(a)-z)^3\,dz\,\frac{dy}{(H(a)-y)^4} \ < +\infty$$

$f_0 \in \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ and satisfies the equation (4.16), but in general does not satisfy the constraint (4.10). We look for a solution of (4.16) of form $f_0(x,a)+C(a)\tilde{u}_{H,\lambda}(x,a)$. For it to satisfy (4.10), we need that

$$-\frac{1}{\alpha}\frac{dC}{da}(a)\tilde{u}_{H,\lambda}(a,a) + C(a)\left(\left(1-\frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right)\tilde{u}_{H,\lambda}(a,a) + h(a) = 0 \tag{4.18}$$

where

$$h(a) = \left(\left(1-\frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right) f_0(a,a)$$

using the fact that $\tilde{u}_{H,\lambda}(a,a) \equiv 1$, (4.18) becomes

$$-\frac{1}{\alpha}\frac{dC}{da}(a) + C(a)\partial_1\tilde{u}_{H,\lambda}(a,a) + h(a) = 0 \tag{4.19}$$

Let U be a primitive of $a \mapsto \partial_1\tilde{u}_{H,\lambda}(a,a)$. Then

$$C(a) := \exp(\alpha U(a)) \int_{\inf I}^{a} h(y)\exp(-\alpha U(y))\,dy$$

is a solution to (4.19). $C(a)$ vanishes for a small enough. $f_0(x,a)+C(a)\tilde{u}_{H,\lambda}(x,a)$ is in $\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$. The condition (ii) follows and hence the existence of a continuous positive contraction semi-group on $\widehat{\mathcal{C}}_0(D_{I,H,\hat{a}})$.

Let $\hat{a}' > \hat{a} \in I$. Let $(\Xi(t))_{0\le t<\zeta_\alpha}$ be a sample path for the semi-group generated by $(L_{H,1}, \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}})$ on $D_{I,H,\hat{a}}$. Let $f \in \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}'}$. Then f restricted to $D_{I,H,\hat{a}}$ is in $\widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$ and

$$\left(f(\Xi(t\wedge\zeta_\alpha)) - \int_0^{t\wedge\zeta_\alpha} L_{H,1}f(\Xi(s))\,ds\right)_{t\ge 0}$$

is a martingale. This implies that $(\Xi(t))_{0\le t<\zeta_\alpha}$ is also a sample path for the semi-group generated by $(L_{H,1}, \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}'})$ on $D_{I,H,\hat{a}'}$ (see theorem 4.1 in [8], section 4.4). □

Proposition 4.11. *Let $x_0 \in I$. The path $(\xi_{\alpha,L,H}(t), \theta_{\alpha,L,H}(t))_{0\le t<\zeta_\alpha}$ starting from (x_0,x_0) obtained by glueing together the loops $\gamma \in \mathcal{L}_{\alpha,L}$ such that $\min\gamma < x_0$ and $\max\gamma < H(\min\gamma)$ is a sample path for the semi-group generated by $(L_{H,1}, \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}})$, for any $\hat{a} \in I$, $\hat{a} \ge x_0$.*

Proof. We need only to show that given $\hat{a} \in I$, $\hat{a} \ge x_0$, and $f \in \widehat{\mathcal{D}}_{\alpha,L,H,\hat{a}}$, then $(Y_t)_{t\ge0}$, defined by

$$(Y_t)_{t\ge0} := \left(f(\xi_{\alpha,L,H}(t\wedge\zeta_\alpha), \theta_{\alpha,L,H}(t\wedge\zeta_\alpha)) - \int_0^{t\wedge\zeta_\alpha} L_{H,1}f(\xi_{\alpha,L,H}(s), \theta_{\alpha,L,H}(s))\,ds\right)_{t\ge0}$$

is a martingale. For this we will use an approximation of H from below by step functions. Consider f fixed. For $n \in \mathbb{N}$, we define function the H_n equal to

$$H_n(a) := \inf\left\{H(a') | x_0 - \frac{1}{2^n}(1+\lfloor 2^n(x_0-a)\rfloor) \le a' \le x_0 - \frac{1}{2^n}\lfloor 2^n(x_0-a)\rfloor\right\}$$

Let $u_{H_n}(x,a)$ be

$$u_{H_n}(x,a) := \frac{u^{-,H_n(a)}(x)}{u^{-,H_n(a)}(a)}$$

$u_{H_n}(x,a)$ may be discontinuous at points where a is of form $x_0 - \dfrac{j}{2^n}$. Let $L_{H_n,1}$ be the second order partial differential operator

$$L_{H_n,1} := \frac{1}{u_{H_n}(x,a)^2 m(x)}\partial_1\left(\frac{u_{H_n}(x,a)^2}{w(x)}\partial_1\right)$$

$L_{H_n,1}f(x,a)$ may be discontinuous at points where a is of form $x_0 - \dfrac{j}{2^n}$. $L_{H_n,1}f$ converges uniformly on compact subsets of $D_{I,H,\hat{a}}$ to $L_{H,1}f$.

Let $(\xi_{\alpha,L,H_n}(t))_{0\le t<\zeta_{n,\alpha}}$ be the path obtained by glueing together the loops in $\gamma \in \mathcal{L}_\alpha$ such that $\min\gamma < x_0$ and $\max\gamma < H_n(\min\gamma)$, rooted at their minimum and ordered in the decreasing sense of their minimum. Let

$$\theta_{\alpha,L,H_n}(t) := \inf_{[0,t]} \xi_{\alpha,L,H_n}$$

Let

$$(Y_{n,t})_{0\le t<\zeta_{n,\alpha}} := \left(f(\xi_{\alpha,L,H_n}(t), \theta_{\alpha,L,H_n}(t)) - \int_0^t L_{H_n,1}f(\xi_{\alpha,L,H_n}(s), \theta_{\alpha,L,H_n}(s))\,ds\right)_{0\le t<\zeta_{n,\alpha}}$$

Let $T_{n,x_0-j2^{-n}}$ be the first time ξ_{α,L,H_n} hits $x_0 - \frac{j}{2^n}$. The loops making up the path $(Y_{n,t})_{0\leq t<T_{n,x_0-(j+1)2^{-n}}-T_{n,x_0-j2^{-n}}}$ are the loops in $\gamma \in \mathcal{L}_\alpha$ such that

$$\min \gamma \in (x_0 - j2^{-n}, x_0 - (j+1)2^{-n})$$

It follows from proposition 4.5 that for $T_{n,x_0-(j+1)2^{-n}} < \zeta_{n,\alpha}$, $(Y_{n,(T_{n,x_0-j2^{-n}}+t)\wedge T_{n,x_0-(j+1)2^{-n}}})_{t\geq 0}$ is a martingale. This implies that for $T_{n,x_0-j2^{-n}} < \zeta_{n,\alpha}$, $(Y_{n,t\wedge T_{n,x_0-(j+1)2^{-n}}})_{t\geq 0}$ is a martingale. For $j \in \mathbb{N}^*$, let

$$K_j := \left\{(x,a) \in D_{I,H} | (1-\frac{1}{j}) \sup I + \frac{a_0}{j} \leq a \leq a_0,\ a \leq x_0 \leq (1-\frac{1}{j})H(a) + \frac{a}{j}\right\}$$

$(K_j)_{j\geq 1}$ is an increasing sequence of compact subsets of D_{I,H,a_0} containing (x_0, x_0) and

$$D_{I,H,a_0} = \bigcup_{j\geq 0} \mathring{K}_j$$

Let

$$T_{n,K_j} := \inf\{t \geq 0 | \xi_{\alpha,L,H_n}(t) \notin K_j\}$$

$$T_{K_j} := \inf\{t \geq 0 | \xi_{\alpha,L,H}(t) \notin K_j\}$$

Then $(Y_{n,t\wedge T_{n,K_j}})_{t\geq 0}$ is a martingale. From lemma 4.2 follows that ξ_{α,L,H_n} converges uniformly on compact time intervals to $\xi_{\alpha,L,H}$ and then that $(Y_{n,t\wedge T_{n,K_j}})_{t\geq 0}$ converges uniformly to $(Y_{t\wedge T_{K_j}})_{t\geq 0}$. Thus $(Y_{t\wedge T_{K_j}})_{t\geq 0}$ is a martingale. Since $\lim_{j\to+\infty} T_{K_j} = \zeta_\alpha$, it follows that $(Y_t)_{t\geq 0}$ is a martingale. □

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