Actions of groups of homeomorphisms on one-manifolds

Emmanuel Militon

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Abstract

In this article, we describe all the group morphisms from the group of compactly-supported homeomorphisms isotopic to the identity of a manifold to the group of homeomorphisms of the real line or of the circle.
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1 Introduction

Fix a connected manifold M (without boundary). For an integer $r \geq 0$, we denote by $\mathrm{Diff}^r(M)$ the group of C^r-diffeomorphisms of M. When $r = 0$, this group will also be denoted by $\mathrm{Homeo}(M)$. For a homeomorphism f of M, the *support* of f is the closure of the set:

$$\{x \in M,\ f(x) \neq x\}.$$

We denote by $\mathrm{Diff}^r_0(M)$ ($\mathrm{Homeo}_0(M)$ if $r = 0$) the identity component of the group of compactly supported C^r-diffeomorphisms of M (for the strong topology). If $r \neq \dim(M) + 1$, these groups are simple by a well-known and difficult theorem (see [1], [2], [4], [9], [10]).

In [6], Étienne Ghys asked whether the following statement was true: if M and N are two closed manifolds and if there exists a non-trivial morphism $\mathrm{Diff}^\infty_0(M) \to \mathrm{Diff}^\infty_0(N)$, then $\dim(M) \geq \dim(N)$. In [8], Kathryn Mann proved the following theorem. Take a connected manifold M of dimension greater than 1 and a one-dimensional connected manifold N. Then any morphism $\mathrm{Diff}^\infty_0(M) \to \mathrm{Diff}^\infty_0(N)$ is trivial: she answers Ghys's question in the case where the manifold N is one-dimensional. Mann also describes all the group morphisms $\mathrm{Diff}^r_0(M) \to \mathrm{Diff}^r_0(N)$

*supported by the Fondation Mathématique Jacques Hadamard. Centre de mathématiques Laurent Schwartz. École Polytechnique. 91128 Palaiseau cedex. e-mail : emmanuel.militon@math.polytechnique.fr

for $r \geq 3$ when M as well as N are one-dimensional. The techniques involved in the proofs of these theorems are Kopell's lemma (see [15] Theorem 4.1.1) and Szekeres's theorem (see [15] Theorem 4.1.11). These theorems are valid only for a regularity at least C^2. In this article, we prove similar results in the case of a C^0 regularity. The techniques used are different.

Theorem 1.1. *Let M be a connected manifold of dimension greater than 2 and let N be a connected one-manifold. Then any group morphism* $\mathrm{Homeo}_0(M) \rightarrow \mathrm{Homeo}(N)$ *is trivial.*

The case where the manifold M is one-dimensional is also well-understood.

Theorem 1.2. *Let N be a connected one-manifold. For any group morphism* $\varphi : \mathrm{Homeo}_0(\mathbb{R}) \rightarrow \mathrm{Homeo}(N)$*, there exists a closed set $K \subset N$ such that:*

1. *The set K is pointwise fixed under any homeomorphism in* $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$.
2. *For any connected component I of $N - K$, there exists a homeomorphism $h_I : \mathbb{R} \rightarrow I$ such that:*

$$\forall f \in \mathrm{Homeo}_0(\mathbb{R}),\ \varphi(f)_{|I} = h_I f h_I^{-1}.$$

Remark: By a theorem by Matsumoto (see [12] Theorem 5.3), every group morphism $\mathrm{Homeo}_0(\mathbb{S}^1) \rightarrow \mathrm{Homeo}_0(\mathbb{S}^1)$ is a conjugation by a homeomorphism of the circle. Moreover, any group morphism $\mathrm{Homeo}_0(\mathbb{S}^1) \rightarrow \mathrm{Homeo}(\mathbb{R})$ is trivial. Indeed, as the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ is simple, such a group morphism is either one-to-one or trivial. However, the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ contains torsion elements whereas the group $\mathrm{Homeo}(\mathbb{R})$ does not: such a morphism cannot be one-to-one.

2 Proofs of Theorems 1.1 and 1.2

Fix an integer $d \geq 1$. For a point p in $\mathbb{R}^d$, we denote by G_p^d the group $\mathrm{Homeo}_0(\mathbb{R}^d - \{p\})$. This group is seen as a subgroup of $\mathrm{Homeo}_0(\mathbb{R}^d)$ consisting of homeomorphisms which pointwise fix a neighbourhood of the point p. We will call embedded $(d-1)$-dimensional ball of $\mathbb{R}^d$ the image of the closed unit ball of $\mathbb{R}^{d-1} = \mathbb{R}^{d-1} \times \{0\} \subset \mathbb{R}^d$ under a homeomorphism of $\mathbb{R}^d$. For an embedded $(d-1)$-dimensional ball $D \subset \mathbb{R}^d$ (which is a single point if $d = 1$), we denote by H_D^d the subgroup of $\mathrm{Homeo}_0(\mathbb{R}^d)$ consisting of homeomorphisms which pointwise fix a neighbourhood of the embedded ball D. Finally, if G denotes a subgroup of $\mathrm{Homeo}(\mathbb{R}^d)$, a point $p \in \mathbb{R}^d$ is said to be fixed under the group G if it is fixed under all the elements of this group. We denote by $\mathrm{Fix}(G)$ the (closed) set of fixed points of G.

The theorems will be deduced from the following propositions. The two first propositions will be proved respectively in Sections 3 and 4.

Proposition 2.1. *Let* $\varphi : \mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{R})$ *be a group morphism. Suppose that no point of the real line is fixed under the group* $\varphi(\mathrm{Homeo}_0(\mathbb{R}^d))$*. Then, for any embedded* $(d-1)$*-dimensional ball* $D \subset \mathbb{R}^d$*, the group* $\varphi(H^d_D)$ *admits at most one fixed point.*

Proposition 2.2. *Let* $\varphi : \mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{R})$ *be a group morphism. Then, for any point* p *in* $\mathbb{R}^d$*, the group* $\varphi(G^d_p)$ *admits at least one fixed point.*

Proposition 2.3. *For any group morphism* $\psi : \mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{S}^1)$*, the group* $\psi(\mathrm{Homeo}_0(\mathbb{R}^d))$ *has a fixed point.*

Proof of Proposition 2.3. Recall that the group $\mathrm{Homeo}_0(\mathbb{R}^d)$ is infinite and simple and that the group $\mathrm{Homeo}(\mathbb{S}^1)/\mathrm{Homeo}_0(\mathbb{S}^1)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence any morphism $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{S}^1)/\mathrm{Homeo}_0(\mathbb{S}^1)$ is trivial. Therefore, the image of a morphism $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{S}^1)$ is contained in $\mathrm{Homeo}_0(\mathbb{S}^1)$.

For some background about the bounded cohomology of groups and the bounded Euler class of a group acting on a circle, see Section 6 in [5]. By [11] and [13]:

$$H^2_b(\mathrm{Homeo}_0(\mathbb{R}^d), \mathbb{Z}) = \{0\}\,.$$

Therefore, the bounded Euler class of a morphism $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}_0(\mathbb{S}^1)$ vanishes: this action has a fixed point. □

Proof of Theorem 1.1. Let $d = \dim(M)$. The theorem will be deduced from the following lemma.

Lemma 2.4. *Any group morphism* $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{R})$ *is trivial.*

Let us see why this lemma implies the theorem. Consider a morphism $\mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(N)$. Take an open set $U \subset M$ homeomorphic to $\mathbb{R}^d$ and let us denote by $\mathrm{Homeo}_0(U)$ the subgroup of $\mathrm{Homeo}_0(M)$ consisting of homeomorphisms supported in U. By Lemma 2.4 and Proposition 2.3, the restriction of this morphism to the subgroup $\mathrm{Homeo}_0(U)$ is trivial. Moreover, as the group $\mathrm{Homeo}_0(M)$ is simple, such a group morphism is either one-to-one or trivial: it is necessarily trivial in this case. □

Proof of Lemma 2.4. Take a group morphism $\varphi : \mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{R})$. Suppose by contradiction that this morphism is nontrivial. Replacing if necessary $\mathbb{R}$ with a connected component of the complement of the closed set $\mathrm{Fix}(\varphi(\mathrm{Homeo}_0(\mathbb{R}^d)))$, we can suppose that the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}^d))$ has no fixed points.

Let us prove that, for any points $p_1 \neq p_2$ in $\mathbb{R}^d$:

$$\mathrm{Fix}(\varphi(G^d_{p_1})) \cap \mathrm{Fix}(\varphi(G^d_{p_2})) = \emptyset.$$

The proof of this fact requires the following lemma.

Lemma 2.5. *Let $d' \geq 1$ be an integer. Let $p_1 \neq p_2$ be two distinct points in $\mathbb{R}^{d'}$. Then, for any homeomorphism f in* $\mathrm{Homeo}_0(\mathbb{R}^{d'})$*, there exist homeomorphisms f_1, f_3 in $G^{d'}_{p_1}$ and f_2 in $G^{d'}_{p_2}$ such that:*

$$f = f_1 f_2 f_3.$$

Proof. Take a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R}^{d'})$. Let f_1 be a homeomorphism in $G^{d'}_{p_1}$ such that f_1^{-1} sends the point $f(p_1)$ to a point which lies in the same connected component of $\mathbb{R}^{d'} - \{p_2\}$ as the point p_1. Let f_2 be a homeomorphism in $G^{d'}_{p_2}$ which is equal to $f_1^{-1}f$ in a neighbourhood of the point p_1. The existence of the homeomorphism f_2 is easy to prove when $d' = 1$, is a consequence of the Schönflies Theorem when $d' = 2$ and of the annulus theorem by Kirby and Quinn when $d' \geq 3$ (see [7] and [16]). Changing if necessary the homeomorphism f_2 into the composition of the homeomorphism f_2 with a homeomorphism supported in a small neighbourhood of the point p_1, the homeomorphism $f_3 = f_2^{-1} f_1^{-1} f$ belongs to $G^{d'}_{p_1}$. □

Take two points p_1 and p_2 in $\mathbb{R}^d$. Suppose by contradiction that $\mathrm{Fix}(\varphi(G^d_{p_1})) \cap \mathrm{Fix}(\varphi(G^d_{p_2})) \neq \emptyset$. By Lemma 2.5, a point in this set is a fixed point of the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}^d))$, a contradiction.

By Proposition 2.2, the sets $\mathrm{Fix}(\varphi(G^d_p))$, for $p \in \mathbb{R}^d$ are nonempty. We just saw that they are pairwise disjoint. Recall that, for any embedded $(d-1)$-dimensional ball D, the set $\mathrm{Fix}(\varphi(H^d_D))$ contains the union of the sets $\mathrm{Fix}(\varphi(G^d_p))$ over the points p in the closed set D. Hence, this set has infinitely many points as $d \geq 2$, a contradiction with Proposition 2.1. □

Proof of Theorem 1.2. Let $\varphi : \mathrm{Homeo}_0(\mathbb{R}) \to \mathrm{Homeo}(N)$ be a nontrivial group morphism. By Proposition 2.3, we can suppose that the manifold N is the real line $\mathbb{R}$. Replacing $\mathbb{R}$ with a connected component of the complement of the closed set $\mathrm{Fix}(\varphi(\mathrm{Homeo}_0(\mathbb{R})))$ if necessary, we can suppose that the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$ has no fixed point. Recall that the group $\mathrm{Homeo}_0(\mathbb{R})$ is simple. Hence any morphism $\mathrm{Homeo}_0(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$ is trivial. Thus, any element of the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$ preserves the orientation of $\mathbb{R}$.

By Propositions 2.1 and 2.2, for any real number x, the group $\varphi(G^1_x)$ has a unique fixed point $h(x)$. Take a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R})$ which sends a point x in $\mathbb{R}$ to a point y in $\mathbb{R}$. Then $fG^1_x f^{-1} = G^1_y$ and, taking the image under φ, $\varphi(f)\varphi(G^1_x)\varphi(f)^{-1} = \varphi(G^1_y)$. Hence $\varphi(f)(\mathrm{Fix}(\varphi(G^1_x))) = \mathrm{Fix}(\varphi(G^1_y))$. Therefore, for any homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R})$, $\varphi(f)h = hf$.

Let us prove that the map h is one-to-one. Suppose by contradiction that there exist real numbers $x \neq y$ such that $h(x) = h(y)$. The point $h(x)$ is fixed under the groups $\varphi(G^1_x)$ and $\varphi(G^1_y)$. However, the groups G^1_x and G^1_y generate the group

$\mathrm{Homeo}_0(\mathbb{R})$ by Lemma 2.5. Therefore, the point $h(x)$ is fixed under the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$, a contradiction.

Now we prove that the map h is either strictly increasing or strictly decreasing. Fix two points $x_0 < y_0$ of the real line. For any two points $x < y$ of the real line, let us consider a homeomorphism $f_{x,y}$ in $\mathrm{Homeo}_0(\mathbb{R})$ such that $f_{x,y}(x_0) = x$ and $f_{x,y}(y_0) = y$. As $\varphi(f_{x,y})h = hf_{x,y}$, the homeomorphism $\varphi(f_{x,y})$ sends the ordered pair $(h(x_0), h(y_0))$ to the ordered pair $(h(x), h(y))$. As the homeomorphism $\varphi(f_{x,y})$ is strictly increasing:

$$h(x) < h(y) \Leftrightarrow h(x_0) < h(y_0)$$

and

$$h(x) > h(y) \Leftrightarrow h(x_0) > h(y_0).$$

Hence the map h is either strictly increasing or strictly decreasing.

Now, it remains to prove that the map h is onto to complete the proof. Suppose by contradiction that the map h is not onto. Notice that the set $h(\mathbb{R})$ is preserved under the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$. If this set had a lower bound or an upper bound, then the supremum of this set or the infimum of this set would provide a fixed point for the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$, a contradiction. This set has neither upper bound nor lower bound. Let C be a connected component of the complement of the set $h(\mathbb{R})$. To simplify the exposition of the proof, we suppose that the map h is increasing. Let us denote by x_0 the supremum of the set of points x such that the real number $h(x)$ is lower than any point in the interval C. Then the point $h(x_0)$ is necessarily in the closure of C: otherwise, there would exist an interval in the complementary of $h(\mathbb{R})$ which strictly contains the interval C. We suppose for instance that the point $h(x_0)$ is the supremum of the interval C. Choose, for each couple (z_1, z_2) of real numbers, a homeomorphism g_{z_1,z_2} in $\mathrm{Homeo}_0(\mathbb{R})$ which sends the point z_1 to the point z_2. Then the sets $g_{x_0,x}(C)$, for x in $\mathbb{R}$, are pairwise disjoint: they are pairwise distinct as their suprema are pairwise distinct (the supremum of the set $g_{x_0,x}(C)$ is the point $h(x)$). Moreover, those sets do not contain any point of $h(\mathbb{R})$ and the infima of those sets are accumulated by points in $h(\mathbb{R})$. Hence, these sets are pairwise disjoint. Then the set C has necessarily an empty interior as the topological space $\mathbb{R}$ is second-countable. Therefore $C = \{h(x_0)\}$, which is not possible. □

3 Proof of Proposition 2.1

The proof of this proposition is similar to the proofs of Lemmas 3.6 and 3.7 in [14]. We need the following lemma. The proof of this lemma is almost identical to the proof of Lemma 2.5 and is omitted.

Lemma 3.1. *Take two disjoint embedded $(d-1)$-dimensional balls D and D' in $\mathbb{R}^d$. For any homeomorphism f in* $\mathrm{Homeo}_0(\mathbb{R}^d)$*, there exist homeomorphisms h_1,*

h_3 *in* H^d_D *and* h_2 *in* $H^d_{D'}$ *such that*

$$h = h_1 h_2 h_3.$$

For such an embedded $(d-1)$-dimensional ball D, let $F_D = \mathrm{Fix}(\varphi(H^d_D))$. Let us prove that these sets are pairwise homeomorphic. Take two embedded $(d-1)$-dimensional balls D an D' and take a homeomorphism h in $\mathrm{Homeo}_0(\mathbb{R}^d)$ which sends the set D onto D'. Observe that $hH^d_D h^{-1} = H^d_{D'}$ and that $\varphi(h)\varphi(H^d_D)\varphi(h)^{-1} = \varphi(H^d_{D'})$. Therefore: $\varphi(h)(F_D) = F_{D'}$.

In the case where these sets are all empty, there is nothing to prove. We suppose in what follows that they are not empty.

Given two disjoint embedded $(d-1)$-dimensional balls D and D', Lemma 3.1 implies, as in the proof of Lemma 2.4:

$$F_D \cap F_{D'} = \emptyset.$$

Lemma 3.2. *Fix an embedded* $(d-1)$*-dimensional ball* D_0 *of* $\mathbb{R}^d$*. Then any connected component* C *of the complement of* F_{D_0} *meets one of the sets* F_D*, where* D *is an embedded* $(d-1)$*-dimensional ball disjoint from* D_0*.*

Proof. Let (a_1, a_2) be a connected component of the complement of F_{D_0}. It is possible that either $a_1 = -\infty$ or $a_2 = +\infty$. Consider a homeomorphism $e : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^d$ such that $e(B^{d-1} \times \{0\}) = D_0$, where B^{d-1} denotes the unit closed ball in $\mathbb{R}^{d-1}$. For any real number x, let $D_x = e(B^{d-1} \times \{x\})$. Given two real $x \neq y$, take a homeomorphism $\eta_{x,y}$ in $\mathrm{Homeo}_0(\mathbb{R})$ which sends the point x to the point y. Consider a homeomorphism $h_{x,y}$ such that the following property is satisfied. The restriction of $eh_{x,y}e^{-1}$ to $B^{d-1} \times \mathbb{R}$ is equal to the map:

$$\begin{array}{rcl} B^{d-1} \times \mathbb{R} & \to & \mathbb{R}^{d-1} \times \mathbb{R} \\ (p, z) & \mapsto & (p, \eta_{x,y}(z)) \end{array}$$

Notice that, for any real numbers x and y, $h_{x,y}(D_x) = D_y$

Let us prove by contradiction that there exists a real number $x \neq 0$ such that $F_{D_x} \cap (a_1, a_2) \neq \emptyset$. Suppose that, for any such embedded ball D_x, $F_{D_x} \cap (a_1, a_2) = \emptyset$. We claim that the open sets $\varphi(h_{0,x})((a_1, a_2))$ are pairwise disjoint. It is not possible as there would be uncountably many pairwise disjoint open intervals in $\mathbb{R}$.

Indeed, suppose by contradiction that there exists real numbers $x \neq y$ such that $\varphi(h_{0,x})((a_1, a_2)) \cap \varphi(h_{0,y})((a_1, a_2)) \neq \emptyset$. Notice that the homeomorphism $h^{-1}_{0,x}h_{0,y}$ and $h^{-1}_{0,y}h_{0,x}$ send respectively the set D_0 to sets of the form D_z and $D_{z'}$, where $z, z' \in \mathbb{R}$. Hence, for $i = 1, 2$, the homeomorphisms $\varphi(h^{-1}_{0,x}h_{0,y})$ (respectively $\varphi(h^{-1}_{0,y}h_{0,x})$) sends the point $a_i \in F_{D_0}$ to a point in F_{D_z} (respectively in $F_{D_{z'}}$). By hypothesis, these points do not belong to (a_1, a_2). Therefore

$$\varphi(h_{0,y}^{-1} h_{0,x})(a_1, a_2) = (a_1, a_2)$$

or

$$\varphi(h_{0,x})(a_1, a_2) = \varphi(h_{0,y})(a_1, a_2).$$

But this last equality cannot hold as the real endpoints of the interval on the left-hand side belong to F_{D_x} and the real endpoints point of the interval on the right-hand side belongs to F_{D_y}. Moreover, we saw that these two closed sets were disjoint, a contradiction. □

Lemma 3.3. *Each set F_D contains only one point.*

Proof. Suppose that there exists an embedded $(d-1)$-dimensional ball D such that the set F_D contains two points $p_1 < p_2$. By Lemma 3.2, there exists an embedded $(d-1)$-dimensional ball D' disjoint from D such that the set $F_{D'}$ has a common point with the open interval (p_1, p_2). Take a real number $r < p_1$. Then, for any homeomorphisms g_1 in G_D, g_2 in $G_{D'}$ and g_3 in G_D,

$$\varphi(g_1) \circ \varphi(g_2) \circ \varphi(g_3)(r) < p_2.$$

By Lemma 3.1, this implies that the following inclusion holds:

$$\left\{ \varphi(g)(r), g \in \mathrm{Homeo}_0(\mathbb{R}^d) \right\} \subset (-\infty, p_2].$$

The supremum of the left-hand set provides a fixed point for the action φ, a contradiction. □

4 Proof of Proposition 2.2

This proof uses the following lemmas. For a subgroup G of $\mathrm{Homeo}_0(\mathbb{R}^d)$, we define the support $\mathrm{Supp}(G)$ of G as the closure of the set:

$$\left\{ x \in \mathbb{R}^d, \ \exists g \in G, \ gx \neq x \right\}.$$

Let $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) = \{ f \in \mathrm{Homeo}(\mathbb{R}), \ \forall x \in \mathbb{R}, f(x+1) = f(x) + 1 \}.$

To prove Proposition 2.2, we need the following lemmas.

Lemma 4.1. *Let G and G' be subgroups of the group $\mathrm{Homeo}_+(\mathbb{R})$ of orientation-preserving homeomorphisms of the circle. Suppose that the following conditions are satisfied.*

1. *The groups G and G' are isomorphic to the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.*
2. *The subgroups G and G' of $\mathrm{Homeo}_+(\mathbb{R})$ commute: $\forall g \in G, g' \in G', \ gg' = g'g$.*

Then $\mathrm{Supp}(G) \subset \mathrm{Fix}(G')$ *and* $\mathrm{Supp}(G') \subset \mathrm{Fix}(G)$.

Lemma 4.2. *Take any nonempty open subset* U *of* $\mathbb{R}^d$. *Then there exists a subgroup of* $\mathrm{Homeo}_0(\mathbb{R}^d)$ *isomorphic to* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ *which is supported in* U.

Lemma 4.1 will be proved in the next section. We now provide a proof of Lemma 4.2.

Proof of Lemma 4.2. Take a closed ball B contained in U. Changing coordinates if necessary, we can suppose that B is the unit closed ball in $\mathbb{R}^d$. Take an orientation-preserving homeomorphism $h : \mathbb{R} \to (-1, 1)$. For any orientation-preserving homeomorphism $f : \mathbb{R} \to \mathbb{R}$, we define the homeomorphism $\lambda_h(f) : \mathbb{R}^d \to \mathbb{R}^d$ in the following way.

1. The homeomorphism $\lambda_h(f)$ is equal to the identity outside the interior of the ball B.
2. For any $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} \cap \mathrm{int}(B)$:

$$\lambda_h(f)(x_1, x') = (\sqrt{1 - \|x'\|^2} h \circ f \circ h^{-1}(\frac{x_1}{\sqrt{1 - \|x'\|^2}}), x').$$

The map λ_h defines an embedding of the group $\mathrm{Homeo}_+(\mathbb{R})$ into the group $\mathrm{Homeo}_0(\mathbb{R}^d)$. The image under λ_h of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is a subgroup of $\mathrm{Homeo}_0(\mathbb{R}^d)$ which is supported in U. □

Let us complete now the proof of Proposition 2.2.

Proof of Proposition 2.2. Fix a point p in $\mathbb{R}^d$. Take a closed ball $B \subset \mathbb{R}^d$ which is centered at p. Let G_B^d be the subgroup of G_p^d consisting of homeomorphisms which pointwise fix a neighbourhood of the ball B. Let us prove that $\mathrm{Fix}(G_B^d) \neq \emptyset$.

Take a subgroup G_1 of $\mathrm{Homeo}_0(\mathbb{R}^d)$ which is isomorphic to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ and supported in B. Such a subgroup exists by Lemma 4.2. This subgroup commutes with any subgroup G_2 of $\mathrm{Homeo}_0(\mathbb{R}^d)$ which is isomorphic to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ and supported outside B.

If the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}^d))$ admits a fixed point, there is nothing to prove. Suppose that this group has no fixed point. As the group $\mathrm{Homeo}_0(\mathbb{R}^d)$ is simple, the morphism φ is one-to-one. Moreover, any morphism $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathbb{Z}/2\mathbb{Z}$ is trivial: the morphism φ takes values in $\mathrm{Homeo}_+(\mathbb{R})$. Hence the subgroups $\varphi(G_1)$ and $\varphi(G_2)$ of $\mathrm{Homeo}(\mathbb{R})$ satisfy the hypothesis of Lemma 4.1. By this lemma:

$$\emptyset \neq \mathrm{Supp}(\varphi(G_1)) \subset \mathrm{Fix}(\varphi(G_2)).$$

We claim that the group G_B^d is generated by the union of its subgroups isomorphic to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$. This claim implies that

$$\emptyset \neq \mathrm{Supp}(\varphi(G_1)) \subset \mathrm{Fix}(\varphi(G_B^d)).$$

For $d \geq 2$, the claim is a direct consequence of the simplicity of the group G_B^d. In the case where $d = 1$, denote by $[a, b]$ the compact interval B. The inclusions of the groups $\mathrm{Homeo}_0((-\infty, a))$ and $\mathrm{Homeo}_0((b, +\infty))$ induce an isomorphism $\mathrm{Homeo}_0((-\infty, a)) \times \mathrm{Homeo}_0((b, +\infty)) \to G_B^d$. The simplicity of each factor of this decomposition implies the claim.

Now, let us prove that the set $\mathrm{Fix}(\varphi(G_B^d))$ is compact. Suppose by contradiction that there exists a sequence $(a_k)_{k\in\mathbb{N}}$ of real numbers in $\mathrm{Fix}(\varphi(G_B^d))$ which tends to $+\infty$ (if we suppose that it tends to $-\infty$, we obtain of course an analogous contradiction). Let us choose a closed ball $B' \subset \mathbb{R}^d$ which is disjoint from B. Observe that the subgroups G_B^d and $G_{B'}^d$ are conjugate in $\mathrm{Homeo}_0(\mathbb{R}^d)$ by a homeomorphism which sends the ball B to the ball B'. Then the subgroups $\varphi(G_B^d)$ and $\varphi(G_{B'}^d)$ are conjugate in the group $\mathrm{Homeo}_+(\mathbb{R})$. Hence the sets $\mathrm{Fix}(\varphi(G_B^d))$ and $\mathrm{Fix}(\varphi(G_{B'}^d))$ are homeomorphic: there exists a sequence $(b_k)_{k\in\mathbb{N}}$ of elements in $\mathrm{Fix}(\varphi(G_{B'}^d))$ which tends to $+\infty$. Take positive integers n_1, n_2 and n_3 such that $a_{n_1} < b_{n_2} < a_{n_3}$. Fix $x_0 < a_{n_1}$. We notice then that for any homeomorphisms $g_1 \in G_B^d$, $g_2 \in G_{B'}^d$ and $g_3 \in G_B^d$, the following inequality is satisfied:

$$\varphi(g_1)\varphi(g_2)\varphi(g_3)(x_0) < a_{n_3}.$$

However, any element g in $\mathrm{Homeo}_0(\mathbb{R}^d)$ can be written as a product

$$g = g_1 g_2 g_3,$$

where g_1 and g_3 belong to G_B^d and g_2 belongs to $G_{B'}^d$. The proof of this fact is similar to that of Lemma 2.5. Therefore:

$$\overline{\{\varphi(g)(x_0),\ g \in \mathrm{Homeo}_c(\mathbb{R})\}} \subset (-\infty, a_{n_3}].$$

The greatest element of the left-hand set is a fixed point of the image of φ: this is not possible as this image was supposed to have no fixed point.

Observe that the group $\varphi(G_p^d)$ is the union of its subgroup of the form $\varphi(G_{B'}^d)$, with B' varying over the set $\mathcal{B}_p$ of closed balls centered at the point p. By compactness, the set

$$\mathrm{Fix}(\varphi(G_p^d)) = \bigcap_{B' \in \mathcal{B}_p} \mathrm{Fix}(G_{B'}^d)$$

is nonempty. Proposition 2.2 is proved. □

5 Proof of Lemma 4.1

We need the following lemmas. The first one will be proved afterwards.

Lemma 5.1. *Let* $\psi : \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(\mathbb{R})$ *be a group morphism. Then there exists a closed set* $F \subset \mathbb{R}$ *such that:*

1. *The set F is pointwise fixed under any element in* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.
2. *For any connected component K of the complement of F, there exists a homeomorphism $h_K : \mathbb{R} \to K$ such that:*

$$\forall f \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}), \forall x \in K,\ \psi(f)(x) = h_K f h_K^{-1}.$$

Lemma 5.2. *Any group morphism* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{Z}$ *is trivial.*

Proof of Lemma 5.2. Actually, any element in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ can be written as a product of commutators, *i.e.* elements of the form $aba^{-1}b^{-1}$, with $a,\ b \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$. For an explicit construction of such a decomposition, see Section 2 in [3]. □

Observe that the center of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is the subgroup generated by the translation $x \mapsto x+1$. Let α (respectively α') be a generator of the center of G (respectively of G'). Let $A_\alpha = \mathbb{R} - \mathrm{Fix}(\alpha)$ and $A_{\alpha'} = \mathbb{R} - \mathrm{Fix}(\alpha')$.

As the homeomorphisms α and α' commute:

$$\begin{cases} \alpha'(A_\alpha) = A_\alpha \\ \alpha(A_{\alpha'}) = A_{\alpha'} \end{cases}.$$

Take any connected component I of A_α and any connected component I' of $A_{\alpha'}$. Then, either I is contained in I', or I' is contained in I, or I and I' are disjoint.

We now prove that only the latter case can occur. Suppose by contradiction that the interval I is strictly contained in the interval I'. Let $\sim$ be the equivalence relation defined on I' by

$$x \sim y \Leftrightarrow (\exists k \in \mathbb{Z},\ x = \alpha'^k(y)).$$

The topological space $I'/\sim$ is homeomorphic to a circle. By Lemma 5.1, the group G' preserves the interval I'. Notice that the group $G'/<\alpha'> \approx \mathrm{Homeo}_0(\mathbb{S}^1)$ acts on the circle $I'/\sim$. As the group G' commutes with the homeomorphism α, this action preserves the nonempty set $(A_\alpha \cap I')/\sim$. As $\alpha'(A_\alpha) = A_\alpha$, the endpoints of the interval I are sent to points in the complement of A_α under the iterates of the homeomorphism α'. Hence the set $(A_\alpha \cap I')/\sim$ is not equal to the whole circle $I'/\sim$. However, by Theorem 5.3 in [12] (see the remark below Theorem 1.2), any non-trivial action of the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ on a circle is transitive. Hence, the group $G'/<\alpha'>$ acts trivially on the circle $I'/\sim$: for any element β' of G', and any point $x \in I'$, there exists an integer $k(x,\beta') \in \mathbb{Z}$ such that $\beta'(x) = \alpha'^{k(x,\beta')}(x)$. Fixing such a point x, we see that the map

$$\begin{array}{ccc} G' & \to & \mathbb{Z} \\ \beta' & \mapsto & k(x,\beta') \end{array}$$

is a group morphism. Such a group morphism is trivial by Lemma 5.2. Therefore, the group G' acts trivially on the interval I', a contradiction.

Of course, the case where the interval I' is strictly contained in I is symmetric and cannot occur.

Suppose now that $I = I'$. Take any element β' in G'. As the homeomorphism β' commutes with α, by Lemma 5.1, the homeomorphism β' is equal to some element of G on I. As the homeomorphism β' commute with any element of G, there exists a unique integer $k(\beta')$ such that $\beta'_{|I} = \alpha_{|I}^{k(\beta')}$. The map $k : G \to \mathbb{Z}$ is a nontrivial group morphism. But such a map cannot exist by Lemma 5.2. Lemma 4.1 is proved.

It remains to prove Lemma 5.1.

Proof of Lemma 5.1. Denote by t a generator of the center of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

Claim 1. The connected components of the complement of $\mathrm{Fix}(\psi(t))$ are each preserved by the group $\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$. Moreover

$$\mathrm{Fix}(\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))) = \mathrm{Fix}(\psi(t)).$$

Claim 2. Any action of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ on $\mathbb{R}$ without fixed points is conjugate to the standard action.

It is clear that these two claims imply Lemma 5.1.

First, let us prove Claim 1. The set $\mathrm{Fix}(\psi(t))$ is preserved under any element in $\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$, because any element of this group commutes with the homeomorphism $\psi(t)$. Moreover, any element in $\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$ preserves the orientation. Hence any connected component of the complement of $\mathrm{Fix}(\psi(t))$ with infinite length is preserved under the action of the group $\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$. We suppose now that any connected component of the complement of $\mathrm{Fix}(\psi(t))$ has a finite length. Let us denote by $\equiv$ the equivalence relation on $\mathbb{R}$ such that $x \equiv y$ if and only if the points x and y belong to the same connected component of $\mathrm{Supp}(\psi(t))$. The morphism ψ induces an action of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})/ < t > \approx \mathrm{Homeo}_0(\mathbb{S}^1)$ on the quotient topological space $\mathbb{R}/\equiv$ which is homeomorphic to $\mathbb{R}$ if the set $\mathrm{Fix}(\psi(t))$ has a nonempty interior. However, such an action is trivial by the remark below Theorem 1.2. Hence, any connected component of $\mathrm{Supp}(\psi(t))$ is preserved by the action. Restricting to one of these connected components if necessary, we can suppose that the closed set $\mathrm{Fix}(\psi(t))$ contains only isolated points. If this set is empty, there is nothing to prove. Otherwise, let

$$\mathrm{Fix}(\psi(t)) = \{x_i, i \in A\},$$

where A is a set contained in $\mathbb{Z}$ and contains 0 and where the sequence $(x_i)_{i \in A}$ is strictly increasing. Then, for any element f in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, there exists an integer $i(f)$ such that $\psi(f)(x_0) = x_{i(f)}$. The map $i : \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{Z}$ is a group morphism: it is trivial by Lemma 5.2. Claim 1 is proved.

By Claim 1, restricting if necessary the action on a connected component of the complement of $\mathrm{Fix}(\psi(t))$, we can suppose that the homeomorphism $\psi(t)$ has no fixed point. Changing coordinates if necessary, we can suppose that the homeomorphism $\psi(t)$ is the translation $x \mapsto x+1$. The morphism ψ induces an action $\hat{\psi}$ of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})/ < t > \approx \mathrm{Homeo}_0(\mathbb{S}^1)$ on the circle $\mathbb{R}/\mathbb{Z}$. This action is nontrivial: otherwise, there would exist a nontrivial group morphism $\mathrm{Homeo}_0(\mathbb{S}^1) \to \mathbb{Z}$. By the remark below Theorem 1.2, there exists a homeomorphism h of the circle $\mathbb{R}/\mathbb{Z}$ such that, for any homeomorphism f in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})/ < t >$ (which can be canonically identified with $\mathrm{Homeo}_0(\mathbb{R}/\mathbb{Z})$):

$$\hat{\psi}(f) = hfh^{-1}.$$

Take a lift $\tilde{h} : \mathbb{R} \to \mathbb{R}$ of h. For any integer n, denote by $T_n : \mathbb{R} \to \mathbb{R}$ the translation $x \mapsto x+n$. For any homeomorphism f in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, there exists an integer $n(f)$ such that

$$\psi(f) = T_{n(f)}\tilde{h}f\tilde{h}^{-1}.$$

However, the map $n : \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{Z}$ is a group morphism: it is trivial by Lemma 5.2. This completes the proof of Claim 2. □

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