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# Actions of groups of homeomorphisms on one-manifolds

Emmanuel Militon \*

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## Abstract

In this article, we describe all the group morphisms from the group of compactly-supported homeomorphisms isotopic to the identity of a manifold to the group of homeomorphisms of the real line or of the circle.

MSC: 37C85.

## 1 Introduction

Fix a connected manifold  $M$  (without boundary). For an integer  $r \geq 0$ , we denote by  $\text{Diff}^r(M)$  the group of  $C^r$ -diffeomorphisms of  $M$ . When  $r = 0$ , this group will also be denoted by  $\text{Homeo}(M)$ . For a homeomorphism  $f$  of  $M$ , the *support* of  $f$  is the closure of the set:

$$\{x \in M, f(x) \neq x\}.$$

We denote by  $\text{Diff}_0^r(M)$  ( $\text{Homeo}_0(M)$  if  $r = 0$ ) the identity component of the group of compactly supported  $C^r$ -diffeomorphisms of  $M$  (for the strong topology). If  $r \neq \dim(M) + 1$ , these groups are simple by a well-known and difficult theorem (see [1], [2], [4], [9], [10]).

In [6], Étienne Ghys asked whether the following statement was true: if  $M$  and  $N$  are two closed manifolds and if there exists a non-trivial morphism  $\text{Diff}_0^\infty(M) \rightarrow \text{Diff}_0^\infty(N)$ , then  $\dim(M) \geq \dim(N)$ . In [8], Kathryn Mann proved the following theorem. Take a connected manifold  $M$  of dimension greater than 1 and a one-dimensional connected manifold  $N$ . Then any morphism  $\text{Diff}_0^\infty(M) \rightarrow \text{Diff}_0^\infty(N)$  is trivial: she answers Ghys's question in the case where the manifold  $N$  is one-dimensional. Mann also describes all the group morphisms  $\text{Diff}_0^r(M) \rightarrow \text{Diff}_0^r(N)$

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for  $r \geq 3$  when  $M$  as well as  $N$  are one-dimensional. The techniques involved in the proofs of these theorems are Kopell's lemma (see [15] Theorem 4.1.1) and Szekeres's theorem (see [15] Theorem 4.1.11). These theorems are valid only for a regularity at least  $C^2$ . In this article, we prove similar results in the case of a  $C^0$  regularity. The techniques used are different.

**Theorem 1.1.** *Let  $M$  be a connected manifold of dimension greater than 2 and let  $N$  be a connected one-manifold. Then any group morphism  $\text{Homeo}_0(M) \rightarrow \text{Homeo}(N)$  is trivial.*

The case where the manifold  $M$  is one-dimensional is also well-understood.

**Theorem 1.2.** *Let  $N$  be a connected one-manifold. For any group morphism  $\varphi : \text{Homeo}_0(\mathbb{R}) \rightarrow \text{Homeo}(N)$ , there exists a closed set  $K \subset N$  such that:*

1. *The set  $K$  is pointwise fixed under any homeomorphism in  $\varphi(\text{Homeo}_0(\mathbb{R}))$ .*
2. *For any connected component  $I$  of  $N - K$ , there exists a homeomorphism  $h_I : \mathbb{R} \rightarrow I$  such that:*

$$\forall f \in \text{Homeo}_0(\mathbb{R}), \varphi(f)|_I = h_I f h_I^{-1}.$$

**Remark:** By a theorem by Matsumoto (see [12] Theorem 5.3), every group morphism  $\text{Homeo}_0(\mathbb{S}^1) \rightarrow \text{Homeo}_0(\mathbb{S}^1)$  is a conjugation by a homeomorphism of the circle. Moreover, any group morphism  $\text{Homeo}_0(\mathbb{S}^1) \rightarrow \text{Homeo}(\mathbb{R})$  is trivial. Indeed, as the group  $\text{Homeo}_0(\mathbb{S}^1)$  is simple, such a group morphism is either one-to-one or trivial. However, the group  $\text{Homeo}_0(\mathbb{S}^1)$  contains torsion elements whereas the group  $\text{Homeo}(\mathbb{R})$  does not: such a morphism cannot be one-to-one.

## 2 Proofs of Theorems 1.1 and 1.2

Fix an integer  $d \geq 1$ . For a point  $p$  in  $\mathbb{R}^d$ , we denote by  $G_p^d$  the group  $\text{Homeo}_0(\mathbb{R}^d - \{p\})$ . This group is seen as a subgroup of  $\text{Homeo}_0(\mathbb{R}^d)$  consisting of homeomorphisms which pointwise fix a neighbourhood of the point  $p$ . We will call embedded  $(d-1)$ -dimensional ball of  $\mathbb{R}^d$  the image of the closed unit ball of  $\mathbb{R}^{d-1} = \mathbb{R}^{d-1} \times \{0\} \subset \mathbb{R}^d$  under a homeomorphism of  $\mathbb{R}^d$ . For an embedded  $(d-1)$ -dimensional ball  $D \subset \mathbb{R}^d$  (which is a single point if  $d = 1$ ), we denote by  $H_D^d$  the subgroup of  $\text{Homeo}_0(\mathbb{R}^d)$  consisting of homeomorphisms which pointwise fix a neighbourhood of the embedded ball  $D$ . Finally, if  $G$  denotes a subgroup of  $\text{Homeo}(\mathbb{R}^d)$ , a point  $p \in \mathbb{R}^d$  is said to be fixed under the group  $G$  if it is fixed under all the elements of this group. We denote by  $\text{Fix}(G)$  the (closed) set of fixed points of  $G$ .

The theorems will be deduced from the following propositions. The two first propositions will be proved respectively in Sections 3 and 4.

**Proposition 2.1.** *Let  $\varphi : \text{Homeo}_0(\mathbb{R}^d) \rightarrow \text{Homeo}(\mathbb{R})$  be a group morphism. Suppose that no point of the real line is fixed under the group  $\varphi(\text{Homeo}_0(\mathbb{R}^d))$ . Then, for any embedded  $(d-1)$ -dimensional ball  $D \subset \mathbb{R}^d$ , the group  $\varphi(H_D^d)$  admits at most one fixed point.*

**Proposition 2.2.** *Let  $\varphi : \text{Homeo}_0(\mathbb{R}^d) \rightarrow \text{Homeo}(\mathbb{R})$  be a group morphism. Then, for any point  $p$  in  $\mathbb{R}^d$ , the group  $\varphi(G_p^d)$  admits at least one fixed point.*

**Proposition 2.3.** *For any group morphism  $\psi : \text{Homeo}_0(\mathbb{R}^d) \rightarrow \text{Homeo}(\mathbb{S}^1)$ , the group  $\psi(\text{Homeo}_0(\mathbb{R}^d))$  has a fixed point.*

*Proof of Proposition 2.3.* Recall that the group  $\text{Homeo}_0(\mathbb{R}^d)$  is infinite and simple and that the group  $\text{Homeo}(\mathbb{S}^1)/\text{Homeo}_0(\mathbb{S}^1)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . Hence any morphism  $\text{Homeo}_0(\mathbb{R}^d) \rightarrow \text{Homeo}(\mathbb{S}^1)/\text{Homeo}_0(\mathbb{S}^1)$  is trivial. Therefore, the image of a morphism  $\text{Homeo}_0(\mathbb{R}^d) \rightarrow \text{Homeo}(\mathbb{S}^1)$  is contained in  $\text{Homeo}_0(\mathbb{S}^1)$ .

For some background about the bounded cohomology of groups and the bounded Euler class of a group acting on a circle, see Section 6 in [5]. By [11] and [13]:

$$H_b^2(\text{Homeo}_0(\mathbb{R}^d), \mathbb{Z}) = \{0\}.$$

Therefore, the bounded Euler class of a morphism  $\text{Homeo}_0(\mathbb{R}^d) \rightarrow \text{Homeo}_0(\mathbb{S}^1)$  vanishes: this action has a fixed point.  $\square$

*Proof of Theorem 1.1.* Let  $d = \dim(M)$ . The theorem will be deduced from the following lemma.

**Lemma 2.4.** *Any group morphism  $\text{Homeo}_0(\mathbb{R}^d) \rightarrow \text{Homeo}(\mathbb{R})$  is trivial.*

Let us see why this lemma implies the theorem. Consider a morphism  $\text{Homeo}_0(M) \rightarrow \text{Homeo}_0(N)$ . Take an open set  $U \subset M$  homeomorphic to  $\mathbb{R}^d$  and let us denote by  $\text{Homeo}_0(U)$  the subgroup of  $\text{Homeo}_0(M)$  consisting of homeomorphisms supported in  $U$ . By Lemma 2.4 and Proposition 2.3, the restriction of this morphism to the subgroup  $\text{Homeo}_0(U)$  is trivial. Moreover, as the group  $\text{Homeo}_0(M)$  is simple, such a group morphism is either one-to-one or trivial: it is necessarily trivial in this case.  $\square$

*Proof of Lemma 2.4.* Take a group morphism  $\varphi : \text{Homeo}_0(\mathbb{R}^d) \rightarrow \text{Homeo}(\mathbb{R})$ . Suppose by contradiction that this morphism is nontrivial. Replacing if necessary  $\mathbb{R}$  with a connected component of the complement of the closed set  $\text{Fix}(\varphi(\text{Homeo}_0(\mathbb{R}^d)))$ , we can suppose that the group  $\varphi(\text{Homeo}_0(\mathbb{R}^d))$  has no fixed points.

Let us prove that, for any points  $p_1 \neq p_2$  in  $\mathbb{R}^d$ :

$$\text{Fix}(\varphi(G_{p_1}^d)) \cap \text{Fix}(\varphi(G_{p_2}^d)) = \emptyset.$$

The proof of this fact requires the following lemma.

**Lemma 2.5.** *Let  $d' \geq 1$  be an integer. Let  $p_1 \neq p_2$  be two distinct points in  $\mathbb{R}^{d'}$ . Then, for any homeomorphism  $f$  in  $\text{Homeo}_0(\mathbb{R}^{d'})$ , there exist homeomorphisms  $f_1, f_3$  in  $G_{p_1}^{d'}$  and  $f_2$  in  $G_{p_2}^{d'}$  such that:*

$$f = f_1 f_2 f_3.$$

*Proof.* Take a homeomorphism  $f$  in  $\text{Homeo}_0(\mathbb{R}^{d'})$ . Let  $f_1$  be a homeomorphism in  $G_{p_1}^{d'}$  such that  $f_1^{-1}$  sends the point  $f(p_1)$  to a point which lies in the same connected component of  $\mathbb{R}^{d'} - \{p_2\}$  as the point  $p_1$ . Let  $f_2$  be a homeomorphism in  $G_{p_2}^{d'}$  which is equal to  $f_1^{-1}f$  in a neighbourhood of the point  $p_1$ . The existence of the homeomorphism  $f_2$  is easy to prove when  $d' = 1$ , is a consequence of the Schönflies Theorem when  $d' = 2$  and of the annulus theorem by Kirby and Quinn when  $d' \geq 3$  (see [7] and [16]). Changing if necessary the homeomorphism  $f_2$  into the composition of the homeomorphism  $f_2$  with a homeomorphism supported in a small neighbourhood of the point  $p_1$ , the homeomorphism  $f_3 = f_2^{-1}f_1^{-1}f$  belongs to  $G_{p_1}^{d'}$ .  $\square$

Take two points  $p_1$  and  $p_2$  in  $\mathbb{R}^d$ . Suppose by contradiction that  $\text{Fix}(\varphi(G_{p_1}^d)) \cap \text{Fix}(\varphi(G_{p_2}^d)) \neq \emptyset$ . By Lemma 2.5, a point in this set is a fixed point of the group  $\varphi(\text{Homeo}_0(\mathbb{R}^d))$ , a contradiction.

By Proposition 2.2, the sets  $\text{Fix}(\varphi(G_p^d))$ , for  $p \in \mathbb{R}^d$  are nonempty. We just saw that they are pairwise disjoint. Recall that, for any embedded  $(d-1)$ -dimensional ball  $D$ , the set  $\text{Fix}(\varphi(H_D^d))$  contains the union of the sets  $\text{Fix}(\varphi(G_p^d))$  over the points  $p$  in the closed set  $D$ . Hence, this set has infinitely many points as  $d \geq 2$ , a contradiction with Proposition 2.1.  $\square$

*Proof of Theorem 1.2.* Let  $\varphi : \text{Homeo}_0(\mathbb{R}) \rightarrow \text{Homeo}(N)$  be a nontrivial group morphism. By Proposition 2.3, we can suppose that the manifold  $N$  is the real line  $\mathbb{R}$ . Replacing  $\mathbb{R}$  with a connected component of the complement of the closed set  $\text{Fix}(\varphi(\text{Homeo}_0(\mathbb{R})))$  if necessary, we can suppose that the group  $\varphi(\text{Homeo}_0(\mathbb{R}))$  has no fixed point. Recall that the group  $\text{Homeo}_0(\mathbb{R})$  is simple. Hence any morphism  $\text{Homeo}_0(\mathbb{R}) \rightarrow \mathbb{Z}/2\mathbb{Z}$  is trivial. Thus, any element of the group  $\varphi(\text{Homeo}_0(\mathbb{R}))$  preserves the orientation of  $\mathbb{R}$ .

By Propositions 2.1 and 2.2, for any real number  $x$ , the group  $\varphi(G_x^1)$  has a unique fixed point  $h(x)$ . Take a homeomorphism  $f$  in  $\text{Homeo}_0(\mathbb{R})$  which sends a point  $x$  in  $\mathbb{R}$  to a point  $y$  in  $\mathbb{R}$ . Then  $fG_x^1f^{-1} = G_y^1$  and, taking the image under  $\varphi$ ,  $\varphi(f)\varphi(G_x^1)\varphi(f)^{-1} = \varphi(G_y^1)$ . Hence  $\varphi(f)(\text{Fix}(\varphi(G_x^1))) = \text{Fix}(\varphi(G_y^1))$ . Therefore, for any homeomorphism  $f$  in  $\text{Homeo}_0(\mathbb{R})$ ,  $\varphi(f)h = hf$ .

Let us prove that the map  $h$  is one-to-one. Suppose by contradiction that there exist real numbers  $x \neq y$  such that  $h(x) = h(y)$ . The point  $h(x)$  is fixed under the groups  $\varphi(G_x^1)$  and  $\varphi(G_y^1)$ . However, the groups  $G_x^1$  and  $G_y^1$  generate the group

$\text{Homeo}_0(\mathbb{R})$  by Lemma 2.5. Therefore, the point  $h(x)$  is fixed under the group  $\varphi(\text{Homeo}_0(\mathbb{R}))$ , a contradiction.

Now we prove that the map  $h$  is either strictly increasing or strictly decreasing. Fix two points  $x_0 < y_0$  of the real line. For any two points  $x < y$  of the real line, let us consider a homeomorphism  $f_{x,y}$  in  $\text{Homeo}_0(\mathbb{R})$  such that  $f_{x,y}(x_0) = x$  and  $f_{x,y}(y_0) = y$ . As  $\varphi(f_{x,y})h = hf_{x,y}$ , the homeomorphism  $\varphi(f_{x,y})$  sends the ordered pair  $(h(x_0), h(y_0))$  to the ordered pair  $(h(x), h(y))$ . As the homeomorphism  $\varphi(f_{x,y})$  is strictly increasing:

$$h(x) < h(y) \Leftrightarrow h(x_0) < h(y_0)$$

and

$$h(x) > h(y) \Leftrightarrow h(x_0) > h(y_0).$$

Hence the map  $h$  is either strictly increasing or strictly decreasing.

Now, it remains to prove that the map  $h$  is onto to complete the proof. Suppose by contradiction that the map  $h$  is not onto. Notice that the set  $h(\mathbb{R})$  is preserved under the group  $\varphi(\text{Homeo}_0(\mathbb{R}))$ . If this set had a lower bound or an upper bound, then the supremum of this set or the infimum of this set would provide a fixed point for the group  $\varphi(\text{Homeo}_0(\mathbb{R}))$ , a contradiction. This set has neither upper bound nor lower bound. Let  $C$  be a connected component of the complement of the set  $h(\mathbb{R})$ . To simplify the exposition of the proof, we suppose that the map  $h$  is increasing. Let us denote by  $x_0$  the supremum of the set of points  $x$  such that the real number  $h(x)$  is lower than any point in the interval  $C$ . Then the point  $h(x_0)$  is necessarily in the closure of  $C$ : otherwise, there would exist an interval in the complementary of  $h(\mathbb{R})$  which strictly contains the interval  $C$ . We suppose for instance that the point  $h(x_0)$  is the supremum of the interval  $C$ . Choose, for each couple  $(z_1, z_2)$  of real numbers, a homeomorphism  $g_{z_1, z_2}$  in  $\text{Homeo}_0(\mathbb{R})$  which sends the point  $z_1$  to the point  $z_2$ . Then the sets  $g_{x_0, x}(C)$ , for  $x$  in  $\mathbb{R}$ , are pairwise disjoint: they are pairwise distinct as their suprema are pairwise distinct (the supremum of the set  $g_{x_0, x}(C)$  is the point  $h(x)$ ). Moreover, those sets do not contain any point of  $h(\mathbb{R})$  and the infima of those sets are accumulated by points in  $h(\mathbb{R})$ . Hence, these sets are pairwise disjoint. Then the set  $C$  has necessarily an empty interior as the topological space  $\mathbb{R}$  is second-countable. Therefore  $C = \{h(x_0)\}$ , which is not possible.  $\square$

### 3 Proof of Proposition 2.1

The proof of this proposition is similar to the proofs of Lemmas 3.6 and 3.7 in [14]. We need the following lemma. The proof of this lemma is almost identical to the proof of Lemma 2.5 and is omitted.

**Lemma 3.1.** *Take two disjoint embedded  $(d - 1)$ -dimensional balls  $D$  and  $D'$  in  $\mathbb{R}^d$ . For any homeomorphism  $f$  in  $\text{Homeo}_0(\mathbb{R}^d)$ , there exist homeomorphisms  $h_1,$*

$h_3$  in  $H_D^d$  and  $h_2$  in  $H_{D'}^d$ , such that

$$h = h_1 h_2 h_3.$$

For such an embedded  $(d-1)$ -dimensional ball  $D$ , let  $F_D = \text{Fix}(\varphi(H_D^d))$ . Let us prove that these sets are pairwise homeomorphic. Take two embedded  $(d-1)$ -dimensional balls  $D$  and  $D'$  and take a homeomorphism  $h$  in  $\text{Homeo}_0(\mathbb{R}^d)$  which sends the set  $D$  onto  $D'$ . Observe that  $hH_D^d h^{-1} = H_{D'}^d$ , and that  $\varphi(h)\varphi(H_D^d)\varphi(h)^{-1} = \varphi(H_{D'}^d)$ . Therefore:  $\varphi(h)(F_D) = F_{D'}$ .

In the case where these sets are all empty, there is nothing to prove. We suppose in what follows that they are not empty.

Given two disjoint embedded  $(d-1)$ -dimensional balls  $D$  and  $D'$ , Lemma 3.1 implies, as in the proof of Lemma 2.4:

$$F_D \cap F_{D'} = \emptyset.$$

**Lemma 3.2.** *Fix an embedded  $(d-1)$ -dimensional ball  $D_0$  of  $\mathbb{R}^d$ . Then any connected component  $C$  of the complement of  $F_{D_0}$  meets one of the sets  $F_D$ , where  $D$  is an embedded  $(d-1)$ -dimensional ball disjoint from  $D_0$ .*

*Proof.* Let  $(a_1, a_2)$  be a connected component of the complement of  $F_{D_0}$ . It is possible that either  $a_1 = -\infty$  or  $a_2 = +\infty$ . Consider a homeomorphism  $e : \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^d$  such that  $e(B^{d-1} \times \{0\}) = D_0$ , where  $B^{d-1}$  denotes the unit closed ball in  $\mathbb{R}^{d-1}$ . For any real number  $x$ , let  $D_x = e(B^{d-1} \times \{x\})$ . Given two real  $x \neq y$ , take a homeomorphism  $\eta_{x,y}$  in  $\text{Homeo}_0(\mathbb{R})$  which sends the point  $x$  to the point  $y$ . Consider a homeomorphism  $h_{x,y}$  such that the following property is satisfied. The restriction of  $eh_{x,y}e^{-1}$  to  $B^{d-1} \times \mathbb{R}$  is equal to the map:

$$\begin{aligned} B^{d-1} \times \mathbb{R} &\rightarrow \mathbb{R}^{d-1} \times \mathbb{R} \\ (p, z) &\mapsto (p, \eta_{x,y}(z)) \end{aligned}$$

Notice that, for any real numbers  $x$  and  $y$ ,  $h_{x,y}(D_x) = D_y$

Let us prove by contradiction that there exists a real number  $x \neq 0$  such that  $F_{D_x} \cap (a_1, a_2) \neq \emptyset$ . Suppose that, for any such embedded ball  $D_x$ ,  $F_{D_x} \cap (a_1, a_2) = \emptyset$ . We claim that the open sets  $\varphi(h_{0,x})((a_1, a_2))$  are pairwise disjoint. It is not possible as there would be uncountably many pairwise disjoint open intervals in  $\mathbb{R}$ .

Indeed, suppose by contradiction that there exists real numbers  $x \neq y$  such that  $\varphi(h_{0,x})((a_1, a_2)) \cap \varphi(h_{0,y})((a_1, a_2)) \neq \emptyset$ . Notice that the homeomorphism  $h_{0,x}^{-1}h_{0,y}$  and  $h_{0,y}^{-1}h_{0,x}$  send respectively the set  $D_0$  to sets of the form  $D_z$  and  $D_{z'}$ , where  $z, z' \in \mathbb{R}$ . Hence, for  $i = 1, 2$ , the homeomorphisms  $\varphi(h_{0,x}^{-1}h_{0,y})$  (respectively  $\varphi(h_{0,y}^{-1}h_{0,x})$ ) sends the point  $a_i \in F_{D_0}$  to a point in  $F_{D_z}$  (respectively in  $F_{D_{z'}}$ ). By

hypothesis, these points do not belong to  $(a_1, a_2)$ . Therefore

$$\varphi(h_{0,y}^{-1}h_{0,x})(a_1, a_2) = (a_1, a_2)$$

or

$$\varphi(h_{0,x})(a_1, a_2) = \varphi(h_{0,y})(a_1, a_2).$$

But this last equality cannot hold as the real endpoints of the interval on the left-hand side belong to  $F_{D_x}$  and the real endpoints point of the interval on the right-hand side belongs to  $F_{D_y}$ . Moreover, we saw that these two closed sets were disjoint, a contradiction.  $\square$

**Lemma 3.3.** *Each set  $F_D$  contains only one point.*

*Proof.* Suppose that there exists an embedded  $(d-1)$ -dimensional ball  $D$  such that the set  $F_D$  contains two points  $p_1 < p_2$ . By Lemma 3.2, there exists an embedded  $(d-1)$ -dimensional ball  $D'$  disjoint from  $D$  such that the set  $F_{D'}$  has a common point with the open interval  $(p_1, p_2)$ . Take a real number  $r < p_1$ . Then, for any homeomorphisms  $g_1$  in  $G_D$ ,  $g_2$  in  $G_{D'}$  and  $g_3$  in  $G_D$ ,

$$\varphi(g_1) \circ \varphi(g_2) \circ \varphi(g_3)(r) < p_2.$$

By Lemma 3.1, this implies that the following inclusion holds:

$$\{\varphi(g)(r), g \in \text{Homeo}_0(\mathbb{R}^d)\} \subset (-\infty, p_2].$$

The supremum of the left-hand set provides a fixed point for the action  $\varphi$ , a contradiction.  $\square$

## 4 Proof of Proposition 2.2

This proof uses the following lemmas. For a subgroup  $G$  of  $\text{Homeo}_0(\mathbb{R}^d)$ , we define the support  $\text{Supp}(G)$  of  $G$  as the closure of the set:

$$\{x \in \mathbb{R}^d, \exists g \in G, gx \neq x\}.$$

Let  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R}) = \{f \in \text{Homeo}(\mathbb{R}), \forall x \in \mathbb{R}, f(x+1) = f(x) + 1\}$ .

To prove Proposition 2.2, we need the following lemmas.

**Lemma 4.1.** *Let  $G$  and  $G'$  be subgroups of the group  $\text{Homeo}_+(\mathbb{R})$  of orientation-preserving homeomorphisms of the circle. Suppose that the following conditions are satisfied.*

1. *The groups  $G$  and  $G'$  are isomorphic to the group  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ .*
2. *The subgroups  $G$  and  $G'$  of  $\text{Homeo}_+(\mathbb{R})$  commute:  $\forall g \in G, g' \in G', gg' = g'g$ .*



Then  $\text{Supp}(G) \subset \text{Fix}(G')$  and  $\text{Supp}(G') \subset \text{Fix}(G)$ .

**Lemma 4.2.** *Take any nonempty open subset  $U$  of  $\mathbb{R}^d$ . Then there exists a subgroup of  $\text{Homeo}_0(\mathbb{R}^d)$  isomorphic to  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  which is supported in  $U$ .*

Lemma 4.1 will be proved in the next section. We now provide a proof of Lemma 4.2.

*Proof of Lemma 4.2.* Take a closed ball  $B$  contained in  $U$ . Changing coordinates if necessary, we can suppose that  $B$  is the unit closed ball in  $\mathbb{R}^d$ . Take an orientation-preserving homeomorphism  $h : \mathbb{R} \rightarrow (-1, 1)$ . For any orientation-preserving homeomorphism  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we define the homeomorphism  $\lambda_h(f) : \mathbb{R}^d \rightarrow \mathbb{R}^d$  in the following way.

1. The homeomorphism  $\lambda_h(f)$  is equal to the identity outside the interior of the ball  $B$ .
2. For any  $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} \cap \text{int}(B)$ :

$$\lambda_h(f)(x_1, x') = (\sqrt{1 - \|x'\|^2} h \circ f \circ h^{-1}(\frac{x_1}{\sqrt{1 - \|x'\|^2}}), x').$$

The map  $\lambda_h$  defines an embedding of the group  $\text{Homeo}_+(\mathbb{R})$  into the group  $\text{Homeo}_0(\mathbb{R}^d)$ . The image under  $\lambda_h$  of the group  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  is a subgroup of  $\text{Homeo}_0(\mathbb{R}^d)$  which is supported in  $U$ .  $\square$

Let us complete now the proof of Proposition 2.2.

*Proof of Proposition 2.2.* Fix a point  $p$  in  $\mathbb{R}^d$ . Take a closed ball  $B \subset \mathbb{R}^d$  which is centered at  $p$ . Let  $G_B^d$  be the subgroup of  $G_p^d$  consisting of homeomorphisms which pointwise fix a neighbourhood of the ball  $B$ . Let us prove that  $\text{Fix}(G_B^d) \neq \emptyset$ .

Take a subgroup  $G_1$  of  $\text{Homeo}_0(\mathbb{R}^d)$  which is isomorphic to  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  and supported in  $B$ . Such a subgroup exists by Lemma 4.2. This subgroup commutes with any subgroup  $G_2$  of  $\text{Homeo}_0(\mathbb{R}^d)$  which is isomorphic to  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  and supported outside  $B$ .

If the group  $\varphi(\text{Homeo}_0(\mathbb{R}^d))$  admits a fixed point, there is nothing to prove. Suppose that this group has no fixed point. As the group  $\text{Homeo}_0(\mathbb{R}^d)$  is simple, the morphism  $\varphi$  is one-to-one. Moreover, any morphism  $\text{Homeo}_0(\mathbb{R}^d) \rightarrow \mathbb{Z}/2\mathbb{Z}$  is trivial: the morphism  $\varphi$  takes values in  $\text{Homeo}_+(\mathbb{R})$ . Hence the subgroups  $\varphi(G_1)$  and  $\varphi(G_2)$  of  $\text{Homeo}(\mathbb{R})$  satisfy the hypothesis of Lemma 4.1. By this lemma:

$$\emptyset \neq \text{Supp}(\varphi(G_1)) \subset \text{Fix}(\varphi(G_2)).$$

We claim that the group  $G_B^d$  is generated by the union of its subgroups isomorphic to  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ . This claim implies that

$$\emptyset \neq \text{Supp}(\varphi(G_1)) \subset \text{Fix}(\varphi(G_B^d)).$$

For  $d \geq 2$ , the claim is a direct consequence of the simplicity of the group  $G_B^d$ . In the case where  $d = 1$ , denote by  $[a, b]$  the compact interval  $B$ . The inclusions of the groups  $\text{Homeo}_0((-\infty, a))$  and  $\text{Homeo}_0((b, +\infty))$  induce an isomorphism  $\text{Homeo}_0((-\infty, a)) \times \text{Homeo}_0((b, +\infty)) \rightarrow G_B^d$ . The simplicity of each factor of this decomposition implies the claim.

Now, let us prove that the set  $\text{Fix}(\varphi(G_B^d))$  is compact. Suppose by contradiction that there exists a sequence  $(a_k)_{k \in \mathbb{N}}$  of real numbers in  $\text{Fix}(\varphi(G_B^d))$  which tends to  $+\infty$  (if we suppose that it tends to  $-\infty$ , we obtain of course an analogous contradiction). Let us choose a closed ball  $B' \subset \mathbb{R}^d$  which is disjoint from  $B$ . Observe that the subgroups  $G_B^d$  and  $G_{B'}^d$  are conjugate in  $\text{Homeo}_0(\mathbb{R}^d)$  by a homeomorphism which sends the ball  $B$  to the ball  $B'$ . Then the subgroups  $\varphi(G_B^d)$  and  $\varphi(G_{B'}^d)$  are conjugate in the group  $\text{Homeo}_+(\mathbb{R})$ . Hence the sets  $\text{Fix}(\varphi(G_B^d))$  and  $\text{Fix}(\varphi(G_{B'}^d))$  are homeomorphic: there exists a sequence  $(b_k)_{k \in \mathbb{N}}$  of elements in  $\text{Fix}(\varphi(G_{B'}^d))$  which tends to  $+\infty$ . Take positive integers  $n_1, n_2$  and  $n_3$  such that  $a_{n_1} < b_{n_2} < a_{n_3}$ . Fix  $x_0 < a_{n_1}$ . We notice then that for any homeomorphisms  $g_1 \in G_B^d$ ,  $g_2 \in G_{B'}^d$  and  $g_3 \in G_B^d$ , the following inequality is satisfied:

$$\varphi(g_1)\varphi(g_2)\varphi(g_3)(x_0) < a_{n_3}.$$

However, any element  $g$  in  $\text{Homeo}_0(\mathbb{R}^d)$  can be written as a product

$$g = g_1 g_2 g_3,$$

where  $g_1$  and  $g_3$  belong to  $G_B^d$  and  $g_2$  belongs to  $G_{B'}^d$ . The proof of this fact is similar to that of Lemma 2.5. Therefore:

$$\overline{\{\varphi(g)(x_0), g \in \text{Homeo}_c(\mathbb{R})\}} \subset (-\infty, a_{n_3}].$$

The greatest element of the left-hand set is a fixed point of the image of  $\varphi$ : this is not possible as this image was supposed to have no fixed point.

Observe that the group  $\varphi(G_p^d)$  is the union of its subgroup of the form  $\varphi(G_{B'}^d)$ , with  $B'$  varying over the set  $\mathcal{B}_p$  of closed balls centered at the point  $p$ . By compactness, the set

$$\text{Fix}(\varphi(G_p^d)) = \bigcap_{B' \in \mathcal{B}_p} \text{Fix}(G_{B'}^d)$$

is nonempty. Proposition 2.2 is proved.  $\square$

## 5 Proof of Lemma 4.1

We need the following lemmas. The first one will be proved afterwards.

**Lemma 5.1.** *Let  $\psi : \text{Homeo}_{\mathbb{Z}}(\mathbb{R}) \rightarrow \text{Homeo}_+(\mathbb{R})$  be a group morphism. Then there exists a closed set  $F \subset \mathbb{R}$  such that:*

1. The set  $F$  is pointwise fixed under any element in  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ .
2. For any connected component  $K$  of the complement of  $F$ , there exists a homeomorphism  $h_K : \mathbb{R} \rightarrow K$  such that:

$$\forall f \in \text{Homeo}_{\mathbb{Z}}(\mathbb{R}), \forall x \in K, \psi(f)(x) = h_K f h_K^{-1}.$$

**Lemma 5.2.** Any group morphism  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R}) \rightarrow \mathbb{Z}$  is trivial.

*Proof of Lemma 5.2.* Actually, any element in  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  can be written as a product of commutators, *i.e.* elements of the form  $aba^{-1}b^{-1}$ , with  $a, b \in \text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ . For an explicit construction of such a decomposition, see Section 2 in [3].  $\square$

Observe that the center of the group  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  is the subgroup generated by the translation  $x \mapsto x + 1$ . Let  $\alpha$  (respectively  $\alpha'$ ) be a generator of the center of  $G$  (respectively of  $G'$ ). Let  $A_\alpha = \mathbb{R} - \text{Fix}(\alpha)$  and  $A_{\alpha'} = \mathbb{R} - \text{Fix}(\alpha')$ .

As the homeomorphisms  $\alpha$  and  $\alpha'$  commute:

$$\begin{cases} \alpha'(A_\alpha) = A_\alpha \\ \alpha(A_{\alpha'}) = A_{\alpha'} \end{cases}.$$

Take any connected component  $I$  of  $A_\alpha$  and any connected component  $I'$  of  $A_{\alpha'}$ . Then, either  $I$  is contained in  $I'$ , or  $I'$  is contained in  $I$ , or  $I$  and  $I'$  are disjoint.

We now prove that only the latter case can occur. Suppose by contradiction that the interval  $I$  is strictly contained in the interval  $I'$ . Let  $\sim$  be the equivalence relation defined on  $I'$  by

$$x \sim y \Leftrightarrow (\exists k \in \mathbb{Z}, x = \alpha'^k(y)).$$

The topological space  $I'/\sim$  is homeomorphic to a circle. By Lemma 5.1, the group  $G'$  preserves the interval  $I'$ . Notice that the group  $G'/\langle \alpha' \rangle \simeq \text{Homeo}_0(\mathbb{S}^1)$  acts on the circle  $I'/\sim$ . As the group  $G'$  commutes with the homeomorphism  $\alpha$ , this action preserves the nonempty set  $(A_\alpha \cap I)/\sim$ . As  $\alpha'(A_\alpha) = A_\alpha$ , the endpoints of the interval  $I$  are sent to points in the complement of  $A_\alpha$  under the iterates of the homeomorphism  $\alpha'$ . Hence the set  $(A_\alpha \cap I)/\sim$  is not equal to the whole circle  $I'/\sim$ . However, by Theorem 5.3 in [12] (see the remark below Theorem 1.2), any non-trivial action of the group  $\text{Homeo}_0(\mathbb{S}^1)$  on a circle is transitive. Hence, the group  $G'/\langle \alpha' \rangle$  acts trivially on the circle  $I'/\sim$ : for any element  $\beta'$  of  $G'$ , and any point  $x \in I'$ , there exists an integer  $k(x, \beta') \in \mathbb{Z}$  such that  $\beta'(x) = \alpha'^{k(x, \beta')}(x)$ . Fixing such a point  $x$ , we see that the map

$$\begin{aligned} G' &\rightarrow \mathbb{Z} \\ \beta' &\mapsto k(x, \beta') \end{aligned}$$

is a group morphism. Such a group morphism is trivial by Lemma 5.2. Therefore, the group  $G'$  acts trivially on the interval  $I'$ , a contradiction.

Of course, the case where the interval  $I'$  is strictly contained in  $I$  is symmetric and cannot occur.

Suppose now that  $I = I'$ . Take any element  $\beta'$  in  $G'$ . As the homeomorphism  $\beta'$  commutes with  $\alpha$ , by Lemma 5.1, the homeomorphism  $\beta'$  is equal to some element of  $G$  on  $I$ . As the homeomorphism  $\beta'$  commutes with any element of  $G$ , there exists a unique integer  $k(\beta')$  such that  $\beta'|_I = \alpha|_I^{k(\beta')}$ . The map  $k : G \rightarrow \mathbb{Z}$  is a nontrivial group morphism. But such a map cannot exist by Lemma 5.2. Lemma 4.1 is proved.

It remains to prove Lemma 5.1.

*Proof of Lemma 5.1.* Denote by  $t$  a generator of the center of the group  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ .

**Claim 1.** The connected components of the complement of  $\text{Fix}(\psi(t))$  are each preserved by the group  $\psi(\text{Homeo}_{\mathbb{Z}}(\mathbb{R}))$ . Moreover

$$\text{Fix}(\psi(\text{Homeo}_{\mathbb{Z}}(\mathbb{R}))) = \text{Fix}(\psi(t)).$$

**Claim 2.** Any action of the group  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$  on  $\mathbb{R}$  without fixed points is conjugate to the standard action.

It is clear that these two claims imply Lemma 5.1.

First, let us prove Claim 1. The set  $\text{Fix}(\psi(t))$  is preserved under any element in  $\psi(\text{Homeo}_{\mathbb{Z}}(\mathbb{R}))$ , because any element of this group commutes with the homeomorphism  $\psi(t)$ . Moreover, any element in  $\psi(\text{Homeo}_{\mathbb{Z}}(\mathbb{R}))$  preserves the orientation. Hence any connected component of the complement of  $\text{Fix}(\psi(t))$  with infinite length is preserved under the action of the group  $\psi(\text{Homeo}_{\mathbb{Z}}(\mathbb{R}))$ . We suppose now that any connected component of the complement of  $\text{Fix}(\psi(t))$  has a finite length. Let us denote by  $\equiv$  the equivalence relation on  $\mathbb{R}$  such that  $x \equiv y$  if and only if the points  $x$  and  $y$  belong to the same connected component of  $\text{Supp}(\psi(t))$ . The morphism  $\psi$  induces an action of the group  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R}) / \langle t \rangle \approx \text{Homeo}_0(\mathbb{S}^1)$  on the quotient topological space  $\mathbb{R} / \equiv$  which is homeomorphic to  $\mathbb{R}$  if the set  $\text{Fix}(\psi(t))$  has a nonempty interior. However, such an action is trivial by the remark below Theorem 1.2. Hence, any connected component of  $\text{Supp}(\psi(t))$  is preserved by the action. Restricting to one of these connected components if necessary, we can suppose that the closed set  $\text{Fix}(\psi(t))$  contains only isolated points. If this set is empty, there is nothing to prove. Otherwise, let

$$\text{Fix}(\psi(t)) = \{x_i, i \in A\},$$

where  $A$  is a set contained in  $\mathbb{Z}$  and contains 0 and where the sequence  $(x_i)_{i \in A}$  is strictly increasing. Then, for any element  $f$  in  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ , there exists an

integer  $i(f)$  such that  $\psi(f)(x_0) = x_{i(f)}$ . The map  $i : \text{Homeo}_{\mathbb{Z}}(\mathbb{R}) \rightarrow \mathbb{Z}$  is a group morphism: it is trivial by Lemma 5.2. Claim 1 is proved.

By Claim 1, restricting if necessary the action on a connected component of the complement of  $\text{Fix}(\psi(t))$ , we can suppose that the homeomorphism  $\psi(t)$  has no fixed point. Changing coordinates if necessary, we can suppose that the homeomorphism  $\psi(t)$  is the translation  $x \mapsto x+1$ . The morphism  $\psi$  induces an action  $\hat{\psi}$  of the group  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R}) / \langle t \rangle \approx \text{Homeo}_0(\mathbb{S}^1)$  on the circle  $\mathbb{R}/\mathbb{Z}$ . This action is nontrivial: otherwise, there would exist a nontrivial group morphism  $\text{Homeo}_0(\mathbb{S}^1) \rightarrow \mathbb{Z}$ . By the remark below Theorem 1.2, there exists a homeomorphism  $h$  of the circle  $\mathbb{R}/\mathbb{Z}$  such that, for any homeomorphism  $f$  in  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R}) / \langle t \rangle$  (which can be canonically identified with  $\text{Homeo}_0(\mathbb{R}/\mathbb{Z})$ ):

$$\hat{\psi}(f) = hfh^{-1}.$$

Take a lift  $\tilde{h} : \mathbb{R} \rightarrow \mathbb{R}$  of  $h$ . For any integer  $n$ , denote by  $T_n : \mathbb{R} \rightarrow \mathbb{R}$  the translation  $x \mapsto x+n$ . For any homeomorphism  $f$  in  $\text{Homeo}_{\mathbb{Z}}(\mathbb{R})$ , there exists an integer  $n(f)$  such that

$$\psi(f) = T_{n(f)}\tilde{h}f\tilde{h}^{-1}.$$

However, the map  $n : \text{Homeo}_{\mathbb{Z}}(\mathbb{R}) \rightarrow \mathbb{Z}$  is a group morphism: it is trivial by Lemma 5.2. This completes the proof of Claim 2.  $\square$

## References

- [1] A. Banyaga, *The structure of classical diffeomorphism groups*, Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1997.
- [2] A. Bounemoura *Simplicité des groupes de transformations de surface*, Ensaos Matematicos, vol. 14 (2008), 1-143.
- [3] D. Eisenbud, U. Hirsch, W. Neumann, *Transverse foliations of Seifert bundles and self-homeomorphisms of the circle*, Comment. Math. Helv. 56 (1981), 638-660.
- [4] G.M. Fischer, *On the group of all homeomorphisms of a manifold*, Trans. of the Amer. Math. Soc. 97 (1960), 193-212.
- [5] É. Ghys, *Groups acting on the circle*, L'Enseignement Mathématique, vol.47 (2001), 329-407.
- [6] É. Ghys, *Prolongements des difféomorphismes de la sphère*, L'Enseignement Mathématique (2) vol.37 (1991), no. 1-2, 45-59.
- [7] R.C. Kirby, *Stable homeomorphisms and the annulus conjecture*, Ann. of Math. 2nd Ser. 89 (3) (1969), 575-582.
- [8] K. Mann, *Homomorphisms between diffeomorphism groups*, preprint available on arXiv:1206.1196.
- [9] J.N. Mather, *Commutators of diffeomorphisms I*, Comment. Math. Helv. 49 (1974), 512-528.

- [10] J.N. Mather, *Commutators of diffeomorphisms II*, Comment. Math. Helv. 50 (1975), 33-40.
- [11] J.N. Mather, *The vanishing of the homology of certain groups of homeomorphisms*, Topology 10 (1971), 297-298.
- [12] S. Matsumoto, *Numerical invariants for semiconjugacy of homeomorphisms of the circle*, Proceedings of the American Mathematical Society, vol. 98 (1986), no 1, 163-168.
- [13] S. Matsumoto, S. Morita, *Bounded cohomology of certain groups of homeomorphisms*, Proc. Amer. Math. Soc. 94 (1985), no 3, 539-544.
- [14] E. Militon, *Continuous actions of the group of homeomorphisms of the circle on surfaces*, preprint available on arXiv:1211.0846.
- [15] A. Navas, *Groups of Circle Diffeomorphisms*, Chicago Lectures in Mathematics Series, The University of Chicago Press.
- [16] F. Quinn, *Ends of maps III : Dimensions 4 and 5*, J. Differential Geom. 17 (1982), 503-521.