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# IRREDUCIBILITY OF THE MODULI SPACE OF STABLE VECTOR BUNDLES OF RANK TWO AND ODD DEGREE ON A VERY GENERAL QUINTIC SURFACE

NICOLE MESTRANO AND CARLOS SIMPSON

ABSTRACT. The moduli space  $M(c_2)$ , of stable rank two vector bundles of degree one on a very general quintic surface  $X \subset \mathbb{P}^3$ , is irreducible for all  $c_2 \geq 4$  and empty otherwise.

## 1. INTRODUCTION

Let  $X \subset \mathbb{P}_{\mathbb{C}}^3$  be a very general quintic hypersurface. Let  $M(c_2) := M_X(2, 1, c_2)$  denote the moduli space [7] of stable rank 2 vector bundles on  $X$  of degree 1 with  $c_2(E) = c_2$ . Let  $\overline{M}(c_2) := \overline{M}_X(2, 1, c_2)$  denote the moduli space of stable rank 2 torsion-free sheaves on  $X$  of degree 1 with  $c_2(E) = c_2$ . Recall that  $\overline{M}(c_2)$  is projective, and  $M(c_2) \subset \overline{M}(c_2)$  is an open set, whose complement is called the *boundary*. Let  $\overline{M}(c_2)$  denote the closure of  $M(c_2)$  inside  $\overline{M}(c_2)$ . This might be a strict inclusion, as will in fact be the case for  $c_2 \leq 10$ .

In [14] we showed that  $M(c_2)$  is irreducible for  $4 \leq c_2 \leq 9$ , and empty for  $c_2 \leq 3$ . In [15] we showed that the open subset  $M(10)^{\text{sn}} \subset M(10)$ , of bundles with seminatural cohomology, is irreducible. In 1995 Nijssse [17] showed that  $M(c_2)$  is irreducible for  $c_2 \geq 16$ .

In the present paper, we complete the proof of irreducibility for the remaining intermediate values of  $c_2$ .

**Theorem 1.1.** *For any  $c_2 \geq 4$ , the moduli space of bundles  $M(c_2)$  is irreducible.*

*For  $c_2 \geq 11$ , the moduli space of torsion-free sheaves  $\overline{M}(c_2)$  is irreducible. On the other hand,  $\overline{M}(10)$  has two irreducible components: the closure  $\overline{M}(10)$  of the irreducible open set  $M(10)$ ; and the smallest stratum  $M(10, 4)$  of the double dual stratification corresponding to torsion-free sheaves whose double dual has  $c_2 = 4$ . Similarly  $\overline{M}(c_2)$  has several irreducible components when  $5 \leq c_2 \leq 9$  too.*

*The moduli space  $\overline{M}(c_2)$  is good for  $c_2 \geq 10$ , generically smooth of the expected dimension  $4c_2 - 20$ , whereas for  $4 \leq c_2 \leq 9$ , the moduli space  $M(c_2)$  is not good. For  $c_2 \leq 3$  it is empty.*

Yoshioka [21, 22, 23], Gomez [5] and others have shown that the moduli space of stable torsion-free sheaves with irreducible Mukai vector (which contains, in particular, the case of bundles of rank 2 and degree 1) is irreducible, over an abelian or K3 surface. Those results use the triviality of the canonical bundle, leading to a

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symplectic structure and implying among other things that the moduli spaces are smooth [16]. Notice that the case of K3 surfaces includes degree 4 hypersurfaces in  $\mathbb{P}^3$ .

We were motivated to look at a next case, of bundles on a quintic or degree 5 hypersurface in  $\mathbb{P}^3$  where  $K_X = \mathcal{O}_X(1)$  is ample but not by very much. This paper is the third in a series starting with [14, 15] dedicated to Professor Maruyama who, along with Gieseker, pioneered the study of moduli of bundles on higher dimensional varieties [2, 3, 10, 11, 12]. Recall that the moduli space of stable bundles is irreducible for  $c_2 \gg 0$  on any smooth projective surface [4, 9, 18, 19], but there exist surfaces, such as smooth hypersurfaces in  $\mathbb{P}^3$  of sufficiently high degree [13], where the moduli space is not irreducible for intermediate values of  $c_2$ .

Our theorem shows that the irreducibility of the moduli space of bundles  $M(c_2)$ , for all values of  $c_2$ , can persist into the range where  $K_X$  is ample. On the other hand, the fact that  $\overline{M}(10)$  has two irreducible components, means that if we consider all torsion-free sheaves, then the property of irreducibility in the good range has already started to fail in the case of a quintic hypersurface.

A possible application of our theorem to the case of Calabi-Yau varieties could be envisioned, by noting that a general hyperplane section of a quintic threefold in  $\mathbb{P}^4$  will be a quintic surface  $X \subset \mathbb{P}^3$ .

#### *Outline of the proof*

Our technique is to use O'Grady's method of deformation to the boundary [18, 19], as it was exploited by Nijssse [17] in the case of a very general quintic hypersurface. We use, in particular, some of the intermediate results of Nijssse who showed, for example, that  $\overline{M}(c_2)$  is connected for  $c_2 \geq 10$ . Application of these results is made possible by the explicit description of the moduli spaces  $M(c_2)$  for  $4 \leq c_2 \leq 9$  obtained in [14] and the partial result for  $M(10)$  obtained in [15].

The boundary  $\partial\overline{M}(c_2) := \overline{M}(c_2) - M(c_2)$  is the set of points corresponding to torsion-free sheaves which are not locally free. We just endow  $\partial\overline{M}(c_2)$  with its reduced scheme structure. There might in some cases be a better non-reduced structure which one could put on the boundary or onto some strata, but that won't be necessary for our argument and we don't worry about it here.

We can further refine the decomposition

$$\overline{M}(c_2) = M(c_2) \sqcup \partial\overline{M}(c_2)$$

by the *double dual stratification* [19]. Let  $M(c_2; c'_2)$  denote the locally closed subset, again with its reduced scheme structure, parametrizing sheaves  $F$  which fit into an exact sequence

$$0 \rightarrow F \rightarrow F^{**} \rightarrow S \rightarrow 0$$

such that  $F \in \overline{M}(c_2)$  and  $S$  is a coherent sheaf of finite length  $d = c_2 - c'_2$  hence  $c_2(F^{**}) = c'_2$ . Notice that  $E = F^{**}$  is also stable so it is a point in  $M(c'_2)$ . The stratum can be nonempty only when  $c'_2 \geq 4$ , which shows by the way that  $\overline{M}(c_2)$  is empty for  $c_2 \leq 3$ . The boundary now decomposes into locally closed subsets

$$\partial\overline{M}(c_2) = \coprod_{4 \leq c'_2 < c_2} M(c_2; c'_2).$$

Let  $\overline{M}(c_2, c'_2)$  denote the closure of  $M(c_2, c'_2)$  in  $\overline{M}(c_2)$ . Notice that we don't know anything about the position of this closure with respect to the stratification; its boundary will not in general be a union of strata. We can similarly denote by

$\overline{M}(c_2)$  the closure of  $M(c_2)$  inside  $\overline{M}(c_2)$ , a subset which might well be strictly smaller than  $\overline{M}(c_2)$ .

The construction  $F \mapsto F^{**}$  provides, by the definition of the stratification, a well-defined map

$$M(c_2; c'_2) \rightarrow M(c'_2).$$

The fiber over  $E \in M(c'_2)$  is the Grothendieck Quot-scheme  $\text{Quot}(E; d)$  of quotients of  $E$  of length  $d := c_2 - c'_2$ .

It follows from Li's theorem [9, Proposition 6.4] that if  $M(c'_2)$  is irreducible, then  $M(c_2; c'_2)$  and hence  $\overline{M}(c_2; c'_2)$  are irreducible, with  $\dim(M(c_2; c'_2)) = \dim(M(c'_2)) + 3(c_2 - c'_2)$ . See Corollary 3.3 below. From the previous papers [14, 15], we know the dimensions of  $M(c'_2)$ , so we can fill in the dimensions of the strata, as will be summarized in Table 2. Furthermore, by [14] and Li's theorem, the strata  $M(c_2; c'_2)$  are irreducible whenever  $c'_2 \leq 9$ .

Nijse [17] proves that  $\overline{M}(c_2)$  is connected whenever  $c_2 \geq 10$ , using O'Grady's techniques [18, 18]. By [14], the moduli space  $\overline{M}(c_2)$  is *good*, that is to say it is generically reduced of the expected dimension  $4c_2 - 20$ , whenever  $c_2 \geq 10$ . In particular, the dimension of the Zariski tangent space, minus the dimension of the space of obstructions, is equal to the dimension of the moduli space. The Kuranishi theory of deformation spaces implies that  $\overline{M}(c_2)$  is locally a complete intersection. Hartshorne's connectedness theorem [6] now says that if two different irreducible components of  $\overline{M}(c_2)$  meet at some point, then they intersect in a codimension 1 subvariety. This intersection has to be contained in the singular locus.

The singular locus in  $M(c_2)$  contains a subvariety denoted  $V(c_2)$ , which is the set of bundles  $E$  with  $h^0(E) > 0$ . For  $c_2 \geq 10$ , the locus  $V(c_2)$  has dimension  $3c_2 - 11$ , and  $V(c_2)$  may be described as the space of bundles fitting into an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_P(1) \rightarrow 0$$

where  $P$  satisfies Cayley-Bacharach for quadrics. For  $c_2 \geq 11$  it is a general set of points, and the extension class is general, from which one can see that the closure of  $V(c_2)$  meets the boundary. For  $c_2 = 10$ ,  $V(10)$  is also irreducible but its general point parametrizes bundles corresponding to subschemes  $P$  consisting of 10 general points on a smooth quadric section  $Y \subset X$  (i.e. the intersection of  $X$  with a divisor in  $|\mathcal{O}_{\mathbb{P}^3}(2)|$ ). A generalization then has seminatural cohomology so is contained in the component constructed in [15], in particular meeting the boundary. On the other hand, any other irreducible components of the singular locus have strictly smaller dimension [14].

These properties of the singular locus and  $V(c_2)$ , together with the connectedness statement of [17], allow us to show that any irreducible component of  $\overline{M}(c_2)$  meets the boundary. O'Grady proves furthermore an important lemma, that the intersection with the boundary must have pure codimension 1.

We explain the strategy for proving irreducibility of  $M(10)$  and  $M(11)$  below, but it will perhaps be easiest to explain first why this implies irreducibility of  $M(c_2)$  for  $c_2 \geq 12$ . Based on O'Grady's method, this is the same strategy as was used by Nijse who treated the cases  $c_2 \geq 16$ .

Suppose  $c_2 \geq 12$  and  $Z \subset \overline{M}(c_2)$  is an irreducible component. Suppose inductively we know that  $M(c_2 - 1)$  is irreducible. Then  $\partial Z := Z \cap \partial \overline{M}(c_2)$  is a nonempty subset in  $Z$  of codimension 1, thus of dimension  $4c_2 - 21$ . However, by looking at Table 2, the boundary  $\partial \overline{M}(c_2)$  is a union of the stratum  $M(c_2, c_2 - 1)$  of dimension

$4c_2 - 21$ , plus other strata of strictly smaller dimension. Therefore,  $\partial Z$  must contain  $M(c_2, c_2 - 1)$ . But, the general torsion-free sheaf parametrized by a point of  $M(c_2, c_2 - 1)$  is the kernel  $F$  of a general surjection  $E \rightarrow S$  from a stable bundle  $E$  general in  $M(c_2 - 1)$ , to a sheaf  $S$  of length 1. We claim that  $F$  is a smooth point of the moduli space  $\overline{M}(c_2)$ . Indeed, if  $F$  were a singular point then there would exist a nontrivial co-obstruction  $\phi : F \rightarrow F(1)$ , see [8, 14, 24]. This would have to come from a nontrivial co-obstruction  $E \rightarrow E(1)$  for  $E$ , but that cannot exist because a general  $E$  is a smooth point since  $M(c_2 - 1)$  is good. Thus,  $F$  is a smooth point of the moduli space. It follows that a given irreducible component of  $M(c_2, c_2 - 1)$  is contained in at most one irreducible component of  $\overline{M}(c_2)$ . On the other hand, by the induction hypothesis  $M(c_2 - 1)$  is irreducible, so  $M(c_2, c_2 - 1)$  is irreducible. This gives the induction step, that  $M(c_2)$  is irreducible.

The strategy for  $M(10)$  is similar. However, due to the fact that the moduli spaces  $M(c'_2)$  are not good for  $c'_2 \leq 9$ , in particular they tend to have dimensions bigger than the expected dimensions, there are several boundary strata which can come into play. Luckily, we know that the  $M(c'_2)$ , hence all of the strata  $M(10, c'_2)$ , are irreducible for  $c'_2 \leq 9$ .

The dimension of  $M(c_2)$ , equal to the expected one, is 20. Looking at the row  $c_2 = 10$  in Table 2 below, one may see that there are three strata  $M(10, 9)$ ,  $M(10, 8)$  and  $M(10, 6)$  with dimension 19. These can be irreducible components of the boundary  $\partial Z$  if we follow the previous argument. More difficult is the case of the stratum  $M(10, 4)$  which has dimension 20. A general point of  $M(10, 4)$  is not in the closure of  $M(10)$ , in other words  $M(10, 4)$ , which is closed since it is the lowest stratum, constitutes a separate irreducible component of  $\overline{M}(10)$ . Now, if  $Z \subset M(10)$  is an irreducible component,  $\partial Z$  could contain a codimension 1 subvariety of  $M(10, 4)$ .

The idea is to use the main result of [15], that the moduli space  $M(10)^{\text{sn}}$  of bundles with seminatural cohomology, is irreducible. To prove that  $M(10)$  is irreducible, it therefore suffices to show that a general point of any irreducible component  $Z$ , has seminatural cohomology. From [15] there are two conditions that need to be checked:  $h^0(E) = 0$  and  $h^1(E(1)) = 0$ . The first condition is automatic for a general point, since the locus  $V(10)$  of bundles with  $h^0(E) > 0$  has dimension  $3 \cdot 10 - 11 = 19$  so cannot contain a general point of  $Z$ . For the second condition, it suffices to note that a general sheaf  $F$  in any of the strata  $M(10, 9)$ ,  $M(10, 8)$  and  $M(10, 6)$  has  $h^1(F(1)) = 0$ ; and to show that the subspace of sheaves  $F$  in  $M(10, 4)$  with  $h^1(F(1)) > 0$  has codimension  $\geq 2$ . This latter result is treated in Section 6, using the dimension results of Ellingsrud-Lehn for the scheme of quotients of a locally free sheaf, generalizing Li's theorem. This is how we will show irreducibility of  $M(10)$ .

The full moduli space of torsion-free sheaves  $\overline{M}(10)$  has two different irreducible components, the closure  $\overline{M}(10)$  and the lowest stratum  $M(10, 4)$ . This distinguishes the case of the quintic surface from the cases of abelian and K3 surfaces, where the full moduli spaces of stable torsion-free sheaves were irreducible [23, 22, 5].

For  $M(11)$ , the argument is almost the same as for  $c_2 \geq 12$ . However, there are now two different strata of codimension 1 in the boundary:  $M(11, 10)$  coming from the irreducible variety  $M(10)$ , and  $M(11, 4)$  which comes from the other 20-dimensional component  $M(10, 4)$  of  $\overline{M}(10)$ . To show that these two can give rise to at most a single irreducible component in  $M(11)$ , completing the proof, we will

note that they do indeed intersect, and furthermore that the intersection contains smooth points.

2. PRELIMINARY FACTS

The moduli space  $\overline{M}(c_2)$  is locally a fine moduli space. The obstruction to existence of a Poincaré universal sheaf on  $\overline{M}(c_2) \times X$  is an interesting question but not considered in the present paper. A universal family exists étale-locally over  $\overline{M}(c_2)$  so for local questions we may consider  $\overline{M}(c_2)$  as a fine moduli space.

The Zariski tangent space to  $\overline{M}(c_2)$  at a point  $E$  is  $\text{Ext}^1(E, E)$ . If  $E$  is locally free, this is the same as  $H^1(\text{End}(E))$ . The space of obstructions  $\text{obs}(E)$  is by definition the kernel of the surjective map

$$\text{Tr} : \text{Ext}^2(E, E) \rightarrow H^2(\mathcal{O}_X).$$

The space of co-obstructions is the dual  $\text{obs}(E)^*$  which is, by Serre duality with  $K_X = \mathcal{O}_X(1)$ , equal to  $\text{Hom}^0(E, E(1))$ , the space of maps  $\phi : E \rightarrow E(1)$  such that  $\text{Tr}(E) = 0$  in  $H^0(\mathcal{O}_X(1)) \cong \mathbb{C}^4$ . Such a map is called a co-obstruction.

Since a torsion-free sheaf  $E$  of rank two and odd degree can have no rank-one subsheaves of the same slope, all semistable sheaves are stable, and Gieseker and slope stability are equivalent. If  $E$  is a stable sheaf then  $\text{Hom}(E, E) = \mathbb{C}$  so the space of trace-free endomorphisms is zero. Notice that  $H^1(\mathcal{O}_X) = 0$  so we may disregard the trace-free condition for  $\text{Ext}^1(E, E)$ . An Euler-characteristic calculation gives

$$\dim(\text{Ext}^1(E, E)) - \dim(\text{obs}(E)) = 4c_2 - 20,$$

and this is called the expected dimension of the moduli space. The moduli space is said to be good if the dimension is equal to the expected dimension.

**Lemma 2.1.** *If the moduli space is good, then it is locally a complete intersection.*

*Proof.* Kuranishi theory expresses the local analytic germ of the moduli space  $\overline{M}(c_2)$  at  $E$ , as  $\Phi^{-1}(0)$  for a holomorphic map of germs  $\Phi : (\mathbb{C}^a, 0) \rightarrow (\mathbb{C}^b, 0)$  where  $a = \dim(\text{Ext}^1(E, E))$  (resp.  $b = \dim(\text{obs}(E))$ ). Hence, if the moduli space has dimension  $a - b$ , it is a local complete intersection.  $\square$

We investigated closely the structure of the moduli space for  $c_2 \leq 9$ , in [14].

**Proposition 2.2.** *The moduli space  $M(c_2)$  is empty for  $c_2 \leq 3$ . For  $4 \leq c_2 \leq 9$ , the moduli space  $M(c_2)$  is irreducible. It has dimension strictly bigger than the expected one, for  $4 \leq c_2 \leq 8$ , and for  $c_2 = 9$  it is generically nonreduced but with dimension equal to the expected one; it is also generically nonreduced for  $c_2 = 7$ . The dimensions of the moduli spaces, the dimensions of the spaces of obstructions at a general point, and the dimensions  $h^1(E(1))$  for a general bundle  $E$  in  $M(c_2)$ , are given in the following table.*

TABLE 1. Moduli spaces for  $c_2 \leq 9$

$c_2$	4	5	6	7	8	9
$\dim(M)$	2	3	7	9	13	16
$\dim(\text{obs})$	6	3	3	3	1	1
$h^1(E(1))$	0	1	0	0	0	0

*Proof.* The bundles  $E$  occurring for  $c_2 \leq 9$  always fit into an extension of  $J_P(1)$  by  $\mathcal{O}_X$ . As in the previous paper, we apologize again for the change of notation with respect to [14] where we considered bundles of degree  $-1$ , but the indexing by second Chern class remains the same. The subscheme  $P \subset X$  is locally a complete intersection of length  $c_2$  and satisfies the Cayley-Bacharach condition for quadrics.

In [14], we considered  $c$  the number of conditions imposed by  $P$  on quadrics. This is related to  $h^1(E(1))$  by the exact sequences

$$H^0(\mathcal{O}_X(2)) \rightarrow H^0(\mathcal{O}_P(2)) \rightarrow H^1(J_{P/X}(2)) \rightarrow 0$$

and

$$0 \rightarrow H^1(E(1)) \rightarrow H^1(J_{P/X}(2)) \rightarrow H^2(\mathcal{O}_X(1)) \rightarrow 0$$

where  $H^2(E(1)) = H^0(E(1))^* = 0$  by stability, and  $H^2(\mathcal{O}_X(1)) = H^2(K_X) = \mathbb{C}$ . The number  $c$  is the rank of the evaluation map of  $H^0(\mathcal{O}_X(2))$  on  $P$ , so  $h^1(J_{P/X}(2)) = c_2 - c$ , and by the second exact sequence we have  $h^1(E(1)) = c_2 - c - 1$ . This helps to extract the remaining values of the last row of the table from the discussion of [14].

Be careful that [14, Lemma 5.2] doesn't discuss  $h^1(E(1))$  but rather speaks of  $h^1(E)$  in our notation.

For the column  $c_2 = 9$ , see [14], Theorem 6.1 and Proposition 7.2 for the dimension 16 and general obstruction space of dimension 1. The proof of Proposition 7.2 starts out by ruling out, for a general point of an irreducible component, all cases of Proposition 7.1 except case (d), for which  $c = 8$ . Thus  $h^1(E(1)) = 9 - 8 - 1 = 0$  for a general bundle.

For the column  $c_2 = 8$ , see [14, Theorem 7.2] for irreducibility and the dimension. The family constructed in [14, Section 6.2] has obstruction space of dimension 1, and the dimension of the whole space is 13, strictly greater than the expected dimension, so it follows that the generic space of obstructions has dimension 1. Note that  $c = 7$  for the 13-dimensional family considered in [14, Section 7.4], so  $h^1(E(1)) = 8 - 7 - 1 = 0$ .

For  $c_2 = 6, 7$  the general bundle corresponds to a subscheme  $P$  contained in, and spanning a unique plane. In this case, the space of obstructions has dimension 3 by [14, Lemma 5.5].

For the column  $c_2 = 7$ , see [14, Proposition 7.3] where the dimension is 9. In the proof there, the biggest stratum of the moduli space corresponds to  $c = 6$ , giving  $h^1(E(1)) = 7 - 6 - 1 = 0$ .

For the column,  $c_2 = 6$ , see [14, Proposition 7.4] where the dimension is 7 and  $c = 5$ , so again  $h^1(E(1)) = 6 - 5 - 1 = 0$ .

For the columns  $c_2 = 4, 5$ , note that the subscheme  $P$  is either 4 or 5 points contained in a line. Both of these configurations impose  $c = 3$  conditions on quadrics, since  $h^0(\mathcal{O}_{\mathbb{P}^1}(2)) = 3$ . This gives values of  $4 - 3 - 1 = 0$  and  $5 - 3 - 1 = 1$  for  $h^1(E(1))$  respectively. The moduli space is generically smooth and its dimension is equal to  $c_2 - 2$  by [14, Lemma 7.7], and we get the dimension of the space of co-obstructions by subtracting the expected dimension.  $\square$

We also proved that the moduli space is good for  $c_2 \geq 10$ , known by Nijse [17] for  $c_2 \geq 13$ .

**Proposition 2.3.** *For  $c_2 \geq 10$ , the moduli space  $M(c_2)$  is good. The singular locus  $M(c_2)^{\text{sing}}$  is the union of the locus  $V(c_2)$  consisting of bundles with  $h^0(E) >$*



0, which has dimension  $3c_2 - 11$ , plus other pieces of dimension  $\leq 13$  which in particular have codimension  $\geq 6$ .

*Proof.* Following O’Grady’s and Nijssse’s terminology  $V(c_2)$  denotes the locus which is the image of the moduli space of bundles together with a section, called  $\Sigma_{c_2}$ . See [14], Theorem 7.1. Any pieces of the singular locus corresponding to bundles which are not in  $V(c_2)$ , have dimension  $\leq 13$  by [14, Corollary 5.1].  $\square$

**Proposition 2.4.** *Let  $M(10)^{\text{sn}} \subset M(10)$  denote the open subset of bundles  $E \in M(10)$  which have seminatural cohomology, that is where for any  $m$  at most one of  $h^i(E(m))$  is nonzero for  $i = 0, 1, 2$ . Then  $E \in M(10)^{\text{sn}}$  if and only if  $h^0(E) = 0$  and  $h^1(E(1)) = 0$ . The moduli space  $M(10)^{\text{sn}}$  is irreducible.*

*Proof.* See [15], Theorem 0.2 and Corollary 3.5.  $\square$

### 3. THE DOUBLE DUAL STRATIFICATION

Our proofs will make use of O’Grady’s techniques [18, 19], as they were recalled and used by Nijssse in [17]. The main idea is to look at the boundary of the moduli spaces. His first main observation is the following [19, Proposition 3.3]:

**Lemma 3.1** (O’Grady). *The boundary of any irreducible component (or indeed, of any closed subset) of  $M(c_2)$  has pure codimension 1, if it is nonempty.*

The boundary is divided up into Uhlenbeck strata corresponding to the “number of instantons”, which in the geometric picture corresponds to the number of points where the torsion-free sheaf is not a bundle, counted with correct multiplicities. A boundary stratum denoted  $M(c_2, c_2 - d)$  parametrizes torsion-free sheaves  $F$  fitting into an exact sequence of the form

$$0 \rightarrow F \rightarrow E \xrightarrow{\sigma} S \rightarrow 0$$

where  $E \in M(c_2 - d)$  is a stable locally free sheaf of degree 1 and  $c_2(E) = c_2 - d$ , and  $S$  is a finite coherent sheaf of finite length  $d$  so that  $c_2(F) = c_2$ . In this case  $E = F^{**}$ . We may think of  $M(c_2, c_2 - d)$  as the moduli space of pairs  $(E, \sigma)$ . Forgetting the quotient  $\sigma$  gives a smooth map

$$M(c_2, c_2 - d) \rightarrow M(c_2 - d),$$

sending  $F$  to its double dual. The fiber over  $E$  is the Grothendieck Quot scheme  $\text{Quot}(E, d)$  parametrizing quotients  $\sigma$  of  $E$  of length  $d$ .

Since we are dealing with sheaves of degree 1, all semistable points are stable and our objects have no non-scalar automorphisms. Hence the moduli spaces are fine, with a universal family existing etale-locally and well-defined up to a scalar automorphism. We may view the double-dual map as being the relative Grothendieck Quot scheme of quotients of the universal object  $E^{\text{univ}}$  on  $M(c_2 - d) \times X/M(c_2 - d)$ . Furthermore, locally on the Quot scheme the quotients are localized near a finite set of points, and we may trivialize the bundle  $E^{\text{univ}}$  near these points, so  $M(c_2, c_2 - d)$  has a covering by, say, analytic open sets which are trivialized as products of open sets in the base  $M(c_2 - d)$  with open sets in  $\text{Quot}(E, d)$  for any single choice of  $E$ . This is all to say that the map  $M(c_2, c_2 - d) \rightarrow M(c_2 - d)$  may be viewed as a fibration in a fairly strong sense, with fiber  $\text{Quot}(E, d)$ .

Li shows in [9, Proposition 6.4] that  $\text{Quot}(E, d)$  is irreducible with a dense open subset  $U$  parametrizing quotients which are given by a collection of  $d$  quotients of length 1 supported at distinct points of  $X$ :



**Theorem 3.2 (Li).** *Suppose  $E$  is a locally free sheaf of rank 2 on  $X$ . Then for any  $d > 0$ ,  $\text{Quot}(E, d)$  is an irreducible scheme of dimension  $3d$ , containing a dense open subset parametrizing quotients  $E \rightarrow S$  such that  $S \cong \bigoplus \mathbb{C}_{y_i}$  where  $\mathbb{C}_{y_i}$  is a skyscraper sheaf of length 1 supported at  $y_i \in X$ , and the  $y_i$  are distinct. This dense open set maps to  $X^{(d)} - \text{diag}$  (the space of choices of distinct  $d$ -uple of points in  $X$ ), with fiber over  $\{y_i\}$  equal to  $\prod_{i=1}^d \mathbb{P}(F_{y_i})$ .*

*Proof.* See Proposition 6.4 in the appendix of [9]. Notice right away that  $U$  is an open subset of  $\text{Quot}(F, d)$ , and that  $U$  fibers over the set  $X^{(d)} - \text{diag}$  of distinct  $d$ -uples of points  $(y_1, \dots, y_d)$  (up to permutations). The fiber over a  $d$ -uple  $(y_1, \dots, y_d)$  is the product of projective lines  $\mathbb{P}(F_{y_i})$  of quotients of the vector spaces  $F_{y_i}$ . As  $X^{(d)} - \text{diag}$  has dimension  $2d$ , and  $\prod_{i=1}^d \mathbb{P}(F_{y_i})$  has dimension  $d$ , we get that  $U$  is a smooth open variety of dimension  $3d$ .

This theorem may also be viewed as a consequence of a more precise bound established by Ellingsrud and Lehn [1], which will be stated as Theorem 6.6 below, needed for our arguments in Section 6.  $\square$

**Corollary 3.3.** *We have*

$$\dim(M(c_2; c'_2)) = \dim(M(c'_2)) + 3(c_2 - c'_2).$$

*If  $M(c'_2)$  is irreducible, then  $M(c_2; c'_2)$  and hence  $\overline{M}(c_2; c'_2)$  are irreducible.*

*Proof.* The fibration  $M(c_2; c'_2) \rightarrow M(c'_2)$  has fiber the Quot scheme whose dimension is  $3(c_2 - c'_2)$  by the previous proposition. Furthermore, these Quot schemes are irreducible so if the base is irreducible, so is the total space.  $\square$

Corollary 3.3 allows us to fill in the dimensions of the strata  $M(c_2; c'_2)$  in the following table. The entries in the second column are the expected dimension  $4c_2 - 20$ ; in the third column the dimension of  $M := M(c_2)$ ; and in the following columns,  $\dim M(c_2, c_2 - d)$  for  $d = 1, 2, \dots$ . The rule is to add 3 as you go diagonally down and to the right by one.

TABLE 2. Dimensions of strata

$c_2$	e.d.	$\dim(M)$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
4	-4	2	-	-	-	-	-	-	-	-
5	0	3	5	-	-	-	-	-	-	-
6	4	7	6	8	-	-	-	-	-	-
7	8	9	10	9	11	-	-	-	-	-
8	12	13	12	13	12	14	-	-	-	-
9	16	16	16	15	16	15	17	-	-	-
10	20	20	19	19	18	19	18	20	-	-
11	24	24	23	22	22	21	22	21	23	-
12	28	28	27	26	25	25	24	25	24	26
$\geq 13$	$4c_2 - 20$	$4c_2 - 20$	$4c_2 - 21$	$\leq 4c_2 - 22$						

The first remark useful for interpreting this information, is that any irreducible component of  $\overline{M}(c_2)$  must have dimension at least equal to the expected dimension  $4c_2 - 20$ . In particular, a stratum with strictly smaller dimension, must be a part

of at least one irreducible component consisting of a bigger stratum. For  $c_2 \geq 11$ , we have

$$\dim(M(c_2, c'_2)) < \dim(\overline{M}(c_2)) = 4c_2 - 20.$$

Hence, for  $c_2 \geq 11$  the closures  $\overline{M}(c_2, c'_2)$  cannot themselves form irreducible components of  $\overline{M}(c_2)$ , in other words the irreducible components of  $\overline{M}(c_2)$  are the same as those of  $M(c_2)$ . Notice, on the other hand, that  $\overline{M}(10)$  contains two pieces of dimension 20, the locally free sheaves in  $M(10)$  and the sheaves in  $M(10, 4)$  whose double duals come from  $M(4)$ .

Recall from [14] that the moduli spaces  $M(c_2)$  are irreducible for  $c_2 = 4, \dots, 9$ . It follows from Corollary 3.3 that the strata  $M(c_2, c'_2)$  are irreducible, for any  $c'_2 \leq 9$ . In particular, the piece  $\overline{M}(10, 4)$  is irreducible, and its general point, representing a non-locally free sheaf, is not confused with any point of  $\overline{M}(10)$ . Since the other strata of  $\overline{M}(10)$  all have dimension  $< 20$ , it follows that  $\overline{M}(10, 4)$  is an irreducible component of  $\overline{M}(10)$ . One similarly gets from the table that  $\overline{M}(c_2)$  has several irreducible components when  $5 \leq c_2 \leq 9$ .

#### 4. HARTSHORNE'S CONNECTEDNESS THEOREM

Hartshorne proves a connectedness theorem for local complete intersections. Here is the version that we need.

**Theorem 4.1** (Hartshorne). *Suppose  $Z$  is a local complete intersection of dimension  $d$ . Then, any nonempty intersection of two irreducible components of  $Z$  has pure dimension  $d - 1$ .*

*Proof.* See [6, 20]. □

**Corollary 4.2.** *If the moduli space  $\overline{M}$  is good, and has two different irreducible components  $Z_1$  and  $Z_2$  meeting at a point  $z$ , then  $Z_1 \cap Z_2$  has codimension 1 at  $z$  and the singular locus  $\text{Sing}(\overline{M})$  contains  $z$  and has codimension 1 at  $z$ .*

*Proof.* If  $\overline{M}$  is good, then by Lemma 2.1 it is a local complete intersection so Hartshorne's theorem applies:  $Z_1 \cap Z_2$  has pure codimension 1. The intersection of two irreducible components is necessarily contained in the singular locus. □

We draw the following conclusions.

**Corollary 4.3.** *Suppose, for  $c_2 \geq 10$ , that two different irreducible components  $Z_1$  and  $Z_2$  of  $\overline{M}$  meet at a point  $z$  not on the boundary. Then  $c_2 = 10$  and both components contain the subscheme  $V(c_2) = \{E, h^0(E) > 0\}$  which in turn contains  $z$  and has codimension  $\leq 1$  at  $z$ .*

*Proof.* We have seen in [14, Theorem 7.1] that for  $c_2 \geq 10$ , a codimension 1 piece of  $\text{Sing}(M)$  has to be in  $V(c_2)$ , cf Proposition 2.3 above. On the other hand  $V(c_2)$  is irreducible, see the proof of Corollary 4.5 below, so a codimension 1 piece of  $\text{Sing}(M)$  has to be equal to  $V(c_2)$ . This is contained in both irreducible components  $Z_1, Z_2$  by Corollary 4.2. One may furthermore note that  $\dim(V(c_2)) = 3c_2 - 11$  whereas the dimension of the moduli space is  $4c_2 - 20$ , thus for  $c_2 \geq 11$  the singular locus has codimension  $\geq 2$ , so the situation of the present corollary can only happen for  $c_2 = 10$ . □

Next, recall one of Nijssse's theorems, connectedness of the moduli space.

**Theorem 4.4** (Nijssse). *For  $c_2 \geq 10$ , the moduli space  $\overline{M}$  is connected.*

*Proof.* See [17], Proposition 3.2.  $\square$

**Corollary 4.5.** *Suppose  $Z$  is an irreducible component of  $\overline{M}(c_2)$  for  $c_2 \geq 10$ . Then  $Z$  meets the boundary in a nonempty codimension 1 subset.*

*Proof.* The codimension 1 property is given by Lemma 3.1, so we just have to show that  $\overline{Z}$  contains a boundary component.

To start with, note that for  $c_2 \geq 10$ , the first boundary stratum  $M(c_2, c_2 - 1)$  has codimension 1, so it must meet at least one irreducible component of  $\overline{M}(c_2)$ , call it  $Z_0$ . Suppose  $Z \subset \overline{M}(c_2)$  is another irreducible component with  $c_2 \geq 10$ . By the connectedness of  $\overline{M}(10)$ , there exist a sequence of irreducible components  $Z_0, \dots, Z_k = \overline{Z}$  such that  $Z_i \cap Z_{i+1}$  is nonempty. By Lemma 4.3, either  $Z_{k-1} \cap Z_k$  meets the boundary, or else it contains  $V(c_2)$ . In the first case we are done.

In the second case, one must have  $c_2 = 10$  by Corollary 4.3 (although here one could alternatively argue that for  $c_2 \geq 11$ ,  $V(11)$  itself meets the boundary as was discussed in the proof of [17, Proposition 3.2]). At  $c_2 = 10$ ,  $V(10)$  is also irreducible. It parametrizes subschemes  $P \subset X$  such that there exists a quadric section  $Y \subset X$ , a divisor in the linear system  $|\mathcal{O}_X(2)|$ , containing  $P$ . A general element of  $V(10)$  corresponds to 10 general points on a general smooth quadric section  $Y$ . One may see that for a general  $E \in V(10)$ , we have  $h^1(E(1)) = 0$ . This implies that any irreducible component of  $M(10)$  containing  $V$  parametrizes, generically, bundles with seminatural cohomology. Thus, any irreducible component of  $M(10)$  containing  $V$  must be the unique component constructed in [15]. See the proof of Corollary 7.2 below for more details of this argument. But that component meets the boundary, indeed the 19-dimensional boundary strata  $M(10, 9)$  etc., are also contained in the same irreducible component—again see the proof of Corollary 7.2 below. It follows that any irreducible component of  $M(10)$  which contains  $V(10)$ , must meet the boundary. This completes the proof.  $\square$

## 5. SEMINATURALITY ALONG THE 19-DIMENSIONAL BOUNDARY COMPONENTS

To treat the case  $c_2 = 10$ , we will apply the main result of our previous paper.

**Proposition 5.1.** *Suppose  $Z$  is an irreducible component of  $M(10)$ . Suppose that  $\overline{Z}$  contains a point corresponding to a torsion-free sheaf  $F$  with  $h^1(F(1)) = 0$ . Then  $Z$  is the unique irreducible component containing the open set of bundles with seminatural cohomology, constructed in [15].*

*Proof.* The locus  $V(c_2)$  of bundles with  $h^0(E) \neq 0$  has dimension  $\leq 19$ , so a general point of  $Z$  must have  $h^0(E) = 0$ . The hypothesis implies that a general point has  $h^1(E(1)) = 0$ . Thus, there is a nonempty dense open subset  $Z' \subset Z$  parametrizing bundles with  $h^0(E) = 0$  and  $h^1(E(1)) = 0$ . By [15, Corollary 3.5], these bundles have seminatural cohomology. Thus, our open set is  $Z' = M(10)^{\text{sn}}$ , the moduli space of bundles with seminatural cohomology, shown to be irreducible in the main Theorem 0.2 of [15].  $\square$

Using this proposition, and since we know by Corollary 4.5 that any irreducible component  $Z$  meets the boundary in a codimension 1 subset, in order to prove irreducibility of  $M(10)$ , it suffices to show that the torsion-free sheaves  $F$  parametrized

by general points on the various irreducible components of the boundary of  $\overline{M}(10)$  have  $h^1(F(1)) = 0$ .

The dimension is  $\dim(Z) = 20$ , so the boundary components will have dimension 19. Looking at the line  $c_2 = 10$  in Table 2, we notice that there are three 19-dimensional boundary pieces, and a 20-dimensional piece which must constitute a different irreducible component. Consider first the 19-dimensional pieces,

$$M(10, 9), \quad M(10, 8) \text{ and } M(10, 6).$$

Recall that  $M(10, 10 - d)$  consists generically of torsion-free sheaves  $F$  fitting into an exact sequence

$$(5.1) \quad 0 \rightarrow F \rightarrow F^{**} \rightarrow S \rightarrow 0$$

where  $F^{**}$  is a general point in the moduli space of stable bundles with  $c_2 = 10 - d$ , and  $S$  is a general quotient of length  $d$ .

**Proposition 5.2.** *For a general point  $F$  in either of the three boundary pieces  $M(10, 9)$ ,  $M(10, 8)$  or  $M(10, 6)$ , we have  $h^1(F(1)) = 0$ .*

*Proof.* Notice that  $\chi(F^{**}(1)) = 15 - c_2(F^{**}) \geq 6$  and by stability  $h^2(F^{**}(1)) = h^0(F^{**}(-1)) = 0$ , so  $F^{**}$  has at least six linearly independent sections. In particular, for a general quotient  $S$  of length 1, 2 or 4, consisting of the direct sum  $S = \bigoplus S_x$  of general rank 1 quotients  $E_x \rightarrow S_x$  at 1, 2 or 4 distinct general points  $x$ , the map

$$H^0(F^{**}(1)) \rightarrow H^0(S)$$

will be surjective.

For a general point  $F^{**}$  in either  $M(9)$ ,  $M(8)$  or  $M(6)$ , we have  $h^1(F^{**}(1)) = 0$ . These results from [14] were recalled in Proposition 2.2, Table 1. The long exact sequence associated to (5.1) now gives  $h^1(F(1)) = 0$ .  $\square$

This treats the 19-dimensional irreducible components of the boundary. There remains the piece  $\overline{M}(10, 4)$  which has dimension 20. This is a separate irreducible component. It could meet  $\overline{M}(10)$  along a 19-dimensional divisor, and we would like to show that  $h^1(F(1)) = 0$  for the sheaves parametrized by this divisor. In particular, we are no longer in a completely generic situation so some further discussion is needed. This will be the topic of the next section.

## 6. THE LOWEST STRATUM

The lowest stratum is  $\overline{M}(10, 4)$ , which is therefore closed. We would like to understand the points in  $\overline{M}(10) \cap M(10, 4)$ . These are singular, so our main tool will be to look at where the singular locus of  $\overline{M}(10)$  meets  $M(10, 4)$ . Denote this by

$$M(10, 4)^{\text{sing}} := \text{Sing}(\overline{M}(10)) \cap M(10, 4).$$

In what follows, we give a somewhat explicit description of the lowest moduli space  $M(4)$ .

**Lemma 6.1.** *For  $E \in M(4)$  we have  $h^1(E) = 0$ ,  $h^0(E) = h^2(E) = 3$ ,  $h^0(E(1)) = 11$ , and  $h^1(E(1)) = h^2(E(1)) = 0$ .*

*Proof.* Choosing an element  $s \in H^0(E)$  gives an exact sequence

$$(6.1) \quad 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_{P/X}(1) \rightarrow 0.$$

In [14] we have seen that  $P \subset X \cap \ell$  is a subscheme of length 4 in the intersection of  $X$  with a line  $\ell \subset \mathbb{P}^3$ . As  $P$  spans  $\ell$ , the space of linear forms vanishing on  $P$  is the same as the space of linear forms vanishing on  $\ell$ , so  $H^0(J_{P/X}(1)) \cong \mathbb{C}^2$ . In the long exact sequence associated to (6.1), note that  $H^1(\mathcal{O}_X) = 0$ , giving

$$0 \rightarrow H^0(\mathcal{O}_X) \rightarrow H^0(E) \rightarrow H^0(J_{P/X}(1)) \rightarrow 0$$

hence  $H^0(E) \cong \mathbb{C}^3$ . By duality,  $H^2(E) \cong \mathbb{C}^3$ , and the Euler characteristic of  $E$  is 6, so  $H^1(E) = 0$ .

For  $E(1)$ , note that  $H^2(E(1)) = 0$  by stability and duality, and (6.1) gives an exact sequence

$$0 \rightarrow H^1(E(1)) \rightarrow H^1(J_{P/X}(2)) \rightarrow H^2(\mathcal{O}_X(1)) \rightarrow 0.$$

On the other hand,  $H^1(J_{P/X}(2)) \cong \mathbb{C}$  corresponding to the length 4 of  $P$ , minus the dimension 3 of the space of sections of  $\mathcal{O}_P(2)$  coming from global quadrics (since the space of quadrics on  $\ell$  has dimension 3). This gives  $H^1(E(1)) = 0$ . The Euler characteristic then gives  $h^0(E(1)) = 11$ . This is also seen in the first part of the exact sequence, where  $H^0(\mathcal{O}_X(1)) = \mathbb{C}^4$  and  $H^0(J_{P/X}(2)) \cong \mathbb{C}^7$ .  $\square$

If  $p \in \mathbb{P}^3$ , let  $G \cong \mathbb{C}^3$  be the space of linear generators of the ideal of  $p$ , that is to say  $G := H^0(J_{p/\mathbb{P}^3}(1))$ , and consider the natural exact sequence of sheaves on  $\mathbb{P}^3$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3} \otimes G^* \rightarrow \mathcal{R}_p \rightarrow 0.$$

Here the cokernel sheaf  $\mathcal{R}_p$  is a reflexive sheaf of degree 1, and  $c_2(\mathcal{R}_p)$  is the class of a line. The restriction  $\mathcal{R}_p|_X$  therefore has  $c_2 = 5$ . If  $p \in X$ , it is torsion-free but not locally free, giving a point in  $M(5, 4)$ . It turns out that these sheaves account for all of  $M(4)$  and  $M(5)$ .

**Theorem 6.2.** *Suppose  $E \in M(4)$ . Then there is a unique point  $p \in X$  such that  $E$  is generated by global sections outside of  $p$ , and  $\mathcal{R}_p|_X$  is isomorphic to the subsheaf of  $E$  generated by global sections. This fits into an exact sequence*

$$0 \rightarrow \mathcal{R}_p|_X \rightarrow E \rightarrow S \rightarrow 0$$

where  $S$  has length 1, in particular  $E \cong (\mathcal{R}_p|_X)^{**}$ . The correspondence  $E \leftrightarrow p$  establishes an isomorphism  $M(4) \cong X$ .

For  $E' \in M(5)$  there exists a unique point  $p \in \mathbb{P}^3 - X$  such that  $E' \cong \mathcal{R}_p|_X$ . This correspondence establishes an isomorphism  $\overline{M}(5) \cong \mathbb{P}^3$  such that the boundary component  $M(5, 4) \cap \overline{M}(5)$  is exactly  $X \subset \mathbb{P}^3$ . Note however that  $M(5, 4)$  itself is bigger and constitutes another irreducible component of  $\overline{M}(5)$ .

*Proof.* Consider the exact sequence (6.1). The space  $H^0(J_{P/X}(1))$  consists of linear forms on  $X$  (or equivalently, on  $\mathbb{P}^3$ ), which vanish along  $P$ . However, a linear form which vanishes on  $P$  also vanishes on  $\ell$ . In particular, elements of  $H^0(J_{P/X}(1))$  generate  $J_{X \cap \ell/X}(1)$ , which has colength 1 in  $J_{P/X}(1)$ .

Let  $R \subset E$  be the subsheaf generated by global sections, and let  $S$  be the cokernel in the exact sequence

$$0 \rightarrow R \rightarrow E \rightarrow S \rightarrow 0.$$

We also have the exact sequence

$$0 \rightarrow J_{X \cap \ell/X}(1) \rightarrow J_{P/X}(1) \rightarrow S \rightarrow 0$$

so  $S$  has length 1. It is supported on a point  $p$ . The sheaf  $R$  is generated by three global sections so we have an exact sequence

$$0 \rightarrow L \rightarrow \mathcal{O}_X^3 \rightarrow R \rightarrow 0.$$

The kernel is a saturated subsheaf, hence locally free, and by looking at its degree we have  $L = \mathcal{O}_X(-1)$ . Thus,  $R$  is the cokernel of a map  $\mathcal{O}_X(-1) \rightarrow \mathcal{O}_X^3$  given by three linear forms; these linear forms are a basis for the space of forms vanishing at the point  $p$ . We see that  $R$  is the restriction to  $X$  of the sheaf  $\mathcal{R}_p$  described above, hence  $E \cong (\mathcal{R}_p|_X)^{**}$ . The map  $E \mapsto p$  gives a map  $M(4) \rightarrow X$ , with inverse  $p \mapsto (\mathcal{R}_p|_X)^{**}$ .

The second paragraph, about  $\overline{M(5)}$ , is not actually needed later and we leave it to the reader.  $\square$

Even though the moduli space  $M(4)$  is smooth, it has much more than the expected dimension, and the space of co-obstructions is nontrivial. It will be useful to understand the co-obstructions, because if  $F \in M(10, 4)$  is a torsion-free sheaf with  $F^{**} = E$  then co-obstructions for  $F$  come from co-obstructions for  $E$  which preserve the subsheaf  $F \subset E$ .

**Lemma 6.3.** *Suppose  $E \in M(4)$ . A general co-obstruction  $\phi : E \rightarrow E(1)$  has generically distinct eigenvalues with an irreducible spectral variety in  $\text{Tot}(K_X)$ .*

*Proof.* It suffices to write down a map  $\phi : E \rightarrow E(1)$  with generically distinct eigenvalues and irreducible spectral variety. To do this, we construct a map  $\phi_R : R \rightarrow R(1)$  using the expression  $R = \mathcal{R}_p|_X$ . The exact sequence defining  $\mathcal{R}_p$  extends to the Koszul resolution, a long exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3}^3 \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^3 \rightarrow J_{p/\mathbb{P}^3}(2) \rightarrow 0.$$

Thus  $\mathcal{R}_p$  may be viewed as the image of the middle map. Without loss of generality,  $p$  is the origin in an affine system of coordinates  $(x, y, z)$  for  $\mathbb{A}^3 \subset \mathbb{P}^3$ , and the coordinate functions are the three coefficients of the maps on the left and right in the Koszul sequence. The  $3 \times 3$  matrix in the middle is

$$K := \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}.$$

Any  $3 \times 3$  matrix of constants  $\Phi$  gives a composed map

$$\phi_R : \mathcal{R}_p \hookrightarrow \mathcal{O}_{\mathbb{P}^3}(1)^3 \xrightarrow{\Phi} \mathcal{O}_{\mathbb{P}^3}(1)^3 \rightarrow \mathcal{R}_p(1).$$

Use the first two columns of  $K$  to give a map  $k : \mathcal{O}_{\mathbb{P}^3}^2 \rightarrow \mathcal{R}_p$  which is an isomorphism over an open set. On the other hand, the projection onto the first two coordinates gives a map  $q : \mathcal{R}_p \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^2$  which is, again, an isomorphism over an open set. The composition of these two is the map given by the upper  $2 \times 2$  square of  $K$ ,

$$qk = K_{2,2} := \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}.$$

We can now analyze the map  $\phi_R$  by noting that  $q\phi_R k = K_{2,3}\Phi K_{3,2}$  where  $K_{2,3}$  and  $K_{3,2}$  are respectively the upper  $2 \times 3$  and left  $3 \times 2$  blocks of  $K$ . Over the open set where  $q$  and  $k$  are isomorphisms,

$$q\phi_R q^{-1} = q\phi_R k (qk)^{-1} = K_{2,3}\Phi K_{3,2} K_{2,2}^{-1}.$$

Now

$$K_{3,2} K_{2,2}^{-1} = \begin{pmatrix} 0 & z \\ -z & 0 \\ y & -x \end{pmatrix} \cdot \begin{pmatrix} 0 & -1/z \\ 1/z & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x/z & -y/z \end{pmatrix}.$$

Suppose

$$\Phi = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \psi \\ \chi & \theta & \rho \end{pmatrix}$$

then

$$\begin{aligned} q\phi_R q^{-1} &= K_{2,3}\Phi K_{3,2} K_{2,2}^{-1} \\ &= \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \psi \\ \chi & \theta & \rho \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x/z & -y/z \end{pmatrix} \\ &= \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \end{pmatrix} \cdot \begin{pmatrix} \alpha - \gamma x/z & \beta - \gamma y/z \\ \delta - \psi x/z & \epsilon - \psi y/z \\ \chi - \rho x/z & \theta - \rho y/z \end{pmatrix} \\ &= \begin{pmatrix} z\delta - \psi x - y\chi + \rho xy/z & z\epsilon - \psi y + y\theta - \rho y^2/z \\ -z\alpha + \gamma x - x\chi + \rho x^2/z & -z\beta + \gamma y + x\theta - \rho xy/z \end{pmatrix}. \end{aligned}$$

Notice that the trace of this matrix is

$$\text{Tr}(\phi) = x(\theta - \psi) + y(\gamma - \chi) + z(\delta - \beta),$$

which is a section of  $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$  vanishing at  $p$ . A co-obstruction should have trace zero, so we should impose three linear conditions

$$\theta = \psi, \quad \chi = \gamma, \quad \delta = \beta$$

which together just say that  $\Phi$  is a symmetric matrix. Our expression simplifies to

$$q\phi_R q^{-1} = \begin{pmatrix} \beta z - \psi x - \gamma y + \rho xy/z & \epsilon z - \rho y^2/z \\ -\alpha z + \rho x^2/z & -\beta z + \psi x + \gamma y - \rho xy/z \end{pmatrix}.$$

Now, restrict  $\mathcal{R}_p$  to  $X$  to get the sheaf  $R$ , take its double dual to get  $E = R^{**}$ , and consider the induced map  $\phi : E \rightarrow E(1)$ . Over the intersection of our open set with  $X$ , this will have the same formula. We can furthermore restrict to the curve  $Y \subset X$  given by the intersection with the plane  $y = 0$ . Note that  $X$  is in general position subject to the condition that it contain the point  $p$ . Setting  $y = 0$  the above matrix becomes

$$(q\phi q^{-1})|_{y=0} = \begin{pmatrix} \beta z - \psi x & \epsilon z \\ -\alpha z + \rho x^2/z & -\beta z + \psi x \end{pmatrix}.$$

Choose for example  $\beta = \psi = 0$  and  $\alpha = \rho = \epsilon = 1$ , giving the matrix whose determinant is

$$\det \begin{pmatrix} 0 & z \\ x^2/z - z & 0 \end{pmatrix} = z^2 - x^2 = (z+x)(z-x).$$



The eigenvalues of  $\phi|_Y$  are therefore  $\pm\sqrt{(z+x)(z-x)}$ , generically distinct. For a general choice of the surface  $X$ , our curve  $Y = X \cap (y = 0)$  will intersect the planes  $x = z$  and  $x = -z$  transversally, so the two eigenvalues of  $\phi|_Y$  are permuted when going around points in the ramification locus different from  $p$ . This provides an explicit example of  $\phi$  for which the spectral variety is irreducible, completing the proof of the lemma. We included the detailed calculations because they look to be useful if one wants to write down explicitly the spectral varieties.  $\square$

Turn now to the study of the boundary component  $M(10, 4)$  consisting of torsion-free sheaves in  $\overline{M}(10)$  which come from bundles in  $M(4)$ . A point in  $M(10, 4)$  consists of a torsion-free sheaf  $F$  in an exact sequence of the form (5.1)

$$0 \rightarrow F \rightarrow E \xrightarrow{\sigma} S \rightarrow 0$$

where  $E = F^{**}$  is a point in  $M(4)$ , and  $S$  is a length 6 quotient.

The basic description of the space of obstructions as dual to the space of  $K_X$ -twisted endomorphisms still holds for torsion-free sheaves. Thus, the obstruction space for  $F$  is  $\text{Hom}^o(F, F(1))^*$ . A co-obstruction is a map  $\phi : F \rightarrow F(1) = F \otimes K_X$  with  $\text{Tr}(\phi) = 0$ , which is a kind of *Higgs field*. Since the moduli space is good, a point  $F$  is in  $\text{Sing}(\overline{M}(10))$  if and only if the obstruction space is nonzero, that is to say, if and only if there exists a nonzero trace-free  $\phi : F \rightarrow F(1)$ .

To give a map  $\phi$  is the same thing as to give a map  $\varphi : E \rightarrow E(1)$  compatible with the quotient map  $E \rightarrow S$ , in other words fitting into a commutative square with  $\sigma$ , for an induced map  $\varphi_S : S \rightarrow S$ . The maps  $\varphi$ , co-obstructions for  $E$ , were studied in Lemma 6.3 above.

Let  $\mathbb{P}(E) \rightarrow X$  denote the Grothendieck projective space bundle. A point in  $\mathbb{P}(E)$  is a pair  $(x, s)$  where  $x \in X$  and  $s : E_x \rightarrow S_x$  is a rank one quotient of the fiber. Suppose given a map  $\varphi : E \rightarrow E(1)$ . We can consider the *internal spectral variety*

$$\text{Sp}_E(\varphi) \subset \mathbb{P}(E)$$

defined as the set of points  $(x, s) \in \mathbb{P}(E)$  such that there exists a commutative diagram

$$\begin{array}{ccc} E_x & \xrightarrow{\varphi(x)} & E_x \\ \downarrow & & \downarrow \\ S_x & \longrightarrow & S_x. \end{array}$$

The term ‘internal’ signifies that it is a subvariety of  $\mathbb{P}(E)$  as opposed to the classical spectral variety which is a subvariety of the total space of  $K_X$ . Here, we have only given  $\text{Sp}_E(\varphi)$  a structure of closed subset of  $\mathbb{P}(E)$ , hence of reduced subvariety. It would be interesting to give it an appropriate scheme structure which could be non-reduced in case  $\varphi$  is nilpotent, but that will not be needed here.

**Corollary 6.4.** *Suppose  $E \in M(4)$  and  $\varphi : E \rightarrow E(1)$  is a general co-obstruction. Then the internal spectral variety  $\text{Sp}_E(\varphi)$  has a single irreducible component of dimension 2. A quotient  $E \rightarrow S$  consisting of a disjoint sum of rank one quotients  $s_i : E_{x_i} \rightarrow S_i$  with  $S = \bigoplus S_i$  and the points  $x_i$  disjoint, is compatible with  $\varphi$  if and only if the points  $(x_i, s_i) \in \mathbb{P}(E)$  lie on the internal spectral variety  $\text{Sp}_E(\varphi)$ .*

*Proof.* Notice that  $z \in X$  is a point such that  $\varphi(z) = 0$ , then the whole fiber  $\mathbb{P}(E)_z \cong \mathbb{P}^1$  is in  $\text{Sp}_E(\varphi)$ . In particular, if such a point exists then the map  $\text{Sp}_E(\varphi) \rightarrow X$  will not be finite.

A first remark is that the zero-set of  $\varphi$  is 0-dimensional. Indeed, if  $\varphi$  vanished along a divisor  $D$ , then  $D \in |\mathcal{O}_X(n)|$  for  $n \geq 1$  and  $\varphi : F \rightarrow F(1-n)$ . This is possible only if  $n = 1$  and  $\varphi : F \rightarrow F$  is a scalar endomorphism (since  $F$  is stable). However, the trace of the co-obstruction vanishes, so the scalar  $\varphi$  would have to be zero, which we are assuming is not the case.

At an isolated point  $z$  with  $\varphi(z) = 0$ , the fiber of the projection  $\mathrm{Sp}_E(\varphi) \rightarrow X$  contains the whole  $\mathbb{P}(E_z) = \mathbb{P}^1$ . However, these can contribute at most irreducible components of dimension  $\leq 1$  (although we conjecture that in fact these fibers are contained in the closure of the 2-dimensional component so that  $\mathrm{Sp}_E(\varphi)$  is irreducible).

Away from such fibers, the internal spectral variety is isomorphic to the external one, a two-sheeted covering of  $X$ , and by Lemma 6.3, for a general  $\varphi$  the monodromy of this covering interchanges the sheets so it is irreducible. Thus,  $\mathrm{Sp}_E(\varphi)$  has a single irreducible component of dimension 2, and it maps to  $X$  by a generically finite (2 to 1) map.

The second statement, that a quotient consisting of a direct sum of rank one quotients, is compatible with  $\varphi$  if and only if the corresponding points lie on  $\mathrm{Sp}_E(\varphi)$ , is immediate from the definition.  $\square$

**Definition 6.5.** *A triple  $(E, \varphi, \sigma)$  where  $E \in M(4)$ ,  $\varphi : E \rightarrow E(1)$  is a non-nilpotent map, and  $\sigma = \bigoplus s_x$  is a quotient composed of six rank 1 quotients over distinct points, compatible with  $\varphi$  as in the previous Corollary 6.4, leads to an obstructed point  $F = F_{(E, \varphi, \sigma)} \in M(10, 4)^{\mathrm{sing}}$  obtained by setting  $F := \ker(\sigma)$ . Such a point will be called usual.*

Ellingsrud and Lehn have given a very nice description of the Grothendieck quotient scheme of a bundle of rank  $r$  on a smooth surface. It extends the basic idea of Li's theorem which we already stated as Theorem 3.2 above, and will allow us to count dimensions of strata in  $M(10, 4)$ .

**Theorem 6.6** (Ellingsrud-Lehn). *The quotient scheme parametrizing quotients of a locally free sheaf  $\mathcal{O}_X^r$  of rank  $r$  on a smooth surface  $X$ , located at a given point  $x \in X$ , and of length  $\ell$ , is irreducible of dimension  $r\ell - 1$ .*

*Proof.* See [1]. We have given the local version of the statement here.  $\square$

In our case,  $r = 2$  so the dimension of the local quotient scheme is  $2\ell - 1$ .

A given quotient  $E \rightarrow S$  decomposes as a direct sum of quotients  $E \rightarrow S_i$  located at distinct points  $x_i \in X$ . Order these by decreasing length, and define the *length vector* of  $S$  to be the sequence  $(\ell_1, \dots, \ell_k)$  of lengths  $\ell_i = \ell(S_i)$  with  $\ell_i \geq \ell_{i+1}$ . This leads to a stratification of the Quot scheme into strata labelled by length vectors. By Ellingsrud-Lehn, the dimension of the space of quotients supported at a single (but not fixed) point  $x_i$  and having length  $\ell_i$ , is  $2\ell_i + 1$ , giving the following dimension count.

**Corollary 6.7.** *For a fixed bundle  $E$  of rank 2, the dimension of the stratum associated to length vector  $(\ell_1, \dots, \ell_k)$  in the Quot-scheme of quotients  $E \rightarrow S$  with total length  $\ell = \sum_{i=1}^k \ell_i$ , is*

$$\sum (2\ell_i + 1) = 2\ell + k.$$

Recall that the moduli space  $M(4)$  has dimension 2, so the dimension of the stratum of  $M(10, 4)$  corresponding to a vector  $(\ell_1, \dots, \ell_k)$  is  $14 + k$ . In particular,  $M(10, 4)$  has a single stratum  $(1, 1, 1, 1, 1, 1)$  of dimension 20, corresponding to quotients which are direct sums of rank one quotients supported at distinct points, and a single stratum  $(2, 1, 1, 1, 1)$  of length 19. This yields the following corollary.

**Corollary 6.8.** *If  $Z' \subset M(10, 4)$  is any 19-dimensional irreducible subvariety, then either  $Z'$  is equal to the stratum  $(2, 1, 1, 1, 1)$ , or else the general point on  $Z'$  consists of a direct sum of six rank 1 quotients supported over six distinct points of  $X$ .*

**Proposition 6.9.** *The singular locus  $M(10, 4)^{\text{sing}}$  has only one irreducible component of dimension 19. This irreducible component has a nonempty dense open subset consisting of the usual points (Definition 6.5). For a usual point, the co-obstruction  $\varphi$  is unique up to a scalar, so this open set may be viewed as the moduli space of usual triples  $(E, \varphi, \sigma)$ , which is irreducible.*

*Proof.* Suppose  $Z' \subset M(10, 4)^{\text{sing}}$  is an irreducible component. Consider the two cases given by Corollary 6.8.

(i)—If  $Z'$  contains an open set consisting of points which are direct sums of six rank 1 quotients supported on distinct points of  $X$ , then this open set parametrizes usual triples. Furthermore, a point in this open set corresponds to a choice of  $(E, \varphi)$  together with six points on the internal spectral variety  $\text{Sp}_E(\varphi)$ . We count the dimension of this piece as follows.

Let  $M'(4)$  denote the moduli space of pairs  $(E, \varphi)$  with  $E \in M(4)$  and  $\varphi$  a nonzero co-obstruction for  $E$ . The space of co-obstructions for any  $E \in M(4)$ , has dimension 6 and the family of these spaces forms a vector bundle over  $M(4)$  (more precisely, a twisted vector bundle twisted by the obstruction class for existence of a universal family over  $M(4)$ ). Thus, the moduli space of pairs has a fibration  $M'(4) \rightarrow M(4)$  whose fibers are  $\mathbb{P}^5$ . In particular,  $M'(4)$  is a smooth irreducible variety of dimension 7.

For a general such  $(E, \varphi)$  the moduli space of usual triples has dimension  $\leq 12$ , with a unique 12 dimensional piece corresponding to a general choice of 6 points on the unique 2-dimensional irreducible component of  $\text{Sp}_E(\varphi)$ . This gives the 19-dimensional component of  $M(10, 4)^{\text{sing}}$  mentioned in the statement of the proposition.

Suppose  $(E, \varphi)$  is not general, that is to say, contained in some subvariety of  $M'(4)$  of dimension  $\leq 6$ . Then, as  $\varphi$  is nonzero, even though we no longer can say that it is irreducible, in any case the internal spectral variety  $\text{Sp}_E(\varphi)$  has dimension 2 so the space of choices of 6 general points on it has dimension  $\leq 12$ , and this contributes at most subvarieties of dimension  $\leq 18$  in  $M(10, 4)^{\text{sing}}$ . This shows that in the first case (i) of Corollary 6.8, we obtain the conclusion of the proposition.

(ii)—Suppose  $Z'$  is equal to the stratum of  $M(10, 4)$  corresponding to length vector  $(2, 1, 1, 1, 1)$ . In this case, we show that a general point of  $Z'$  has no non-zero co-obstructions, contradicting the hypothesis that  $Z' \subset M(10, 4)^{\text{sing}}$  and showing that this case cannot occur.

Fix  $E \in M(4)$ . The space of co-obstructions of  $E$  has dimension 6. Suppose  $E \rightarrow S_1$  is a quotient of length 2. If it is just the whole fiber of  $E$  over  $x_1$ , then it is automatically compatible with any co-obstruction. However, these quotients contribute only a 2-dimensional subspace of the space of such quotients which has dimension 5 by Ellingsrud-Lehn. Thus, these points don't contribute general points.

On the other hand, a general quotient of length 2 corresponds to an infinitesimal tangent vector in  $\mathbb{P}(E)$ , and the condition that this vector be contained in  $\mathrm{Sp}_E(\varphi)$  imposes two conditions on  $\varphi$ . Therefore, the space of co-obstructions compatible with  $S_1$  has dimension  $\leq 4$ . Next, given a nonzero co-obstruction in that subspace, a general quotient  $E \rightarrow S_2$  of length 1 will not be compatible, so imposing compatibility with  $S_1$  and  $S_2$  leads to a space of co-obstructions of dimension  $\leq 3$ . Continuing in this way, we see that imposing the condition of compatibility of  $\varphi$  with a general quotient  $S = S_1 \oplus \cdots \oplus S_5$  in the stratum  $(2, 1, 1, 1, 1)$  leads to  $\varphi = 0$ . Thus, a general point of this stratum has no non-zero co-obstructions as we have claimed, and this case (ii) cannot occur.

Hence, the only case from Corollary 6.8 which can contribute a 19-dimensional stratum, contributes the single irreducible component described in the statement of the proposition. One may note that  $\varphi$  is uniquely determined for a general set of six points on its internal spectral variety, since the first 5 points are general in  $\mathbb{P}(E)$  and impose linearly independent conditions.  $\square$

**Corollary 6.10.** *Suppose  $M(10, 4) \cap \overline{M(10)}$  is nonempty. Then it is the unique 19-dimensional irreducible component of usual triples in  $M(10, 4)^{\mathrm{sing}}$  identified by Proposition 6.9.*

*Proof.* By Hartshorne's theorem, the intersection  $M(10, 4) \cap \overline{M(10)}$  has pure dimension 19 if it is nonempty. This could also be seen from O'Grady's lemma that the boundary of  $\overline{M(10)}$  has pure dimension 19. However, any point in this intersection is singular. By Proposition 6.9, the singular locus  $M(10, 4)^{\mathrm{sing}}$  has only one irreducible component of dimension 19, and it is the closure of the space of usual triples.  $\square$

If the intersection  $M(10, 4) \cap \overline{M(10)}$  is nonempty, the torsion-free sheaves  $F$  parametrized by general points satisfy  $h^1(F(1)) = 0$ . We show this by a dimension estimate using Ellingsrud-Lehn. The more precise information about  $M(10, 4)^{\mathrm{sing}}$  given in Proposition 6.9, while not really needed for the proof at  $c_2 = 10$ , will be useful in treating the case of  $c_2 = 11$  in Section 8.

**Proposition 6.11.** *The subspace of  $M(10, 4)$  consisting of points  $F$  such that  $h^1(F(1)) \geq 1$ , has codimension  $\geq 2$ .*

*Proof.* Use the exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow S \rightarrow 0$$

where  $E \in M(4)$ . One has  $h^1(E(1)) = 0$  for all  $E \in M(4)$ , see Lemma 6.1. Therefore,  $h^1(F(1)) = 0$  is equivalent to saying that the map

$$(6.2) \quad H^0(E(1)) \rightarrow H^0(S(1)) \cong \mathbb{C}^6$$

is surjective.

Considering the theorem of Ellingsrud-Lehn, there are two strata to be looked at: the case of a direct sum of six quotients of rank 1 over distinct points, to be treated below; and the case of a direct sum of four quotients of rank 1 and one quotient of rank 2. However, this latter stratum already has codimension 1, and it is irreducible. So, for this stratum it suffices to note that a general quotient  $E \rightarrow S$  in it leads to a surjective map (6.2), which may be seen by a classical general position argument, placing first the quotient of rank 2.

Consider now the stratum of quotients which are the direct sum of six rank 1 quotients  $s_i$  at distinct points  $x_i \in X$ . Fix the bundle  $E$ . The space of choices of the six quotients  $(x_i, s_i)$  has dimension 18. We claim that the space of choices such that (6.2) is not surjective, has codimension  $\geq 2$ .

Note that  $h^0(E(1)) = 11$ . Given six quotients  $(x_i, s_i)$ , if the map (6.2) (with  $S = \bigoplus S_i$ ) is not surjective, then its kernel has dimension  $\geq 6$ , so if we choose five additional points  $(y_j, t_j) \in \mathbb{P}(E)$  with  $t_j : E_{y_j} \rightarrow T_j$  for  $T_i$  of length 1, the total evaluation map

$$(6.3) \quad H^0(E(1)) \rightarrow \bigoplus_{i=1}^6 S_i(1) \oplus \bigoplus_{j=1}^5 T_j(1)$$

has a nontrivial kernel. Consider the variety

$$W := \{(u, \dots (x_i, s_i) \dots, \dots (y_j, t_j) \dots) \text{ s.t. } 0 \neq u \in H^0(E(1)), s_i(u) = 0, t_j(u) = 0\}$$

with the nonzero section  $u$  taken up to multiplication by a scalar.

Let  $Q'_6(E)$  and  $Q'_5(E)$  denote the open subsets of the quotient schemes of length 6 and length 5 quotients of  $E$  respectively, open subsets consisting of quotients which are direct sums of rank one quotients over distinct points. Let  $K \subset Q'_6(E)$  denote the locus of quotients  $E \rightarrow S$  such that the kernel sheaf  $F$  has  $h^1(F(1)) \geq 1$ . It is a proper closed subset, since it is easy to see that a general quotient  $E \rightarrow S$  leads to a surjection (6.2). The above argument with (6.3) shows that  $K \times Q'_5(E) \subset p(W)$  where  $p : W \rightarrow Q'_6(E) \times Q'_5(E)$  is the projection forgetting the first variable  $u$ . Our goal is to show that  $K$  has dimension  $\leq 16$ .

We claim that  $W$  has dimension  $\leq 32$  and has a single irreducible component of dimension 32. To see this, start by noting that the choice of  $u$  lies in the projective space  $\mathbb{P}^{10}$  associated to  $H^0(E(1)) \cong \mathbb{C}^{11}$ .

For a section  $u$  which is special in the sense that its scheme of zeros has positive dimension, the locus of choices of  $(x_i, s_i)$  and  $(y_j, t_j)$  has dimension  $\leq 22$ , but might have several irreducible components depending on whether the points are on the zero-set of  $u$  or not. However, the space of sections  $u$  which are special in this sense, is equal to the space of pairs  $u' \in H^0(E)$ ,  $u'' \in H^0(\mathcal{O}_X(1))$  up to scalars for both pieces, and this has dimension  $2 + 3 = 5$ , which is much smaller than the dimension of the space of all sections  $u$ . Therefore, these pieces don't contribute anything of dimension higher than 27.

For a section  $u$  which is not special in the sense of the previous paragraph, the space of choices of a single rank 1 quotient  $(x, s)$  which vanishes on the section, has a single irreducible component of dimension 2. It might possibly have some pieces of dimension 1 corresponding to quotients located at the zeros of  $u$  (although we don't think so). Hence, the space of choices of point in  $W$  lying over the section  $u$ , has dimension  $\leq 22$  and has a single irreducible component of dimension 22.

Putting these together over  $\mathbb{P}^{10}$ , the dimension of  $W$  is  $\leq 32$  and it has a single irreducible component of dimension 32, as claimed. Its image  $p(W)$  therefore also has dimension  $\leq 32$ , and has at most one irreducible component of dimension 32. Denote this component, if it exists, by  $p(W)'$ .

Suppose now that  $K$  had an irreducible component  $K'$  of dimension 17. Then  $K' \times Q'_5(E) \subset p(W)$ , but  $\dim(Q'_5(E)) = 15$  so  $p(W)'$  would exist and would be equal to  $K' \times Q'_5(E)$ . However,  $p(W)'$  is symmetric under permutation of the 11 different variables  $(x, s)$  and  $(y, t)$ , but that would then imply that  $P(W)'$  was the

whole of  $Q'_6(E) \times Q'_5(E)$  which is not the case. Therefore,  $K$  must have codimension  $\geq 2$ . This completes the proof of the proposition.  $\square$

**Corollary 6.12.** *Suppose  $M(10, 4) \cap \overline{M(10)}$  is nonempty. Then a general point of this intersection corresponds to a torsion-free sheaf with  $h^1(F(1)) = 0$ .*

*Proof.* By Hartshorne's or O'Grady's theorem, if the intersection is nonempty then it has pure dimension 19. However, the space of torsion-free sheaves  $F \in M(10, 4)$  with  $h^1(F(1)) > 0$  has dimension  $\leq 18$  by Proposition 6.11. Thus, a general point in any irreducible component of  $M(10, 4) \cap \overline{M(10)}$  must have  $h^1(F(1)) = 0$ . In fact there can be at most one irreducible component, by Corollary 6.10.  $\square$

## 7. IRREDUCIBILITY FOR $c_2 = 10$

**Corollary 7.1.** *Suppose  $Z$  is an irreducible component of  $M(10)$ . Then, for a general point  $F$  in any irreducible component of the intersection of  $\overline{Z}$  with the boundary, we have  $h^1(F(1)) = 0$ .*

*Proof.* By O'Grady's lemma, the intersection of  $\overline{Z}$  with the boundary has pure dimension 19. By considering the line  $c_2 = 10$  in the Table 2, this subset must be a union of some of the irreducible subsets  $\overline{M(10, 9)}$ ,  $\overline{M(10, 8)}$ ,  $\overline{M(10, 6)}$ , and the unique 19-dimensional irreducible component of  $M(10, 4)^{\text{sing}}$  given by Proposition 6.9. Combining Proposition 5.2 and Lemma 6.12, we conclude that  $Z$  contains a point  $F$  such that  $h^1(F(1)) = 0$ . Thus,  $h^1(E(1)) = 0$  for a general bundle  $E$  parametrized by a point of  $Z$ .  $\square$

**Corollary 7.2.** *Suppose  $Z$  is an irreducible component of  $M(10)$ . Then the bundle  $E$  parametrized by a general point of  $Z$  has seminatural cohomology, and  $Z$  is the closure of the irreducible open set  $M(10)^{\text{sn}}$ .*

*Proof.* The closure of  $Z$  meets the boundary in a nonempty subset, by Corollary 4.5. By the previous Corollary 7.1, there exists a point  $F$  in  $\overline{Z}$  with  $h^1(F(1)) = 0$ , thus the general bundle  $E$  in  $Z$  also satisfies  $h^1(E(1)) = 0$ . By Proposition 5.1, the irreducible moduli space  $M(10)^{\text{sn}}$  of bundles with seminatural cohomology is an open set of  $Z$ .  $\square$

**Theorem 7.3.** *The moduli space  $M(10)$  of stable bundles of degree 1 and  $c_2 = 10$ , is irreducible.*

*Proof.* By Corollary 7.2, any irreducible component of  $M(10)$  contains a dense open set parametrizing bundles with seminatural cohomology. By the main theorem of [15], there is only one such irreducible component.  $\square$

**Theorem 7.4.** *The full moduli space of stable torsion-free sheaves  $\overline{M(10)}$  of degree 1 and  $c_2 = 10$ , has two irreducible components,  $\overline{M(10)}$  and  $M(10, 4)$  meeting along the irreducible component of usual triples in  $M(10, 4)^{\text{sing}}$ . These two components have the expected dimension, 20, hence the moduli space is good and connected.*

*Proof.* Recall that we know  $M(10, 4)$  is irreducible by the results of [14]. Also  $M(10)$  is irreducible. Any component has dimension  $\geq 20$ , and by looking at the dimensions in Table 2, these are the only two possible irreducible components. Since they have dimension 20 which is the expected dimension, it follows that the moduli space is good.



It remains to be proven that these two components do indeed intersect in a nonempty subset, which then by Corollary 6.10 has to be the irreducible component of usual triples in  $M(10, 4)^{\text{sing}}$ . Notice that Corollary 6.10 did not say that the intersection was necessarily nonempty, since it started from the hypothesis that there was a meeting point. It is a consequence of Nijssse's connectedness theorem that the intersection is nonempty, but this may be seen more concretely as follows.

Consider the stratum  $M(10, 5)$ . Recall from [14] that the moduli space  $M(5)$  consists of bundles which fit into an exact sequence of the form

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_{P/X}(1) \rightarrow 0,$$

such that  $P = \ell \cap X$  for  $\ell \subset \mathbb{P}^3$  a line. In what follows, choose  $\ell$  general so that  $P$  consists of 5 distinct points.

The space of extensions  $\text{Ext}^1(J_{P/X}(1), \mathcal{O}_X)$  is dual to  $\text{Ext}^1(\mathcal{O}_X, J_{P/X}(2)) = H^1(J_{P/X}(2))$ . We have the exact sequence

$$H^0(\mathcal{O}_X(2)) \rightarrow H^0(\mathcal{O}_P(2)) \rightarrow H^1(J_{P/X}(2)) \rightarrow 0.$$

However,  $H^0(\mathcal{O}_X(2)) = H^0(\mathcal{O}_{\mathbb{P}^3}(2))$  and the map to  $H^0(\mathcal{O}_P(2))$  factors through  $H^0(\mathcal{O}_\ell(2))$ , the space of degree two forms on  $\ell \cong \mathbb{P}^1$ , which has dimension 3. Hence, the cokernel  $H^1(J_{P/X}(2))$  has dimension 2. The extension classes which correspond to bundles, are the linear forms on  $H^1(J_{P/X}(2))$  which don't vanish on any of the images of the lines in  $H^0(\mathcal{O}_P(2))$  corresponding to the 5 different points. Since  $H^1(J_{P/X}(2))$  has dimension two, we can find a family of extension classes whose limiting point is an extension which vanishes on one of the lines corresponding to a point in  $P$ . This gives a degeneration towards a torsion-free sheaf with a single non-locally free point, still sitting in a nontrivial extension of the above form. We conclude that the limiting bundle is still stable, so we have constructed a degeneration from a point of  $M(5)$ , to the single boundary stratum  $M(5, 4)$ .

Notice that the dimension of  $M(5, 4)$  is bigger than that of  $M(5)$ , so the set of limiting points is a strict subvariety of  $M(5, 4)$ . We have  $\overline{M(5)} = M(5) \cup M(5, 4)$ , and we have shown that the closures of these two strata have nonempty intersection. This fact is also a consequence of the more explicit description of  $\overline{M(5)}$  stated in Theorem 6.2 above (but where the proof was left to the reader).

Moving up to  $c_2 = 10$ , it follows that the closure of the stratum  $M(10, 5)$  intersects  $M(10, 4)$ . However,  $M(10, 4)$  is closed, and the remaining strata of the boundary have dimension  $\leq 19$ , so all of the other strata in the boundary, in particular  $\overline{M(10, 5)}$ , are contained in the closure of the locus of bundles  $\overline{M(10)}$ . Thus,  $\overline{M(10, 5)} \subset \overline{M(10)}$ , but  $M(10, 4) \cap \overline{M(10, 5)} \neq \emptyset$ , proving that the intersection  $M(10, 4) \cap \overline{M(10)}$  is nonempty.  $\square$

*Physics discussion:* From this fact, we see that there are degenerations of stable bundles in  $M(10)$ , near to boundary points in  $M(10, 4)$ . Donaldson's Yang-Mills metrics then degenerate towards Uhlenbeck boundary points, metrics where 6 instantons appear. However, these degenerations go not to all points in  $M(10, 4)$  but only to ones which are in the irreducible subvariety  $M(10, 4)^{\text{sing}} \subset M(10, 4)$  consisting of points on the internal spectral variety of a nonzero Higgs field  $\varphi : E \rightarrow E \otimes K_X$ . It gives a constraint of a global nature on the 6-tuples of instantons which can appear in Yang-Mills metrics on a stable bundle  $F \in M(10)$ . It would be interesting to understand the geometry of the Higgs field which shows up, somewhat virtually, in the limit.



8. IRREDUCIBILITY FOR  $c_2 \geq 11$ 

Consider next the moduli space  $\overline{M}(11)$  of stable torsion-free sheaves of degree one and  $c_2 = 11$ . The moduli space is good, of dimension 24. From Table 2, the dimensions of the boundary strata are all  $\leq 23$ , so the set of irreducible components of  $\overline{M}(11)$  is the same as the set of irreducible components of  $M(11)$ . Suppose  $Z$  is an irreducible component. By Corollary 4.5,  $Z$  meets the boundary in a nonempty subset of codimension 1, i.e. dimension 23. From Table 2, the only two possibilities are  $M(11, 10)$  and  $M(11, 4)$ . Note that  $M(11, 4)$  is closed since it is the lowest stratum; it is irreducible by Li's theorem and irreducibility of  $M(4)$ . The stratum  $M(11, 10)$  is irreducible because of Theorem 7.3.

**Lemma 8.1.** *The intersection  $M(11, 4) \cap \overline{M}(11, 10)$  is a nonempty subset containing, in particular, points which are torsion-free sheaves  $F'$  entering into an exact sequence of the form*

$$0 \rightarrow F' \rightarrow F \rightarrow S_x \rightarrow 0$$

where  $F$  is a usual point of  $M(10, 4)^{\text{sing}}$ ,  $x \in X$  is a general point, and  $F \rightarrow S_x$  is a general rank one quotient.

*Proof.* Theorem 7.4 shows that the intersection  $M(10, 4) \cap \overline{M}(10)$  is nonempty. It is the unique 19-dimensional irreducible component of  $M(10, 4)^{\text{sing}}$ , containing the usual points. Starting with a general point  $F \in M(10, 4) \cap \overline{M}(10)$  and taking an additional general rank 1 quotient  $S_x$ , the subsheaf  $F'$  gives a point in  $M(11, 4) \cap \overline{M}(11, 10)$ .  $\square$

Let  $Y \subset M(10, 4)$  be the unique 19-dimensional irreducible component of the singular locus  $M(10, 4)^{\text{sing}}$ . It contains a dense open set where the quotient  $S$  is a direct sum of six quotients  $(x_i, s_i)$  of rank 1. Choose a quasi-finite surjection  $Y' \rightarrow Y$  such that  $(x_i, s_i)$  are well defined as functions  $Y' \rightarrow \mathbb{P}(E)$ .

Forgetting the quotients and considering only the bundle  $E$  gives a map  $Y' \rightarrow M(4)$ . Fix a bundle  $E$  in the image of  $Y' \rightarrow M(4)$ . Let  $Y'_E$  denote the fiber of  $Y'$  over  $E$ , which has dimension  $\geq 17$ .

We claim that for any  $0 \leq k \leq 5$ , there exists a choice of  $k$  out of the 6 points such that the map  $Y'_E \rightarrow \mathbb{P}(E)^4$  is surjective. For  $k = 0$  this is automatic, so assume that  $k \leq 4$  and it is known for  $k$ ; we need to show that it is true for  $k+1$  points. Reorder so that the  $k$  points to be chosen, are the first ones. For a general point  $q \in \mathbb{P}(E)^k$ , let  $Y'_{E,q}$  denote the fiber of  $Y'_E \rightarrow \mathbb{P}(E)^k$  over  $q$ . We have  $\dim(Y'_{E,q}) \geq 17 - 3k$ . We get an injection

$$Y'_{E,q} \rightarrow \mathbb{P}(E)^{6-k}.$$

Suppose that the image mapped into a proper subvariety of each factor; then it would map into a subvariety of dimension  $\leq 2(6-k)$ , which would give  $\dim(Y'_{E,q}) \leq 12 - 2k$ . However, for  $k \leq 4$  we have  $12 - 2k < 17 - 3k$ , a contradiction. Therefore, at least one of the projections must be a surjection  $Y'_{E,q} \rightarrow \mathbb{P}(E)$ . Adding this point to our list, gives a list of  $k+1$  points such that the map  $Y'_E \rightarrow \mathbb{P}(E)^{k+1}$  is surjective. This completes the induction, yielding the following lemma.

**Lemma 8.2.** *Suppose  $Y \subset M(10, 4)$  is as above. Then for a fixed bundle  $E \in M(4)$  corresponding to some points in  $Y$ , and for a general point in the fiber  $Y_E$  over  $E$ , some 5 out of the 6 quotients correspond to a general point of  $\mathbb{P}(E)^5$ .*

$\square$

**Lemma 8.3.** *Suppose  $F$  is the torsion-free sheaf parametrized by a general point of  $Y$ , and let  $F'$  be defined by an exact sequence*

$$0 \rightarrow F' \rightarrow F \xrightarrow{(x_7, s_7)} S_7 \rightarrow 0$$

where  $S_7$  has length 1 and  $(x_7, s_7)$  is general (with respect to the choice of  $F$ ) in  $\mathbb{P}(E)$ . Then  $F'$  has no nontrivial co-obstructions:  $\text{Hom}(F', F'(1)) = 0$ .

*Proof.* The space of co-obstructions for the bundle  $E$  has dimension 6. Imposing a condition of compatibility with a general rank-1 quotient  $(x_i, s_i)$  cuts down the dimension of the space of co-obstructions by at least 1.

By Lemma 8.2 above, we may assume after reordering that the first five points  $(x_1, s_1), \dots, (x_5, s_5)$  constitute a general vector in  $\mathbb{P}(E)^5$ . Adding the 7th general point given by the statement of the proposition, we obtain a general point  $(x_1, s_1), \dots, (x_5, s_5), (x_7, s_7)$  in  $\mathbb{P}(E)^6$ . As this 6-tuple of points is general with respect to  $E$ , it imposes vanishing on the 6-dimensional space of co-obstructions, giving  $\text{Hom}(F', F'(1)) = 0$ .  $\square$

**Corollary 8.4.** *There exists a point*

$$F' \in \overline{M(11, 10)} \cap M(11, 4)$$

in the boundary of  $\overline{M(11)}$ , such that  $F$  is a smooth point of  $\overline{M(11)}$ .

*Proof.* By Lemma 8.3, choosing a general quotient  $(x_7, s_7)$  gives a torsion-free sheaf  $F'$  with no co-obstructions, hence corresponding to a smooth point of  $\overline{M(11)}$ . By construction we have  $F' \in \overline{M(11, 10)} \cap M(11, 4)$ .  $\square$

**Theorem 8.5.** *The moduli space  $\overline{M(11)}$  is irreducible.*

*Proof.* Suppose  $Z$  is an irreducible component. Then  $Z$  meets the boundary in a codimension 1 subset; but by looking at Table 2, there are only two possibilities:  $\overline{M(11, 10)}$  and  $M(11, 4)$ . The co-obstructions vanish for general points of  $M(10, 4)$  since those correspond to 6 general quotients of rank 1, and the co-obstructions vanish for general points of  $\overline{M(10)}$  by goodness. It follows that there are no co-obstructions at general points of  $\overline{M(11, 10)}$  or  $M(11, 4)$ , so each of these is contained in at most a single irreducible component of  $\overline{M(11)}$ . However, in the previous corollary, there is a unique irreducible component containing  $F'$ , which shows that the irreducible components containing  $\overline{M(11, 10)}$  and  $M(11, 4)$  must be the same. Hence,  $\overline{M(11)}$  has only one irreducible component.  $\square$

The cases  $c_2 \geq 12$  are now easy to treat.

**Theorem 8.6.** *For any  $c_2 \geq 12$ , the moduli space  $\overline{M}(c_2)$  of stable torsion-free sheaves of degree 1 and second Chern class  $c_2$ , is irreducible.*

*Proof.* By Corollary 4.5, any irreducible component of  $\overline{M}(c_2)$  meets the boundary in a subset of codimension 1. However, for  $c_2 \geq 12$ , the only stratum of codimension 1 is  $M(c_2, c_2 - 1)$ . By induction on  $c_2$ , starting at  $c_2 = 11$ , we may assume that  $M(c_2, c_2 - 1)$  is irreducible. Furthermore, if  $E$  is a general point of  $M(c_2 - 1)$  then  $E$  admits no co-obstructions, since  $M(c_2 - 1)$  is good. Hence, a general point  $F$  in  $M(c_2, c_2 - 1)$ , which is the kernel of a general length-1 quotient  $E \rightarrow S$ , doesn't admit any co-obstructions either. Therefore,  $\overline{M}(c_2)$  is smooth at a general point of  $M(c_2, c_2 - 1)$ . Thus, there is a unique irreducible component containing  $M(c_2, c_2 - 1)$ , which completes the proof that  $\overline{M}(c_2)$  is irreducible.  $\square$

We have finished proving our main statement, Theorem 1.1 of the introduction: for any  $c_2 \geq 4$ , the moduli space  $M(c_2)$  of stable vector bundles of degree 1 and second Chern class  $c_2$  on a very general quintic hypersurface  $X \subset \mathbb{P}^3$ , is nonempty and irreducible.

For  $4 \leq c_2 \leq 9$ , this is shown in [14]. For  $c_2 = 10$  it is Theorem 7.3, for  $c_2 = 11$  it is Theorem 8.5, and  $c_2 \geq 12$  it is Theorem 8.6. Note that for  $c_2 \geq 16$  it is Nijse's theorem [17].

It was shown in [14] that the moduli space is good for  $c_2 \geq 10$  (shown by Nijse for  $c_2 \geq 13$ ), and from Table 1 we see that it isn't good for  $4 \leq c_2 \leq 9$ . The moduli space of torsion-free sheaves  $\overline{M}(c_2)$  is irreducible for  $c_2 \geq 11$ , as may be seen by looking at the dimensions of boundary strata in Table 2. Whereas  $M(4) = \overline{M}(4)$  is irreducible, the dimensions of the strata in Table 2 imply that  $\overline{M}(c_2)$  has several irreducible components for  $5 \leq c_2 \leq 9$ , although we haven't answered the question as to their precise number. By Theorem 7.4,  $\overline{M}(10)$  has two irreducible components  $\overline{M}(10)$  and  $M(10, 4)$ .

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