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# BIFURCATION ANALYSIS OF DAMPED VISCO-ELASTIC PLANAR BEAMS UNDER SIMULTANEOUS GRAVITATIONAL AND FOLLOWER FORCES

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The mechanical behavior of a non-conservative non-linear beam, internally and externally damped, undergoing codimension-1 (static or dynamic) and codimension-2 (double-zero) bifurcations, is analyzed. The system consists of a purely flexible, planar, visco-elastic beam, fixed at one end, loaded at the tip by a follower force and a dead load, acting simultaneously. An integro-differential equation of motion in the transversal displacement, with relevant boundary conditions, is derived. Then, the linear stability diagram of the trivial rectilinear configuration is built-up in the space of the two loading parameters. Attention is then focused on the double-zero bifurcation, for which a post-critical analysis is carried out without any *a-priori* discretization. An adapted version of the Multiple Scale Method, based on a fractional series expansion in the perturbation parameter, is employed to derive the bifurcation equations. Finally, bifurcation charts are evaluated, able to illustrate the system behavior around the codimension-2 bifurcation point.

*Keywords:* Bifurcations; damping effects on stability; non-conservative loads; multiple scale method; visco-elastic beam.

## 1. Introduction

Stability of columns subjected to follower forces, after the pioneering paper by Beck, Ref. 1, has recently attracted the attention of many researchers, particularly in aerospace, where tangential forces are produced by jets and rocket motors (see Refs. 2, 3). In many studies, damping is mainly considered as an *external* cause, resulting from the interaction occurring between the structure and the surrounding air. However, damping is also due by an *internal* dissipation of the material, to

be modelled by a proper rheological model. A general treatment of the effect of distributed, internal and external small dampings, on the linear stability of a continuous beam under non-conservative loads, can be found in Ref. 4.

Stability is strongly influenced by damping, which is responsible of a well-known phenomenon, called “the destabilization paradox” (see Ref. 5), according to which the loss of stability of a non-conservative system with vanishingly small damping occurs at a load significantly lower than the critical value relevant to the undamped system.

On the other hand, follower forces can also act simultaneously to gravitational forces, as for example analyzed in Refs. 6 and 7, this interaction resulting in richer bifurcation scenarios. Therefore, it seems interesting, to investigate nonlinear systems under both types of forces, and in presence of both types of damping, in order to analyze the mutual interactions.

In this paper bifurcations of beams, internally and externally damped, loaded by a gravitational and a follower force acting simultaneously, are analysed. The paper is thus organized. In Section 2 the equations of motions are illustrated. In Section 3 the critical scenario is depicted. In Section 4 a post-critical analysis around a double-zero bifurcation point is carried out, similarly to what was done in Refs. 8–10. Finally, in Section 5 some conclusions are drawn.

## 2. Model

A planar beam is considered, fixed at the end  $A$  and, simultaneously loaded at the tip  $B$  by a follower force of intensity  $F$  (tangential to the actual configuration of the beam axis) and by a dead load of intensity  $P$  (acting in the direction of the originally rectilinear axis, Fig. 1). The material behavior of the beam obeys to the Kelvin-Voigt rheological model, with elastic modulus  $E$  and viscous coefficient  $\eta$  (acting as an internal damping); moreover, the beam is considered to lie on a purely viscous linear soil of constant  $c$  (simulating the external damping). The beam is assumed to be inextensible and shear-indeformable.

The following nondimensional equations in the variable  $u(s, t)$  are derived:

$$\begin{aligned} \ddot{u} + u^{IV} + \left[ u'(u'u'')' \right]' + \alpha \dot{u}^{IV} + \alpha \left\{ \left[ u'(u'u'')' \right]' \right\}^\bullet \\ + 2 \left[ \mu \left( 1 - \frac{u_B'^2}{2} \right) + \nu \right] u'' + \beta \dot{u} + \frac{1}{2} \left[ \left( \int_1^s \left( \int_0^s u'^2 ds \right) ds \right)'' \right] u' = 0, \\ u_A = 0, \quad u_A' + \frac{1}{6} u_A^3 = 0, \\ - (u_B''' + u_B'' u_B'^2 + u_B'' u_B') - 2 \left( \nu - \mu \frac{u_B'^2}{2} \right) u_B' - \alpha (u_B''' + u_B'' u_B'^2 + u_B'' u_B')^\bullet = 0, \\ (u_B'' + u_B' u_B'^2) + \alpha (u_B'' + u_B' u_B'^2)^\bullet = 0 \end{aligned} \quad (1)$$

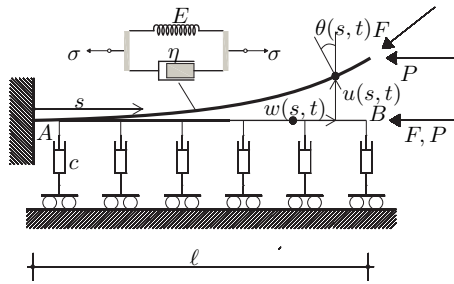


Fig. 1. Visco-elastic beam on viscous soil under follower force and dead load: model and displacements.

in which a dash denotes differentiation with respect to  $s$ , a dot denotes time-differentiation, the index  $A$  evaluation at  $s = 0$ , the index  $B$  evaluation at  $s = \ell$ , and the following non-dimensional quantities have been introduced:  $\omega^2 = EI/ml^4$ ,  $\alpha = \eta\omega/E$ ,  $\beta = c\omega\ell^4/EI$ ,  $\mu = F\ell^2/2EI$ ,  $\nu = P\ell^2/2EI$ .

### 3. Linear Stability Diagram

The linear stability diagram of the beam, is depicted in Fig. 2. All quadrants of the  $(\nu, \mu)$ -plane have been displayed, to account for both tensile/compressive forces. The divergence locus consists in a family of curves (independent of damping) labelled with  $\mathcal{D}$ . An additional straight line  $\mathcal{N}$ , of equation  $\mu = -\nu$ , corresponding to unstressed beams in the undeformed configuration, is founded, but it is *not* a bifurcation locus, since the transversality condition of the eigenvalues is not satisfied on it. Hopf locus consists in a second family of curves labelled with  $\mathcal{H}$  (depending on damping). Of this family, the curves relevant to the undamped system ( $\alpha = \beta = 0$ , also referred as *circulatory system*) have been denoted by  $\mathcal{H}^u$ , and curves relevant to a slightly damped sample system, ( $\alpha = 0.01, \beta = 0.1$ ) by  $\mathcal{H}^d$ . Loci  $\mathcal{D}$  and  $\mathcal{H}$  represent codimension-1 bifurcations. The divergence-locus  $\mathcal{D}$  intersects the  $\nu$ -axis at points  $E_1, E_2, \dots$ , each corresponding to an Eulerian critical load,  $\nu_{E_1} = \pi^2/8$ ,  $\nu_{E_2} = 9\pi^2/8, \dots$ . The Hopf-locus  $\mathcal{H}$  intersects the  $\mu$ -axis at the Beck's loads. Only the lower intersection is depicted in Fig. 2, both for the undamped system,  $B_1^u := (0, 10.025)$ , and for the damped system,  $B_1^d := (0, 6.464)$ . Consistently with the destabilizing effect phenomenon, a small amount of damping considerably reduces the critical load.

The Hopf-curves die at intersections with the divergence curves, according to the well-known mechanism of the double-zero (or Takens-Bogdanov) bifurcation. Such codimension-2 bifurcation occurs at points  $DZ_1^u := (5.51, 3.02)$ ,  $DZ_2^u := (45.87, 11.76)$ , in the undamped case, and  $DZ_1^d := (2.88, 2.81)$ ,  $DZ_2^d := (22.26, 22.26)$ , in the damped case. The stable zone of the  $(\nu, \mu)$ -plane is denoted in grey in the figure; it undergoes a contraction when a small damping is added.

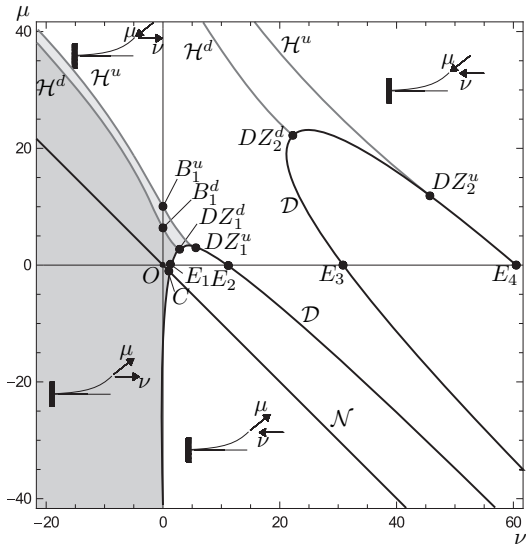


Fig. 2. Linear stability diagram for the undamped system ( $\alpha, \beta = 0$ ;  $u$  superscript) and a sample damped system ( $\alpha = 0.01, \beta = 0.1$ ;  $d$  superscript);  $\mathcal{D}$ : divergence locus,  $\mathcal{H}$ : Hopf locus,  $\mathcal{N}$ : zero-stress locus;  $E$ : Eulerian bifurcations;  $B$ : Beck's bifurcations;  $DZ$ : double-zero bifurcations.

#### 4. Bifurcation Analysis Around a Double-Zero Point

A nonlinear bifurcation analysis is carried out around a double-zero bifurcation point. The partial, integro-differential equations (1) are directly attacked by an adapted version of the Multiple Scale Method and the bifurcation equation for double-zero bifurcation, in the Bogdanov normal form, is obtained.

Two sample systems have been considered: S1 for which  $\alpha = 0.01, \beta = 0.1$  and S2 for which  $\alpha = 0.01, \beta = 10$ . Results relevant to S1 are reported in Fig. 3(a), displaying the bifurcation chart in the neighborhood of double-zero point and sketches of the two-dimensional phase-plane ( $a, \dot{a}$ ) for the bifurcation equation,

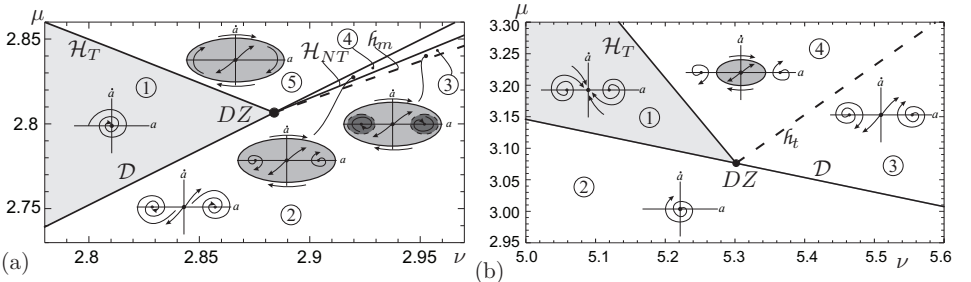


Fig. 3. Bifurcation chart in a small neighborhood around the bifurcation point and sketches of the phase-plan for: (a) system S1; (b) system S2.

in each significant region. In region 1 the trivial solution is stable; due to the supercritical static bifurcation, it loses stability in region 2, where two stable nontrivial equilibria take place; due to supercritical Hopf bifurcation, it loses stability in region 5, where a stable limit cycle exists, causing periodic motion of the beam. In region 4 two equilibria appear, but, in spite of the supercritical character of the static bifurcation, they are unstable, as an effect of the interaction with the dynamic bifurcation; in region 3 two small unstable limit cycles arise, which render stable the nontrivial equilibria. Then, at the straight line  $\mu_m$ , a homoclinic bifurcation occurs; after that, all cycles disappear, so that only stable equilibria survive in region 2.

Results relevant to system S2 are reported in Fig. 3(b). The static bifurcation is subcritical and the bifurcated equilibria do not suffer Hopf bifurcation, so that no a curve  $\mathcal{H}_{NT}$  exists and, consequently, no homoclinic bifurcation  $\mu_m$  occurs. In contrast, a new heteroclinic bifurcation  $\mu_t$  manifests itself. In region 1 the trivial equilibrium is stable, but two unstable equilibrium points coexist. In region 2 the trivial equilibrium loses stability by divergence, and no other local attractors exist. In region 4 the equilibrium loses stability by supercritical Hopf bifurcation, giving rise to a stable limit cycle internal to the nontrivial equilibria. In region 5, however, due to a heteroclinic bifurcation, the cycle itself disappears.

## 5. Conclusions

A nonlinear, visco-elastic, externally damped beam, subjected to two independent axial loads, one gravitational, the other tangential, has been studied.

The position of the bifurcation point and the angle of attack between the incident, divergence and Hopf, bifurcation loci depend on the damping coefficients. The properties of the undamped system are recovered only for evanescent external damping, not for internal damping, this case being *in discontinuity* with the circulatory case. Therefore, some new features of the well-known “destabilization paradox” are revealed.

Also the nonlinear scenario around the double-zero bifurcation is strongly affected by damping. When the external damping is small, the static bifurcation is *supercritical*. In contrast, when the external damping is large, the static bifurcation is *subcritical*. In the whole range studied, instead, the Hopf bifurcation has supercritical character.

The interaction between static and dynamic bifurcations manifests itself via *homoclinic* or *heteroclinic* bifurcations, due to the collision between limit cycles and equilibria, or between cycles.

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