Computing the torsion of the p-ramified module

Frédéric Pitoun, Firmin Varescon

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Abstract. We fix a prime number p and K a number field, we denote by M the maximal abelian p-extension of K unramified outside p. The aim of this paper is to study the $\mathbb{Z}_p$-module $\mathfrak{X} = \mathrm{Gal}(M/K)$ and to give a method to effectively compute its structure as a $\mathbb{Z}_p$-module. Then we give numerical results, for real quadratic fields, cubic fields and quintic fields, together with interpretations via Cohen-Lenstra's heuristics.

1 Introduction

We fix a prime number p and a number field K. We denote by M the maximal abelian p-extension of K unramified outside p. The aim of this paper is to study the $\mathbb{Z}_p$-module $\mathfrak{X} = \mathrm{Gal}(M/K)$ and give an algorithm to compute its $\mathbb{Z}_p$-structure. This module is described by the following exact sequence from class field theory ([Gra1, p. 294])

$$\overline{U}_K \longrightarrow \prod_{v|p} U_v^1 \longrightarrow \mathfrak{X} \longrightarrow \mathrm{Gal}(\mathcal{H}/K) \longrightarrow 1, \qquad (1)$$

where $\overline{U_K}$ is the pro-p-completion of the group of units U_K, U_v^1 is the group of principal units at the place v above p of K, and $\mathcal{H}$ is the maximal p-sub-extension of the Hilbert class field of K. Leopold't conjecture for K and p is equivalent to injectivity of $\overline{U}_K \to \prod_{v|p} U_v^1$. Therefore, from this exact sequence, we deduce that the $\mathbb{Z}_p$-rank r of $\mathfrak{X}$ is greater or equal to $r_2 + 1$ and is equal $r_2 + 1$ if and only if K and p satisfy Leopoldt's conjecture. Hence $\mathfrak{X}$ is the direct product of a free part isomorphic to $\mathbb{Z}_p^r$ and of a torsion part, that we denote by $\mathcal{T}_p$. Our algorithm checks if K satify Leopoldt's conjecture at p and then compute the torsion $\mathcal{T}_p$.

We propose a method which is based on the fact that the $\mathbb{Z}_p$-module $\mathfrak{X}$ is the projective limit of the p-parts of the ray class groups modulo p^n, $\mathcal{A}_{p^n}(K)$. We then study the stabilization of these groups with respect to n and the behaviour of invariants of $\mathcal{A}_{p^n}(K)$, as n is increasing. This approach leads us to our algorithm.

Before addressing the technical part of this article, we recall the definition and some basic properties of the ray class groups modulo p^n. Then, we use our algorithm to compute some cases and propose an heuristic explanation of the statistical data, using the Cohen-Lenstra philosophy ([C-L]).

2 Background from class field theory ([Gra1],[Ser])

In this section, we recall the basic notions from class field theory that we will need later. We fix v a place of K above p and π_v a local uniformiser of K_v, the completion of K at v.

Definition 2.1.

1. *The conductor of an abelian extension of local fields L_v/K_v is the minimum of integers c such that $U_v^c \subset N_{L_v/K_v}(L_v^\times)$ (we recall that $U_v^c = 1 + (\pi_v^c)$ and we use the convention $U_v^0 = U_v$).*

2. *The conductor of an abelian extension L/K of a global field is the ideal $\mathfrak{m} = \prod_v \mathfrak{p}_v^{c_v}$, where v runs through all finite places of K and where c_v is the conductor of the local extension L_v/K_v.([Gra1, p. 126-127] Theorem and Definition 4.1 + Lemma 4.2.1).*

We start with 2 lemmas.

Lemma 2.2. *([Ser] p. 219). Let K_v be the completion of K at the valuation v normalized by $v(p) = 1$ and $v(\pi_v) = \frac{1}{e_v}$, where e_v is the ramification index of the extension $K_v/\mathbb{Q}_p$. If $m > \frac{e_v}{p-1}$, then the application $x \to x^p$ is an isomorphism from U_v^m to $U_v^{m+e_v}$.*

Lemma 2.3. *Let $K_v \subset L_v \subset M_v$ be a tower of extensions of $\mathbb{Q}_p$, such that the extension M_v/K_v is abelian and the extension M_v/L_v is of degree p. We denote respectively by $c_{M,v}$ and $c_{L,v}$ the conductors of the extensions M_v/K_v and L_v/K_v. If $c_{L,v} > \frac{e_v}{p-1}$, then we have*

$$c_{M,v} \leq c_{L,v} + e_v.$$

Proof. By definition $c_{L,v}$ is the smallest integer n such that $U_v^n \subset N_{L_v/K_v}(L_v^\times)$. Local class field theory gives the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_{M_v/K_v}(M_v^\times) & \longrightarrow & K_v^\times & \longrightarrow & \mathrm{Gal}(M_v/K_v) & \longrightarrow & 1 \\ & & \downarrow & & \| & & \downarrow & & \\ 1 & \longrightarrow & N_{L_v/K_v}(L_v^\times) & \longrightarrow & K_v^\times & \longrightarrow & \mathrm{Gal}(L_v/K_v) & \longrightarrow & 1 \end{array}$$

Applying snake lemma we get the exact sequence

$$1 \longrightarrow N_{M_v/K_v}(M_v^\times) \longrightarrow N_{L_v/K_v}(L_v^\times) \longrightarrow \mathrm{Gal}(M_v/L_v) = \mathbb{Z}/p\mathbb{Z} \longrightarrow 1.$$

Consequently $N_{M_v/K_v}(M_v^\times)$ is a subgroup of index p of $N_{L_v/K_v}(L_v^\times)$. Let $n \in \mathbb{N}, n \geq c_{L,v} + e_v$ and $x \in U_v^n$. We have to show that $x \in N_{M_v/K_v}(M_v^\times)$. By lemma 2.2, $x^{\frac{1}{p}}$ is a well defined element of $\in U_v^{n-e_v}$. Yet $n - e_v \geq c_{L,v}$ therefore $x^{\frac{1}{p}} \in N_{L_v/K_v}(L_v^\times)$. Now, as $N_{M_v/K_v}(M_v^\times)$ is of index p in $N_{L_v/K_v}(L_v^\times)$, we deduce that $x \in N_{M_v/K_v}(M_v^\times)$. We have therefore $U_v^n \subset N_{M_v/K_v}(M_v^\times)$ for all integers n such that $n \geq c_L + e_v$. By definition of the conductor, this proves $c_{M,v} \leq c_{L,v} + e_v$.

□

Definition 2.4. *Let n be a positive integer. We denote by*

- *H the maximal abelian unramified extension of K;*
- *H_{p^n} the compositum of all abelian extensions of K whose conductors divide p^n;*
- *$\mathcal{H}_{p^n}$ the compositum of all abelian p-extensions of K whose conductors divide p^n;*
- *M the maximal extension of K which is abelian and unramified outside p.*

So the Galois groups $\mathrm{Gal}(\mathcal{H}/K)$ and $\mathrm{Gal}(\mathcal{H}_{p^n}/K)$ are respectively isomorphic to the p-parts of $\mathrm{Gal}(H/K)$ and $\mathrm{Gal}(H_{p^n}/K)$.

Proposition 2.5. *([Gra1, p. 47] corollary 5.1.1) We have the following exact sequences,*

$$1 \longrightarrow K^\times \prod_{v\nmid p} U_v \prod_{v|p} U_v^{ne_v} \longrightarrow \mathcal{I}_K \longrightarrow \mathrm{Gal}(H_{p^n}/K) \longrightarrow 1$$

$$1 \longrightarrow K^\times \prod_{v} U_v \longrightarrow \mathcal{I}_K \longrightarrow \mathrm{Gal}(H/K) \longrightarrow 1,$$

where $\mathcal{I}_K$ is the group of idèles of K.

We denote the Galois group $\mathrm{Gal}(\mathcal{H}_{p^n}/K)$ by $\mathcal{A}_{p^n}(K)$. It is the p-part of the Galois group $\mathrm{Gal}(H_{p^n}/K)$ which, in turn, is isomorphic to the ray class group modulo p^n of K. By definition, we have a natural inclusion $\mathcal{H}_{p^n} \subset \mathcal{H}_{p^{n+1}}$, the union $\bigcup_n \mathcal{H}_{p^n}$ is equal to M and the projective limit $\varprojlim \mathcal{A}_{p^n}(K)$ is canonically isomorphic to $\mathfrak{X}$.

Proposition 2.6. *For any integer $n > 0$, the Galois groups of the extensions M and H_{p^n} of K are related by the following exact sequence*

$$1 \longrightarrow U_K^{(p^n)} \longrightarrow \prod_{v|p} U_v^{ne_v} \longrightarrow \mathrm{Gal}(M/K) \longrightarrow \mathrm{Gal}(H_{p^n}/K) \longrightarrow 1,$$

where $U_K^{(p^n)} = \{u \in U_K \text{ such that } \forall v|p, \quad u \in U_v^{ne_v}\}$ and

$$\overline{U}_K^{(p^n)} \longrightarrow \prod_{v|p} U_v^{ne_v} \longrightarrow \mathfrak{X} \longrightarrow \mathcal{A}_{p^n}(K) \longrightarrow 1,$$

where $\overline{U}_K^{(p^n)}$ is the pro-p-completion of $U_K^{(p^n)}$, i.e $\varprojlim_m U_K^{(p^n)}/p^m$. If Moreover K and p satisfy Leopoldt' conjecture, then $\overline{U}_K^{(p^n)} \to \prod_{v|p} U_v^{ne_v}$ is injective.

Proof. To obtain the second exact sequence, we apply pro-p-completion process to the first. Note that injectivity of $\overline{U}_K^{(p^n)} \to \prod_{v|p} U_v^{ne_v}$ is equivalent to Leopoldt's conjecture. Now we prove exactness of the first sequence.
From the definition of the extensions M and H_{p^n}, we deduce the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K^\times \prod_{v\nmid p} U_v \prod_{v|p} 1 & \longrightarrow & \mathcal{I}_K & \longrightarrow & \mathrm{Gal}(M/K) & \longrightarrow & 1 \\
 & & \downarrow & & \| & & \downarrow & & \\
1 & \longrightarrow & K^\times \prod_{v\nmid p} U_v \prod_{v|p} U_v^{ne_v} & \longrightarrow & \mathcal{I}_K & \longrightarrow & \mathrm{Gal}(H_{p^n}/K) & \longrightarrow & 1
\end{array}$$

It follows immediately from the snake lemma that

$$\ker(\mathrm{Gal}(M/K) \to \mathrm{Gal}(H_{p^n}/K)) = (K^\times \prod_{v\nmid p} U_v \prod_{v|p} U_v^{ne_v})/(K^\times \prod_{v\nmid p} U_v \prod_{v|p} 1).$$

Now, we define the application

$$\theta : (K^\times \prod_{v\nmid p} U_v \prod_{v|p} U_v^{ne_v}) \to (\prod_{v|p} U_v^{ne_v})/U_K^{(p^n)},$$

by setting for $k(u_v)_v \in K^\times \prod_{v\nmid p} U_v \prod_{v|p} U_v^{ne_v}$, $\theta(k(u_v)_v) = \overline{(u_v)_{v|p}}$, where $\overline{(u_v)_{v|p}}$ is the class of $(u_v)_{v|p}$ in $(\prod_{v|p} U_v^{ne_v})/U_K^{(p^n)}$.

We first check that the application θ is well defined, i.e. that if $k(u_v)_v = k'(u'_v)_v$ in $K^\times \prod_{v\nmid p} U_v \prod_{v|p} U_v^{ne_v}$, then $\theta(k(u_v)_v) = \theta(k'(u'_v)_v)$. By definition, for all v, $k(u_v)_v = k'(u'_v)_v$ if and only if $i_v(k)u_v = i_v(k')u'_v$, where i_v is the embedding of K in K_v. We deduce that for all v, $i_v(k'k^{-1}) \in U_v$ and that for all $v|p$, $i_v(k'k^{-1}) \in U_v^{ne_v}$. So we get $k'k^{-1} \in U_K^{(p^n)}$ and $\overline{(u_v)_{v|p}} = \overline{(u'_v)_{v|p}}$.

It is clear that $(K^\times \prod_{v\nmid p} U_v \prod_{v|p} 1) \subset \ker(\theta)$ and that the application θ is surjective. We will show that $(K^\times \prod_{v\nmid p} U_v \prod_{v|p} 1) = \ker(\theta)$. Let $k(u_v) \in \ker(\theta)$, there exists $x \in U_K^{(p^n)}$ such that for all $v|p$, $u_v = i_v(x)$. We consider the element $x(u'_v)_v$, where $u'_v = 1$ if $v|p$ and $u'_v = i_v(x)^{-1}u_v$ if $v \nmid p$. We have $(u_v)_v = x(u'_v)_v \Rightarrow k(u_v)_v = kx(u'_v)_v$ and as $kx(u'_v)_v \in (K^\times \prod_{v\nmid p} U_v \prod_{v|p} 1)$, we have $\ker(\theta) \subset (K^\times \prod_{v\nmid p} U_v \prod_{v|p} 1)$ and finally

$$(K^\times \prod_{v\nmid p} U_v \prod_{v|p} U_v^{ne_v})/(K^\times \prod_{v\nmid p} U_v \prod_{v|p} 1) \simeq (\prod_{v|p} U_v^{ne_v})/U_K^{(p^n)}.$$

The result follows. □

3 Explicit Computation of $\mathcal{T}_p$

In this section, we present our method to check that K verify Leopoldt's conjectue at p and then to compute $\mathcal{T}_p$. The main point is that, for n large enough, $\mathcal{A}_{p^n}(K)$ determines $\mathfrak{X}$.

3.1 Stabilization of $\mathcal{A}_{p^n}(K)$

For simplicity we note $Y_n = \ker(\mathcal{A}_{p^{n+1}}(K) \to \mathcal{A}_{p^n}(K))$. Let $\tilde{K}$ be the compositum of all the $\mathbb{Z}_p$-extensions of K. We denote by r the $\mathbb{Z}_p$-rank de $\mathfrak{X}$, so $r \geq r_2 + 1$.

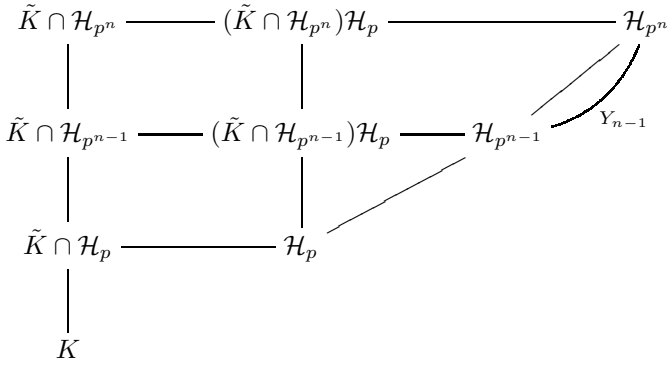
Proposition 3.1. *There exists n_0 such that $\tilde{K} \cap \mathcal{H}_{p^{n_0}}/\tilde{K} \cap \mathcal{H}_p$ is ramified at all places above of p and for all $n \geqslant n_0$, Y_n surjects on $(\mathbb{Z}/p\mathbb{Z})^r$.*

Before proving the proposition we need a lemma.

Lemma 3.2. *If the extension $\tilde{K} \cap \mathcal{H}_{p^n}/\tilde{K} \cap \mathcal{H}_p$ is ramified at a place v above p, then $c_{n,v} > \frac{e_v}{p-1}$, where $c_{n,v}$ is the conductor of the local extension $(\tilde{K} \cap \mathcal{H}_{p^n})_w/K_v$.*

Proof of the Lemma 3.2. As M contains the cyclotomic $\mathbb{Z}_p$-extension, there exists a n_0 such that $\tilde{K} \cap \mathcal{H}_{p^{n_0}}/\tilde{K} \cap \mathcal{H}_p$ is ramified at all places v above of p. As $\tilde{K} \cap \mathcal{H}_{p^{n_0}}/\tilde{K} \cap \mathcal{H}_p$ is ramified at v then, for $n \geq n_0$, $\tilde{K} \cap \mathcal{H}_{p^n}/\tilde{K} \cap \mathcal{H}_p$ is ramified at v, so that there exists m such that $n \geq m \geq 2$ and that $\tilde{K} \cap \mathcal{H}_{p^{m-1}}/\tilde{K} \cap \mathcal{H}_p$ is unramified at v and such that $\tilde{K} \cap \mathcal{H}_{p^m}/\tilde{K} \cap \mathcal{H}_p$ is ramified at v. Then, the local conductor $c_{m,v}$ is greater than $(m-1)e_v$, yet $m \geq 2$ so $c_{m,v} > (m-1)e_v \geqslant e_v \geqslant \frac{e_v}{p-1}$. As the conductor of the local extension $\tilde{K} \cap \mathcal{H}_{p^m}/K$ divides the conductor of $\tilde{K} \cap \mathcal{H}_{p^n}/K$, we have $c_{n,v} \geqslant cm, v > \frac{e_v}{p-1}$. □

Proof of the Proposition 3.1. We consider the following diagram.



(2)

We have $\mathrm{Gal}(\tilde{K}/K) = \mathbb{Z}_p^r$. It is clear that $Y_n \twoheadrightarrow \mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^{n+1}}/\tilde{K} \cap \mathcal{H}_{p^n})$. Yet $\mathrm{Gal}(\tilde{K}/\tilde{K} \cap \mathcal{H}_{p^n})$ is a $\mathbb{Z}_p$-sub-module of $\mathrm{Gal}(\tilde{K}/K) = \mathbb{Z}_p^r$ of finite index, therefore it is isomorphic to $\mathbb{Z}_p^r$. Hence there exists r extensions, say $M_1, M_2, \cdots, M_r$ of $\tilde{K} \cap \mathcal{H}_{p^n}$, contained in $\tilde{K}$ such that $\mathrm{Gal}(M_i/\tilde{K} \cap \mathcal{H}_{p^n}) \simeq \mathbb{Z}/p\mathbb{Z}$ and $\mathrm{Gal}(M_1 \cdots M_r/\tilde{K} \cap \mathcal{H}_{p^n}) \simeq (\mathbb{Z}/p\mathbb{Z})^r$. Yet the conductor of the extension $\tilde{K} \cap \mathcal{H}_{p^n}/K$ divides $p^n = \prod_{v|p} \mathfrak{p}_v^{ne_v}$. Moreover the hypothesis on $\tilde{K} \cap \mathcal{H}_{p^n}/\tilde{K} \cap \mathcal{H}_p$ ensures that we can use Lemma 2.3 and consequently the conductor of the extension M_i/K divides $\prod_{v|p} \mathfrak{p}_v^{ne_v+e_v} = p^{n+1}$, i.e. $M_i \subset \mathcal{H}_{p^{n+1}}$ for all $i \in \{1, \cdots, r\}$. Hence the map is surjective. □

We deduce immediately the corollary.

Corollary 3.3. *We assume that for a naturel number n the extension $\tilde{K} \cap \mathcal{H}_{p^n}/\tilde{K} \cap \mathcal{H}_p$ is ramified at all places above of p, and that the cardinal of Y_n is exactly p^{r_2+1}. Then $Y_n \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}$ and K verify the Leopoldt's conjecture at p.*

From now on, as we can numerically check that K sastify the Leopoldt's conjecture at p, we assume it, and we use it to compute $\mathcal{T}_p$. Note that if the Leopodt's conjecture is false, then $r > r_2 + 1$ and our algorithm never stops.

Corollary 3.4. *We assume that, for some integer n such that the extension $\tilde{K} \cap \mathcal{H}_{p^n}/\tilde{K} \cap \mathcal{H}_p$ is ramified at all places above of p, the cardinal of Y_n is exactly p^{r_2+1}. Then, $Y_n \simeq \mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^{n+1}}/\tilde{K} \cap \mathcal{H}_{p^n})$.*

It remains to check that if $Y_{n_0} \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}$ for some n_0, then $Y_n \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}$ for all integer $n \geq n_0$. For this purpose, we consider the exact sequence defining the p-part of the ray class group.

$$1 \longrightarrow \overline{U}_K^{(p^n)} \longrightarrow \prod_{v|p} U_v^{ne_v} \longrightarrow \mathfrak{X} \longrightarrow \mathcal{A}_{p^n}(K) \longrightarrow 1,$$

and we note $\mathcal{Q}_n = \prod_{v|p} U_v^{ne_v}/\overline{U}_K^{(p^n)}$. We have $\mathcal{Q}_n = \mathrm{Gal}(M/\mathcal{H}_{p^n})$ and consequently $\mathcal{Q}_n/\mathcal{Q}_{n+1} = Y_n \simeq \mathrm{Gal}(\mathcal{H}_{p^{n+1}}/\mathcal{H}_{p^n})$.

Proposition 3.5. *For $n \geq 2$, raising to the p^{th} power induces, via the Artin map, a surjection from Y_n to Y_{n+1}.*

Proof. We recall that $\mathcal{Q}_n = \prod_{v|p} U_v^{ne_v}/\overline{U}_K^{(p^n)} = \ker(\mathfrak{X} \to \mathcal{A}_{p^n}(K))$. We have that $n > \frac{1}{p-1}$. Raising to the p^{th} power realizes an isomorphism of $\prod_{v|p} U_v^{ne_v}$ onto $\prod_{v|p} U_v^{ne_v+e_v}$. This isomorphism induces a surjection from $\mathcal{Q}_n$ onto $\mathcal{Q}_{n+1}$. We consider finally the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{Q}_{n+1} & \longrightarrow & \mathcal{Q}_n & \longrightarrow & \mathcal{Q}_n/\mathcal{Q}_{n+1} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle (.)^p} & & \downarrow{\scriptstyle (.)^p} & & \downarrow{\scriptstyle (.)^p} & & \\ 1 & \longrightarrow & \mathcal{Q}_{n+2} & \longrightarrow & \mathcal{Q}_{n+1} & \longrightarrow & \mathcal{Q}_{n+1}/\mathcal{Q}_{n+2} & \longrightarrow & 1 \end{array}$$

We deduce from the snake lemma that the vertical arrow on the right side is a surjection from $\mathcal{Q}_n/\mathcal{Q}_{n+1}$ onto $\mathcal{Q}_{n+1}/\mathcal{Q}_{n+2}$, i.e. from Y_n onto Y_{n+1}. □

Corollary 3.6. *We denote $q_n = \#(Y_n)$. For all $n \geqslant 2$, $q_n \geqslant q_{n+1}$. Therefore the sequence $(q_n)_{n\geq 1}$ is ultimately constant.*

We recall that Y_n is $\ker(\mathcal{A}_{p^{n+1}}(K) \to \mathcal{A}_{p^n}(K))$.

Theorem 3.7. *There exists an integer n_0 such that $Y_{n_0} \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}$. Moreover for all integers $n \geq n_0$, the modules $\mathcal{Q}_n = \mathrm{Gal}(M/\mathcal{H}_{p^n})$ are $\mathbb{Z}_p$-free of rank r_2+1 and*

$$Y_n \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}.$$

Proof. The $\mathbb{Z}_p$-module $\mathfrak{X}$ is isomorphic to the direct product of its torsion part and of $\mathbb{Z}_p^{r_2+1}$. An isomorphism being chosen, we can identify $\mathbb{Z}_p^{r_2+1}$ with a subgroup of $\mathfrak{X}$ and therefore define, via Galois theory, an extension M' of K such that $\mathrm{Gal}(M'/K) \simeq \mathcal{T}_p$ and $\tilde{K}M' = M$.

This extension being unramified outside p, there exists an integer n_1 such that $M' \subset \mathcal{H}_{p^{n_1}}$ and consequently $\mathcal{H}_{p^{n_1}}\tilde{K} = M$. Moreover for all integer $n \geq n_1$, $\mathrm{Gal}(M/\mathcal{H}_{p^n})$ is a sub-module of finite index of $\mathrm{Gal}(M/M') = \mathbb{Z}_p^{r_2+1}$, consequently $\mathcal{Q}_n = \mathrm{Gal}(M/\mathcal{H}_{p^n}) \simeq \mathbb{Z}_p^{r_2+1}$. The $\mathbb{Z}_p$-module $\mathcal{Q}_n$ is therefore free of rank r_2+1.

About the other kernel Y_n we saw that there exists an integer n_2 such that Y_n maps surjectively onto $(\mathbb{Z}/p\mathbb{Z})^{r_2+1}$ for all integer $n \geq n_2$ (we can choose n_2 to be the minimum of all integers n such that for all p-places v the conductors of $(\tilde{K} \cap \mathcal{H}_{p^n})_w/K_v$ are greater than or equal to $\frac{e}{p-1}$). Finally we note that raising to the p^{th} power realizes an isomorphism between $U_v^{ne_v}$ and $U_v^{ne_v+e_v}$, hence the quotient $\mathcal{Q}_n/\mathcal{Q}_{n+1}$, which is isomorphic to Y_n, is killed by p. Define $n_0 = \mathrm{Max}(n_1, n_2)$ and fix an integer $n \geq n_0$. The kernel Y_n is therefore a quotient of $\mathbb{Z}_p^{r_2+1}$, which maps surjectively onto $(\mathbb{Z}/p\mathbb{Z})^{r_2+1}$ and is killed by p. Hence we get $Y_n \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}$. □

3.2 Computing the invariants of $\mathcal{T}_p$

We start by recalling the definition of invariant factors of an abelian group G.

Definition 3.8. *Let G a finite abelian group, there exists a unique sequence $a_1, \cdots, a_t$ such that for all i, $a_i | a_{i+1}$ for $i \in \{1, \cdots, t-1\}$ and $G \simeq \prod_{i=1}^{t} \mathbb{Z}/a_i\mathbb{Z}$. These a_i are the invariant factors of the group G.*

In what follows we will note them $\mathcal{FI}(G) = [a_1, \cdots, a_t]$. If G is a p-group, these invariant factors are all powers of p. In practice, we are able to determine the invariant factors of $\mathcal{A}_{p^n}(K)$. We will see in this section that the knowledge of invariant factors of $\mathcal{A}_{p^n}(K)$, for n large enough, combined with the stabilizing properties of $\mathcal{A}_{p^n}(K)$, does determine explicitly the invariant factors of, and thus $\mathcal{T}_p$. We recall that for n large enough, $\mathcal{A}_{p^n}(K)$ is isomorphic to the direct product of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^n}/K)$ and of $\mathrm{Gal}(\mathcal{H}_{p^n}/\tilde{K} \cap \mathcal{H}_{p^n}) = \mathcal{T}_p$. So we will first explore the structure of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^n}/K)$.

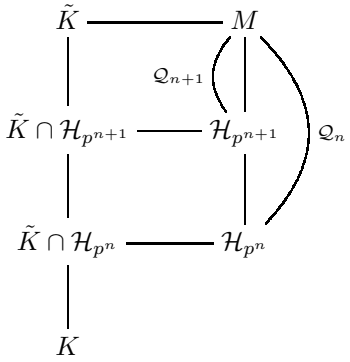
Proposition 3.9. *Let n_0 be such that $\tilde{K} \cap \mathcal{H}_{p^{n_0}}/\tilde{K} \cap \mathcal{H}_p$ is ramified at all places above of p and*

$$Y_{n_0} \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}.$$

Then for all integer $n \geq n_0$, we have

$$\mathrm{Gal}(\tilde{K}/\tilde{K} \cap \mathcal{H}_{p^{n+1}}) = p\,\mathrm{Gal}(\tilde{K}/\tilde{K} \cap \mathcal{H}_{p^n}).$$

Proof. By Theorem 3.7, on one hand, $\mathcal{Q}_n$ is $\mathbb{Z}_p$-free of rank $r_2 + 1$ and on the other hand $Y_n = \mathcal{Q}_n/\mathcal{Q}_{n+1} \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}$. This gives $\mathcal{Q}_{n+1} = p\mathcal{Q}_n$. As $\tilde{K} \cap \mathcal{H}_{p^{n_0}}/\tilde{K} \cap \mathcal{H}_p$ is ramified at all places above of p and $Y_{n_0} \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}$, we have $\mathcal{T}_p \subset \mathcal{A}_{p^{n_0}}(K)$, so $\tilde{K}\mathcal{H}_{p^{n_0}} = M$. Then, considering the following diagram,



we get the required isomorphism. □

Corollary 3.10. *Let n_0 be an integer such that $\tilde{K} \cap \mathcal{H}_{p^{n_0}}/\tilde{K} \cap \mathcal{H}_p$ is ramified at all places above of p and such that $Y_{n_0} \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1}$. Then for all integers $n \geq n_0$, the invariant factors of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^{n+1}}/K)$ are obtained by multiplying by p each invariant factor of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^n}/K)$.*

From the fact that $\mathfrak{X} \simeq \mathbb{Z}_p^{r_2+1} \times \mathcal{T}_p$, the ray class group, $\mathrm{Gal}(\mathcal{H}_{p^n}/K)$, is isomorphic to the direct product of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^n}/K)$ and of $\mathrm{Gal}(\mathcal{H}_{p^n}/\tilde{K} \cap \mathcal{H}_{p^n})$. The invariant factors of $\mathrm{Gal}(\mathcal{H}_{p^n}/K)$ are then simply obtained by concatenating those of the two groups forming the direct product. We now state the result that explicitly determines $\mathcal{T}_p$.

Theorem 3.11. *Let n such that $Y_n = (\mathbb{Z}/p\mathbb{Z})^{r_2+1}$ and $\tilde{K} \cap \mathcal{H}_{p^n}/\tilde{K} \cap \mathcal{H}_p$ is ramified at all places above of p. We assume that*

$$\mathcal{FI}(\mathcal{A}_{p^n}(K)) = [b_1, \cdots, b_t, a_1, \cdots, a_{r_2+1}]$$

with $(v_p(a_1)) > (v_p(b_t)) + 1$, and that

$$\mathcal{FI}(\mathcal{A}_{p^{n+1}}(K)) = [b_1, \cdots, b_t, pa_1, \cdots, pa_{r_2+1}].$$

Then, we have

$$\mathcal{FI}(\mathcal{T}_p) = [b_1, \cdots, b_t].$$

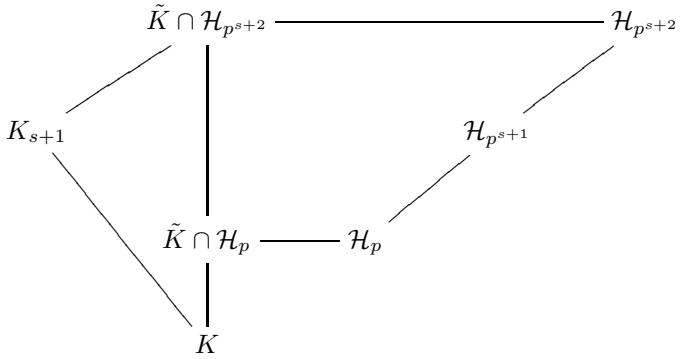
Proof. Indeed, as

$$Y_n \simeq (\mathbb{Z}/p\mathbb{Z})^{r_2+1},$$

we have $\mathcal{A}_{p^i}(K) \simeq tor_{\mathbb{Z}_p}(\mathfrak{X}) \times \mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^i}/K)$ for $i \in \{n, n+1\}$. We saw that the invariant factors of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^{n+1}}/K)$ are exactly equals to p times those of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^n}/K)$. Consequently, if a is an invariant factor of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^{n+1}}/K)$, we have necessarily $a = pa_i$ or $a = pb_i$.
But as $\mathrm{Min}(v_p(a_i)) > \mathrm{Max}(v_p(b_i)) + 1$, none of the invariants factors of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^{n+1}}/K)$ is of the form pb_i. The invariant factors of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^{n+1}}/K)$ are therefore exactly $pa_1, \cdots, pa_{r_2+1}$. The result follows from the fact that $\mathcal{A}_{p^{n+1}}(K)$ is isomorphic to the direct product of $\mathcal{T}_p$ and of $\mathrm{Gal}(\tilde{K} \cap \mathcal{H}_{p^{n+1}}/K)$. □

4 Explicit computation of bounds

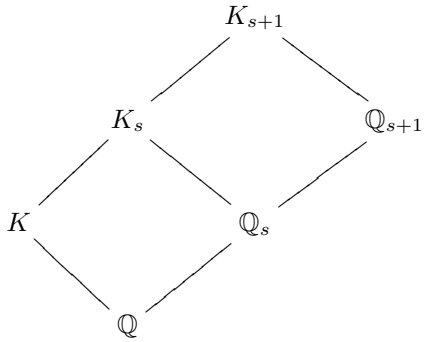
More generally if we note $e = \max_{v|p}\{e_v\}$ the ramification index of $K/\mathbb{Q}$ and s the p-adic valuation of e, then we start to check whether $\mathcal{A}_{p^n}(K)$ stabilizes from rank $n = 2 + s$. To show that $n = 2 + s$ is the proper starting point we consider the diagram,



where K_j is the j^{th}-step of the $\mathbb{Z}_p$-extension of K.

The places above of p are totally ramified in K_{s+1}/K_s therefore $\tilde{K} \cap \mathcal{H}_{p^s}/\tilde{K} \cap \mathcal{H}_p$ is ramified at all places above of p and we start the computation checking whether $\mathcal{A}_{p^n}(K)$ stabilizes from $n = s + 2$. We first prove that all places above

p are totally ramified in K_{s+1}/K_s.
Considering the following diagram,



the ramification index of p in $\mathbb{Q}_{s+1}/\mathbb{Q}$ is p^{s+1} while the one in $K/\mathbb{Q}$ is $p^s a$ with $p \nmid a$. Therefore the extension K_{s+1}/K is ramified and K_{s+1}/K_s is totally ramified at all places above p.

Corollary 4.1. *Let e be the ramification index of p in $K/\mathbb{Q}$ and s be the p-adic valuation of e. Let $n \geq 2+s$, we assume that*

$$\mathcal{FI}(\mathcal{A}_{p^n}(K)) = [b_1, \cdots, b_t, a_1, \cdots, a_{r_2+1}],$$

with $\mathrm{Min}(v_p(a_i)) > \mathrm{Max}(v_p(b_i)) + 1$, *and moreover that*

$$\mathcal{FI}(\mathcal{A}_{p^{n+1}}(K)) = [b_1, \cdots, b_t, pa_1, \cdots, pa_{r_2+1}].$$

Then, we have

$$\mathcal{FI}(\mathcal{T}_p) = [b_1, \cdots, b_t].$$

All the computations are been done using the PARI/GPsystem [PARI-GP].

Example 4.2. *We consider the field $K = \mathbb{Q}(\sqrt{-129})$ and $p = 3$. We have: $\mathcal{FI}(\mathcal{A}_{p^2}(K)) = [3, 3, 9]$, $\mathcal{FI}(\mathcal{A}_{p^3}(K)) = [3, 9, 27]$ and $\mathcal{FI}(\mathcal{A}_{p^4}(K)) = [3, 27, 81]$. We deduce that $\mathcal{T}_p = (\mathbb{Z}/3\mathbb{Z})$.*

5 Numerical results

In the section, we give an explanation of some numerical results that we have computed.

5.1 Heuristic approach

We first recall some results on Cohen-Lenstra Heuristics. The main reference on the subject is the seminal paper of Cohen-Lenstra [C-L], see also [Del]. If we assume that the extension $K/\mathbb{Q}$ is Galois with $\Delta = \mathrm{Gal}(K/\mathbb{Q})$, then the module $\mathcal{T}_p$ is a $\mathbb{Z}[\Delta]$-module. In this section, we assume that Δ is cyclic of cardinality l, for some prime number l. Then $\mathcal{T}_p$ is O_l-module, where O_l is the ring of integers of $\mathbb{Q}(\zeta_l)$. In general, we know that all O_l-module G can be written in a non-canonical way as $\oplus_{i=1}^{q} O_l/\mathfrak{a}_i$, where the $\mathfrak{a}_i$ are ideals of O_l. Yet the Fitting ideal $\mathfrak{a} = \prod_{i=1}^{q} \mathfrak{a}_i$ depends only of the isomorphism class of G, considered as a

O_l-module. This invariant, denoted by $\mathfrak{a}(G)$, can be considered as a generalization of the order of G. We also have $N(\mathfrak{a}(G)) = \#G$.

To simplify the notation we set.

- $\sum_{G,N} = \sum_{G,N(\mathfrak{a}(G))\leq N}$, where the sum is over all isomorphism classes of O_l-module G;
- $\sum_{\mathfrak{a},N} = \sum_{\mathfrak{a},N(\mathfrak{a})\leq N}$;
- $\sum_{\mathfrak{a}',N} = \sum_{\mathfrak{a}',N(\mathfrak{a}')\leq N \text{ and } \mathfrak{a}'\wedge p=1}$;
- $\sum_{\mathfrak{p},N} = \sum_{\mathfrak{p},N(\mathfrak{p})\leq N \text{ and } \mathfrak{p}\in S_p}$, where S_p designed the set of all p-places of O_l.

We consider a function g, defined on the set of the isomorphism classes of O_l-modules (typically g is a characteristic function). We then put

$$\begin{array}{rcl} S_N(g) & = & \sum_{G,N} \frac{g(G)}{\#\operatorname{Aut}_{O_l}(G)}; \\ S_N & = & \sum_{G,N} \frac{1}{\#\operatorname{Aut}_{O_l}(G)}. \end{array}$$

Definition 5.1. *The average of g, if it exists is, the limit when $N \to \infty$ of the quotient*

$$\frac{S_N(g)}{S_N}.$$

This average is denoted by $M_{l,0}(g)$.

As in [C-L], we denote by $w(\mathfrak{a}) = \sum_{G,\mathfrak{a}(G)=\mathfrak{a}} \frac{1}{\#\operatorname{Aut}_{O_l}(G)}$, where $\mathfrak{a}$ is an ideal of O_l.

Proposition 5.2. *([C-L] p.40 corollary 3.8) Let $n \in \mathbb{N}$, then*

$$w(\mathfrak{a}) = \frac{1}{N_{\mathbb{Q}(\zeta_l)}(\mathfrak{a})} \left(\prod_{\mathfrak{p}^\alpha || \mathfrak{a}} \prod_{k=1}^{\alpha} (1 - \frac{1}{N_{O_l}(\mathfrak{p})^k}) \right)^{-1}.$$

The notation $\mathfrak{p}^\alpha || \mathfrak{a}$ meaning $\mathfrak{p}^\alpha | \mathfrak{a}$ and that $\mathfrak{p}^{\alpha+1} \nmid \mathfrak{a}$. Consequently the function w, defined on the set of ideals of O_l, is multiplicative.

Notation. We denote by Π_p the characteristic function of the set of the isomorphism classes of groups whose p-part is non trivial.

Proposition 5.3. *([C-L] p. 47 example 5.10) We note $\mathfrak{p}_1, \cdots, \mathfrak{p}_g$ the p-places of O_l, the average of Π_p exist and we have*

$$M_{l,0}(\Pi_p) = 1 - \prod_{i=1}^{g} \prod_{k\geq 1} \left(1 - \frac{1}{p^{kf_i}}\right),$$

where the f_i designed the degree of the residual extensions $O_l/\mathfrak{p}_i$ over $\mathbb{F}_p$.

Corollary 5.4. *If the extension K is a Galois extension, all residual degrees are equals to f and in this case*

$$M_{l,0}(\Pi_p) = 1 - \left(\prod_{k \geq 1} 1 - \frac{1}{p^{kf}} \right)^g .$$

Remark. *The real $M_{l,0}(\Pi_p)$ is called the 0-average. This notion can be generalized to u-average. The expression to compute the u-average is obtained by replacing k by $k+u$ in the expression of the 0-average.*

5.2 Somes numerical results

5.2.1 Case of the quadratic fields

We consider all quadratic fields of the type $\mathbb{Q}(\sqrt{d})$ with d square-free and $0 < d \leq 10^9$. Then, we compute the proportion of fields with non trivial $\mathbb{Z}_p$-torsion of $\mathfrak{X}$. We note this proportion f_{exp}. The relative error $|f_{exp} - M_{2,0}(\Pi_p)|/M_{2,0}(\Pi_p)$ is denoted by δ.

p	$M_{2,0}(\Pi_p)$	f_{exp}	δ
2	0,71118	0,93650	0,31683
3	0,43987	0,50120	0,13942
5	0,23967	0,23854	0,00470
7	0,16320	0,16280	0,00247
11	0,09916	0,09893	0,00243
13	0,08284	0,08266	0,00212
17	0,06228	0,06214	0,00233
19	0,05540	0,05526	0,00260
23	0,04537	0,04527	0,00207
29	0,03375	0,03560	0,00193
31	0,03330	0,03323	0,00219
37	0,02776	0,02770	0,00198
41	0,02499	0,02493	0,00207
43	0,02380	0,02376	0,00152
47	0,02173	0,02168	0,00207

We consider now the quadratic field of the type $\mathbb{Q}(\sqrt{d})$ with $-10^9 \leqslant d \leqslant 0$. One uses the 1-average that denoted by $M_{2,1}(\Pi_p)$.

p	$M_{2,1}(\Pi_p)$	f_{exp}	δ
2	0,42235	0,93650	1.12734
3	0,15981	0,25718	0,60926
5	0,04958	0,04909	0,00989
7	0,02374	0,02365	0,00374
11	0,00908	0,00905	0,00416
13	0,00641	0,00638	0,00360
17	0,00368	0,00365	0,00445
19	0,00292	0,00291	0,00589
23	0,00198	0,00197	0,00510
29	0,00123	0,00122	0,00916
31	0,00108	0,00107	0,00929
37	0,00075	0,00074	0,00813
41	0,00061	0,00060	0,00982
43	0,00055	0,00055	0,00998
47	0,00046	0,00046	0,01626

We have also computed these proportions for other fields and we consider the distribution of torsion modules with respect to invariants factors but they will not be given here.

5.2.2 Explanation of numerical results

In this section we explain our numerical result. Looking at two tables in §5.2.1 we remark that the proportion f_{exp} for real quadratic fields seems to be a 0-average and a 1-average for the imaginary quadratic. We remark also that the default δ for $p = 2, 3$ increases with the number of fields computed. To explain these phenomena we recall a computation of Gras [Gra2] p. 94-97. Let k be a number field, we denote by $K = k(\zeta_p)$ and ω the idempotent associated with the action of $\mathrm{Gal}(K/k)$ on μ_p.

Theorem 5.5 (Corollaire 1 p. 96 [Gra2])**.** *Let p be a prime, $p \neq 2$. If $\mu_p \not\subset k$ then the torsion of $\mathfrak{X}$ is trivial if and only if any prime ideal of k dividing p is totally split in K/k and $(Cl_K)^\omega$ is trivial where Cl_K is the p-part of the class group of K.*

In case of quadratic fields, if $p > 3$ then $\mu_p \not\subset k$ and the ramification index of p in $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ is $p-1$; then all prime ideal of k dividing p ramifies in K, therefore they are not totally split, hence the torsion is trivial if and only if $(Cl_K)^\omega$ is trivial. So in case k is a real quadratic field the computation of $\mathcal{T}_p$ reduces to the computation of a class group of imaginary quadratic field and we use the 0-average following Cohen-Lenstra Heuristics. In case of imaginary quadratic the remark [Gra2] p.96-97 explains the 1-average. In case $p = 3$, if $d \equiv 6 \bmod 9$ then the ideal of k above p is totally split in K, so the torsion is non trivial. It

explains why the frequency obtained is greater. If we consider the other average $M_2'(\Pi_3) = M_{2,0}(\Pi_3) \times \frac{7}{8} + \frac{1}{8}$; then we obtain

N	$M_2'(\Pi_3)$	f_{exp}	δ
10^6	0,50989	0,48094	0,05678
10^7	0,50989	0,49054	0,03794
10^8	0,50989	0,49697	0,02533
10^9	0,50809	0,50120	0,01704

We now make the computation without the $d \equiv 6 \bmod 9$.

N	$M_{2,0}(\Pi_3)$	f_{exp}	δ
10^6	0,43987	0,40679	0,07521
10^7	0,43987	0,41776	0,05027
10^8	0,43987	0,42511	0,03356
10^9	0,43987	0,42995	0,02257

It remains to study the 9-rank in case where $d \equiv 6 \bmod 9$, and to try and find density formulas for the 9-rank. Finally the discrepancy in the case $p = 2$ is explained by genus theory. Indeed if the discriminant is divided by enough primes then the torsion is not trivial, this explains that the frequency tends to 1.

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