Robust updating of uncertain computational models using experimental modal analysis

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In this paper, a methodology is presented to perform the robust updating of complex uncertain dynamical systems with respect to modal experimental data in the context of structural dynamics. Since both model uncertainties and parameter uncertainties must be considered in the computational model, then the uncertain computational model is constructed by using the nonparametric probabilistic approach. We present an extension to the probabilistic case of the input error methodology for modal analysis adapted to the deterministic updating problem. It is shown that such an extension to the robust updating context induces some conceptual difficulties and is not straightforward. The robust updating formulation leads us to solve a mono-objective optimization problem in presence of inequality probabilistic constraints. A numerical application is presented in order to show the efficiency of the proposed method.

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Nomenclature

$\mathbb{C}$	=	complex space
$\mathcal{D}_{\Lambda,\delta}$	=	admissible set for (β,ϵ) for optimal updating dispersion parameter related to the probabilistic eigenvalue constraint
$\mathcal{D}_{\Phi,\delta}$	=	admissible set for (β,ϵ) for optimal updating dispersion parameter related to the probabilistic eigenvector constraint
$\mathcal{D}_{\Phi-\Lambda,\delta}$	=	admissible set for (β,ϵ) for optimal updating dispersion parameter related to both probabilistic constraints
$\mathcal{D}_{\Lambda,s}$	=	admissible set for (β,ϵ) for optimal updating mean parameter related to the probabilistic eigenvalue constraint
$\mathcal{D}_{\Phi,s}$	=	admissible set for (β,ϵ) for optimal updating mean parameter related to the probabilistic eigenvector constraint
$\mathcal{D}_{\Phi-\Lambda,s}$	=	admissible set for (β,ϵ) for optimal updating mean parameter related to both probabilistic constraints
E_0	=	Young modulus
$\mathcal{E}$	=	mathematical expectation
$\mathcal{F}_{\Lambda,\delta}$	=	admissible set for optimal updating dispersion parameter related to probabilistic eigenvalue constraint
$\mathcal{F}_{\Phi,\delta}$	=	admissible set for optimal updating dispersion parameter related to the probabilistic eigenvector constraint
$\mathcal{F}_{\Phi-\Lambda,\delta}$	=	admissible set for optimal updating dispersion parameter related to both probabilistic constraints
$\mathcal{F}_{\Lambda,s}$	=	admissible set for optimal updating mean parameter related to the probabilistic eigenvalue constraint
$\mathcal{F}_{\Phi,s}$	=	admissible set for optimal updating mean parameter related to the probabilistic eigenvector constraint
$\mathcal{F}_{\Phi-\Lambda,s}$	=	admissible set for optimal updating mean parameter related to both probabilistic constraints
$\mathbf{g}$	=	vector-valued function defining all the probabilistic constraints
g_Λ	=	function defining the probabilistic constraint on the eigenvalues
$g_{\tilde{\Phi}}$	=	function defining the probabilistic constraint on the eigenvectors
$\underline{j}$	=	cost function for the deterministic updating
j	=	cost function for the robust updating
m	=	number of rigid body modes
N	=	number of generalized coordinates
n	=	number of DOF in the computational model
n_2	=	number of non measured DOF
n_{obs}	=	number of measured DOF
$\mathbb{n}$	=	dimension of the mean reduced computational model

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$Proba$	=	probability
$\mathbf{Q}$	=	random vector of the generalized coordinates related to the uncertain computational model
$\underline{\mathbf{q}}$	=	vector of the generalized coordinates related to the mean computational model
$\mathbf{R}_\alpha$	=	random vector of the nonzero components of the residue vector related to the uncertain computational model
$\mathbb{R}$	=	real space
$\mathbb{R}^+$	=	real positive space
r	=	number of experimental eigenvalues and eigenmodes
$\underline{\mathtt{r}}_\alpha$	=	residue vector of the mean computational model related to experimental eigenvalue number α
$\underline{\mathbf{r}}_\alpha$	=	vector of the nonzero components of the residue vector related to the mean computational model
$\mathcal{S}$	=	admissible set for the updating mean parameters
$\mathbf{s}$	=	updating mean parameter
$\mathbf{s}^{opt}$	=	updated mean parameter
S_0	=	section of the beam
β	=	given probability level
Δ	=	admissible set for the updating dispersion parameters
$\Delta\Lambda$	=	norm of the random error with respect to the experimental eigenvalues
$\Delta\Lambda_\alpha$	=	random error with respect to the experimental eigenvalue number α
$\Delta\widetilde{\Phi}$	=	norm of the random error with respect to the experimental eigenvectors
$\Delta\widetilde{\Phi}_\alpha$	=	random error with respect to the experimental to eigenvector number α
δ_M	=	mass updating dispersion parameter
δ_K	=	stiffness updating dispersion parameter
$\delta_{\alpha\beta}$	=	Kronecker Symbol
$\boldsymbol{\delta}$	=	updating dispersion parameter
$\boldsymbol{\delta}^{opt}$	=	updated dispersion parameter
ϵ	=	given error level
Λ_α	=	random eigenvalue number α related to the uncertain computational model
$\underline{\lambda}_\alpha$	=	eigenvalue number α of the mean computational model
$\underline{\lambda}_\alpha^{exp}$	=	experimental eigenvalue number α
$\underline{\lambda}_\alpha^0$	=	eigenvalue number α of the mean computational model with fixed measured DOF
ρ_0	=	mass density of the non updated mean computational model
ν_0	=	Poisson ratio
$\underline{\nu}_\alpha$	=	eigenfrequency number α of the mean computational model
$\underline{\nu}_\alpha^{exp}$	=	experimental eigenfrequency number α
$\underline{\nu}_\alpha^0$	=	eigenfrequency number α of the mean computational model with fixed measured DOF
$\widetilde{\Phi}_\alpha$	=	random eigenvector number α in the physical space restricted to the experimental DOF
$\underline{\varphi}_\alpha$	=	eigenvector number α of the mean computational model
$\underline{\varphi}_\alpha^{exp}$	=	experimental eigenvector number α
$\underline{\varphi}_{2,\alpha}$	=	extrapolation of the experimental eigenvector number α on the non measured DOF

$\boldsymbol{\Psi}_\alpha$	=	random eigenvector number α related to the uncertain computational model
$\underline{\boldsymbol{\psi}}_\alpha$	=	eigenvector number α of the mean computational model with fixed measured DOF
$[\mathbf{G}_M]$	=	random germ of the random mass matrix
$[\mathbf{G}_K]$	=	random germ of the random stiffness matrix
$[\underline{H}_\alpha]$	=	projection basis matrix
$[I]$	=	identity matrix
$[\underline{K}]$	=	finite element stiffness matrix of the mean computational model
$[\underline{K}_{red,\alpha}]$	=	mean reduced stiffness matrix
$[\mathbf{K}_{red,\alpha}]$	=	random stiffness matrix
$[\underline{L}_{M,\alpha}]$	=	factorization of the mean reduced mass matrix
$[\underline{L}_{K,\alpha}]$	=	factorization of the mean reduced stiffness matrix
$[\underline{M}]$	=	finite element mass matrix of the mean computational model
$[\underline{M}_{red,\alpha}]$	=	mean reduced mass matrix
$[\mathbf{M}_{red,\alpha}]$	=	random mass matrix
$[\mathcal{R}]$	=	modal residue matrix related to the mean computational model
$[\underline{\mathcal{R}}]$	=	modal residue matrix related to the uncertain computational model
$[S_\alpha]$	=	modal liftings matrix
$[\Lambda^{exp}]$	=	real diagonal matrix of the experimental eigenvalues
$[\underline{\Phi}^{exp}]$	=	experimental modal matrix
$[\underline{\Psi}]$	=	modal matrix of the mean computational model with measured DOF fixed

I. Introduction

The updating of computational models using experimental data is currently a challenge of interest in structural dynamics. The updating formulation involves an optimization problem for which the cost function can be defined from the operator of the computational model (input error formulation) or from the inverse of the operator of the computational model (output error formulation). These last decades, such an updating has been carried out using deterministic computational models (see for instance,[?] for the input error formulations and[?,?,?] for the output error formulations). It is well known that deterministic computational models are not sufficient to accurately predict the dynamical behavior of complex structures. The uncertainties have then to be taken into account in the computational models by using probabilistic models as soon as the probability theory can be used. More recently, the terminology of robust updating has been introduced. The robust updating is defined as the updating of the parameters of the computational model which contains uncertainties. The uncertainties are taken into account in the computational model which is then called the uncertain computational model. Let us recall that

there exist two classes of uncertainties: (1) the system parameter uncertainties which are the uncertainties on the parameters of the computational model (system parameters), (2) the model uncertainties which are induced by the mathematical-mechanical process used for the construction of the computational model and which, by definition, cannot be taken into account by variations of the system parameters. In general, system parameter uncertainties can be taken into account by using the parametric probabilistic approach, see for instance [?,?,?] and [?] for rotating structures. Both model uncertainties and system parameter uncertainties can be taken into account by using the nonparametric probabilistic approach recently introduced. [?,?,?] We then can distinguish the robust updating of the updating parameters in presence of system parameter uncertainties [?,?,?] (the uncertain computational model is then constructed with the parametric probabilistic approaches) from the robust updating of the updating parameters in presence of both model uncertainties and system parameter uncertainties [?,?] (the uncertain computational model is then constructed with the nonparametric probabilistic approach). Until now, all these robust updating formulations involve cost functions which are defined from the observations of the uncertain computational model (output error formulations). The motivation of this paper is to propose a robust updating methodology of the updating parameters in presence of both model uncertainties and system parameter uncertainties using modal experimental data and defining a cost function relative to the operators of the uncertain computational model (input error formulation). The deterministic underlying methodology is based on the modal updating formulation proposed. [?] In this paper, we propose to extend such a deterministic updating formulation to the case of an uncertain computational model for which uncertainties are modeled using the probability theory. Note that this extension is not trivial. The paper is organized as follows. Section II is a brief summarizing of the deterministic updating methodology. [?] This deals with the updating of a mean computational model for which the updating parameters are called the mean updating parameters. The main idea is to modify the generalized eigenvalue problem of the mean computational model in order to calculate a residue which characterizes the good matching between the mean computational model and the available experimental data. The cost function is defined from this residue and is then optimized with respect to the admissible set of the mean updating parameters. Section III deals with the robust updating formulation. In this robust updating context, there are model uncertainties which are such that the available experimental data can not exactly be reproduced by any computational model. This context does not allow the strategy of deterministic updating to be effective. The main idea is thus to implement the nonparametric probabilistic approach in a mean computational model in order to take into account both model uncertainties and parameter uncertainties. First of

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all, a modified Craig and Bampton dynamical substructuring method [?,?,?,?] is introduced in order to construct a mean reduced matrix equation allowing the deterministic residue to be calculated. In a second step, the generalized matrices of this mean reduced equation are replaced by random matrices for which the probability model is explicitly constructed. With such an approach, the uncertainty level of each random matrix is controlled by a dispersion parameter. We then obtain a random residue which is defined as a function of the updating parameters which are the updating mean parameters related to the mean computational model and the dispersion parameters which allow the uncertainty level in the computational model to be controlled. In a third step, the cost function is defined as the second-order moment of the norm of the random residue. Difficulties arise from a conceptual point of view. A straightforward generalization of the deterministic optimization problem which would consist in optimizing the cost function with respect to the admissible set of the updating parameters would yield a deterministic updated computational model which would not be compatible with the existence of model uncertainties in the computational model. The formulation is then modified by adding probabilistic constraints related to the nonreducible gap between the uncertain computational model and the experiments due to the presence of model uncertainties. In Section IV, a numerical example is presented in order to validate the methodology proposed.

II.   Summarizing the input error methodology for the deterministic updating of a computational dynamical model using experimental modal data analysis

The assumptions concerning the available experimental data are given below. It is assumed that experimental modal analysis is carried out on only one manufactured dynamical system with free free boundary conditions. Consequently, there are $m = 6$ rigid-body modes associated with 6 zero eigenvalues which are not taken into account in the analysis. The experimental data consists in r experimental elastic eigenvalues denoted by $0 < \underline{\lambda}_1^{exp} < \ldots < \underline{\lambda}_r^{exp}$ and r corresponding experimental eigenmodes denoted as $\underline{\varphi}_\alpha^{exp}$ which are measured at n_{obs} observation points. Moreover, it is assumed that the manufactured dynamical system can be modeled by a deterministic computational model which is called the mean computational model. The usual methodology for the updating of a deterministic computational model using modal analysis is the output error formulation (see for instance[?]) which consists in solving a multi-objective optimization problem in order to simultaneously minimize the distance between each experimental eigenvalue / eigenvector and between each eigenvalue / eigenvector of

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the deterministic computational model. The alternative formulation used in this Section belongs to the input error formulation. This means that the cost function which quantifies the gap between the mean computational model and the experimental data is directly defined from the operators of the mean computational model so that the eigenfrequencies and the eigenmodes are simultaneously treated with a coherent way. This deterministic updating yields to solve a mono-objective optimization problem with respect to the admissible set of the updating parameters of the deterministic computational model. Since the robust updating proposed for modal analysis is based on the method proposed in,[?] we briefly summarize it below in order to improve the readability of the manuscript.

The mean computational model of the dynamical system is constructed using the finite element method and has n DOF (degrees of freedom). It is assumed that the finite element mesh is compatible with the n_{obs} experimental measurement points. Let $\mathbf{s}$ be the $\mathbb{R}^s$-vector of the updating parameters of the mean computational model called the updating mean parameters. Vector $\mathbf{s}$ belongs to an admissible set $\mathcal{S}$ corresponding to a given family of mean computational models. Assuming the dynamical system to be linear, for fixed $\mathbf{s}$ in $\mathcal{S}$, the generalized eigenvalue problem related to the conservative dynamical system is written as: find $(\underline{\lambda}_\alpha, \underline{\varphi}_\alpha)$ belonging to $\mathbb{R}^+ \times \mathbb{R}^n$ such that

$$\mathbf{0} = \left([\underline{K}(\mathbf{s})] - \underline{\lambda}_\alpha [\underline{M}(\mathbf{s})]\right) \underline{\varphi}_\alpha \quad , \quad \alpha = 1,\ldots,r \quad , \tag{1}$$

in which the matrices $[\underline{M}(\mathbf{s})]$ and $[\underline{K}(\mathbf{s})]$ are the finite element mass and stiffness matrices. Since the dynamical system has free free boundary conditions, matrices $[\underline{M}(\mathbf{s})]$ and $[\underline{K}(\mathbf{s})]$ are positive-definite and semi-positive-definite symmetric $(n \times n)$ real matrices whose bloc decomposition with respect to the n_{obs} experimental measured DOF and the $n_2 = n - n_{obs}$ unmeasured DOF is written as

$$[\underline{M}(\mathbf{s})] = \begin{bmatrix} [\underline{M}_{11}(\mathbf{s})] & [\underline{M}_{12}(\mathbf{s})] \\ [\underline{M}_{12}(\mathbf{s})]^T & [\underline{M}_{22}(\mathbf{s})] \end{bmatrix} \quad , \quad [\underline{K}(\mathbf{s})] = \begin{bmatrix} [\underline{K}_{11}(\mathbf{s})] & [\underline{K}_{12}(\mathbf{s})] \\ [\underline{K}_{12}(\mathbf{s})]^T & [\underline{K}_{22}(\mathbf{s})] \end{bmatrix} \quad . \tag{2}$$

The matrix formulation which allows the deterministic updating to be solved is written as follows :

$$\underline{\mathbf{r}}_\alpha(\mathbf{s}) = \left(\begin{bmatrix} [\underline{K}_{11}(\mathbf{s})] & [\underline{K}_{12}(\mathbf{s})] \\ [\underline{K}_{12}(\mathbf{s})]^T & [\underline{K}_{22}(\mathbf{s})] \end{bmatrix} - \underline{\lambda}_\alpha^{exp} \begin{bmatrix} [\underline{M}_{11}(\mathbf{s})] & [\underline{M}_{12}(\mathbf{s})] \\ [\underline{M}_{12}(\mathbf{s})]^T & [\underline{M}_{22}(\mathbf{s})] \end{bmatrix} \right) \begin{bmatrix} \underline{\varphi}_\alpha^{exp} \\ \underline{\varphi}_{2,\alpha}(\mathbf{s}) \end{bmatrix} \tag{3}$$

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In Eq. (3), the unknown quantities are then vectors $\underline{r}_\alpha(\mathbf{s})$ and $\underline{\varphi}_{2,\alpha}(\mathbf{s})$. For a given updating mean parameter $\mathbf{s}$ belonging to $\mathcal{S}$, the component $\underline{r}_{\alpha,k}(\mathbf{s})$ of the residue vector $\underline{r}_\alpha(\mathbf{s})$ quantifies the errors of the mean computational model induced by the experimental eigenvalue number α and its associated elastic eigenmode for the DOF number k. It should be noted that vector $\underline{\varphi}_{2,\alpha}(\mathbf{s})$ is not the restriction of eigenvector $\underline{\varphi}_\alpha(\mathbf{s})$ to the unmeasured DOF. Note that vector $\underline{\varphi}_{2,\alpha}(\mathbf{s})$ is calculated from the mean computational model and from the experimental modal measurements by solving Eq. (3). Following the reference[?] for the deterministic updating methodology, two assumptions are introduced which ensure the existence and the uniqueness of a solution: (1) Since the information concerning the experimental eigenmodes are only available for the measured DOF, then it is assumed that the residue is zero for the unmeasured DOF, that is to say,

$$\underline{r}_\alpha(\mathbf{s}) \;=\; \begin{bmatrix} \mathbf{r}_\alpha(\mathbf{s}) \\ \\ \mathbf{0} \end{bmatrix} \quad . \tag{4}$$

(2) For each α in $\{1,\dots,r\}$, the matrix $[\underline{B}_\alpha(\mathbf{s})]$ defined by

$$[\underline{B}_\alpha(\mathbf{s})] \;=\; [\underline{K}_{22}(\mathbf{s})] \;-\; \underline{\lambda}_\alpha^{exp}[\underline{M}_{22}(\mathbf{s})] \quad , \tag{5}$$

is assumed to be invertible. With such an assumption, the eigenvalues of the generalized eigenvalue problem related to the mean computational model for which the measured DOF are fixed have to differ from the experimental eigenvalues. In practice, such a condition is verified in the low-frequency domain for which we are only interested in the first smallest eigenfrequencies and if the measured DOF are regularly distributed through the structure. Let us introduce the first eigenfrequency of the structure for which all the measured DOF are fixed. Then, if this eigenvalue is much larger than the experimental frequency band of analysis, then the assuption is satisfied. It is assumed that the number r of experimental pairs of eigenvalues and eigenmodes which are considered for the deterministic updating is chosen in order to fulfill this condition. Finally, it should be noted that Eq.(3) is coherent with Eq.(1) if the experimental data matches with the mean computational model.

The deterministic updating is solved by simultaneously minimizing the residue vectors $\underline{r}_\alpha(\mathbf{s})$ for all α belonging to $\{1,\dots,r\}$. The cost function is defined as a function of the updating mean parameters $\mathbf{s}$ by

$$\underline{j}(\mathbf{s}) \;=\; ||[\underline{\mathcal{R}}(\mathbf{s})]||_F^2 \quad , \tag{6}$$

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in which the $(r \times r)$ real matrix $[\underline{\mathcal{R}}(\mathbf{s})]$ is defined by $[\underline{\mathcal{R}}(\mathbf{s})]_{\alpha\beta} = \underline{\boldsymbol{\varphi}}_\alpha^{exp,T} \underline{\mathbf{r}}_\beta(\mathbf{s})$. In Eq. (6), $||[\underline{X}]||_F^2 = tr([\underline{X}] [\underline{X}]^T)$.

Note that in the input error methodology, the sum of the Euclidean norms of the residue vectors are generally used. Since a change of basis can always be performed to express the cost function relative to the residue vectors, we have chosen to express the residue vectors in the modal basis. The solution of this deterministic updating problem is then given by

$$\underline{\mathbf{s}}^{opt} = arg \min_{\mathbf{s}\in\mathcal{S}} \underline{j}(\mathbf{s}) \quad . \tag{7}$$

Note that the components of vector $\mathbf{s}$ can represent any physical parameters of the mechanical system and that the dimension of this vector is s. Eliminating $\underline{\boldsymbol{\varphi}}_{2,\alpha}(\mathbf{s})$ in Eq. (3) and considering the r experimental elastic modes, it can be deduced that there are $r \times n_{obs}$ independent nonlinear algebraic equations to identify the vector parameter $\mathbf{s}$. Consequently, the inverse problem is *a priori* well posed if $r \times n_{obs} > s$ that is assumed. Clearly, such an assumption is not sufficient to guarantee the existence of a unique solution. Nevertheless, the problem is not to construct the global optimum for this updating problem but to improve the computational model in the neighborhood of the nominal design. In addition, in the methodology proposed, the cost function is evaluated by solving direct eigenvalue problem and consequently, there are no potential difficulties related to the inverse problem.

III.   Robust input error methodology for experimental modal data analysis

In this Section, it is assumed that the computational model used for modeling the manufactured dynamical system for which experimental modal data are available contains significant model uncertainties. Consequently, the deterministic updating formulation presented in Section III can be improved in taking in to account the presence of model uncertainties. It should be noted that in general, the optimization of a deterministic computational model can produce a non optimal result with respect to the robust optimization of an uncertain computational model as it is shown for instance in[?] . We then propose to adapt the deterministic updating formulation presented in Section III to the robust updating context as explained in Section I. The nonparametric probabilistic approach is then implemented in the mean reduced matrix model. The formulation of the optimization problem is then discussed in order to capture the largest possible class of uncertain computational models.

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A.   Mean reduced computational model

The proposed dynamical substructuring method is based on the Craig and Bampton method. [?,?,?,?] Let us recall that this method consists in decomposing the displacement vector of a substrucre as the direct sum of the displacement vector of the substructure with fixed coupling interface and of the static lifting relative to the coupling interface. In the present context, the coupling interface is defined by the n_{obs} measured DOF. The extension of the Craig and Bampton method consists in replacing the static liftings by the "modal liftings" related to each experimental eigenvalue. Note that in this context, the projection basis depends on α. For a given α belonging to $\{1, \ldots, r\}$, the projection basis is given by

$$
\begin{bmatrix} \underline{\boldsymbol{\varphi}}_{\alpha}^{exp} \\ \underline{\boldsymbol{\varphi}}_{2,\alpha}(\mathbf{s}) \end{bmatrix} = [\underline{H}_{\alpha}(\mathbf{s})] \begin{bmatrix} \underline{\boldsymbol{\varphi}}_{\alpha}^{exp} \\ \underline{\mathbf{q}}_{\alpha}(\mathbf{s}) \end{bmatrix} \quad , \quad [\underline{H}_{\alpha}(\mathbf{s})] = \begin{bmatrix} [I] & [\mathbb{0}] \\ [\underline{S}_{\alpha}(\mathbf{s})] & [\underline{\Psi}(\mathbf{s})] \end{bmatrix}.
\tag{8}
$$

In Eq. (8), $\underline{\mathbf{q}}_{\alpha}(\mathbf{s})$ is the $\mathbb{R}^N$-vector of the generalized coordinates. The matrix $[\underline{\Psi}(\mathbf{s})]$ is the $(n_2 \times N)$ real modal matrix whose columns are the eigenvectors $\underline{\boldsymbol{\psi}}_1(\mathbf{s}), \ldots, \underline{\boldsymbol{\psi}}_N(\mathbf{s})$ corresponding to the lowest eigenvalues of the generalized eigenvalue problem related to the mean computational model with fixed coupling interface (n_{obs} measured DOF): find $(\underline{\lambda}_{\beta}^0, \underline{\boldsymbol{\psi}}_{\beta})$ belonging to $\mathbb{R}^+ \times \mathbb{R}^{n_2}$ such that

$$
\mathbf{0} = \left([\underline{K}_{22}(\mathbf{s})] - \underline{\lambda}_{\beta}^0(\mathbf{s}) [\underline{M}_{22}(\mathbf{s})] \right) \underline{\boldsymbol{\psi}}_{\beta}(\mathbf{s})
\tag{9}
$$

.

In Eq. (8), the matrix $[\underline{S}_{\alpha}(\mathbf{s})]$ is the $(n_2 \times n_{obs})$ real matrix of the "modal" boundary functions defined by

$$
[\underline{S}_{\alpha}(\mathbf{s})] = -[\underline{B}_{\alpha}(\mathbf{s})]^{-1} \left([\underline{K}_{12}(\mathbf{s})]^T - \underline{\lambda}_{\alpha}^{exp} [\underline{M}_{12}(\mathbf{s})]^T \right) \quad , \quad \alpha = 1, \ldots, r \quad ,
\tag{10}
$$

in which $[\underline{B}_{\alpha}(\mathbf{s})]$ is defined by Eq. (5) It should be noted that the usual Craig and Bampton method corresponds to Eq. (10) for which the mass dynamic term is not taken into account. Let $\mathbb{n} = N + n_{obs}$. The mean reduced matrix

equation which allows $\underline{\mathbf{r}}_\alpha(\mathbf{s})$ and $\underline{\mathbf{q}}_\alpha(\mathbf{s})$ to be calculated is then written as

$$
\begin{bmatrix} \underline{\mathbf{r}}_\alpha(\mathbf{s}) \\ \\ \mathbf{0} \end{bmatrix} = \left([\underline{K}_{red,\alpha}(\mathbf{s})] - \underline{\lambda}_\alpha^{exp} [\underline{M}_{red,\alpha}(\mathbf{s})] \right) \begin{bmatrix} \underline{\varphi}_\alpha^{exp} \\ \\ \underline{\mathbf{q}}_\alpha(\mathbf{s}) \end{bmatrix} \quad , \tag{11}
$$

in which the matrices $[\underline{M}_{red,\alpha}(\mathbf{s})]$ and $[\underline{K}_{red,\alpha}(\mathbf{s})]$ are the $(\mathsf{n} \times \mathsf{n})$ positive-definite and positive symmetric real mass and stiffness matrices defined by $[\underline{M}_{red,\alpha}(\mathbf{s})] = [\underline{H}_\alpha(\mathbf{s})]^T [\underline{M}(\mathbf{s})] [\underline{H}_\alpha(\mathbf{s})]$ and $[\underline{K}_{red,\alpha}(\mathbf{s})] = [\underline{H}_\alpha(\mathbf{s})]^T [\underline{K}(\mathbf{s})] [\underline{H}_\alpha(\mathbf{s})]$. It should be noted that a convergence analysis with respect to the numerical parameter N is systematically carried out for every application (see subsection D of Section V).

B. Stochastic computational model

The nonparametric probabilistic approach[?,?,?] recently introduced is used to model both data uncertainties and model uncertainties in Eq. (11). Briefly, the method consists in replacing the deterministic matrices $[\underline{M}_{red,\alpha}(\mathbf{s})]$ and $[\underline{K}_{red,\alpha}(\mathbf{s})]$ by random matrices $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ and $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$ for which the probability distribution is constructed using the maximum entropy principle under the constraints defined by the available information. The scalar parameters δ_M and δ_K are the dispersion parameters which allow the amount of uncertainty of the random matrices to be quantified. The random matrices $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ and $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$ are written as $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)] = [\underline{L}_{M,\alpha}(\mathbf{s})]^T [\mathbf{G}_M(\delta_M)] [\underline{L}_{M,\alpha}(\mathbf{s})]$ and $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)] = [\underline{L}_{K,\alpha}(\mathbf{s})]^T [\mathbf{G}_K(\delta_K)] [\underline{L}_{K,\alpha}(\mathbf{s})]$ in which the matrices $[\underline{L}_{M,\alpha}(\mathbf{s})]$ and $[\underline{L}_{K,\alpha}(\mathbf{s})]$ are $(\mathsf{n} \times \mathsf{n})$ and $((\mathsf{n} - m) \times \mathsf{n})$ real matrices such that $[\underline{M}_{red,\alpha}(\mathbf{s})] = [\underline{L}_{M,\alpha}(\mathbf{s})]^T [\underline{L}_{M,\alpha}(\mathbf{s})]$ and $[\underline{K}_{red,\alpha}(\mathbf{s})] = [\underline{L}_{K,\alpha}(\mathbf{s})]^T [\underline{L}_{K,\alpha}(\mathbf{s})]$ and where the matrices $[\mathbf{G}_M(\delta_M)]$ and $[\mathbf{G}_K(\delta_K)]$ are full $(\mathsf{n} \times \mathsf{n})$ and $((\mathsf{n} - m) \times (\mathsf{n} - m))$. Below, the algebraic representation of these random matrices adapted to the Monte Carlo numerical simulation is briefly recalled. Let $[\mathbf{G}(\delta)]$ denotes one of the random matrix $[\mathbf{G}_M(\delta_M)]$ or $[\mathbf{G}_K(\delta_K)]$ for which the dimension is denoted by μ. From the probability distribution constructed with the Maximum Entropy Principle, it can be deduced that $[\mathbf{G}] = [\mathbf{L}_G]^T [\mathbf{L}_G]$ in which $[\mathbf{L}_G]$ is a real upper triangular random matrix such that

(1) random variables $\{[\mathbf{L}_G]_{jj'}, j \leq j'\}$ are independent;

(2) for $j < j'$, real-valued random variable $[\mathbf{L}_G]_{jj'}$ can be written as $[\mathbf{L}_G]_{jj'} = \sigma_\mu U_{jj'}$ in which $\sigma_\mu = \delta (\mu + 1)^{-1/2}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1;

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(3) for $j = j'$, positive-valued random variable $[\mathbf{L}_G]_{jj}$ can be written as $[\mathbf{L}_G]_{jj} = \sigma_\mu \sqrt{2V_j}$ in which σ_μ is defined above and where V_j is a positive-valued gamma random variable whose probability density function $p_{V_j}(v)$ with respect to dv is written as

$$p_{V_j}(v) = \mathbb{1}_{\mathbb{R}^+}(v) \frac{1}{\Gamma(\frac{\mu+1}{2\delta^2} + \frac{1-j}{2})} v^{\frac{\mu+1}{2\delta^2} - \frac{1+j}{2}} e^{-v}. \tag{12}$$

All the details concerning the construction of the probability model of these random matrices can be found in [?,?,?] . It should be noted that the random matrices $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ and $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$ have the same algebraic properties than the deterministic matrices $[\underline{M}_{red,\alpha}(\mathbf{s})]$ and $[\underline{K}_{red,\alpha}(\mathbf{s})]$. In particular random matrix $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ (or $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$) is with values in the set of all the positive-definite (or semi-positive-definite) symmetric real matrices. Let $\delta = (\delta_M, \delta_K)$ be the vector of the dispersion parameters which has to be updated. It can be shown from the construction of the probability model that dispersion parameter δ must belong to the admissible set $\Delta = \left\{ [0, \sqrt{\frac{n+1}{n+5}}] \times [0, \sqrt{\frac{n-m+1}{n-m+5}}] \right\}$. It should also be noted that the same random matrices $[\mathbf{G}_M(\delta_M)]$ and $[\mathbf{G}_K(\delta_K)]$ are used to construct the random matrices $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ and $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$ for all α belonging to $\{1, \ldots, r\}$. The stochastic matrix equation whose unknowns are the random residue vector $\mathbf{R}_\alpha(\mathbf{s}, \delta)$ and the random vector $\mathbf{Q}_\alpha(\mathbf{s})$ of the random generalized coordinates is written as

$$\begin{bmatrix} \mathbf{R}_\alpha(\mathbf{s}, \delta) \\ \\ \mathbf{0} \end{bmatrix} = \left([\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)] - \underline{\lambda}_\alpha^{exp} [\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)] \right) \begin{bmatrix} \underline{\varphi}_\alpha^{exp} \\ \\ \mathbf{Q}_\alpha(\mathbf{s}, \delta) \end{bmatrix} \quad , \tag{13}$$

C. Estimation of $\mathbf{Q}_\alpha(\mathbf{s}, \delta)$

The matrices $[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)]$ and $[\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)]$ are block decomposed with respect to the number of experimental measured DOF and with respect to the number of generalized coordinates N such that

$$[\mathbf{K}_{red,\alpha}(\mathbf{s}, \delta_K)] = \begin{bmatrix} [\mathcal{K}_{1,\alpha}(\mathbf{s}, \delta_K)] & [\mathbf{K}_{c,\alpha}(\mathbf{s}, \delta_K)] \\ [\mathbf{K}_{c,\alpha}(\mathbf{s}, \delta_K)]^T & [\mathcal{K}_{2,\alpha}(\mathbf{s}, \delta_K)] \end{bmatrix} , \; [\mathbf{M}_{red,\alpha}(\mathbf{s}, \delta_M)] = \begin{bmatrix} [\mathcal{M}_{1,\alpha}(\mathbf{s}, \delta_M)] & [\mathbf{M}_{c,\alpha}(\mathbf{s}, \delta_M)] \\ [\mathbf{M}_{c,\alpha}(\mathbf{s}, \delta_M)]^T & [\mathcal{M}_{2,\alpha}(\mathbf{s}, \delta_M)] \end{bmatrix} . \tag{14}$$

The random residue vector $\mathbf{R}_\alpha(\mathbf{s}, \delta)$ and the random vector $\mathbf{Q}_\alpha(\mathbf{s})$ of the random generalized coordinates solution

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of the random matrix equation (13) are then given by

$$\mathbf{R}_\alpha(\mathbf{s}, \delta) \quad = \quad [\boldsymbol{\mathcal{B}}_{1,\alpha}(\mathbf{s}, \delta)]\, \underline{\boldsymbol{\varphi}}_\alpha^{exp} + [\mathbf{B}_{c,\alpha}(\mathbf{s}, \delta)]\, \mathbf{Q}_\alpha(\mathbf{s}, \delta) \tag{15}$$

$$\mathbf{Q}_\alpha(\mathbf{s}, \delta) \quad = \quad -[\boldsymbol{\mathcal{B}}_{2,\alpha}(\mathbf{s}, \delta)]^{-1}\, [\mathbf{B}_{c,\alpha}(\mathbf{s}, \delta)]^T\, \underline{\boldsymbol{\varphi}}_\alpha^{exp} \quad , \tag{16}$$

in which $[\boldsymbol{\mathcal{B}}_{1,\alpha}(\mathbf{s}, \delta)] = [\boldsymbol{\mathcal{K}}_{1,\alpha}(\mathbf{s}, \delta)] - \underline{\lambda}_\alpha^{exp}\, [\boldsymbol{\mathcal{M}}_{1,\alpha}(\mathbf{s}, \delta)]$, $[\mathbf{B}_{c,\alpha}(\mathbf{s}, \delta)] = [\mathbf{K}_{c,\alpha}(\mathbf{s}, \delta)] - \underline{\lambda}_\alpha^{exp}\, [\mathbf{M}_{c,\alpha}(\mathbf{s}, \delta)]$ and

$[\boldsymbol{\mathcal{B}}_{2,\alpha}(\mathbf{s}, \delta)] = [\boldsymbol{\mathcal{K}}_{2,\alpha}(\mathbf{s}, \delta)] - \underline{\lambda}_\alpha^{exp}\, [\boldsymbol{\mathcal{M}}_{2,\alpha}(\mathbf{s}, \delta)]$. The calculation of random vector $\mathbf{Q}_\alpha$ requires the inversion of

the random matrix $[\boldsymbol{\mathcal{B}}_{2,\alpha}(\mathbf{s}, \delta)]$ for all α belonging to $\{1, \ldots, r\}$. It is assumed that the number r of experimental

eigenvalues is chosen under the assumption that random matrix $[\boldsymbol{\mathcal{B}}_{2,\alpha}(\mathbf{s}, \delta)]$ is invertible almost surely.

D. Robust updating formulation

The robust updating formulation requires to define the cost function from the uncertain computational model as a

function of the updating mean parameter $\mathbf{s}$ and of the dispersion parameter δ. In coherence with Eq. (6), the cost

function denoted by $j(\mathbf{s}, \delta)$ is written as

$$j(\mathbf{s}, \delta) = \mathcal{E}\{||[\boldsymbol{\mathcal{R}}(\mathbf{s}, \delta)]||_F^2\} \quad , \tag{17}$$

in which the $(r \times r)$ real matrix $[\boldsymbol{\mathcal{R}}(\mathbf{s}, \delta)]$ is defined by

$$[\boldsymbol{\mathcal{R}}(\mathbf{s}, \delta)]_{\alpha\beta} = \underline{\boldsymbol{\varphi}}_\alpha^{exp,T}\, \mathbf{R}_\beta(\mathbf{s}, \delta) \quad . \tag{18}$$

Note that the cost function $j(\mathbf{s}, \delta)$ tends to the cost function $\underline{j}(\mathbf{s})$ as δ_M and δ_K go to zero, which means as the

structure tends to be deterministic. The straightforward generalization of Eq. (7) to the random case yields the

solution $(\mathbf{s}^{opt}, \delta^{opt}) = arg \min_{\mathbf{s} \in \mathcal{S}} j(\mathbf{s}, \delta)$. The following comment shows that this formulation is not adapted

to the robust updating context. If the deterministic updating context assumed that there were no model uncertain-

ties and no parameter uncertainties, then it would mean that the family of deterministic models would be able to

exactly reproduce the experimental data. In that case, the deterministic cost function would be zero for the updated

solution. In the present context of robust updating, there are model uncertainties which are then taken into account

by a class of computational model generated with the nonparametric probabilistic approach. The above formula-

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tion for robust updating tends to minimize the model uncertainties ($\delta \to 0$) which means that this formulation is equivalent to the deterministic updating formulation. However, since it is assumed that there are significant model uncertainties, the class of deterministic computational models is not able to reproduce the experiments. Consequently, the cost function is doubtlessly minimized but is nonzero and there still exists an irreducible distance between each eigenvalue /eigenvector of the updated computational model and each experimental eigenvalue / eigenvector. The above formulation for robust updating is then not correct. In order to generate a larger class of uncertain computational models, additional probabilistic constraints involving these distances are added in the formulation of the robust updating optimization problem. Let $\Delta\Lambda$ and $\Delta\widetilde{\Phi}$ be the positive-valued random variables defined by

$$\Delta\Lambda(\mathbf{s},\delta) = \sqrt{\frac{1}{r}\sum_{\alpha=1}^{r}\{\Delta\Lambda_\alpha(\mathbf{s},\delta)\}^2} \quad , \quad \Delta\Lambda_\alpha(\mathbf{s},\delta) = \frac{|\Lambda_\alpha(\mathbf{s},\delta) - \underline{\lambda}_\alpha^{exp}|}{\underline{\lambda}_\alpha^{exp}} \quad , \tag{19}$$

$$\Delta\widetilde{\Phi}(\mathbf{s},\delta) = \sqrt{\frac{1}{r}\sum_{\alpha=1}^{r}\{\Delta\widetilde{\Phi}_\alpha(\mathbf{s},\delta)\}^2} \quad , \quad \Delta\widetilde{\Phi}_\alpha(\mathbf{s},\delta) = \frac{||\widetilde{\Phi}_\alpha(\mathbf{s},\delta) - \underline{\varphi}_\alpha^{exp}||}{||\underline{\varphi}_\alpha^{exp}||} \quad . \tag{20}$$

In Eqs. (19) and (20), for each α belonging to $\{1,\ldots,r\}$, the positive-valued random eigenvalue $\Lambda_\alpha(\mathbf{s},\delta)$ and the $\mathbb{R}^{n_{obs}}$-valued random eigenvector $\widetilde{\Phi}_\alpha(\mathbf{s},\delta)$ restricted to the measurement DOF are defined by the generalized eigenvalue problem related to the uncertain computational model which is written as: find $(\Lambda_\alpha(\mathbf{s},\delta), \Psi_\alpha(\mathbf{s},\delta))$

$$\mathbf{0} = \left([\mathbf{K}_{red,\alpha}(\mathbf{s},\delta_K)] - \Lambda_\alpha(\mathbf{s},\delta)\,[\mathbf{M}_{red,\alpha}(\mathbf{s},\delta_M)]\right)\Psi_\alpha(\mathbf{s},\delta) \quad , \alpha = 1,\ldots,r \quad , \tag{21}$$

for which random eigenvector $\widetilde{\Phi}_\alpha(\mathbf{s},\delta)$ is reconstructed by

$$\widetilde{\Phi}_\alpha(\mathbf{s},\delta) = [\underline{H}_1]\,\Psi_\alpha(\mathbf{s},\delta) \quad , \tag{22}$$

where $[\underline{H}_1] = [[I] \quad [\mathbb{0}]]$ is the first row bloc of matrix $[\underline{H}_\alpha(\mathbf{s})]$. We now introduce the probabilistic constraints. Let $g_\Lambda(\mathbf{s},\delta;\beta_\Lambda,\varepsilon_\Lambda)$ and $g_{\widetilde{\Phi}}(\mathbf{s},\delta;\beta_\Phi,\varepsilon_\Phi)$ be the functions defined by

$$g_\Lambda(\mathbf{s},\delta;\beta_\Lambda,\varepsilon_\Lambda) = \beta_\Lambda - Proba\left(\Delta\Lambda(\mathbf{s},\delta) < \varepsilon_\Lambda\right) \tag{23}$$

$$g_{\widetilde{\Phi}}(\mathbf{s},\delta;\beta_\Phi,\varepsilon_\Phi) = \beta_\Phi - Proba\left(\Delta\Phi(\mathbf{s},\delta) < \varepsilon_\Phi\right) \quad , \tag{24}$$

in which $Proba$ denotes the probability and where ε_Λ, ε_Φ and β_Λ, β_Φ denote a given error level and a given probability level respectively. The robust updating formulation consists in defining, for a given $\boldsymbol{\beta} = (\beta_\Lambda, \beta_\Phi)$ belonging to $[0\,,\,1[\times[0\,,\,1[$ and for a given $\boldsymbol{\varepsilon} = (\epsilon_\Lambda, \epsilon_\Phi)$ belonging to $]0,\,+\infty[\times]0,\,+\infty[$, the solution $(\mathbf{s}^{opt}, \boldsymbol{\delta}^{opt})$ as

$$(\mathbf{s}^{opt}, \boldsymbol{\delta}^{opt}) = arg \min_{\substack{(\mathbf{s}, \boldsymbol{\delta}) \in \{\mathcal{S} \times \Delta\} \\ \mathbf{g}(\mathbf{s}, \boldsymbol{\delta}; \boldsymbol{\beta}, \boldsymbol{\varepsilon}) < \mathbf{0}}} j(\mathbf{s}, \boldsymbol{\delta}) \quad, \tag{25}$$

in which $\mathbf{g}(\mathbf{s}, \boldsymbol{\delta}; \boldsymbol{\beta}, \boldsymbol{\varepsilon}) = (g_\Lambda(\mathbf{s}, \boldsymbol{\delta}; \beta_\Lambda, \varepsilon_\Lambda), g_{\widetilde{\Phi}}(\mathbf{s}, \boldsymbol{\delta}; \beta_\Phi, \varepsilon_\Phi))$. The existence of a solution for this optimization problem cannot be proven in the general case. A specific analysis must be carried out for every application (see Section V).

IV. Numerical Validation

A. Description of the mean finite element model

The numerical validation is carried out using the truss system presented in.[?] This structure is located in the plane $(OX\,,\,OY)$ of a Cartesian coordinate system. The truss is constituted of 4 vertical bars, 4 diagonal bars and 2 horizontal beams. For the non updated truss, all the bars and beams are made up of a homogeneous isotropic elastic material with mass density $\rho_0 = 2\,800\,kg \times m^{-3}$, Poisson ratio $\nu_0 = 0.3$ and Young modulus $E_0 = 0.75 \times 10^{11}\,N \times m^{-2}$. The vertical bars have a constant cross-section of $0.6 \times 10^{-2}\,m^2$ and a length of $3\,m$. The diagonal bars have a constant cross-section of $0.3 \times 10^{-2}\,m^2$ and a length of $5.83\,m$. The horizontal beams have a constant cross-section of $S_0 = 0.4 \times 10^{-2}\,m^2$, a constant beam inertia of $0.756 \times 10^{-1}\,m^4$ and a length of $15\,m$. The truss has free-free boundary conditions. The mean finite element model of this truss is constituted of 41 bar elements (with two nodes) and 42 beam elements (with two nodes) yielding $n = 166$ DOF (see Fig. 1). There is only one updating parameter $s = \rho\,S_0$ with ρ the mass density of the upper beam which has to be updated. It should be noted that for this non updated truss, $s_0 = 11.2\,kg/m$. The admissible set $\mathcal{S}$ for the updating parameter s of the mean computational model is taken as $\mathcal{S} = [10\,,\,40]\,kg/m$.

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B. Description of the data basis

Since no experiment has been carried out on this truss, a numerical experiment is generated to represent the experimental data basis. The experimental data are simulated as follows. We consider the stochastic computational model corresponding to the mean computational model with uncertainties and defined by Eqs. (21) and (22). For $s = s_0$ and $\delta_K = \delta_M = \delta_0$ with $\delta_0 = 0.3$, one realization $\Lambda_\alpha(s_0, \delta_0; \theta)$ of the random eigenvalues $\Lambda_\alpha(s_0, \delta_0)$, and the corresponding realization $\widetilde{\boldsymbol{\Phi}}_\alpha(s_0, \delta_0; \theta)$ of the random eigenvectors $\widetilde{\boldsymbol{\Phi}}_\alpha(s_0, \delta_0)$ are calculated using the stochastic computational model. Then, an arbitrary finite perturbation is applied to every eigenvalues $\Lambda_\alpha(s_0, \delta_0; \theta)$ without modifying the eigenmodes $\widetilde{\boldsymbol{\Phi}}_\alpha(s_0, \delta_0; \theta)$ and thus defining the experimental data. Consequently, this experimental data cannot be obtained with a deterministic updating of the truss ($\delta_M = \delta_K = 0$) for which the mass density ρ of the upper beam is the updating parameter. The experimental data basis is thus constituted of (1) $r = 3$ elastic experimental eigenfrequencies $\underline{\nu}_1^{exp} = 93\,Hz$, $\underline{\nu}_2^{exp} = 110\,Hz$ and $\underline{\nu}_3^{exp} = 170\,Hz$ and (2) the translational components corresponding to $n_{obs} = 28$ translational measured DOF and representing the corresponding experimental eigenmodes (see Fig. 1).
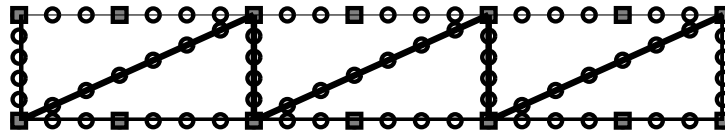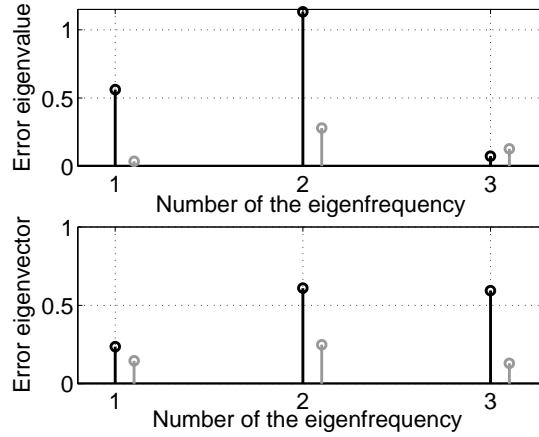

Figure 1. Finite element mesh of the truss. Symbol ○: nodes of the mesh, symbol □ measured nodes, thick solid line: elements with fixed properties, thin solid line: elements whose properties have to be updated.

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C. Deterministic updating

The results concerning the deterministic updating formulation (see Section 2) are presented in order to construct a reference solution. The deterministic updating optimization problem yields $s^{opt,det} = 31 \, kg/m$ for which cost function $\underline{j}(s^{opt,det})$ is normalized to 1. Figure 2 and Table 1 quantify the differences with respect to each eigenfrequency and with respect to each eigenmode for the non updated mean computational model and for the updated mean computational model. For a given α belonging to $\{1, \ldots, r\}$, we introduce $\Delta \underline{\lambda}_\alpha(s) \, \underline{\lambda}_\alpha^{exp} = |\underline{\lambda}_\alpha(s) - \underline{\lambda}_\alpha^{exp}|$ and $\Delta \underline{\tilde{\phi}}_\alpha(s) \, \|\underline{\varphi}_\alpha^{exp}\| = \|\underline{\tilde{\varphi}}_\alpha(s) - \underline{\varphi}_\alpha^{exp}\|$. Let $\Delta \underline{\lambda}_\alpha^{ini} = \Delta \underline{\lambda}_\alpha(s_0)$, $\Delta \underline{\lambda}_\alpha^{opt,det} = \Delta \underline{\lambda}_\alpha(s^{opt,det})$, $\Delta \underline{\tilde{\phi}}_\alpha^{ini} = \Delta \underline{\tilde{\phi}}_\alpha(s_0)$, $\Delta \underline{\tilde{\phi}}_\alpha^{opt,det} = \Delta \underline{\tilde{\phi}}_\alpha(s^{opt,det})$ the similar quantities to those defined in Eq. (13) but for the deterministic case. Figure 2 shows the graphs $\alpha \mapsto \Delta \underline{\lambda}_\alpha^{ini}$, $\alpha \mapsto \Delta \underline{\lambda}_\alpha^{opt,det}$, $\alpha \mapsto \Delta \underline{\tilde{\phi}}_\alpha^{ini}$ and $\alpha \mapsto \Delta \underline{\tilde{\phi}}_\alpha^{opt,det}$. The results show the efficiency of the deterministic updating formulation to reduce the gap between the experiments and between the computational model. Nevertheless, the cost function is not zero which means that model uncertainties have to be taken into account in the modeling of the computational model which has to be updated.

Figure 2. Quantification of the errors between the non updated and the updated mean computational model with the experimental data. Upper graph : graph of $\alpha \mapsto \Delta \underline{\lambda}_\alpha^{ini}$ (black line), $\alpha \mapsto \Delta \underline{\lambda}_\alpha^{opt,det}$ (gray line). Lower graph : graph of $\alpha \mapsto \Delta \underline{\tilde{\phi}}_\alpha^{ini}$ (black line), $\alpha \mapsto \Delta \underline{\tilde{\phi}}_\alpha^{opt,det}$ (gray line).

	$\Delta \underline{\lambda}_1$	$\Delta \underline{\lambda}_2$	$\Delta \underline{\lambda}_3$	$\Delta \underline{\tilde{\phi}}_1$	$\Delta \underline{\tilde{\phi}}_2$	$\Delta \underline{\tilde{\phi}}_3$
non updated	56.1%	113%	7.3%	23.6%	60.9%	59.4%
deterministic updating	3.6%	27.9%	12.7%	14.6%	24.9%	13%

Table 1. Quantification of the errors between the non updated and the updated mean computational model with the experimental data.

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D. Convergence analysis with respect to the numerical parameters
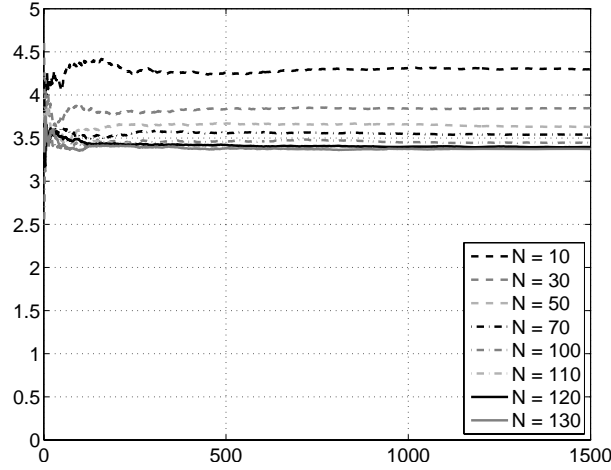
In the context of the robust updating, the stochastic equations of the uncertain computational model are solved by using the Monte Carlo numerical simulation. In order to simplify the calculations, the same level of uncertainties is considered for the mass and for the stiffness terms, that is to say $\delta = \delta_M = \delta_K$. A convergence analysis is carried out in order to calculate the number N of eigenmodes to be kept in the modal reduction and the number n_s of realizations. The mean square convergence is analyzed by studying the function $(N, n_s) \mapsto Conv(N, n_s)$ defined by

$$Conv^2(N, n_s) = \frac{1}{n_s} \sum_{i=1}^{n_s} ||[\mathcal{R}(s, \delta; \theta_i)]||_F^2 \quad , \tag{26}$$

in which $[\mathcal{R}(s, \delta; \theta_i)]$ is the realization number i of random matrix $[\mathcal{R}(s, \delta)]$ given by Eq. (18). The convergence analysis is carried out with $s = 11.2\,kg/m$ and with $\delta = 0.3$. Figure 3 shows the graph $n_s \mapsto Conv(N, n_s)$ for different values of N. It can be seen that a reasonable convergence is reached for $N = 110$ ans $n_s = 600$. From now on, the numerical calculations are carried out with the numerical parameters $N = 110$ ans $n_s = 600$. It should be noted that a more precise convergence analysis could be performed in studying the convergence on the robust updating solution. However, such an analysis would imply that the optimization problem should be solved many times and this is time consuming. That is why the convergence analysis has been carried out on the objective function for a given fixed set of updating parameters. In particular, the value 0.3 of the updating dispersion parameter has been set to a sufficiently high value in order to ensure that the values of the optimal numerical parameters be also valid for smaller values of the updating dispersion parameter.
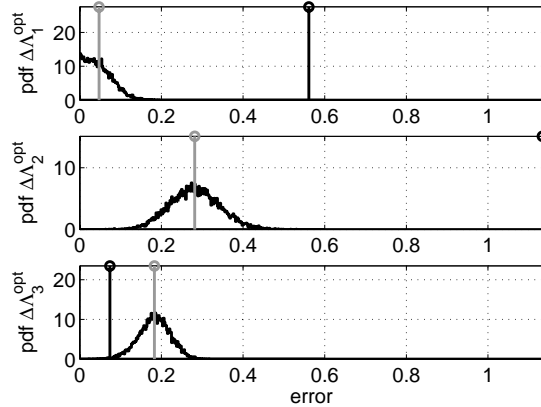
E. Robust updating formulation without inequality constraints

As we have explained in Section IV, the robust updating formulation without inequality constraints does not allow the updating to be improved with respect to the presence of model uncertainties. In this subsection, we prove this result by using the numerical example. First, the case for which the level of uncertainty in the structure is assumed to be known is considered with $\delta = \delta^{fix} = 0.3$. The updated uncertain computational model is characterized by updating parameters $(s^{opt}, \delta^{fix}) = (26.2, 0.3)$ for which $j(s^{opt}, \delta^{fix}) = 1.18$. The generalized eigenvalue problem related to the updated uncertain computational model is then solved by using $n_s = 10\,000$ realizations in order to characterize, for each α belonging to $\{1, 2, 3\}$ the probability density functions of the random variables

Figure 3. Convergence analysis : graph of function $n_s \mapsto Conv(N, n_s)$ **for the truss structure with updating parameter** $s = 11.2\,Kg/m$ **and** $\delta = 0.3$.

$\Delta\Lambda_\alpha^{opt} = \Delta\Lambda(s^{opt}, \delta^{fix})$ and $\Delta\widetilde{\Phi}_\alpha^{opt} = \Delta\widetilde{\Phi}(s^{opt}, \delta^{fix})$. For each α belonging to $\{1, 2, 3\}$, Table 2 shows the

mean values $\mu_{\Delta\lambda_\alpha}$ and $\mu_{\Delta\tilde{\Phi}_\alpha}$, and the standard deviations $\sigma_{\Delta\lambda_\alpha}$ and $\sigma_{\Delta\tilde{\Phi}_\alpha}$ of the random variables $\Delta\Lambda_\alpha^{opt}$ and

$\Delta\widetilde{\Phi}_\alpha^{opt}$. Figures 4 and 5 show the probability density functions of the random variables $\Delta\Lambda_\alpha^{opt}$ and $\Delta\widetilde{\Phi}_\alpha^{opt}$. It can

be seen that the mean error committed on each eigenvalue is lower than 29% and the mean error committed on

each eigenvector is lower than 19%. Figure 6 shows the family of graphs corresponding to the function $\delta \mapsto j(s, \delta)$

for the admissible set $\mathcal{S}$. Clearly, it can be seen that if the uncertainty level is unknown, then the robust updating

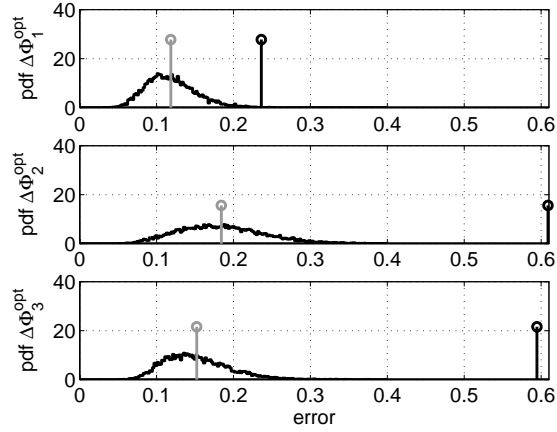optimization problem goes to the deterministic solution presented in subsection C.

Figure 4. Updated uncertain computational model corresponding to $(s^{opt}, \delta^{fix}) = (26.2, 0.3)$**. Graph of the probability density functions** $\Delta\Lambda_\alpha^{opt}$ **(black line), of its first order moment** $\mathcal{E}\{\Delta\Lambda_\alpha^{opt}\}$ **(vertical gray line), of** $\Delta\underline{\lambda}_\alpha^{ini}$ **(vertical black line) for** $\alpha = 1$ **(upper graph),** $\alpha = 2$ **(middle graph),** $\alpha = 3$ **(lower graph).**
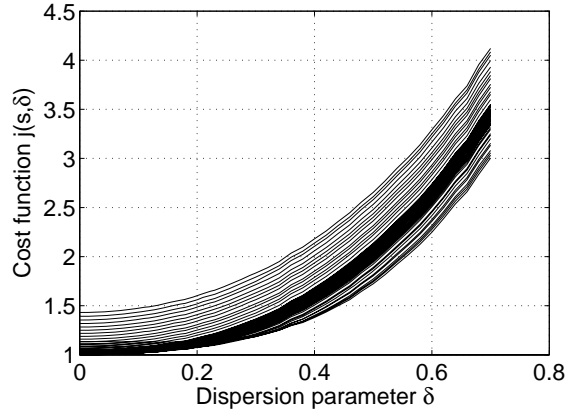
Figure 5. **Updated uncertain computational model corresponding to** $(s^{opt}, \delta^{fix}) = (26.2, 0.3)$. **Graph of the probability density functions** $\Delta\widetilde{\Phi}_\alpha^{opt}$ **(black line), of its first order moment** $\mathcal{E}\{\Delta\widetilde{\Phi}_\alpha^{opt}\}$ **(vertical gray line), of** $\Delta\underline{\widetilde{\phi}}_\alpha^{ini}$ **(vertical black line) for** $\alpha = 1$ **(upper graph),** $\alpha = 2$ **(middle graph),** $\alpha = 3$ **(lower graph).**

Figure 6. **Family of graphs** $\delta \mapsto j(s, \delta)$ **for** s **belonging to** $\mathcal{S}$.

F. Robust updating formulation with inequality constraints

We now present the results concerning the robust updating formulation in presence of inequality constraints obtained with Eq. (25). The updated mean parameter s^{opt} and the updated parameter δ^{opt} are analyzed as a function of the probability level and of the error level. Three cases are considered : (1) the case for which there is only one probabilistic constraint for the eigenvalue corresponding to $\beta_\Phi = 0$ and $\varepsilon_\Phi = +\infty$. We then study the function $(\beta_\Lambda, \varepsilon_\Lambda) \mapsto \delta^{opt}$ defined from the domain $\mathcal{D}_{\Lambda,\delta}$ into the set $\mathcal{F}_{\Lambda,\delta}$ and the function $(\beta_\Lambda, \varepsilon_\Lambda) \mapsto s^{opt}$ defined from the domain $\mathcal{D}_{\Lambda,s}$ into the set $\mathcal{F}_{\Lambda,s}$; (2) the case for which there is one probabilistic constraint for the eigenvector corre-

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sponding to $\beta_\Lambda = 0$ and $\varepsilon_\Lambda = +\infty$. We then study the function $(\beta_\Phi, \varepsilon_\Phi) \mapsto \delta^{opt}$ defined from the domain $\mathcal{D}_{\Phi,\delta}$ into the set $\mathcal{F}_{\Phi,\delta}$ and the function $(\beta_\Phi, \varepsilon_\Phi) \mapsto s^{opt}$ defined from the domain $\mathcal{D}_{\Phi,s}$ into the set $\mathcal{F}_{\Phi,s}$; and (3) the case for which there are two probabilistic constraints with $\beta = \beta_\Lambda = \beta_\Phi$ and $\varepsilon = \varepsilon_\Lambda = \varepsilon_\Phi$. We then study the function $(\beta, \varepsilon) \mapsto \delta^{opt}$ defined from the domain $\mathcal{D}_{\Lambda-\Phi,\delta}$ into the set $\mathcal{F}_{\Lambda-\Phi,\delta}$ and the function $(\beta, \varepsilon) \mapsto s^{opt}$ defined from the domain $\mathcal{D}_{\Lambda-\Phi,s}$ into the set $\mathcal{F}_{\Lambda-\Phi,s}$. Figures 7 and 8 show a bidimensional representation of the graph of the functions $(\beta_\Lambda, \varepsilon_\Lambda) \mapsto \delta^{opt}$ and $(\beta_\Lambda, \varepsilon_\Lambda) \mapsto s^{opt}$ (case 1). Figures 9 and 10 show the graph of the functions $(\beta_\Phi, \varepsilon_\Phi) \mapsto \delta^{opt}$ and $(\beta_\Phi, \varepsilon_\Phi) \mapsto s^{opt}$ (case 2). Figure 11 and 12 show the graph of the functions $(\beta, \varepsilon) \mapsto \delta^{opt}$ and $(\beta, \varepsilon) \mapsto s^{opt}$ (case 3). In these figures, the blank zone corresponds to the values of the probability level and of the error level for which the optimization problem defined by Eq. (25) has no solution. By comparing figures 7 and 9 with figures 8 and 10, it can be seen that $\mathcal{D}_{\Lambda,\delta} \subset \mathcal{D}_{\Phi,\delta}$ and that $\mathcal{D}_{\Lambda,s} \subset \mathcal{D}_{\Phi,s}$ which means that the robust updating methodology allows the random eigenvectors to be better updated than the random eigenvalues. In addition, Figure 7 shows that significant model uncertainties ($\delta^{opt} > 0.1$) are obtained for small values of probability level ($\beta < 0.2$). In opposite, Figure 9 shows that significant model uncertainties on the eigenvectors ($\delta^{opt} > 0.1$) are obtained for large values of the probability level ($\beta < 0.6$). These results are coherent because we have introduced in the experimental data model errors only on the eigenvalues. From figures 7 to 12 show that $\mathcal{F}_{\Lambda,\delta} = [0, 0.25]$, $\mathcal{F}_{\Phi,\delta} = [0, 0.18]$, $\mathcal{F}_{\Lambda-\Phi,\delta} = [0, 0.34]$ and $\mathcal{F}_{\Lambda,s} = [31, 36.4]$, $\mathcal{F}_{\Phi,s} = [22.4, 31.1]$, $\mathcal{F}_{\Lambda-\Phi,s} = [28, 31.1]$. Clearly, the sets $\mathcal{F}_{\Lambda,s}$ and $\mathcal{F}_{\Phi,s}$ are almost disjoint which means that the optimal uncertain computational model strongly depends on the nature of the constraints used in the robust updating formulation. It can also be seen that the updated uncertain computational model related to the eigenvector probabilistic constraint is more sensitive to the updated mean parameter s^{opt} than to the updated dispersion parameter δ^{opt} whereas the contrary is observed when using the robust updating formulation related to the eigenvalue probabilistic constraint. Moreover, it can be seen that $\mathcal{F}_{\Lambda-\Phi,s} \subset \mathcal{F}_{\Lambda,s} \cup \mathcal{F}_{\Phi,s}$ and that $\mathcal{F}_{\Phi,\delta} \subset \mathcal{F}_{\Lambda,\delta} \subset \mathcal{F}_{\Lambda-\Phi,\delta}$. This means that when both probabilistic constraints are used in the robust updating formulation, the updated uncertain computational model is mainly sensitive to updated dispersion parameter δ^{opt}.
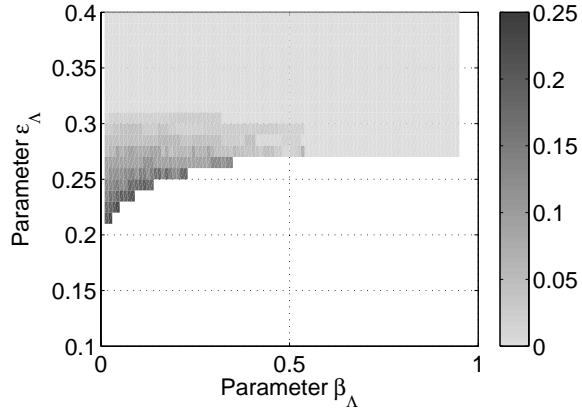
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Figure 7. Graph of δ^{opt} with respect to β_Λ and ε_Λ for $\beta_\Phi = 0, \varepsilon_\Phi = +\infty$.
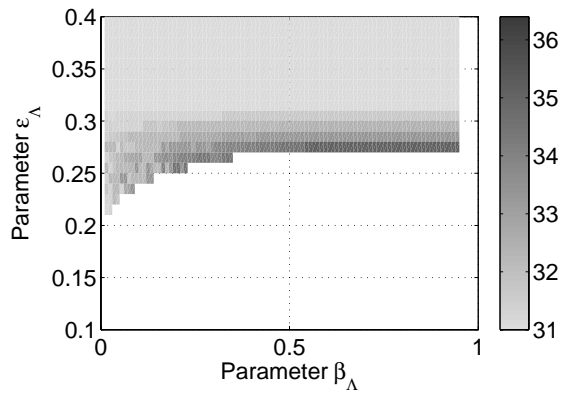

Figure 8. Graph of s^{opt} with respect to β_Λ and ε_Λ for $\beta_\Phi = 0, \varepsilon_\Phi = +\infty$.
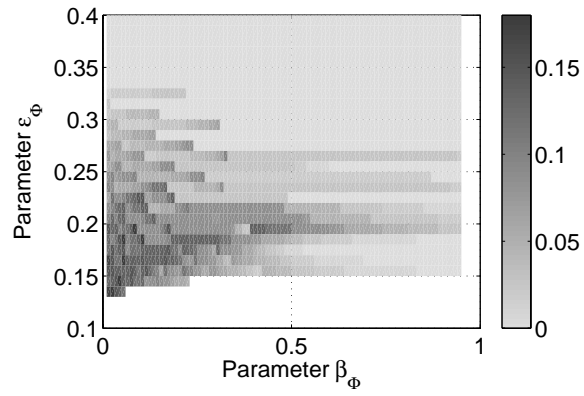
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Figure 9. Graph of δ^{opt} with respect to β_Φ and ε_Φ for $\beta_\Lambda = 0, \varepsilon_\Lambda = +\infty$.

Figure 10. Graph of s^{opt} with respect to β_Φ and ε_Φ for $\beta_\Lambda = 0, \varepsilon_\Lambda = +\infty$.
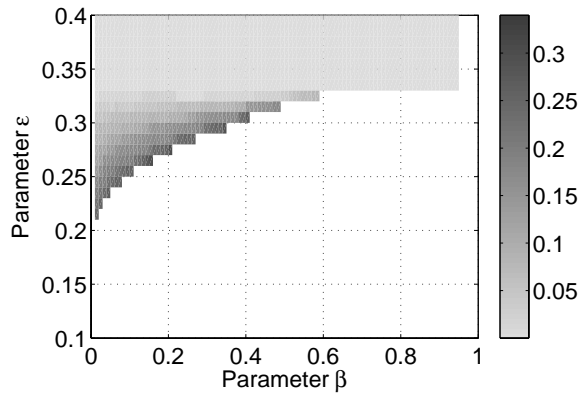
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Figure 11. Graph of δ^{opt} with respect to $\beta = \beta_\Phi = \beta_\Lambda$ and $\varepsilon = \varepsilon_\Phi = \varepsilon_\Lambda$
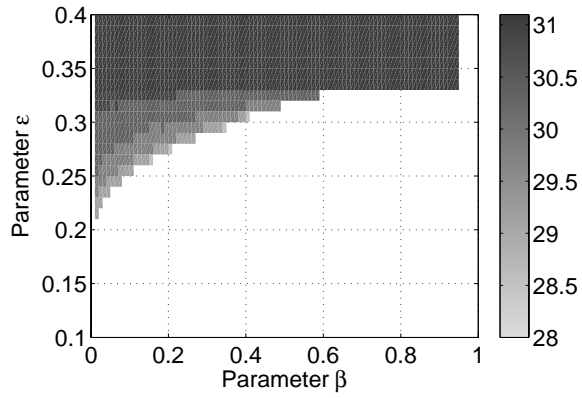

Figure 12. Graph of s^{opt} with respect to $\beta = \beta_\Phi = \beta_\Lambda$ and $\varepsilon = \varepsilon_\Phi = \varepsilon_\Lambda$

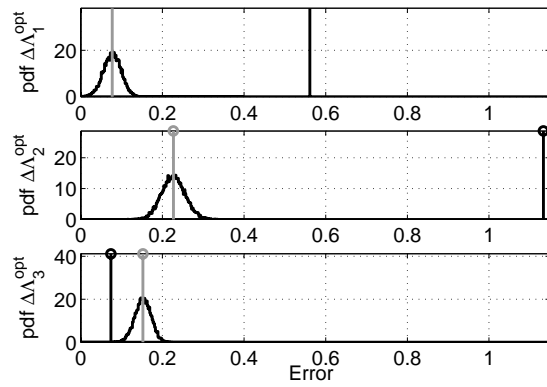American Institute of Aeronautics and Astronautics

In order to analyze more precisely the results presented in the Fig. 7 to 12, we reanalyze the three cases for an error level equal to 0.25 with a probability level equal to 0.1. For α belonging to $\{1, 2, 3\}$, let $\mu_{\Delta\Lambda_\alpha}$, $\mu_{\Delta\tilde{\Phi}_\alpha}$ and $\sigma_{\Delta\Lambda_\alpha}$, $\sigma_{\Delta\tilde{\Phi}_\alpha}$ be the mean value and the standard deviation of random variable $\Delta\Lambda_\alpha$ and $\Delta\tilde{\Phi}_\alpha$ defined by Eqs. (19) and (20). For each case, the main characteristics of the updated uncertain computational model are summarized in Tables 2 and 3. In order to characterize the efficiency of the proposed robust updating methodology, Figs. 13 to 18 show the probability density functions of the random variables $\Delta\Lambda_\alpha^{opt}$ and $\Delta\tilde{\Phi}_\alpha^{opt}$ for the three cases. These figures show that the updating is improved in the probabilistic context because the value of the error is smaller than for the non updated mean computational model. It can be seen that if only one constraint is considered, then the other one is not verified which means that there can remain an important error (for instance $\mu_{\Delta\Lambda_\alpha} = 0.33$ for case 2 for which there is only one eigenvector probability constraint). Moreover, it can be seen that the robust updating using both constraints guarantees that the mean error committed for each eigenvalue and eigenvector with respect to the experimental data is lower than 23.5%.

	$\mu_{\Delta\lambda_1}$	$\mu_{\Delta\lambda_2}$	$\mu_{\Delta\lambda_3}$	$\sigma_{\Delta\lambda_1}$	$\sigma_{\Delta\lambda_2}$	$\sigma_{\Delta\lambda_3}$
constraint on eigenvalue	7.7%	22.7%	15.2%	2.1%	2.8%	2%
constraint on eigenvector	1%	33%	11%	0.7%	1.2%	0.8%
both constraints	6.9%	23.5%	17.7%	3.5%	5%	3.3%
no constraint , $\delta^{fix} = 0.3$	4.7%	28.2%	18.2%	3.3%	6.1%	3.8%
	$\mu_{\Delta\tilde{\Phi}_1}$	$\mu_{\Delta\tilde{\Phi}_2}$	$\mu_{\Delta\tilde{\Phi}_3}$	$\sigma_{\Delta\tilde{\Phi}_1}$	$\sigma_{\Delta\tilde{\Phi}_2}$	$\sigma_{\Delta\tilde{\Phi}_3}$
constraint on eigenvalue	16%	27.5%	15.1%	1.7%	3%	2.6%
constraint on eigenvector	12.2%	19.9%	11.3%	0.7%	1.2%	1%
both constraints	13.9%	22.7%	15.2%	2.9%	5.1%	3.9%
no constraint , $\delta^{fix} = 0.3$	11.8%	18.4%	15.2%	3.2%	5.5%	4.2%

Table 2. Quantification of the errors induced by the updated computational model with respect to the experimental data.

	s^{opt}	δ^{opt}	$j(s^{opt}, \delta^{opt})$	$-g_\Lambda(s^{opt}, \delta^{opt}, 0.25, 0.1)$	$-g_{\tilde{\Phi}}(s^{opt}, \delta^{opt}, 0.25, 0.1)$
constraint on eigenvalue	32.2	0.15	1.06	0.014	< 0
constraint on eigenvector	28.6	0.06	1.03	< 0	0.024
both constraints	29.2	0.26	1.14	0.005	0.009
no constraint , $\delta^{fix} = 0.3$	26.2	0.3	1.18	< 0	0.27

Table 3. Characteristics of the updated computational model for each case.
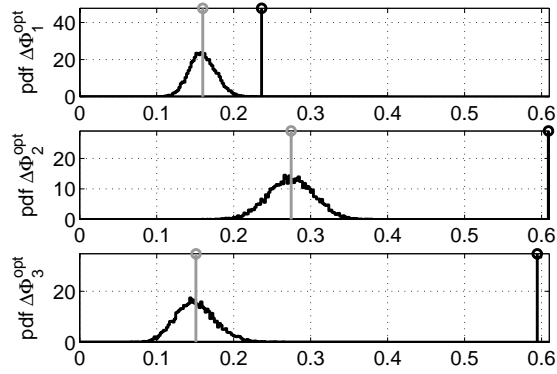
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Figure 13. Updated uncertain computational model corresponding to $\beta_\Phi = 0$, $\epsilon_\Phi = +\infty$, $\beta_\Lambda = 0.1$, $\epsilon_\Lambda = 0.25$ and yielding $(s^{opt}, \delta^{opt}) = (32.2, 0.15)$. Graph of the probability density functions $\Delta\Lambda_\alpha^{opt}$ (black line), of its first order moment $\mathcal{E}\{\Delta\Lambda_\alpha^{opt}\}$ (vertical gray line), of $\Delta\underline{\lambda}_\alpha^{ini}$ (vertical black line) for $\alpha = 1$ (upper graph), $\alpha = 2$ (middle graph), $\alpha = 3$ (lower graph).

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Figure 14. Updated uncertain computational model corresponding to $\beta_\Phi = 0$, $\epsilon_\Phi = +\infty$, $\beta_\Lambda = 0.1$, $\epsilon_\Lambda = 0.25$ **and yielding** $(s^{opt}, \delta^{opt}) = (32.2, 0.15)$**. Graph of the probability density functions** $\Delta\widetilde{\Phi}_\alpha^{opt}$ **(black line), of its first order moment** $\mathcal{E}\{\Delta\widetilde{\Phi}_\alpha^{opt}\}$ **(vertical gray line), of** $\Delta\widetilde{\underline{\phi}}_\alpha^{ini}$ **(vertical black line) for** $\alpha = 1$ **(upper graph),** $\alpha = 2$ **(middle graph),** $\alpha = 3$ **(lower graph).**
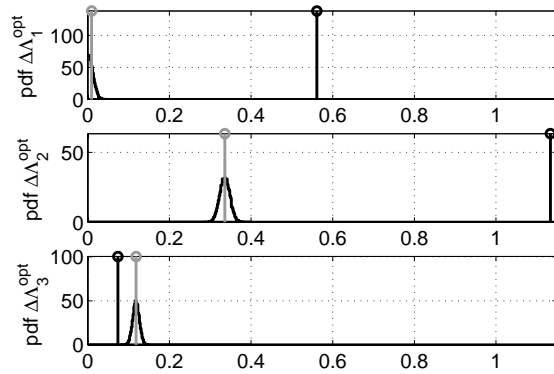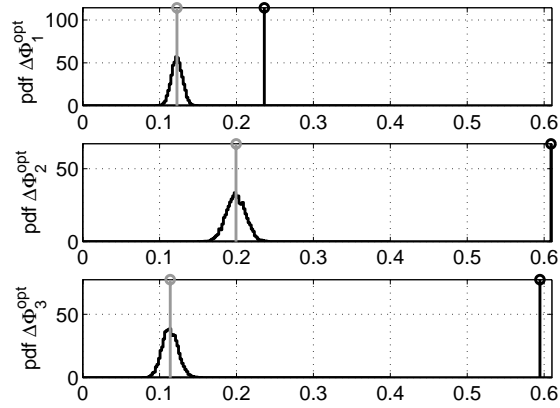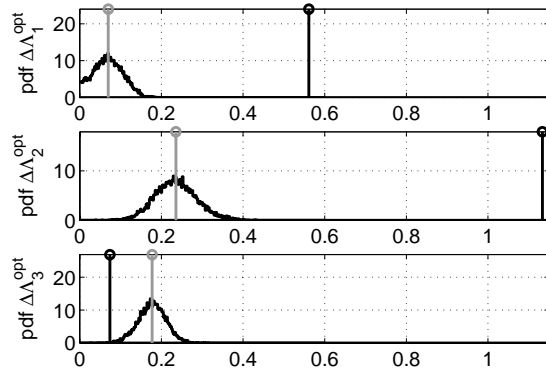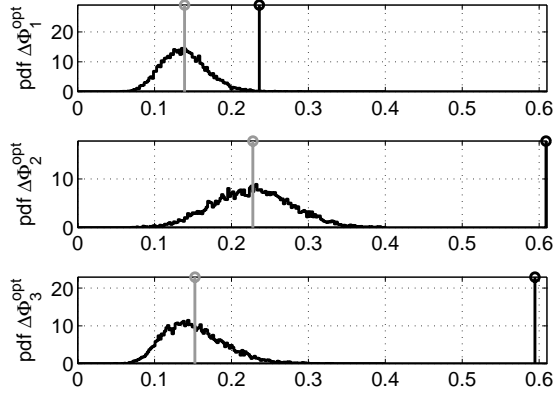

Figure 15. Updated uncertain computational model corresponding to $\beta_\Lambda = 0$, $\epsilon_\Lambda = +\infty$, $\beta_\Phi = 0.1$, $\epsilon_\Phi = 0.25$ **and yielding** $(s^{opt}, \delta^{opt}) = (28.6, 0.06)$**. Graph of the probability density functions** $\Delta\Lambda_\alpha^{opt}$ **(black line), of its first order moment** $\mathcal{E}\{\Delta\Lambda_\alpha^{opt}\}$ **(vertical gray line), of** $\Delta\underline{\lambda}_\alpha^{ini}$ **(vertical black line) for** $\alpha = 1$ **(upper graph),** $\alpha = 2$ **(middle graph),** $\alpha = 3$ **(lower graph).**

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Figure 16. Updated uncertain computational model corresponding to $\beta_\Lambda = 0, \epsilon_\Lambda = +\infty, \beta_\Phi = 0.1, \epsilon_\Phi = 0.25$ **and yielding** $(s^{opt}, \delta^{opt}) = (28.6, 0.06)$. **Graph of the probability density functions** $\Delta\widetilde{\Phi}_\alpha^{opt}$ **(black line), of its first order moment** $\mathcal{E}\{\Delta\widetilde{\Phi}_\alpha^{opt}\}$ **(vertical gray line), of** $\Delta\underline{\widetilde{\phi}}_\alpha^{ini}$ **(vertical black line) for** $\alpha = 1$ **(upper graph),** $\alpha = 2$ **(middle graph),** $\alpha = 3$ **(lower graph).**

Figure 17. Updated uncertain computational model corresponding to $\beta_\Phi = \beta_\Lambda = \beta = 0.1, \varepsilon_\Phi = \varepsilon_\Lambda = \varepsilon = 0.25$ **and yielding** $(s^{opt}, \delta^{opt}) = (29.2, 0.26)$. **Graph of the probability density functions** $\Delta\Lambda_\alpha^{opt}$ **(black line), of its first order moment** $\mathcal{E}\{\Delta\Lambda_\alpha^{opt}\}$ **(vertical gray line), of** $\Delta\underline{\lambda}_\alpha^{ini}$ **(vertical black line) for** $\alpha = 1$ **(upper graph),** $\alpha = 2$ **(middle graph),** $\alpha = 3$ **(lower graph).**

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Figure 18. **Updated uncertain computational model corresponding to** $\beta_\Phi = \beta_\Lambda = \beta = 0.1$, $\varepsilon_\Phi = \varepsilon_\Lambda = \varepsilon = 0.25$ **and yielding** $(s^{opt}, \delta^{opt}) = (29.2, 0.26)$. **Graph of the probability density functions** $\Delta\widetilde{\Phi}_\alpha^{opt}$ **(black line), of its first order moment** $\mathcal{E}\{\Delta\widetilde{\Phi}_\alpha^{opt}\}$ **(vertical gray line), of** $\Delta\underline{\widetilde{\phi}}_\alpha^{ini}$ **(vertical black line) for** $\alpha = 1$ **(upper graph),** $\alpha = 2$ **(middle graph),** $\alpha = 3$ **(lower graph).**

V.   Conclusions

A not straightforward methodology to perform the robust updating of complex uncertain dynamical systems with respect to modal experimental data in the context of structural dynamics has been presented. The present formulation based on an input error methodology adapted to the deterministic updating problem has been extended to the robust updating context required in presence of model uncertainties in the computational model. The robust updating formulation leads a mono-objective optimization problem to be solved in presence of inequality probabilistic constraints. An application is presented in order to validate the proposed approach.

Acknowledgements