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ON THE SELF-DECOMPOSABILITY OF THE FRÉCHET DISTRIBUTION

PIERRE BOSCH AND THOMAS SIMON

ABSTRACT. Let $\{\Gamma_t, t \geq 0\}$ be the Gamma subordinator. Using a moment identification due to Bertoin-Yor (2002), we observe that for every $t > 0$ and $\alpha \in (0, 1)$ the random variable $\Gamma_t^{-\alpha}$ is distributed as the exponential functional of some spectrally negative Lévy process. This entails that all size-biased samplings of Fréchet distributions are self-decomposable and that the extreme value distribution F_ξ is infinitely divisible if and only if $\xi \notin (0, 1)$, solving problems raised by Steutel (1973) and Bondesson (1992). We also review different analytical and probabilistic interpretations of the infinite divisibility of $\Gamma_t^{-\alpha}$ for $t, \alpha > 0$.

1. INTRODUCTION

The extreme value theorem - see e.g. Theorem 8.13.1 in [4] - states that non-degenerate distribution functions arising as limits of properly renormalized running maxima of i.i.d. random variables belong to one of the families

$$F_0(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}, \quad \text{or} \quad F_\xi(x) = \begin{cases} 1 - e^{-x^{1/\xi}} & \text{if } \xi > 0 \\ e^{-x^{1/\xi}} & \text{if } \xi < 0 \end{cases}, \quad x > 0.$$

The distribution F_0 is known as the Gumbel distribution, whereas F_ξ is called a Weibull distribution for $\xi > 0$ and a Fréchet distribution for $\xi < 0$. In the following, we denote by X_ξ the random variable with distribution function F_ξ . Observe that

$$\frac{1 - X_\xi}{\xi} \xrightarrow{d} X_0 \quad \text{as } \xi \rightarrow 0,$$

so that the above parametrization is continuous in ξ . In the present paper we are interested in the self-decomposability (SD) of X_ξ , referring e.g. to Section 15 in [14] for an account on self-decomposability. The Gumbel distribution is SD because of the identities

$$X_0 \stackrel{d}{=} -\log L \stackrel{d}{=} -\alpha \log L + \alpha \log S_\alpha$$

for every $\alpha \in (0, 1)$, where here and throughout L stands for the standard exponential variable and S_α for the standard positive α -stable variable - see e.g. Exercise 29.16 in [14] for a proof of the second identity. If $\xi \in (0, 1)$ then the variable X_ξ is not infinitely divisible (ID) and hence not SD, because of its superexponential distribution tails - see e.g. Theorem 26.1 in [14]. When $\xi \geq 1$, the variable X_ξ has a completely monotone density and is ID by Goldie's criterion - see e.g. Theorem 4.2 in [17], or by the ME property which makes it the

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first-passage time of some continuous time Markov chain - see e.g. Chapter 9 in [5] for an account. When $|\xi| \geq 1$, the identity in law

$$X_\xi \stackrel{d}{=} L^\xi$$

and the HCM theory of Thorin and Bondesson [5] show that the distribution of X_ξ is a generalized Gamma convolution (GGC) and is hence SD - see Example 4.3.4 in [5]. The natural question whether X_ξ is SD or even ID for $\xi \in (-1, 0)$ was first raised by Steutel in 1973 - see Section 3.4 in [17], and has remained open ever since. In section 4.5 of [5] - see also the Appendix B.3 of [18], this problem is rephrased in the broader context of generalized Gamma distributions. The latter are power transformations of Γ_t where $\{\Gamma_t, t \geq 0\}$ is the Gamma subordinator, and can be thought of as size-biased samplings of X_ξ when $\xi < 0$, in view of the formulæ

$$\mathbb{E}[f(\Gamma_t^\xi)] = \frac{\mathbb{E}[f(X_\xi)X_\xi^u]}{\mathbb{E}[X_\xi^u]}$$

valid for every f bounded continuous and $t > 0$, with $u = (t - 1)/\xi$. Recall in passing that Steutel's equation - see e.g. Theorem 51.1 in [14] - establishes a precise link between size-biased sampling of order one and infinite divisibility for integrable positive random variables. In this note, we provide an answer to the above questions of [17, 5].

Theorem. *For every $\xi \in (-1, 0)$ and $t > 0$, the random variable Γ_t^ξ is SD.*

As a direct consequence of this result, all Fréchet distributions are SD and the extreme value distribution F_ξ is ID if and only if $\xi \notin (0, 1)$. Contrary to the case $|\xi| \geq 1$, our argument is probabilistic and consists in showing that Γ_t^ξ is distributed as the exponential functional of some spectrally negative Lévy process. This extends a classical result of Dufresne [6] for the case $\xi = -1$. The identification is made possible thanks to an entire moment method due to Bertoin-Yor [3], which applies in our context as a case study. The proof is given in the next section.

In Section 3, we review the possible interpretations of the infinite divisibility of Γ_t^ξ for $\xi < 0$. The classical case $\xi = -1$ allows at least four different formulations in terms of processes, and also an explicit computation of the Lévy density which shows the GGC property without the HCM argument. For $\xi < -1$ the ID property is only known by analytical means and there is no direct probabilistic explanation, save for the case $t = 1$ by subordination or, tentatively, the spectral theory of a certain spectrally positive Markov processes. The situation for $\xi \in (-1, 0)$ is exactly the opposite since in addition to the exponential functional argument, the ID property can also be obtained rigorously by a first-passage time argument for a spectrally positive Markov processes. On the other hand there is no analytic proof of the ID property for $\xi \in (-1, 0)$. In this situation the GGC character of Γ_t^ξ remains in particular an open question, which we plan to tackle in some further research.

2. PROOF OF THE THEOREM

We begin with a computation on the Gamma function.

Lemma. For every $\alpha \in (0, 1)$ and $u, t > 0$ one has

$$\frac{u\Gamma(t + \alpha(u + 1))}{\Gamma(t + \alpha u)} = \left(\frac{\Gamma(t + \alpha)}{\Gamma(t)} \right) u + \int_{-\infty}^0 (e^{ux} - 1 - ux) f_{\alpha,t}(x) dx,$$

where

$$f_{\alpha,t}(x) = \frac{e^{(1+t/\alpha)x} (\alpha + e^{x/\alpha} + t(1 - e^{x/\alpha}))}{\alpha\Gamma(1 - \alpha)(1 - e^{x/\alpha})^{\alpha+2}}$$

is the density of a Lévy measure on $(-\infty, 0)$.

Proof. We set $\lambda = t + \alpha u > 0$ and compute

$$\begin{aligned} \frac{\Gamma(\lambda + \alpha)}{\Gamma(\lambda)} &= \frac{\lambda\beta(\lambda + \alpha, 1 - \alpha)}{\Gamma(1 - \alpha)} \\ &= \frac{\lambda}{\Gamma(1 - \alpha)} \int_0^{+\infty} \frac{e^{-(\alpha+\lambda)x}}{(1 - e^{-x})^\alpha} dx \\ &= \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^{+\infty} (1 - e^{-\lambda x}) \frac{e^{-\alpha x}}{(1 - e^{-x})^{\alpha+1}} dx \end{aligned}$$

where the second equality comes from a change of variable and the third from an integration by parts. This yields

$$\begin{aligned} \frac{u\Gamma(t + \alpha(u + 1))}{\Gamma(t + \alpha u)} &= \frac{\alpha u}{\Gamma(1 - \alpha)} \int_0^{+\infty} (1 - e^{-(t+\alpha u)x}) \frac{e^{-\alpha x}}{(1 - e^{-x})^{\alpha+1}} dx \\ &= \left(\frac{\Gamma(t + \alpha)}{\Gamma(t)} \right) u + \frac{\alpha u}{\Gamma(1 - \alpha)} \int_{-\infty}^0 (1 - e^{\alpha u x}) \frac{e^{(\alpha+t)x}}{(1 - e^x)^{\alpha+1}} dx \\ &= \left(\frac{\Gamma(t + \alpha)}{\Gamma(t)} \right) u + \int_{-\infty}^0 (e^{ux} - 1 - ux) f_{\alpha,t}(x) dx \end{aligned}$$

where again, the second equality comes from a change of variable and the third from an integration by parts. □

Remarks. (a) The above proof follows [2] p. 102. Notice in passing that some computations performed in [2] are slightly erroneous. For example the subordinator whose exponential functional is distributed as $\tau_\alpha^{-\alpha}$ (with the notation of [2]) has no drift, but it is also killed at rate $1/\Gamma(1 - \alpha)$.

(b) The above decomposition extends to $\alpha = 1$ since

$$\frac{u\Gamma(t + (u + 1))}{\Gamma(t + u)} = u(t + u)$$

is the Lévy-Khintchine exponent of a drifted Brownian motion (the latter was already noticed in [3] - see Example 3 therein - in order to recover Dufresne's identity). However, such a formula does not seem to exist for $\alpha > 1$.

End of the proof. Fix $\xi \in (-1, 0)$, $t > 0$, and set $\alpha = -\xi \in (0, 1)$ for simplicity. The entire moments of Γ_t^α are given for every $n \geq 1$ by

$$\begin{aligned} \mathbb{E}[\Gamma_t^{\alpha n}] &= \frac{\Gamma(t + \alpha n)}{\Gamma(t)} \\ &= \frac{\Gamma(t + \alpha)}{\Gamma(t)} \times \dots \times \frac{\Gamma(t + \alpha n)}{\Gamma(t + \alpha(n-1))} = m \frac{\psi(1) \dots \psi(n-1)}{(n-1)!} \end{aligned}$$

with the notation

$$\psi(u) = \frac{u\Gamma(t + \alpha(u+1))}{\Gamma(t + \alpha u)} = \left(\frac{\Gamma(t + \alpha)}{\Gamma(t)} \right) u + \int_{-\infty}^0 (e^{ux} - 1 - ux) f_{\alpha, t}(x) dx$$

by the Lemma, and

$$m = \frac{\Gamma(t + \alpha)}{\Gamma(t)} = \psi'(0+).$$

It is clear that ψ is the Lévy-Khintchine exponent of a spectrally negative Lévy process Z with infinite variation and mean $m > 0$. By Proposition 2 in [3], this entails

$$\mathbb{E}[\Gamma_t^{\alpha n}] = \mathbb{E}[I^{-n}]$$

for every $n \geq 1$, where I is the exponential functional of Z :

$$I = \int_0^\infty e^{-Z_s} ds.$$

Since Z has no positive jumps, Proposition 2 in [3] shows also that the random variable $1/I$ is moment-determinate, whence

$$\Gamma_t^\xi \stackrel{d}{=} I.$$

The self-decomposability of I is a direct consequence of the Markov property. More precisely, introducing the stopping-time $T_y = \inf\{s > 0, Z_s = y\}$ for every $y > 0$, the fact that $Z_s \rightarrow +\infty$ a.s. as $s \rightarrow +\infty$ and the absence of positive jumps entail that $T_y < +\infty$ a.s. Decomposing, we get

$$I = \int_0^{T_y} e^{-Z_s} ds + \int_{T_y}^\infty e^{-Z_s} ds \stackrel{d}{=} \int_0^{T_y} e^{-Z_s} ds + e^{-y} \int_0^\infty e^{-Z'_s} ds$$

where Z' is an independent copy of Z and the second equality follows from the Markov property at T_y . This shows that for every $c \in (0, 1)$ there is an independent factorization

$$I = cI + I_c$$

for some random variable I_c , in other words that $I \stackrel{d}{=} \Gamma_t^\xi$ is self-decomposable. □

Remarks. (a) By the above Remark 1 (b), it does not seem that Γ_t^ξ is distributed as the exponential functional of a Lévy process for $\xi < -1$. It would be interesting to have an explanation of the infinite divisibility of Γ_t^ξ in terms of processes when $\xi < -1$. See next section for a more precise conjecture in the case $t = 1$.

(b) The self-decomposability of S_α^α for every $\alpha \in (0, 1)$ has been shown by Patie [12] in using the same kind of argument. Specifically, one can write

$$\mathbb{E}[S_\alpha^{n\alpha}] = \frac{\Gamma(1+n)}{\Gamma(1+\alpha n)} = m \frac{\psi(1) \dots \psi(n-1)}{(n-1)!}$$

where we use the same notation as above and, correcting small mistakes made in Paragraph 3.2 of [12],

$$\psi(u) = \frac{u}{\Gamma(1+\alpha)} + \int_{-\infty}^0 (e^{ux} - 1 - ux) \frac{(1-\alpha)e^{x/\alpha}((2-\alpha)e^{x/\alpha} + (1-e^{x/\alpha}))}{\alpha^2 \Gamma(1+\alpha)(1-e^{x/\alpha})^{3-\alpha}} dx$$

is the Lévy-Khintchine exponent of some spectrally negative Lévy process with positive mean. Setting $\alpha = t = 1/2$ and comparing the above expression to the one in the Lemma, one can check the well-known identity in law

$$(2.1) \quad \sqrt{S_{1/2}} = \frac{1}{2\sqrt{\Gamma_{1/2}}}.$$

The present paper shows that all positive powers of $S_{1/2}$ are SD and one may wonder if the same is true for S_α with any $\alpha \in (0, 1)$. See [9] for related results and also for a characterization of the SD property of negative powers of S_α when $\alpha \leq 1/2$.

3. FURTHER REMARKS AND OPEN QUESTIONS

In this section we would like to review several existing or tentative approaches for the ID, SD and GGC properties of the distribution of Γ_t^ξ or X_ξ when $\xi \leq 0$.

3.1. The case $\xi = 0$. This is a rather specific situation but we include it here for completeness. As mentioned in Section 3.4 of [17], the SD property of the two-sided X_0 is a direct consequence of the extreme value theory because

$$(3.1) \quad L_1 + \frac{L_2}{2} + \dots + \frac{L_n}{n} - \log n \stackrel{d}{=} \max(L_1, \dots, L_n) - \log n \xrightarrow{d} X_0$$

as $n \rightarrow +\infty$, where L_1, \dots, L_n are independent copies of $L \sim \text{Exp}(1)$. The above identity and convergence in law, readily obtained in comparing Laplace transforms and distribution functions, yield after some further computations the following closed expression for the Laplace transform of X_0 :

$$\mathbb{E}[e^{-\lambda X_0}] = \Gamma(1+\lambda) = \exp \left[-\gamma\lambda + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x) \frac{dx}{x(e^x - 1)} \right],$$

where γ is Euler's constant. The complete monotonicity of $1/(e^x - 1)$ shows then that X_0 is an extended GGC in the sense of Chapter 7 in [5]. See also Exercise 18.19 in [14] and Example 7.2.3 in [5] for another argument based on Pick functions, recovering (3.1).

3.2. **The case $\xi = -1$.** This is the classical situation, very well-known, but we give some details for comparison purposes. The ID property of X_{-1} can first be understood by the sole fact that

$$\lim_{n \rightarrow +\infty} \left(\frac{nx}{1+nx} \right)^n = e^{-1/x}$$

because the left-hand side is the first-passage time distribution function of a certain birth and death process - see Theorem 3.1 and (3.3) in [17]. The random variable Γ_t^{-1} is also a GIG and is hence distributed as the unilateral first-passage time of a diffusion [1], which explains its infinite divisibility by continuity and the Markov property. More precisely one has

$$(3.2) \quad \frac{1}{4\Gamma_t} \stackrel{d}{=} \inf\{u > 0, X_u^t = 0\}$$

where $\{X_u^t, u \geq 0\}$ is a Bessel process of dimension $2(1-t)$ starting from one. The SD property follows as above from Dufresne's identity [6], which reads

$$(3.3) \quad \frac{2}{\Gamma_t} \stackrel{d}{=} \int_0^\infty e^{B_u - tu/2} du$$

where $\{B_u, u \geq 0\}$ is a standard linear Brownian motion. Also, Exercise 16.4 in [14] shows that Γ_t^{-1} is the one-dimensional marginal of a certain self-similar additive process, whence its self-decomposability by Theorem 16.1 in [14]. The link between this latter interpretation and (3.2) and (3.3) has been explained in depth in [19].

It does not seem that these four interpretations can provide any explicit information on the Lévy-Khintchine exponent of Γ_t^{-1} . But in this case analytical computations can also be performed. More precisely, taking for simplicity the same normalization as in (3.2) and setting $\varphi_t(\lambda) = -\log \mathbb{E}[e^{-\lambda/4\Gamma_t}]$, formulæ (7.12.23), (7.11.25) and (7.11.26) in [7] entail

$$(3.4) \quad \varphi'_t(\lambda) = \frac{K_{t-1}(\sqrt{\lambda})}{2\sqrt{\lambda}K_t(\sqrt{\lambda})}$$

where K_t is the Macdonald function. This shows $\varphi'_{1/2}(\lambda) = 1/2\sqrt{\lambda}$ viz. $\varphi_{1/2}(\lambda) = \sqrt{\lambda}$ when $t = 1/2$, and one recovers the identity (2.1). For $t = 3/2$, one obtains

$$2\varphi'_{3/2}(\lambda) = \frac{1}{1+\sqrt{\lambda}} = \mathbb{E}[e^{-\lambda(L^2 \times S_{1/2})}] = \int_0^\infty \left(\frac{1}{\lambda+x} \right) \frac{\sqrt{x} dx}{\pi(1+x)}$$

where the first equality follows from Formula (7.2.40) in [7], and the third equality from Exercise 29.16 in [14] and (2.2.5) in [5]. This means precisely - see (3.1.1) in [5] - that the distribution of $1/4\Gamma_{3/2}$ is a GGC with zero drift and Thorin measure

$$U_{3/2}(dx) = \frac{\sqrt{x} dx}{2\pi(1+x)}.$$

The latter property can be extended to *all* values of t thanks to a result originally due to Grosswald [8] on Student's distribution. Together with (3.4), the main theorem in [8] entails

namely that the distribution of $1/4\Gamma_t$ is a GGC with zero drift and Thorin measure

$$U_t(dx) = \frac{1}{\pi^2 x (J_t^2(\sqrt{x}) + Y_t^2(\sqrt{x}))}$$

where J_t and Y_t are the usual Bessel functions of the first kind - see [7] p. 4.

3.3. The case $\xi \in (-1, 0)$. In this situation, the present paper yields an interpretation of the self-decomposability of Γ_t^ξ by the identification

$$\Gamma_t^\xi \stackrel{d}{=} \int_0^\infty e^{-Z_u} du,$$

where Z is a spectrally negative Lévy process. Another explanation, similar to (3.2), can then be obtained by the Lamperti transformation - see e.g. the introduction of [3] for an account and references. More precisely, setting

$$Y_u = \exp[-Z_{\tau_u}]$$

with the notation $\tau_u = \inf\{s > 0, \int_0^s e^{-Z_v} dv > u\}$, then $Y = \{Y_u, 0 \leq u < \Gamma_t^\xi\}$ is a spectrally positive Markov process (which is also self-similar) starting from one and we have

$$\Gamma_t^\xi \stackrel{d}{=} \inf\{u > 0, Y_u = 0\},$$

so that the infinite divisibility of Γ_t^ξ (but not, or at least not directly, its self-decomposability) is a consequence of the Markov property and the absence of negative jumps for Y . It would be interesting to see if Γ_t^ξ could be embedded in some self-similar additive process analogous to the Brownian escape process of the case $\xi = 1$, described in Exercise 16.4 of [14].

Our main result can also be interpreted analytically in terms of generalized Bessel functions. Setting $\alpha = -\xi$ and writing down

$$(3.5) \quad \mathbb{E}[e^{-\lambda \Gamma_t^\xi}] = \frac{1}{\alpha \Gamma(t)} \int_0^\infty x^{-t\alpha-1} e^{-\lambda x + x^{-1/\alpha}} dx = \frac{Z_{1/\alpha}^{t/\alpha}(\lambda)}{\alpha \Gamma(t)}$$

with the notation (1.7.42) of [11], the infinite divisibility of Γ_t^ξ entails that the function

$$(3.6) \quad \lambda \mapsto - \left(\frac{Z_\rho^{\nu'}(\lambda)}{Z_\rho^\nu(\lambda)} \right)$$

is completely monotone for any $\rho > 1$ and $\nu > 0$. One might ask if the latter function is also a Stieltjes transform, which is equivalent to the GGC property for the distribution of Γ_t^ξ - see Chapter 3 in [5]. Indeed, it is very natural to conjecture such a property for $\xi \in (-1, 0)$ in view of the above cases $\xi = 0$ and $\xi = -1$. Compared to classical Bessel functions, the theory of generalized Bessel functions is however rather incomplete, and proving like in [8] that the function (3.6) is a Stieltjes transform is believed to be challenging.

3.4. The case $\xi < -1$. In this situation the GGC property of the distribution of Γ_t^ξ is most quickly obtained from the HCM character of the density function - see Chapter 5 and especially Example 5.5.2 in [5]. Notice that this analytical argument extends to $\xi = -1$ but not to $\xi \in (-1, 0)$ since otherwise $\Gamma_t^{-\xi}$ would also have a HCM density and hence be ID, which is false. This entails that the function in (3.6) is indeed a Stieltjes transform for any $\rho \in (0, 1)$ and $\nu > 0$, and it would be interesting to identify the underlying Thorin measure as in Grosswald's theorem.

A probabilistic interpretation of the self-decomposability of $\Gamma_1^\xi = L^\xi$ can also be given by Bochner's subordination. Setting $\alpha = -1/\xi \in (0, 1)$, one has indeed

$$L^\xi \stackrel{d}{=} L^{-1} \times S_\alpha \stackrel{d}{=} S_{L^{-\alpha}}^\alpha.$$

where $\{S_u^\alpha, u \geq 0\}$ stands for the α -stable subordinator with marginal $S_1^\alpha \stackrel{d}{=} S_\alpha$. Since $L^{-\alpha}$ is SD by our result, this means that L^ξ is the marginal of some subordinator which is itself subordinated to the α -stable one, and Proposition 4.1. in [15] shows that L^ξ is SD. Besides, setting φ_α resp. φ_ξ for the Lévy-Khintchine exponent of $L^{-\alpha}$ resp. L^ξ , one deduces from Theorem 30.4 in [14] the following relationship

$$\varphi_\xi(\lambda) = \varphi_\alpha(\lambda^\alpha).$$

Another, tentative, probabilistic interpretation of the self-decomposability of L^ξ could be given in terms of a certain spectrally positive Markov process. Setting $\alpha = -1/\xi$ and $y_\alpha(\lambda) = \mathbb{E}[e^{-\lambda L^\xi}]$, Theorem 4.17 p. 258 in [11] and (3.5) above show that y_α is a solution to the fractional differential equation

$$xD_-^{\alpha+1}y_\alpha - \alpha y_\alpha = 0,$$

where $D_-^{\alpha+1}$ is a fractional Riemann-Liouville derivative - see Section 2.1 in [11]. When $\alpha = 1$ viz. $\xi = -1$ the above amounts to a Bessel equation and Feller's theory applies, making L^{-1} the first-passage time of a Bessel process of index 0 - see [10]. When $\alpha \in (0, 1)$ the operator $D_-^{\alpha+1}$ is the infinitesimal generator of a spectrally positive $(1 + \alpha)$ -stable Lévy process reflected at its minimum, which is a spectrally positive Markov process - see Section 3 in [13] and the references therein. By downward continuity, one may wonder if L^ξ cannot be viewed as the first-passage time of some scale-transformation of the latter, eventhough no Feller's theory is available for fractional operators whose order lies in $(1, 2)$.

The above probabilistic interpretations do not seem to hold for $t \neq 1$. On the one hand, Theorem 4.17 in [11] yields then an equation with *two* fractional derivatives of different order for the Laplace transform of Γ_t^ξ . On the other hand, keeping the notation $\alpha = -1/\xi$, Theorem 1 in [16] shows the factorization

$$\Gamma_t^\xi = \Gamma_{\alpha t}^{-1} \times S_\alpha^{(t)}$$

where $S_\alpha^{(t)}$ is the size-biased sampling of S_α of order $-\alpha t$, viz.

$$\mathbb{E}[f(S_\alpha^{(t)})] = \frac{\mathbb{E}[f(S_\alpha)S_\alpha^{-\alpha t}]}{\mathbb{E}[S_\alpha^{-\alpha t}]}.$$

The GGC character of $S_\alpha^{(t)}$ follows from that of S_α and Theorem 6.2.4. in [5], which shows by the above case $\xi = -1$ that Γ_t^ξ is the independent product of two SD random variables. By Theorem 16.1 in [14], this entails that there exist two independent 1-self-similar additive increasing processes Y and Z such that

$$\Gamma_t^\xi \stackrel{d}{=} Y_{Z_1}.$$

Unfortunately, contrary to Bochner's subordination the independent composition of two additive processes is not necessarily an additive process anymore, so that the above identity does not provide a probabilistic proof of the self-decomposability of Γ_t^ξ .

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