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Asymptotic spectral gap and Weyl law for Ruelle resonances of open partially expanding maps

Jean Francois Arnoldi*, Frédéric Faure[†], Tobias Weich[‡]

01-06-2013

Abstract

We consider a simple model of an open partially expanding map. Its trapped set \mathcal{K} in phase space is a fractal set. We first show that there is a well defined discrete spectrum of Ruelle resonances which describes the asymptotic of correlation functions for large time and which is parametrized by the Fourier component ν on the neutral direction of the dynamics. We introduce a specific hypothesis on the dynamics that we call “minimal captivity”. This hypothesis is stable under perturbations and means that the dynamics is univalued on a neighborhood of \mathcal{K} . Under this hypothesis we show the existence of an asymptotic spectral gap and a fractal Weyl law for the upper bound of density of Ruelle resonances in the semiclassical limit $\nu \rightarrow \infty$. Some numerical computations with the truncated Gauss map illustrate these results.

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1 Introduction

A “partially expanding map” is a map which is expanding except in some directions which are “neutral”. An “open map” is a map for which the non wandering set (or trapped set) is not the full manifold but a relatively compact subset. The aim of this paper is to study the dynamics of a class of open partially expanding maps from the spectral approach initiated by D. Ruelle and R. Bowen. In this approach, the pull back operator by the map, called transfer operator, is shown to have some discrete spectrum in some specific functional spaces. These eigenvalues called “Ruelle resonances” are very useful to describe the effective long time behavior of the dynamics: to express dynamical correlation functions and deduce statistical properties of the dynamics such as mixing and central limit theorems.

In Section 2 we define the model of expanding maps on some union of intervals $I \subset \mathbb{R}$ precisely called an iterated function scheme (I.F.S.). This is a well studied class of dynamical systems for which the trapped set $K \subset I$ is a Cantor set and has some Hausdorff dimension $\dim_H K \in [0, 1[$. In Section 2.4 we extend this model by adding a neutral direction and obtain a “partially expanding map”. The transfer operator is defined in Section 2.5. We can decompose the transfer operator into its Fourier components $\nu \in \mathbb{Z}$ with respect to the neutral direction and obtain a family of operators \hat{F}_ν also written \hat{F}_\hbar with $\hbar := 1/(2\pi\nu)$ (if $\nu \neq 0$).

In Section 3 we present the main new results of this paper. Theorem 3.1 shows that each transfer operator \hat{F}_ν has some discrete spectrum of Ruelle resonances in specific Sobolev spaces. Then Theorem 3.9 shows that in the limit of large frequencies $|\nu| \rightarrow \infty$ the spectral radius of \hat{F}_ν is bounded by some expression, under some condition that we call “minimal captivity”. In order to derive this result we use a semiclassical approach which relies on microlocal properties of the family of operators $\left(\hat{F}_\nu\right)_\nu$ in phase space T^*I (precisely it is a Fourier integral operator). This allows to consider the associated canonical map F on T^*I . This canonical map F is multivalued and has a trapped set \mathcal{K} which is also a Cantor set (which projects on K). We also obtain an upper bound on the number of Ruelle resonances in the limit $|\nu| \rightarrow \infty$ in Theorem 3.11. This upper bound involves the Hausdorff dimension $\dim_H K$ and is usually called “fractal Weyl law” after the work of J. Sjöstrand in [29]. The “minimal captivity” condition means that the dynamics of the canonical map F restricted to its trapped set \mathcal{K} is one-to-one (whereas the map F on T^*I is multivalued).

In Section 4 we illustrate our results by numerical computations with two particular models: the truncated Gauss map and Bowen-Series map for Schottky groups. We show

that these models fulfill the minimal captivity property. Sections 5 to 8 are devoted to the proofs of the results.

The same semiclassical approach has been used before for “closed dynamical systems” in [11] and [18], i.e. for systems in which the trapped set was the full manifold. In these latter papers as well as in [33] a similar result for the asymptotic spectral radius has been obtained. Technically the open aspect here is overcome by using a truncation function χ as explained in Section 2.6. In [18] the author considers models for which the neutral direction is a non commutative compact Lie group and shows discrete spectrum of Ruelle resonances, asymptotic spectral radius and Weyl law. Let us remark that we could extend the present results similarly by considering extensions with compact groups.

As explained in Section 4.1 our results can be applied to “Bowen Series maps” and “Bowen Series transfer operators” associated to the geodesic flow of “convex co-compact hyperbolic surfaces” also called “Schottky surfaces”. So our results give some results for the zeroes of the Selberg zeta function and resonances of the Laplacian of these surfaces. In that case the Weyl law of Theorem 3.11 is in close relation with the results obtained by Lin, Guilloppe and Zworski in [15] where they give an upper bound on the density of resonances for the Laplace-Beltrami operator on open hyperbolic surfaces. We can also apply our results to the quadratic maps and recover results already obtained in [30, 23].

Also let us remark that with the condition of “minimal captivity”, the dynamics of the canonical map F in the vicinity of the trapped set \mathcal{K} is univalued and can be identified with the classical dynamics of a “open quantum map”. Within this identification, the results of S. Nonnenmacher et M. Zworski [24] about asymptotic spectral radius and Weyl law of these open quantum maps are very similar to the results presented in this paper.

2 The transfer operator

2.1 Iterated function scheme

The transfer operator studied in this paper is constructed from a simple model of chaotic dynamics called “an iterated function scheme, I.F.S.”[10, chap.9]. We give the definition below and refer to Section 4 where many standard examples are presented.

Definition 2.1. “An iterated function scheme (I.F.S.)”. Let $N \in \mathbb{N}$, $N \geq 1$. Let $I_1, \dots, I_N \subset \mathbb{R}$ be a finite collection of **disjoint and closed** intervals. Let A be a $N \times N$ matrix, called adjacency matrix, with $A_{i,j} \in \{0, 1\}$. We will note $i \rightsquigarrow j$ if $A_{i,j} = 1$. Assume that for each pair $i, j \in \{1, \dots, N\}$ such that $i \rightsquigarrow j$, we have a smooth invertible map $\phi_{i,j} : I_i \rightarrow \phi_{i,j}(I_i)$ with $\phi_{i,j}(I_i) \subset \text{Int}(I_j)$. Assume that the map $\phi_{i,j}$ is a **strict contraction**, i.e. there exists $0 < \theta < 1$ such that for every $x \in I_i$,

$$|\phi'_{i,j}(x)| \leq \theta \quad (2.1)$$

We suppose that different images of the maps $\phi_{i,j}$ do not intersect (this is the “strong separation condition” in [9, p.35]):

$$\phi_{i,j}(I_i) \cap \phi_{k,l}(I_k) \neq \emptyset \quad \Rightarrow \quad i = k \text{ and } j = l. \quad (2.2)$$

Remark that the derivative $\phi'_{i,j}(x)$ may be negative. Figure 2.1 illustrates Definition 2.1 on a specific example.

2.2 Example with the truncated Gauss map

The Gauss map is

$$G : \begin{cases}]0, 1[& \rightarrow]0, 1[\\ y & \rightarrow \left\{ \frac{1}{y} \right\} \end{cases} \quad (2.3)$$

where $\{a\} := a - [a] \in [0, 1[$ denotes the fractional part of $a \in \mathbb{R}$. For $j \in \mathbb{N} \setminus \{0\}$, and $y \in \mathbb{R}$ such that $\frac{1}{j+1} < y \leq \frac{1}{j}$ then $G(y) = G_j(y) := \frac{1}{y} - j$. Notice that $dG/dy < 0$. The inverse map is $y = G_j^{-1}(x) = \frac{1}{x+j}$.

Let $N \geq 1$. We will consider only the first N “branches” $(G_j)_{j=1, \dots, N}$. In order to have a well defined I.F.S according to definition 2.1, for $1 \leq i \leq N$, let $\alpha_i := G_i^{-1}\left(\frac{1}{N+1}\right)$, $a_i = \frac{1}{1+i}$, b_i such that $\alpha_i < b_i < \frac{1}{i}$, and intervals $I_i := [a_i, b_i]$. On these intervals $(I_i)_{i=1, \dots, N}$, we define the maps

$$\phi_{i,j}(x) = G_j^{-1}(x) = \frac{1}{x+j}, \quad j = 1, \dots, N. \quad (2.4)$$

The adjacency matrix is $A = (A_{i,j})_{i,j}$, the full $N \times N$ matrix with all entries $A_{i,j} = 1$.

2.3 The trapped set K

We define

$$I := \bigcup_{i=1}^N I_i \quad (2.5)$$

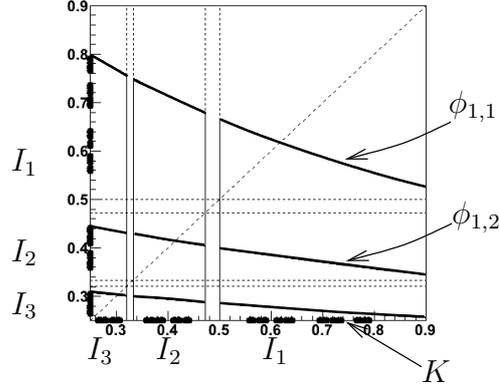


Figure 2.1: The iterated functions scheme (IFS) defined from the truncated Gauss map (2.3). Here we have $N = 3$ branches. The maps $\phi: \phi_{i,j} : I_i \rightarrow I_j$, $i, j = 1 \dots N$ are contracting and given by $\phi_{i,j}(x) = \frac{1}{x+j}$. The trapped set K defined in (2.9) is a N -adic Cantor set. It is obtained as the limit of the sets $K_0 = (I_1 \cup I_2 \dots \cup I_N) \supset K_1 = \phi(K_0) \supset K_2 = \phi(K_1) \supset \dots \supset K$.

and the multivalued map:

$$\phi : I \rightarrow I, \quad \phi = (\phi_{i,j})_{i,j}.$$

ϕ can be iterated and generates a multivalued map $\phi^n : I \rightarrow I$ for $n \geq 1$. From hypothesis (2.2) the inverse map

$$\phi^{-1} : \phi(I) \rightarrow I$$

is uni-valued. Throughout the paper we will use the “**unstable Jacobian function**”

$$J(x) := \log \left| \frac{d\phi^{-1}}{dx}(x) \right| \quad (2.6)$$

defined on $\phi(I)$. From (2.1), one has

$$\forall x, \quad J(x) > \log \frac{1}{\theta} > 0. \quad (2.7)$$

Let

$$K_n := \phi^n(I) \quad (2.8)$$

and $K_0 = I$. Since $K_{n+1} \subset K_n$ we can defined the limit set

$$K := \bigcap_{n \in \mathbb{N}} K_n \quad (2.9)$$

called the **trapped set**. Then the map

$$\phi^{-1} : K \rightarrow K \quad (2.10)$$

is well defined and uni-valued.

2.3.1 The fractal dimension of the trapped set K and the topological pressure

In this paper we will use the following definition of fractal dimension.

Definition 2.2. [20, p.76],[9, p.20] If $B \subset \mathbb{R}^d$ is a non empty bounded set, its **upper Minkowski dimension** (or box dimension) is

$$\dim_M B := d - \text{codim}_M B \quad (2.11)$$

with

$$\text{codim}_M B := \sup \left\{ s \in \mathbb{R} \mid \limsup_{\delta \downarrow 0} \delta^{-s} \cdot \text{Leb}(B_\delta) < +\infty \right\}. \quad (2.12)$$

where $B_\delta := \{x \in \mathbb{R}^d, \text{dist}(x, B) \leq \delta\}$ and $\text{Leb}(\cdot)$ is the Lebesgue measure.

Remark 2.3. In general

$$\limsup_{\delta \downarrow 0} \delta^{-\text{codim}_M B} \cdot \text{Leb}(B_\delta) < +\infty \quad (2.13)$$

does not hold, but if it does, B is said to be of **pure dimension**². It is known that the trapped set K defined in (2.9) has pure dimension and that the above definition of Minkowski dimension coincides with the more usual **Hausdorff dimension** of K [9, p.68]:

$$\dim_M K = \dim_H K \in [0, 1[\quad (2.14)$$

An efficient way to calculate the fractal dimension $\dim_H K$ is given by the topological pressure. The topological pressure can be defined from the periodic points as follows. A **periodic point** of period $n \geq 1$ is $x \in K$ such that $x = \phi^{-n}(x)$. The topological pressure can be defined in terms of periodic points.

Definition 2.4. [9, p.72] The **topological pressure** of a continuous function $\varphi \in C(I)$ is

$$\text{Pr}(\varphi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{x=\phi^{-n}(x)} e^{\varphi_n(x)} \right)$$

where $\varphi_n(x)$ is the Birkhoff sum of φ along the periodic orbit:

$$\varphi_n(x) := \sum_{k=0}^{n-1} \varphi(\phi^{-k}(x))$$

²see [29] for comments and further references.

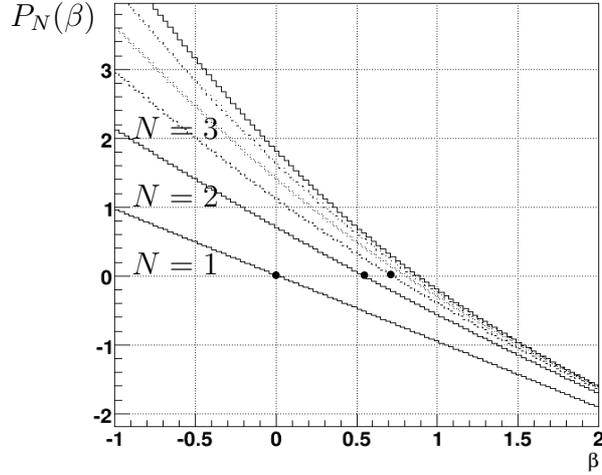


Figure 2.2: Topological Pressure $P_N(\beta)$ defined by (2.15) for the truncated Gauss map example of Section 2.2 for each value of $N = 1, 2, 3 \dots$ being the number of branches. The black points mark the zero of $P_N(\beta) = 0$ giving the fractal dimension of the trapped set K_N for each value of N : $\dim_H K_1 = 0$, $\dim_H K_2 = 0.531 \dots$, $\dim_H K_3 = 0.705 \dots$, and $\dim_H K_N \xrightarrow{N \rightarrow \infty} 1$.

It is interesting to consider the special case of the function $\varphi(x) = -\beta J(x)$ with some $\beta \in \mathbb{R}$ and $J(x)$ defined in (2.6). This gives the function $P : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$P(\beta) := \Pr(-\beta J) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{x=\phi^{-n}(x)} \prod_{k=0}^{n-1} |\phi'(\phi^{-k}(x))|^\beta \right) \quad (2.15)$$

The following lemma provides an easy way to compute (numerically) the dimension of K . See figure 2.2.

Lemma 2.5. [9, p.77] $P(\beta)$ is continuous and strictly decreasing in β and its unique zero is given by $\beta = \dim_H K$.

2.4 An extended partially expanding map and reduced transfer operators

The map $\phi^{-1} : \phi(I) \rightarrow I$ is univalued and expanding. Let $\tau \in C^\infty(I; \mathbb{R})$ be a smooth real valued function called **roof function**. We define the map

$$f : \begin{cases} \phi(I) \times S^1 & \rightarrow I \times S^1 \\ (x, y) & \rightarrow (\phi^{-1}(x), y + \tau(x)) \end{cases} \quad (2.16)$$

with $S^1 := \mathbb{R}/\mathbb{Z}$. Notice that the map f is expanding in the x variable whereas it is neutral in the y variable in the sense that $\frac{\partial f}{\partial y} = 1$. This is called a partially expanding map and is a very simple model of more general partially hyperbolic dynamics [26].

Let $V \in C^\infty(I; \mathbb{C})$ be a smooth complex valued function called a **potential** function.

Definition 2.6. The transfer operator of the map f with potential V is

$$\hat{\mathcal{F}} : \begin{cases} C^\infty(I \times S^1) & \rightarrow C^\infty(\phi(I) \times S^1) \\ \psi(x, y) & \rightarrow e^{V(x)}\psi(f(x, y)) \end{cases}. \quad (2.17)$$

Notice that $\psi(x, y)$ can be decomposed into Fourier modes in the y direction. For $\nu \in \mathbb{Z}$, a Fourier mode is

$$\psi_\nu(x, y) = \varphi(x) e^{i2\pi\nu y}$$

and we have

$$\begin{aligned} \left(\hat{\mathcal{F}}\psi_\nu\right)(x, y) &= e^{V(x)}\psi_\nu(f(x, y)) = e^{V(x)}\varphi(\phi^{-1}(x)) e^{i2\pi\nu(y+\tau(x))} \\ &= \left(\hat{F}_\nu\varphi\right)(x) e^{i2\pi\nu y} \end{aligned}$$

where the Standard reduced transfer operator $\hat{F}_\nu : C^\infty(I) \rightarrow C^\infty(\phi(I))$ defined by

$$\left(\hat{F}_\nu\varphi\right)(x) := e^{V(x)} e^{i2\pi\nu\tau(x)} \varphi(\phi^{-1}(x)). \quad (2.18)$$

So the operator $\hat{\mathcal{F}}$ is the direct sum of operators $\bigoplus_{\nu \in \mathbb{Z}} \hat{F}_\nu$. From the next section we will study the individual operator \hat{F}_ν in (2.19). Since our main interest is the limit $\nu \rightarrow \infty$ of large frequencies in the neutral direction, we will suppose $\nu \neq 0$ and write $\hbar := \frac{1}{2\pi\nu}$. In Section 3.3 we will deduce from our principal results, some asymptotic expansions for time correlation functions of the map (2.16).

2.5 The transfer operator

Notations: We denote $C_0^\infty(\mathbb{R})$ the space of smooth function on \mathbb{R} with compact support. If $B \subset \mathbb{R}$ is a compact set, we denote $C_B^\infty(\mathbb{R}) \subset C_0^\infty(\mathbb{R})$ the space of smooth functions on \mathbb{R} with support included in B . Recall that the inverse map $\phi^{-1} : \phi(I) \rightarrow I$ is uni-valued.

Definition 2.7. Let $\tau \in C^\infty(I; \mathbb{R})$ and $V \in C^\infty(I; \mathbb{C})$ be smooth functions called respectively **roof function** and **potential function**. Let $\hbar > 0$. We define the **transfer operator**:

$$\hat{F} : \begin{cases} C_I^\infty(\mathbb{R}) & \rightarrow C_I^\infty(\mathbb{R}) \\ \varphi = (\varphi_i)_i & \rightarrow \left(\sum_{i=1}^N \hat{F}_{i,j} \varphi_i \right)_j \end{cases} \quad (2.19)$$

with

$$\hat{F}_{i,j} : \begin{cases} C_{I_i}^\infty(\mathbb{R}) & \rightarrow C_{I_j}^\infty(\mathbb{R}) \\ \varphi_i & \rightarrow \left(\hat{F}_{i,j} \varphi_i \right)(x) = \begin{cases} e^{V(x)} e^{i\frac{1}{\hbar} \tau(x)} \varphi_i(\phi_{i,j}^{-1}(x)) & \text{if } i \rightsquigarrow j \text{ and } x \in \phi_{i,j}(I_i) \\ 0 & \text{otherwise} \end{cases} \end{cases} \quad (2.20)$$

Remark:

- From assumption (2.2), for any $x \in I$, the sum $\sum_{i=1}^N \left(\hat{F}_{i,j} \varphi_i \right)(x)$ which appears on the right hand side of (2.19) contains at most one non vanishing term. See figure 2.3.
- For short we can write that

$$\hat{F} : \begin{cases} C_I^\infty(\mathbb{R}) & \rightarrow C_I^\infty(\mathbb{R}) \\ \varphi & \rightarrow \begin{cases} e^{V(x)} e^{i\frac{1}{\hbar} \tau(x)} \varphi(\phi^{-1}(x)) & \text{if } x \in \phi(I) \\ 0 & \text{otherwise} \end{cases} \end{cases} \quad (2.21)$$

Remark 2.8.

- For any $\varphi \in C_I^\infty(\mathbb{R})$, $n \geq 0$ we have

$$\text{supp} \left(\hat{F}^n \varphi \right) \subset K_n \quad (2.22)$$

with K_n defined in (2.8).

- In the definition (10) we can write $e^{V(x)} e^{i\frac{1}{\hbar} \tau(x)} = \exp \left(i\frac{1}{\hbar} \mathcal{V}(x) \right)$ with $\mathcal{V}(x) := \tau(x) + \hbar(-iV(x))$. More generally we may consider a finite series $\mathcal{V}(x) = \sum_{j=0}^n \hbar^j \mathcal{V}_j(x)$ with leading term $\mathcal{V}_0(x) = \tau(x)$ and complex valued sub-leading terms $\mathcal{V}_j : I \rightarrow \mathbb{C}$, $j \geq 1$.

Proof. Using the change of variables $x = \phi_{i,j}(y)$ and definition (2.24), we write

$$\begin{aligned} \langle \varphi_i | \hat{F}_{i,j,\chi}^* \psi_j \rangle &= \int \overline{\varphi_i}(y) \left(\hat{F}_{i,j,\chi}^* \psi_j \right)(y) dy \\ &= \langle \hat{F}_{i,j,\chi} \varphi_i | \psi_j \rangle = \int_{\phi_{i,j}(I_i)} \overline{e^{V(x)} e^{i\frac{1}{\hbar}\tau(x)} \varphi_i(\phi_{i,j}^{-1}(x)) \chi(\phi_{i,j}^{-1}(x))} \psi_j(x) dx \\ &= \int_{I_i} \overline{\varphi_i}(y) \chi(y) |\phi'_{i,j}(y)| e^{\overline{V(x)}} e^{-i\frac{1}{\hbar}\tau_j(x)} \psi_j(x) dy \end{aligned}$$

and deduce (2.25). \square

Remark 2.10.

- Without the cut-off function χ the image of $\hat{F}_{i,j}^*$ may not be continuous on the boundary of I_i .
- An other more general possibility would have been to consider $\chi \in C_I^\infty(\mathbb{R})$ such that $0 < \chi(x)$ for $x \in \text{Int}(I)$ (without assumption that $\chi \equiv 1$ on K_1) and define

$$\hat{F}_{i,j,\chi} := \hat{\chi}^{-1} \hat{F}_{i,j} \hat{\chi} : C_{I_i}^\infty(\mathbb{R}) \rightarrow C_{I_j}^\infty(\mathbb{R}) \quad (2.26)$$

which is well defined since $\text{supp}(\hat{F}_{i,j} \hat{\chi} \varphi) \subset \text{Int}(I_j)$ where χ does not vanish. This more general definition (2.26) may be more useful in some cases, e.g. we use it in numerical computation. We recover the previous definition (2.23) if we make the additional assumption that $\chi \equiv 1$ on K_1 .

Proposition 2.11. *By duality the transfer operator (2.23) extends to distributions:*

$$\hat{F}_\chi : \mathcal{D}'(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R}) \quad (2.27)$$

$$\hat{F}_\chi^* : \mathcal{D}'(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$$

Similarly to (2.22) we have that for any $n \geq 1$, any $\alpha \in \mathcal{D}'(\mathbb{R})$,

$$\text{supp}(\hat{F}_\chi^n \alpha) \subset K_n \quad (2.28)$$

with K_n defined in (2.8).

Proof. The extension is defined by³

$$\hat{F}_{i,j,\chi}(\alpha_i)(\psi_j) = \alpha_i \left(\hat{F}_{i,j,\chi}^* \overline{\psi_j} \right), \quad \alpha_i \in \mathcal{D}'(\mathbb{R}), \psi_j \in C_0^\infty(\mathbb{R}), \quad (2.29)$$

³The complex conjugation appears in (2.29) because duality is related to scalar product on L^2 by $\alpha(\overline{\varphi}) := \int \overline{\varphi} \alpha = \langle \varphi, \alpha \rangle_{L^2}$.

Then the transfer operator extends to: $\hat{F}_\chi : \mathcal{D}'(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$.

If $\psi_j(\phi_{i,j}(y)) = 0, \forall y \in I_i$, Eq.(2.25) shows that $\hat{F}_{i,j,\chi}^* \psi_j \equiv 0$. More generally let $\psi \in C_0^\infty(\mathbb{R})$ with $\text{supp}(\psi) \cap K_n = \emptyset$ with $n \geq 1$ and K_n defined in (2.8). Then

$$\left(\hat{F}_\chi^*\right)^n \psi \equiv 0. \quad (2.30)$$

For any $\alpha \in \mathcal{D}'(\mathbb{R})$, we deduce that $\left(\hat{F}_\chi^n \alpha\right)(\bar{\psi}) = \alpha\left(\overline{\left(\hat{F}_\chi^*\right)^n \psi}\right) = 0$. By definition, this means that $\text{supp}\left(\hat{F}_\chi^n \alpha\right) \subset K_n$. \square

3 Main results

3.1 Discrete spectrum of Ruelle resonances

Theorem 3.1 below shows that the transfer operator \hat{F}_χ (for any \hbar) has discrete spectrum called ‘‘Ruelle resonances’’ in ordinary Sobolev spaces with negative order and that the spectrum does not depend on the choice of χ . Recall that for $m \in \mathbb{R}$, the **Sobolev space** $H^{-m}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ is defined by ([31] p.271)

$$H^{-m}(\mathbb{R}) := \langle \hat{\xi} \rangle^m (L^2(\mathbb{R})) \quad (3.1)$$

with the differential operator $\hat{\xi} := -i\frac{d}{dx}$ and the notation $\langle x \rangle := (1 + x^2)^{1/2}$.

Theorem 3.1. ‘‘Discrete spectrum of resonances’’. *For any fixed \hbar , any $m \in \mathbb{R}$, the transfer operator \hat{F}_χ in (2.27) is bounded in the Sobolev space $H^{-m}(\mathbb{R})$ and can be written*

$$\hat{F}_\chi = \hat{K} + \hat{R} \quad (3.2)$$

where \hat{K} is a compact operator and \hat{R} is such that:

$$\left\| \hat{R} \right\|_{H^{-m}(\mathbb{R})} \leq r_m, \quad r_m := C\theta^m \quad (3.3)$$

where $0 < \theta < 1$ is given in (2.1) and C does not depend on m . Notice that $r_m \rightarrow 0$ as $m \rightarrow +\infty$ and that the operator \hat{F}_χ has discrete spectrum on the domain $|z| > r_m$. These eigenvalues of \hat{F}_χ and their eigenspace do not depend on m nor on χ . The support of the eigendistributions is contained in the trapped set K . These discrete eigenvalues are denoted

$$\text{Res}\left(\hat{F}\right) := \{\lambda_i\}_i \subset \mathbb{C}^* \quad (3.4)$$

and are called **Ruelle resonances**.

In Section 4 we show the discrete spectrum of Ruelle resonances computed numerically for different examples.

3.2 Asymptotic spectral gap and Fractal Weyl law

We will give some partial description of the discrete spectrum of Ruelle resonances of the operator $\hat{F}_{\chi, \hbar}$, Eq.(2.27), in the limit $\hbar \rightarrow 0$. For brevity we will drop the index χ and simply write \hat{F}_{\hbar} . In Theorem 3.9 below we present a result giving an upper bound for the spectral radius of \hat{F}_{\hbar} in the semiclassical limit $\hbar \rightarrow 0$. In Theorem 3.11 we provide an upper bound for the number of resonances outside any radius $\varepsilon > 0$ as $\hbar \rightarrow 0$. This is called ‘‘fractal Weyl law’’. These results rely on the study of the dynamics of a symplectic map or canonical map $F : T^*I \rightarrow T^*I$ associated to the family of operators $(\hat{F}_{\hbar})_{\hbar}$, that we describe first.

Lemma 3.2. *The family of operators $(\hat{F}_{\hbar})_{\hbar}$ restricted to $C_I^{\infty}(\mathbb{R})$ is a \hbar -Fourier integral operator (FIO). Its **canonical transform** is a multi-valued symplectic map $F : T^*I \rightarrow T^*I$ (with $T^*I \cong I \times \mathbb{R}$) given by:*

$$F : \begin{cases} T^*I & \rightarrow T^*I \\ (x, \xi) & \rightarrow \{F_{i,j}(x, \xi) \quad \text{with } i, j \text{ s.t. } x \in I_i, i \rightsquigarrow j\} \end{cases} \quad (3.5)$$

with

$$F_{i,j} : \begin{cases} x' & = \phi_{i,j}(x) \\ \xi' & = \frac{1}{\phi'_{i,j}(x)}\xi + \tau'(x') \end{cases}. \quad (3.6)$$

The proof of Lemma 3.2 will be given in the beginning of Section 6.

Remark 3.3. For short, we can write

$$F : \begin{cases} T^*I & \rightarrow T^*I \\ (x, \xi) & \rightarrow \left(\phi(x), \frac{1}{\phi'(x)}\xi + \tau'(\phi(x)) \right) \end{cases}. \quad (3.7)$$

We will study the dynamics of F in detail in later Sections, but we can already make some remarks. The term $\frac{d\tau_j}{dx}(x')$ in the expression of ξ' , Eq.(3.6), complicates significantly the dynamics near the zero section $\xi = 0$. However the next Lemma shows that a trajectory from an initial point (x, ξ) with $|\xi|$ large enough, escape towards infinity:

Lemma 3.4. *For any $1 < \kappa < 1/\theta$, there exists $R \geq 0$ such that for any $|\xi| > R$ and any $i \rightsquigarrow j$,*

$$|\xi'| > \kappa |\xi| \quad (3.8)$$

where $(x', \xi') = F_{i,j}(x, \xi)$.

Proof. From (3.6), one has $\xi' = \frac{1}{\phi'_{i,j}(x)}\xi + \tau'(x')$. Also $\left| \frac{1}{\phi'_{i,j}(x)} \right| \geq \theta$ hence

$$\begin{aligned} |\xi'| - \kappa |\xi| &= \left| \frac{1}{\phi'_{i,j}(x)}\xi + \tau'(x') \right| - \kappa |\xi| \geq \left| \frac{1}{\phi'_{i,j}(x)}\xi \right| - |\tau'(x')| - \kappa |\xi| \\ &\geq \left(\frac{1}{\theta} - \kappa \right) |\xi| - \max_x |\tau'(x)| > 0. \end{aligned}$$

The last inequality holds true if $|\xi| > R := \left(\frac{1}{\theta} - \kappa \right)^{-1} \max_x |\tau'|$. \square

Definition 3.5. The **trapped set** in phase space T^*I is defined as

$$\mathcal{K} = \{(x, \xi) \in T^*I, \quad \exists C \Subset T^*I \text{ compact}, \forall n \in \mathbb{Z}, F^n(x, \xi) \cap C \neq \emptyset\}. \quad (3.9)$$

Remark 3.6. Since the map $F : T^*I \rightarrow T^*I$ is a lift of the map, we have $\mathcal{K} \subset (K \times \mathbb{R})$. For any R given from Lemma 3.4 we can precise this and obtain

$$\mathcal{K} \subset (K \times [-R, R]).$$

For $\varepsilon > 0$, let \mathcal{K}_ε denote a ε -neighborhood of the trapped set \mathcal{K} , namely

$$\mathcal{K}_\varepsilon := \{(x, \xi) \in T^*I, \quad \exists (x_0, \xi_0) \in \mathcal{K}, \quad \max(|x - x_0|, |\xi - \xi_0|) \leq \varepsilon\}.$$

From now on we will make the following hypothesis on the multi-valued map F .

Assumption 3.7. We assume the following property called “*minimal captivity*”:

$$\exists \varepsilon > 0, \quad \forall (x, \xi) \in \mathcal{K}_\varepsilon, \quad \# \left\{ F(x, \xi) \cap \mathcal{K}_\varepsilon \right\} \leq 1. \quad (3.10)$$

This means that the dynamics of F is univalued on the trapped set \mathcal{K} .

Remark 3.8. In the paper [11] we introduced the property of “**partial captivity**” which is weaker than “**minimal captivity**”: partial captivity roughly states that most of trajectories escape from the trapped set \mathcal{K} whereas minimal captivity states that every trajectory except one, escapes from the trapped set \mathcal{K} .

In Section 6 we provide more details on the dynamics of the map $F : T^*I \rightarrow T^*I$, namely we provide a more precise description of the trapped set \mathcal{K} , a detailed symbolic coding for this dynamics and some equivalent statements to the property of minimal captivity.

For the next theorem, let us define the function $V_0 \in C^\infty(I)$

$$V_0(x) := \frac{1}{2}J(x) \quad (3.11)$$

called “**potential of reference**” with $J(x)$ defined in (2.6). Recall from (2.7) that

$$V_0(x) \geq \frac{1}{2} \log \left(\frac{1}{\theta} \right) > 0.$$

Let us define the function $D \in C^\infty(I)$

$$D(x) := \operatorname{Re}(V(x)) - V_0(x) \quad (3.12)$$

called “**effective damping function**”.

Theorem 3.9. Spectral gap in the semi-classical limit. *With assumption 3.7 of “minimal captivity” (and m sufficiently large so that $r_m \ll 1$ in (3.3)), the spectral radius of the operators $\hat{F}_\hbar : H^{-m}(\mathbb{R}) \rightarrow H^{-m}(\mathbb{R})$ satisfies in the semi-classical limit $\hbar \rightarrow 0$:*

$$r_s(\hat{F}_\hbar) \leq e^{\gamma_+} + o(1) \quad (3.13)$$

with

$$\gamma_+ := \limsup_{n \rightarrow \infty} \left(\sup_{x, w_{0,n}} \frac{1}{n} D_{w_{0,n}}(x) \right) \quad (3.14)$$

where $D_{w_{0,n}}(x) := \sum_{k=1}^n D(\phi_{w_{k,n}}(x))$ is the Birkhoff average of the damping function D along a trajectory of length n , starting from the point x . Moreover the norm of the resolvent is controlled uniformly with respect to \hbar : for any $\rho > e^{\gamma_+}$, there exist $C_\rho > 0$, $\hbar_\rho > 0$ such that $\forall \hbar < \hbar_\rho, \forall |z| > \rho$ then

$$\left\| \left(z - \hat{F}_\hbar \right)^{-1} \right\|_{H^{-m}(\mathbb{R})} \leq C_\rho. \quad (3.15)$$

Remark

- Notice that Theorem 3.9 depends on the roof function τ only implicitly through assumption 3.7. The value of the upper bound (3.13) does not depend on τ .

- Eq.(3.15) implies (3.13) and is equivalent⁴ to the following property that the norm $\left\| \hat{F}_h^n \right\|$ is controlled uniformly with respect to \hbar : For any $\rho > e^{\gamma+}$, there exist $c_\rho > 0$, $\hbar_\rho > 0$ such that $\forall \hbar < \hbar_\rho$, for any $n \in \mathbb{N}$,

$$\left\| \hat{F}_h^n \right\|_{H^{-m}(\mathbb{R})} \leq c_\rho \cdot \rho^n \quad (3.16)$$

We will use (3.16) later to compute asymptotics of correlation functions.

Lemma 3.10. *If assumption 3.7 holds true and if the adjacency matrix A is symmetric then*

$$\dim_M \mathcal{K} = 2 \dim_M K = 2 \dim_H K \quad (3.17)$$

where $\dim_M B$ stands for the Minkowski dimension of a set B as defined in Eq.(2.11).

Recall from (2.14) that $\dim_H K = \dim_M K$.

Theorem 3.11. "Fractal Weyl upper bound". *Suppose that the assumption of minimal captivity 3.7 holds and that the adjacency matrix A is symmetric. For any $\varepsilon > 0$, any $\eta > 0$, we have for $\hbar \rightarrow 0$*

$$\#\left\{ \lambda_i^h \in \text{Res} \left(\hat{F}_h \right) \mid |\lambda_i^h| \geq \varepsilon \right\} = \mathcal{O} \left(\hbar^{-\dim_H(K) - \eta} \right). \quad (3.18)$$

3.3 Decay of correlations

In this subsection we present a quite immediate consequence of the existence of an asymptotic spectral radius $e^{\gamma+}$ obtained in Theorem 3.9: we obtain a finite expansion for correlation functions $\langle v | \hat{\mathcal{F}}^n u \rangle$ of the extended transfer operator $\hat{\mathcal{F}}$ defined in (2.17), with $u, v \in C^\infty(I \times S^1)$.

⁴Let us show the equivalence. In one sense, let $\rho_2 > \rho_1 > e^{\gamma+}$, suppose that $\left\| \hat{F}_h^n \right\|_{H^{-m}(\mathbb{R})} \leq c_{\rho_1} \cdot \rho_1^n$. Let $|z| > \rho_2$. The relation $(z - \hat{F}_h)^{-1} = z^{-1} \sum_{n \geq 0} \left(\frac{\hat{F}_h}{z} \right)^n$ gives that

$$\left\| (z - \hat{F}_h)^{-1} \right\| \leq |z|^{-1} \sum_{n \geq 0} \frac{\left\| \hat{F}_h^n \right\|}{|z|^n} \leq |z|^{-1} c_{\rho_1} \sum_{n \geq 0} \frac{\rho_1^n}{|z|^n} = \frac{c_{\rho_1}}{|z| - \rho_1} \leq \frac{c_{\rho_1}}{\rho_2 - \rho_1} =: C_{\rho_2}$$

For the other sense, suppose that for $|z| > \rho$, $\left\| (z - \hat{F}_h)^{-1} \right\| \leq C_\rho$. From the Cauchy formula $\hat{F}_h^n = \frac{1}{2\pi i} \oint_\gamma z^n (z - \hat{F}_h)^{-1} dz$ where γ is the circle of radius ρ one deduces that $\left\| \hat{F}_h^n \right\| \leq \rho C_\rho \rho^n$.

We first introduce a notation: for a given $\nu \in \mathbb{Z}$, we have seen in Theorem 3.1 that the transfer operator $\hat{\mathcal{F}}_\nu \equiv \hat{F}_N$, that the transfer operator $\hat{\mathcal{F}}_\nu \equiv \hat{F}_N$ has a discrete spectrum of resonances. For $\rho > 0$ such that there is no eigenvalue on the circle $|z| = \rho$ for any $\nu \in \mathbb{Z}$, we denote by $\Pi_{\rho,\nu}$ the spectral projector of the operator \hat{F}_ν on the domain $\{z \in \mathbb{C}, |z| > \rho\}$. These projection operators have obviously finite rank and commute with \hat{F}_ν .

Theorem 3.12. *For any $\rho > e^{\gamma^+}$, there exists $\nu_0 \in \mathbb{N}$ such that for any $u, v \in C^\infty(I \times S^1)$, in the limit $n \rightarrow \infty$,*

$$\langle v | \hat{\mathcal{F}}^n u \rangle = \sum_{|\nu| \leq \nu_0} \langle v | \left(\hat{F}_\nu \Pi_{\rho,\nu} \right)^n u \rangle + O(\rho^n) \quad (3.19)$$

Remark 3.13. In the right hand side of (3.19) there is a finite sum and each operator $\hat{F}_\nu \Pi_{\rho,\nu}$ has finite rank. Using the spectral decomposition of \hat{F}_ν we get an expansion of the correlation function $\langle v | \hat{\mathcal{F}}^n u \rangle$ with a finite number of terms which involve the leading Ruelle resonances (i.e. those with modulus greater than ρ) plus the error term $O(\rho^n)$.

Proof. Let $\rho > e^{\gamma^+}$. Recall that $\hbar = \frac{1}{2\pi\nu}$ and that we note $\hat{F}_\hbar = \hat{F}_\nu$. In Theorem 3.1 we have for $\hbar \rightarrow 0$ that $r_s(\hat{F}_\hbar) \leq e^{\gamma^+} + o(1)$. Let the value of ν_0 be such that $r_s(\hat{F}_\hbar) < \rho$ for every $\nu > \nu_0$. Then

$$\langle v | \hat{\mathcal{F}}^n u \rangle = \sum_{|\nu| \leq \nu_0} \langle v | \left(\hat{F}_\nu \Pi_{\rho,\nu} \right)^n u \rangle + O_{\nu_0}(\rho^n) + \sum_{|\nu| > \nu_0} \langle v | \hat{F}_\nu^n u \rangle \quad (3.20)$$

But $\left| \langle v | \hat{F}_\nu^n u \rangle \right| \leq \|u_\nu\|_{H^m} \|v_\nu\|_{H^{-m}} \left\| \hat{F}_\nu^n \right\|_{H^{-m}}$ where $u_\nu, v_\nu \in C^\infty(S^1)$ stand for the Fourier components of the smooth functions $u, v \in C^\infty(I \times S^1)$. On one hand, for smooth functions one has fast decay $\|u_\nu\|, \|v_\nu\| = O(\nu^{-\infty})$. On the other hand from (3.16), $\left\| \hat{F}_\nu^n \right\|_{H^{-m}} = O(\rho^n)$. So $\left| \sum_{|\nu| > \nu_0} \langle v | \hat{F}_\nu^n u \rangle \right| = O(\rho^n)$. Then (3.20) gives (3.19). \square

We recall the following result called ‘‘Perron-Frobenius Lemma’’:

Lemma 3.14. *For real potential V , if the map ϕ^{-1} is ergodic then the transfer operator $\hat{\mathcal{F}}$ has a leading and simple eigenvalue $\lambda_0 > 0$ in the Fourier mode $\nu = 0$, i.e.*

$$\hat{\mathcal{F}}_{\nu=0} = \lambda_0 \Pi_{\lambda_0} + \hat{\mathcal{F}}'$$

with Π_{λ_0} being the rank 1 spectral projector associated to λ_0 , the remainder operator has $r_s(\hat{\mathcal{F}}') < \lambda_0$ and for any $\nu \neq 0$, we also have $r_s(\hat{\mathcal{F}}_\nu) < \lambda_0$.

For example without potential, i.e. $V = 0$, then $\lambda_0 = \exp(\text{Pr}(-J))$ with J given in (2.6). As a consequence of Lemma 3.14 and Theorem 3.12 we obtain (a result already obtained by Dolgopyat [8]):

Theorem 3.15. *Let $u, v \in C^\infty(I \times S^1)$, then for $n \rightarrow \infty$,*

$$\langle v | \hat{\mathcal{F}}^n u \rangle = \lambda_0^n \langle v | \Pi_{\lambda_0} u \rangle + O(|\lambda_1|^n)$$

where λ_1 is the second eigenvalue with $|\lambda_1| < \lambda_0$. This gives that the extended map $f : I \times S^1 \rightarrow I \times S^1$ is mixing with exponential decay of correlations.

4 Numerical results for the truncated Gauss map and Bowen-Series maps

In this section we will present numerical results for two important classes of I.F.S.: the truncated Gauss map and the Bowen-Series maps for convex co-compact hyperbolic surfaces. We will show that both examples satisfy the partially captive property. We will then give some numerical illustrations of the main theorems presented in this paper and finally discuss the connection between the spectrum of these transfer operators with the resonance spectrum of the Laplacian on hyperbolic surfaces.

4.1 The truncated Gauss map

In this section we consider the example of the I.F.S. defined from the truncated Gauss map with N intervals presented in Section 2.2. We choose the roof function τ and the potential function V which enter in the definition of the transfer operator (2.20) to be:

$$\tau(x) = -J(x), \quad V(x) = (1-a)J(x), \quad a \in \mathbb{R}. \quad (4.1)$$

where $J(x) = \log(|(\phi^{-1})'(x)|) = \log(|G'(x)|) = -2 \log(x)$ has been defined in 2.6. Let us write

$$s = a + ib \in \mathbb{C}, \quad b = \frac{1}{\hbar} > 0.$$

Then for every $s \in \mathbb{C}$, the transfer operator \hat{F} given in (2.19) will be written $\hat{L}_s = \hat{F}$ and is given by:

$$\hat{L}_s \varphi = \hat{F} \varphi = e^{V(x)} e^{i\frac{1}{\hbar}\tau(x)} \varphi \circ \phi^{-1} = e^{(1-s)J} \varphi \circ \phi^{-1} \quad (4.2)$$

As explained in Section 4.1.1 below, this choice is interesting due its relation with the dynamics on the modular surface. The (adjoint of the) transfer operator \hat{F} constructed in this way is usually called the **Gauss-Kuzmin-Wirsing transfer operator** or “Dieter-Mayer transfer operator” for the truncated Gauss map.

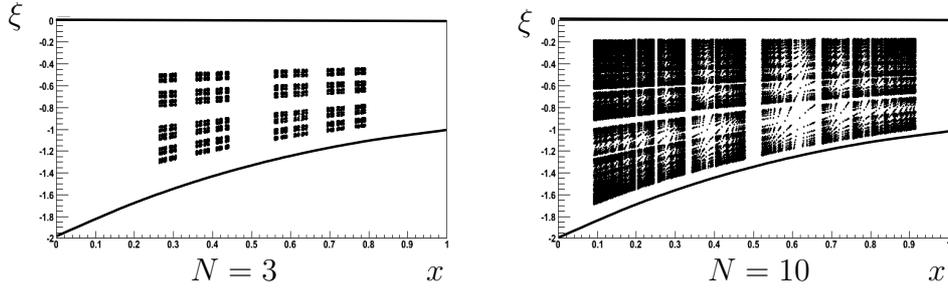


Figure 4.1: The trapped set $\mathcal{K}_N := \mathcal{K}$ for the truncated Gauss map with functions (4.1), for the cases of $N = 3$ and $N = 10$ branches. This corresponds to the Gauss-Kuzmin-Wirsing transfer operator (4.2). We have $\mathcal{K}_N \subset \mathcal{K}_{N+1}$ and for $N \rightarrow \infty$, the limit trapped set $\mathcal{K}_\infty = \bigcup_{N \geq 0} \mathcal{K}_N = \{(x, \xi), x \in]0, 1[, -\frac{2}{1+x} < \xi < 0\}$ is the band between the marked black lines. (More precisely, we have represented the periodic points with period $n = 6$. That explains the sparse aspect of the trapped set).

Proposition 4.1. *For every $N \geq 1$, the minimal captivity assumption 3.7 holds true for the truncated Gauss transfer operator defined by (4.2).*

The proof is given in Section 4.3 below. In this proof we explain the structure of the trapped set \mathcal{K} with more details.

Consequently, we can apply Theorem 3.9 and deduce that there is an asymptotic spectral gap. See figure 4.2 for numerical results and comments. We can also apply Theorem 3.11 and deduce an fractal Weyl upper bound for the density of resonances. See figure 4.3 for numerical results and comments.

4.1.1 Relation with the zeroes of the Selberg zeta function

For the geodesic flow on the modular surface $\mathrm{SL}_2\mathbb{Z} \backslash \mathrm{SL}_2\mathbb{R}$ it is possible to define the Selberg zeta function (see Section 4.2.1 below for more comments and references):

$$\zeta_{\mathrm{Selberg}}(s) = \prod_{\gamma} \prod_{m \geq 0} (1 - e^{-(s+m)|\gamma|}), \quad s \in \mathbb{C},$$

where the product is over the primitives periodic orbits γ of the geodesic flow and $|\gamma|$ denotes the length of the orbit. This zeta function is absolutely convergent for $\mathrm{Re}(s) > 1$. Using the Gauss map and continued fractions, C. Series has shown that a periodic orbit γ is in one to one correspondence with a periodic sequence $(w_j)_{j \in \mathbb{Z}} \in (\mathbb{N} \setminus \{0\})^{\mathbb{Z}}$ where $w_j \in \mathbb{N} \setminus \{0\}$ is the index of the branch of the Gauss map $G_{w_j}^{-1}$ in (2.4). Given $N \geq 1$, we can restrict the product \prod_{γ} over periodic orbits above to orbits for which $w_j \leq N, \forall j \in \mathbb{Z}$,

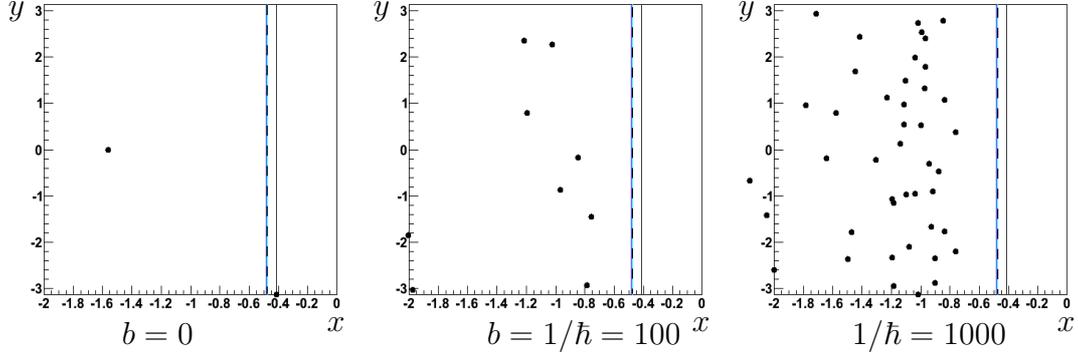


Figure 4.2: The discrete spectrum of Ruelle resonances λ_j (in log scale writing: $\log \lambda = x + iy$) for the truncated Gauss-Kuzmin-Wirsing transfer operator (4.2) associated to the Gauss map, for $N = 3$ branches and parameters $a = 1$, $b = 0, 100, 1000$. For $b = 0$ there is the eigenvalue $\lambda = e^{\text{Pr}(-J)}$ at $x = \text{Pr}(-J)$, $y = -\pi$ corresponding to the “equilibrium measure”. The full vertical line is at $x = \text{Pr}(-J)$. The dashed vertical line is at $x = \gamma_+$ which is shown in (3.13) to be an asymptotic upper bound for $b = 1/\hbar \rightarrow \infty$.

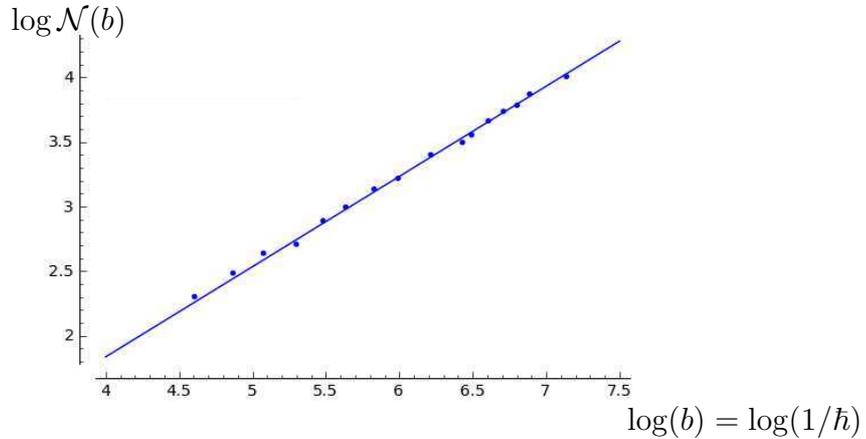


Figure 4.3: This is the Weyl law for the model of Gauss map with $N = 3$ branches. The points represent the number of resonances $\mathcal{N}(b) = \#\left\{\lambda_j \in \text{Res}\left(\hat{L}_s\right), \log |\lambda_j| > -3.5\right\}$ computed numerically, as a function of the semiclassical parameter $b = 1/\hbar$ in log scale. The linear fit gives $\log \mathcal{N}(b) = -0.70 \cdot \log b - 0.96$ which has to be compared to the fractal Weyl law (3.18) giving $\log \mathcal{N}(b) \leq -\dim_H(K) \cdot \log b + \text{cste}$. From (2.2) we have $\dim_H K_3 = 0.705$ giving an excellent agreement with the numerical results and suggesting that the upper bound is in fact optimal.

and define a truncated Selberg zeta function as follows:

$$\zeta_{\text{Selberg},N}(s) = \prod_{\gamma, w_j \leq N, \forall j} \prod_{m \geq 0} (1 - e^{-(s+m)|\gamma|}), \quad s \in \mathbb{C},$$

On the other hand, for fixed $s \in \mathbb{C}$, we have from Theorem 3.1 that the operator \hat{L}_s has discrete spectrum of Ruelle resonances. It is possible to define the dynamical determinant of \hat{L}_s by

$$d(z, s) := \text{Det} \left(1 - z \hat{L}_s \right) := \exp \left(- \sum_{n \geq 1} \frac{z^n}{n} \text{Tr}^b \left(\hat{L}_s^n \right) \right), \quad z \in \mathbb{C}$$

where $\text{Tr}^b \left(\hat{L}_s^n \right)$ stands for the flat trace of Atiyah-Bott. The sum is convergent for $|z|$ small enough. It is known that for fixed s , the zeroes of $d(z, s)$ (as a function of z) coincide with multiplies with the Ruelle resonances of \hat{L}_s [1]. In the case $z = 1$, we also have that $d(1, s)$ coincides with the truncated Selberg zeta function [27][2]:

$$\text{Det} \left(1 - \hat{L}_s \right) = \zeta_{\text{Selberg},N}(s) \tag{4.3}$$

which means that the zeroes of $\zeta_{\text{Selberg},N}(s)$ are given (with multiplicity) by the event that 1 is a Ruelle resonance of the transfer operator \hat{L}_s . This also shows that $\zeta_{\text{Selberg},N}(s)$ has a holomorphic extension to the complex plane $s \in \mathbb{C}$.

Remark 4.2. in [27][2, p.306] they consider the adjoint operator \hat{L}_s^* called the Perron-Frobenius operator.

4.2 Bowen Series maps for Schottky surfaces

The second class of examples that we consider in this section are Bowen-Series maps for Schottky surfaces [4]. We will follow the notation of D. Borthwick's book [2, chap.15] and recall the definition of a Schottky group given there. Recall that an element $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2\mathbb{R}$ acts on $\mathbb{H}^2 = \text{SL}_2\mathbb{R}/\text{SO}_2$ and $\bar{\mathbb{R}} = \partial\mathbb{H}^2$ by $S(x) := \frac{ax+b}{cx+d}$.

Definition 4.3. Let D_1, \dots, D_{2r} be disjoint closed half discs in the Poincaré half plane $\mathbb{H}^2 = \text{SL}_2\mathbb{R}/\text{SO}_2$ with center in $\mathbb{R} = \partial\mathbb{H}^2 \setminus \{\infty\}$. There exist elements $S_i \in \text{SL}_2\mathbb{R}$, $i = 1, \dots, r$ such that $S_i(\partial D_i) = \partial D_{i+r}$ and $S_i(\text{Int}(D_i)) = \mathbb{C} \setminus D_{i+r}$. The group generated by the S_i is called a **Schottky group** $\Gamma = \langle S_1, \dots, S_r \rangle$.

Remark 4.4. For convenience we will use a cyclic notation for the indices $i = 1, \dots, 2r$. Then one can also define S_i for $i = r + 1, \dots, 2r$ as in the definition above and obtains $S_{i+r} = S_i^{-1}$.

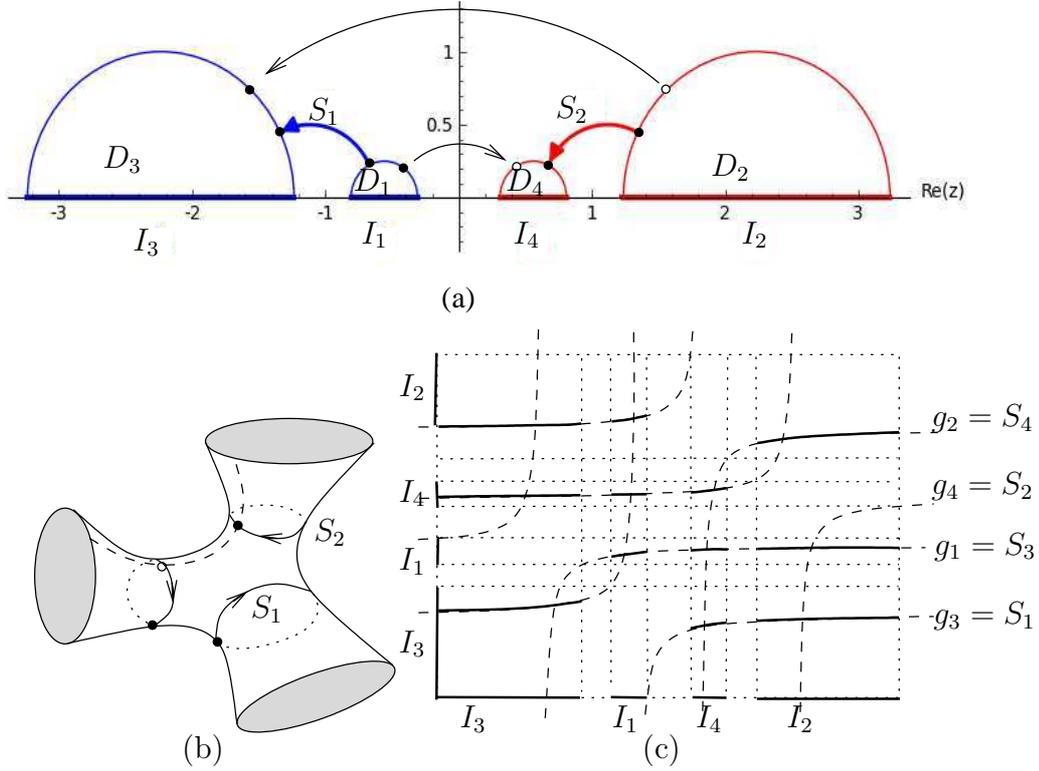


Figure 4.4: In this (arbitrary) example, we have $r = 2$ hyperbolic matrices of $\mathrm{SL}_2\mathbb{R}$: $S_1 = \begin{pmatrix} 4 & \sqrt{5} \\ -\sqrt{5} & -1 \end{pmatrix}$ and $S_2 = \begin{pmatrix} -1 & \sqrt{5} \\ -\sqrt{5} & 4 \end{pmatrix}$ that generate a Schottky group $\Gamma = \langle S_1, S_2 \rangle$. Figure (a) shows the Dirichlet fundamental domain $\mathbb{H}^2 \setminus (D_1 \cup D_2)$ with the intervals $I_i, i = 1, 2$, on which the I.F.S. is defined. Figure (b) shows the resulting Schottky surface $\Gamma \backslash \mathbb{H}^2$. It has three funnels. Figure (c) shows the graph of the generating functions $\phi_{i,j} = g_j = S_{j+r} : I_i \rightarrow I_j$ of the associated iterated function system.

Let $I_i := D_i \cap \partial\mathbb{H}$. Then $(I_i)_{i=1,\dots,2r}$ are $N = 2r$ disjoint closed intervals. One has $S_j(\mathrm{Int}(I_j)) = \partial\mathbb{H} \setminus I_{j+r}$ and we assume that S_j is expanding on I_j (this can always be obtained by taking iterations if necessary and localizing further to the trapped set, see [2, prop.15.4]). The maps S_j are usually called the Bowen series maps. Considering the inverse maps one obtains an iterated function system according to definition 2.1 associated to this Schottky group in the following way. For any $j = 1, \dots, N$ and $i \neq j + r$ let:

$$\phi_{i,j} := S_j^{-1} = S_{j+r} : I_i \rightarrow S_j^{-1}(I_i) \subset \mathrm{Int}(I_j)$$

The adjacency matrix $A_{i,j}$ has all entries are equal to one except $A_{i,i+r} = 0$. See Figure 4.4.

As in (4.1) we make the following choice for the potential and the roof function for $x \in I_j$

$$\tau(x) = -J(x), \quad V(x) = (1-a)J(x), \quad a \in \mathbb{R}. \quad (4.4)$$

where $J(x) = \log \left(\left| (\phi_{i,j}^{-1})'(x) \right| \right) = \log \left(|g_j'(x)| \right)$ has been defined in 2.6. Let us write

$$s = a + ib \in \mathbb{C}, \quad b = \frac{1}{\hbar} > 0.$$

Then for every $s \in \mathbb{C}$, the transfer operator \hat{F} given in (2.19) will be written $\hat{L}_s = \hat{F}$ and is given by:

$$\hat{L}_s \varphi = \hat{F} \varphi = e^{V(x)} e^{i \frac{1}{\hbar} \tau(x)} \varphi \circ \phi^{-1} = e^{(1-s)J} \varphi \circ \phi^{-1} \quad (4.5)$$

The adjoint of our transfer operator $L_s^* = \hat{F}^*$ is exactly the Ruelle transfer operator defined in [2, p.304] and as we will discuss below, its spectrum is in a close connection to the spectrum of the Laplace operator on the Schottky surface.

Proposition 4.5. *The minimal captivity assumption 3.7 holds true for the Bowen-Series transfer operator defined by (4.5).*

The proof is given in Section 4.3 below.

Consequently, we can apply Theorem 3.9 and deduce that there is an asymptotic spectral gap. We can also apply Theorem 3.11 and deduce an fractal Weyl upper bound for the density of resonances.

4.2.1 Selberg zeta function and resonances of the Laplacian

For the geodesic flow on a hyperbolic surface it is possible to define the Selberg zeta function

$$\zeta_{\text{Selberg}}(s) = \prod_{\gamma} \prod_{m \geq 0} (1 - e^{-(s+m)|\gamma|})$$

where the product is over primitive periodic orbits γ of the geodesic flow and $|\gamma|$ denotes the length of the orbit. This zeta function is absolutely convergent for $\text{Re}(s) > 1$ and has a meromorphic continuation to the whole complex plane. This continuation is particularly interesting as its zeros are either “topological zeros” located on the real axis or resonances of the Laplace operator Δ on the corresponding hyperbolic surface $\Gamma \backslash \mathbb{H}^2$. These resonances $s \in \text{Res}(\Delta)$ are defined as the poles of the meromorphic extension of the resolvent[2]:

$$R(s) := (\Delta - s(1-s))^{-1}, \quad s \in \mathbb{C} \quad (4.6)$$

This correspondence follows from the Selberg trace formula for finite-area surfaces, and has been shown by Patterson-Perry [25] for infinite volume surfaces without cusps and Borthwick, Judge and Perry [3] for infinite volume surfaces with cusps (see also [2] for an overview).

For the transfer operators as defined above, one can define a dynamical zeta function by [2, p.305]

$$d(z, s) := \text{Det} \left(1 - z \hat{L}_s \right)$$

The dynamical and the Selberg zeta function are equal $\zeta_{\text{Selberg}}(s) = d(1, s)$ (see [2, th.15.8]). This implies immediately that if $s \in \mathbb{C}$ is a resonance of the Laplacian on the Schottky surface, then 1 has to be an eigenvalue of \hat{L}_s :

$$s \in \text{Res}(\Delta) \Leftrightarrow 1 \in \text{Spec} \left(\hat{L}_s \right) \quad (4.7)$$

Remark 4.6. For the full Gauss map (i.e. with infinitely many branches) the same correspondence between the resonances of the Laplacian on the modular surface $\text{SL}_2\mathbb{Z} \backslash \mathbb{H}^2$ and the Dieter-Mayer transfer operator \hat{L}_s is true and has been developed by Dieter Mayer [21]. For the truncated Gauss map considered in Section 4.2.1, to our knowledge, no such corresponding surfaces are known.

Using the relation (4.7) between the Ruelle spectrum of the transfer operator \hat{L}_s and the resonances of the Laplacian, it is possible to deduce from Theorem 3.9 some estimate on the “asymptotic spectral gap” for the resonances of the Laplacian as follows.

Definition 4.7. The **asymptotic spectral gap** of resonances of the Laplacian Δ is defined by

$$a_{\text{asymp}} := \limsup_{b \rightarrow \infty} \{ \text{Re}(s) \text{ s.t. } s \in \text{Res}(\Delta), |\text{Im}(s)| > b \}$$

The setting (4.4) gives $D(x) = V - \frac{1}{2}J = \left(\frac{1}{2} - a\right) J(x)$ hence our estimate (3.14) gives that $a_{\text{asymp}} \leq \frac{1}{2}$. However this result concerning the resonances of the hyperbolic Laplacian is not new: from the self-adjoint properties of the Laplacian Δ in L^2 space we have that $\text{Im}(s(1-s)) \leq 0$ and this gives that

$$a_{\text{asymp}} \leq \frac{1}{2}. \quad (4.8)$$

Remark 4.8. If δ denotes the dimension of the limit set (equal to the dimension of the trapped set K) a result from F. Naud gives [22]: $\exists \varepsilon > 0$ s.t.

$$a_{\text{asymp}} \leq (1 - \varepsilon) \delta$$

which improves (4.8) if $\delta \leq 1/2$.

4.3 Proof of minimal captivity for both models

We give now the proof of Propositions 4.1 and 4.5. Note first that in both models the contracting maps are Möebius maps i.e. of the form $x'_j = \phi_{i,j}(x) = \frac{a_j x + b_j}{c_j x + d_j} = g_j(x)$ with 2×2 matrices $g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ with $D_j := \det g_j = \pm 1$. For the truncated Gauss map these matrices are

$$g_j = \begin{pmatrix} 0 & 1 \\ 1 & j \end{pmatrix} = G_j^{-1} \quad (4.9)$$

with $j = 1, \dots, N$ and $D := D_j = -1$. For the Bowen-Series maps we have

$$g_j = S_j^{-1} \in \mathrm{SL}_2 \mathbb{R} \quad (4.10)$$

with $j = 1, \dots, 2r$ and $D := D_j = +1$.

The following proposition shows that there exists coordinates (x, η) on phase space such that the canonical map $F = (F_j)_{j=1 \dots N}$ is decoupled in a product of identical maps.

Lemma 4.9. *The canonical map F defined in (3.5) is the union of the following maps F_j , with $j = 1 \dots N$:*

$$(x'_j, \xi'_j) = F_j(x, \xi) = \left(g_j(x), (g_j^{-1})'(x'_j) \xi + \tau'(x'_j) \right) \quad (4.11)$$

$$= \left(g_j(x), D_j \cdot (c_j x + d_j)^2 \xi - 2c_j (c_j x + d_j) \right) \quad (4.12)$$

Using the change of variables $(x, \eta) = \Phi(x, \xi) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ with $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and

$$\eta := x - \frac{2D}{\xi}, \quad (4.13)$$

the map F_j gets the simpler “decoupled expression”

$$(x'_j, \eta'_j) = (\Phi \circ F_j \circ \Phi^{-1})(x, \eta) = (g_j(x), g_j(\eta)) \quad (4.14)$$

Remark 4.10. geometrically these new variables (x, η) can be interpreted as the limit points $(x, \eta) \in \partial \mathbb{H}$ of a geodesic. The map $(x', \eta') = (\Phi \circ F \circ \Phi^{-1})(x, \eta)$ is simply the Poincaré map of the geodesic flow [6].

Proof. One has $g_j^{-1} = D_j \cdot \begin{pmatrix} d_j & -b_j \\ -c_j & a_j \end{pmatrix}$, $(g_j^{-1})(y) = \frac{d_j y - b_j}{-c_j y + a_j}$ and $(g_j^{-1})'(y) = D_j \cdot (a_j - c_j y)^{-2} = D_j \cdot (c_j x + d_j)^2$ if $y = g_j(x)$. The roof function is given by (4.1):

$$\begin{aligned} \tau(y) &= -J(y) = -\log \left(\left| (\phi_{i,j}^{-1})'(y) \right| \right) = -\log \left(\left| (g_j^{-1})'(y) \right| \right) \\ &= 2 \log(a_j - c_j y) \end{aligned}$$

So $\tau'(y) = -2c_j(a_j - c_j y)^{-1} = -2c_j(c_j x + d_j)$ and

$$(x'_j, \xi'_j) = F_j(x, \xi) \stackrel{(4.11)}{=} (g_j(x), D \cdot (c_j x + d_j)^2 \xi - 2c_j(c_j x + d_j)) \quad (4.15)$$

giving (4.12). Now we use the change of variable

$$\xi = \frac{2D}{x - \eta}. \quad (4.16)$$

So

$$\begin{aligned} \xi'_j &= D \cdot (c_j x + d_j)^2 \xi - 2c_j(c_j x + d_j) \\ &= D \cdot (c_j x + d_j)^2 \frac{2D}{(x - \eta)} - 2c_j(c_j x + d_j) \\ &= \frac{2(c_j x + d_j)}{(x - \eta)} (c_j \eta + d_j) \end{aligned}$$

Then

$$\begin{aligned} \eta'_j &= x'_j - \frac{2D}{\xi'_j} = \frac{a_j x + b_j}{c_j x + d_j} - \frac{D(x - \eta)}{(c_j x + d_j)(c_j \eta + d_j)} \\ &= \frac{(a_j x + b_j)(c_j \eta + d_j) - (a_j d_j - b_j c_j)(x - \eta)}{(c_j x + d_j)(c_j \eta + d_j)} = \frac{a_j \eta + b_j}{c_j \eta + d_j} = g_j(\eta) \end{aligned}$$

□

Recall that the multivalued map $\phi = (\phi_{i,j} = g_j)$ has a trapped set K defined in (2.9) as $K = \bigcap_{n \geq 1} \phi^n(I)$. The basin of K on $\overline{\mathbb{R}}$ is $\mathcal{B}(K) := \{x \in \overline{\mathbb{R}}, \exists n \geq 0, \phi^n(x) \in I\} \subset \overline{\mathbb{R}}$.

Lemma 4.11. *The trapped set in phase space \mathcal{K} defined in (3.9) is contained in the following set:*

$$\mathcal{K} \subset \{(x, \xi), x \in I, \eta \notin \mathcal{B}(K) \text{ with } (x, \eta) = \Phi(x, \xi)\} \quad (4.17)$$

Proof. Let $(x, \xi) \in I \times \mathbb{R}$ which does not belong to the set defined on the right hand side of (4.17). Then $\eta \in \mathcal{B}(K)$. Hence for every admissible word $w \in \mathcal{W}$, we have that $|\phi_{w_0,n}(x) - \phi_{w_0,n}(\eta)| \leq C \cdot \theta^n \xrightarrow{n \rightarrow +\infty} 0$. From the change of variable (4.16) and the expression (4.14) with the new variables, this gives that $(x_n, \xi_n) := F_{w_0,n}(x, \xi)$ satisfies

$$|\xi_n| = \frac{2}{|\phi_{w_0,n}(x) - \phi_{w_0,n}(\eta)|} \geq C' \cdot \theta^{-n} \rightarrow +\infty$$

hence $(x, \xi) \notin \mathcal{K}$. We deduce (4.17). □

Finally, we show minimal captivity of the canonical map F . According to (3.10), we have to show that there exists a neighborhood B of \mathcal{K} such that $\forall (x, \xi) \in B, \# \{F(x, \xi) \cap B\} \leq 1$. This is true if $B_j := F_j^{-1}(B), j = 1 \dots N$ are disjoint sets. Using the coordinates (x, η) which decouple the map F_j , in (4.14), it is equivalent to show that there exists a neighborhood \mathcal{B} of K in $\overline{\mathbb{R}}$ such that $\mathcal{B}_j := g_j^{-1}(\mathcal{B}) \subset \overline{\mathbb{R}}, j = 1, \dots, N$ are disjoint sets. For this we consider both cases:

Minimal captivity of the truncated Gauss map. For this map, let $\mathcal{B} :=] - \infty, -1[$. Then the sets $g_j^{-1}(] - \infty, -1[) =] - j - 1, j[$, with $j = 1 \dots N$, are mutually disjoint. From the argument above this implies that the truncated Gauss map is minimal captive, i.e. Proposition 4.1. Notice that, from (4.16), in variables $(x, \xi) \in T^*[0, 1]$ we have

$$B = \{x \in [0, 1], \eta \in] - \infty, -1[\} = \left\{ (x, \xi), x \in [0, 1], \frac{-2}{x+1} < \xi < 0 \right\}$$

This set B contains the trapped set \mathcal{K}_N and is depicted in figure (4.1).

Minimal captivity of the Bowen Series map. For this case, let $\mathcal{B} := I = \bigcup_{j=1}^{2r} I_j$. Then $\mathcal{B}_j = g_j^{-1}(\mathcal{B}) = g_{j+r}(I) \subset I_{j+r}$. Since the sets I_{j+r} are mutually disjoint, the sets \mathcal{B}_j are also disjoint. From the argument above this implies that the Bowen Series map on phase space is minimal captive, i.e. Proposition 4.5.

Figure (4.5) shows the sets $B_j = F_j^{-1}(B)$ with $B := \{x \in I, \eta \in \mathcal{B}\}$ and

$$B_j = \{x \in I, \eta \in \mathcal{B}_j\}$$

that we have used in the proof of minimal captivity.

5 Proof of Theorem 3.1 about the discrete spectrum

For this proof we follow closely the proof⁵ of Theorem 2 in the paper [11] which uses semiclassical analysis.

5.1 Dynamics on the cotangent space T^*I

In order to study the spectral properties of the transfer operator, we have first to study the dynamics of the map $\phi : I \rightarrow I$ lifted on the cotangent space T^*I .

⁵see also theorem 4 in [12] although we are dealing in this paper with expanding maps instead of hyperbolic maps which simplifies the analysis, since we can work with ordinary Sobolev spaces and not anisotropic Sobolev spaces.

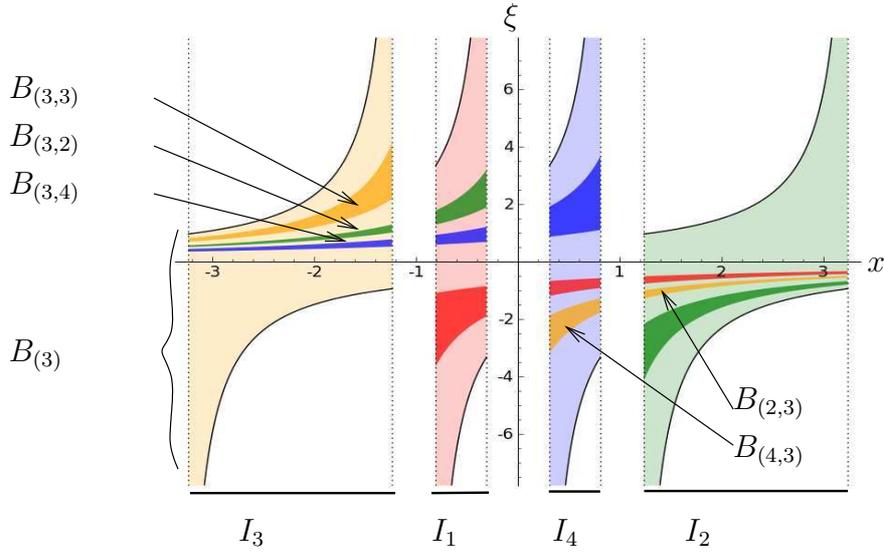


Figure 4.5: This figure illustrates the choice of the bounding functions in the proof of the minimal captive property for the example of a Schottky surface shown in figure 4.4. The light shaded regions indicate the set $B_{(j)} := B \cap (I_j \times \mathbb{R})$ while the darker shaded regions indicate the different pre-images $B_{(i,j)} := F_{ij}^{-1}(B \cap (I_j \times \mathbb{R})) \subset B_{(i)}$, $i \neq j + 2 \pmod{4}$. For example, that dark orange shaded regions $B_{(3,3)}, B_{(4,3)}, B_{(2,3)}$ shows the three preimages of the light orange region $B_{(3)}$. The trapped set \mathcal{K} is contained in the union of these $B_{(i,j)}$.

Proposition 5.1. *Considering $\hbar > 0$ fixed, the transfer operator \hat{F}_χ restricted to $C_I^\infty(\mathbb{R})$ is a Fourier integral operator (FIO). Its **canonical transform** is a multi-valued symplectic map $\mathfrak{F} : T^*I \rightarrow T^*I$ on the cotangent space $T^*I \equiv I \times \mathbb{R}$ given by:*

$$\mathfrak{F} : \begin{cases} T^*I & \rightarrow T^*I \\ (x, \xi) & \rightarrow \{\mathfrak{F}_{i,j}(x, \xi), \text{ with } i, j \text{ s.t. } x \in I_i, i \rightsquigarrow j\} \end{cases}$$

with

$$\mathfrak{F}_{i,j} : \begin{cases} x' & = \phi_{i,j}(x) \\ \xi' & = \frac{1}{\phi'_{i,j}(x)} \xi \end{cases}. \quad (5.1)$$

Remarks:

- For short, we can write

$$\mathfrak{F} : \begin{cases} T^*I & \rightarrow T^*I \\ (x, \xi) & \rightarrow \left(\phi(x), \frac{1}{\phi'(x)} \xi \right) \end{cases}. \quad (5.2)$$

- Notice that the map \mathfrak{F} differs from the canonical map F introduced in (3.7). The reason is that this latter map is used in the asymptotic limit $\hbar \rightarrow 0$, whereas the study in this section is for fixed \hbar .

Proof. of Proposition 5.1. The operator $\varphi \rightarrow \varphi \circ \phi_{i,j}^{-1}$ is a pull back operator, one of the simplest example of Fourier integral operator in the sense of semiclassical homogeneous theory [32], see also [19] example 2 p.150. In that case the canonical map $\mathfrak{F}_{i,j}$ is the map $\phi_{i,j}$ lifted on the cotangent space T^*I in the canonical way. In particular the action on ξ is linear. The term $e^{V(x)} e^{\frac{i}{\hbar}\tau(x)} \chi(\phi_{i,j}^{-1}(x))$ in (2.20) does not contribute to the expression of \mathfrak{F} because it acts as a pseudodifferential operator, equivalently as a FIO whose canonical map is the identity. \square

Remarks

- Observe that the dynamics of the map \mathfrak{F} on T^*I has a quite simple property: the zero section $\{(x, \xi) \in I \times \mathbb{R}, \xi = 0\}$ is globally invariant and any other point (x, ξ) with $\xi \neq 0$ escapes towards infinity ($\xi \rightarrow \pm\infty$) in a controlled manner, because $|\phi'_{i,j}(x)| < \theta < 1$, with θ given in (2.1), hence:

$$|\xi'| \geq \frac{1}{\theta} |\xi| \quad (5.3)$$

- Due to hypothesis (2.2) the map $\phi_{i,j}^{-1}$ is uni-valued (when it is defined). Therefore the map \mathfrak{F}^{-1} is also uni-valued and one has

$$\mathfrak{F}^{-1} \circ \mathfrak{F} = \text{Id}_{T^*I} \quad (5.4)$$

5.2 The escape function

Definition 5.2. [32, p.2] For $m \in \mathbb{R}$, the **class of symbols** $S^{-m}(T^*\mathbb{R})$, with **order** m , is the set of functions on the cotangent space $A \in C^\infty(T^*\mathbb{R})$ such that for any $\alpha, \beta \in \mathbb{N}$, there exists $C_{\alpha,\beta} > 0$ such that

$$\forall (x, \xi) \in T^*\mathbb{R}, \quad \left| \partial_x^\alpha \partial_\xi^\beta A(x, \xi) \right| \leq C_{\alpha,\beta} \langle \xi \rangle^{-m-|\beta|}, \quad \text{with } \langle \xi \rangle = (1 + \xi^2)^{1/2}. \quad (5.5)$$

Lemma 5.3. *Let $m > 0$ and let*

$$A_m(x, \xi) := \langle \xi \rangle^{-m} \in S^{-m}(T^*\mathbb{R}).$$

We have

$$\forall R > 0, \forall |\xi| > R, \quad \forall i \rightsquigarrow j, \forall x \in I_i, \quad \frac{A_m(\mathfrak{F}_{i,j}(x, \xi))}{A_m(x, \xi)} \leq C^m < 1, \quad (5.6)$$

*with $C = \sqrt{\frac{R^2+1}{R^2/\theta^2+1}} < 1$. We say that A_m is an **escape function**: (5.6) shows that A_m decreases strictly along the trajectories of \mathfrak{F} outside the zero section.*

Proof. From Eq. (5.1) and (5.3) we have

$$\frac{A_m(\mathfrak{F}_{i,j}(x, \xi))}{A_m(x, \xi)} = \frac{(1 + \xi^2)^{m/2}}{(1 + (\xi')^2)^{m/2}} \leq \frac{(1 + \xi^2)^{m/2}}{(1 + \xi^2/\theta^2)^{m/2}} \leq \left(\frac{1 + R^2}{1 + R^2/\theta^2} \right)^{m/2} = C^m.$$

The last inequality is because the function decreases with $|\xi|$. □

Using the standard quantization rule [32, p.2] the symbol A_m can be quantized into a pseudodifferential operator \hat{A}_m (PDO for short) which is self-adjoint and invertible on $C_0^\infty(\mathbb{R})$:

$$\left(\hat{A}_m \varphi \right) (x) = \frac{1}{2\pi} \int A_m(x, \xi) e^{i(x-y)\xi} \varphi(y) dy d\xi. \quad (5.7)$$

Conversely A_m is called the symbol of the PDO \hat{A}_m . In our simple case, this is very explicit: in Fourier space, \hat{A}_m is simply the multiplication by $\langle \xi \rangle^m$. Its inverse \hat{A}_m^{-1} is the multiplication by $\langle \xi \rangle^{-m}$.

5.3 Use of the Egorov Theorem

Let

$$\hat{Q}_m := \hat{A}_m \hat{F}_\chi \hat{A}_m^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

which is unitarily equivalent to $\hat{F}_\chi : H^{-m}(\mathbb{R}) \rightarrow H^{-m}(\mathbb{R})$ (from the definition of $H^{-m}(\mathbb{R})$, Eq.(3.1)). This is expressed by the following commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\hat{Q}_m} & L^2(\mathbb{R}) \\ \downarrow \hat{A}_m^{-1} & & \downarrow \hat{A}_m^{-1} \\ H^{-m}(\mathbb{R}) & \xrightarrow{\hat{F}_\chi} & H^{-m}(\mathbb{R}) \end{array} .$$

We will therefore study the operator \hat{Q}_m on $L^2(\mathbb{R})$. Notice that \hat{Q}_m is defined a priori on a dense domain $C_0^\infty(\mathbb{R})$. Define

$$\hat{P} := \hat{Q}_m^* \hat{Q}_m = \hat{A}_m^{-1} \left(\hat{F}_\chi^* \hat{A}_m^2 \hat{F}_\chi \right) \hat{A}_m^{-1} = \hat{A}_m^{-1} \hat{B} \hat{A}_m^{-1} \quad (5.8)$$

with

$$\hat{B} := \hat{F}_\chi^* \hat{A}_m^2 \hat{F}_\chi = \hat{\chi} \hat{F}^* \hat{A}_m^2 \hat{F} \hat{\chi}. \quad (5.9)$$

Now, the crucial step in the proof is to use the Egorov Theorem.

Lemma 5.4. (Egorov theorem). \hat{B} defined in (5.9) is a pseudo-differential operator with symbol in $S^{-2m}(T^*\mathbb{R})$ given by:

$$B(x, \xi) = \left(\chi^2(x) \sum_{j \text{ s.t. } i \rightsquigarrow j} |\phi'_{i,j}(x)| e^{2\text{Re}(V(\phi_{i,j}(x)))} A_m^2(\mathfrak{F}_{i,j}(x, \xi)) \right) + R \quad (5.10)$$

where $R \in S^{-2m-1}(T^*\mathbb{R})$ has a lower order, $x \in I_i$, $\xi \in \mathbb{R}$.

Proof. \hat{F} and \hat{F}^* are Fourier integral operators (FIO) whose canonical maps are respectively \mathfrak{F} and \mathfrak{F}^{-1} . The pseudodifferential operator (PDO) \hat{A}_m can also be considered as a FIO whose canonical map is the identity. By composition we deduce that $\hat{B} = \hat{\chi} \hat{F}^* \hat{A}_m^2 \hat{F} \hat{\chi}$ is a FIO whose canonical map is the identity since $\mathfrak{F}^{-1} \circ \mathfrak{F} = I$ from (5.4). Therefore \hat{B} is a PDO. Using (2.20), (2.25) we obtain that the principal symbol of \hat{B} is the first term of (5.10). \square

Remark: contrary to (5.9), $\hat{F} \hat{A}_m \hat{F}^*$ is not a PDO, but a FIO whose canonical map $\mathfrak{F} \circ \mathfrak{F}^{-1}$ is multivalued.

Now by **theorem of composition of PDO** [32, p.11], Eq.(5.8) and Eq.(5.10) imply that \hat{P} is a PDO with symbol in $S^0(\mathbb{R})$ and for $x \in I_i$, $\xi \in \mathbb{R}$ the principal symbol is given by

$$P(x, \xi) = \frac{B(x, \xi)}{A_m^2(x, \xi)} = \left(\chi^2(x) \sum_{j \text{ s.t. } i \rightsquigarrow j} |\phi'_{i,j}(x)| e^{2\text{Re}(V(\phi_{i,j}(x)))} \frac{A_m^2(\mathfrak{F}_{i,j}(x, \xi))}{A_m^2(x, \xi)} \right). \quad (5.11)$$

The estimate (5.6) gives the following upper bound for any $R > 0$, $x \in I$ and $|\xi| > R$:

$$|P(x, \xi)| \leq \chi^2(x) C^{2m} \sum_{j, i \rightsquigarrow j} |\phi'_{i,j}(x)| e^{2\text{Re}(V(\phi_{i,j}(x)))} \leq C^{2m} N \theta e^{2V_{\max}}$$

with $V_{\max} = \max_{x \in I} \text{Re}(V(x))$.

We apply⁶ the L^2 -continuity theorem for PDO to \hat{P} as given in [14, th 4.5 p.42]. The result is that for any $\varepsilon > 0$,

$$\hat{P} = \hat{k}_\varepsilon + \hat{p}_\varepsilon$$

with \hat{k}_ε a smoothing operator (hence compact) and $\|\hat{p}_\varepsilon\| \leq C^{2m} N \theta e^{2V_{\max}} + \varepsilon$.

If $\hat{Q}_m = \hat{U} \left| \hat{Q}_m \right|$ is the polar decomposition of \hat{Q}_m , with \hat{U} unitary, then from (5.8), $\hat{P} = \left| \hat{Q}_m \right|^2$, hence $\left| \hat{Q}_m \right| = \sqrt{\hat{P}}$ and the spectral theorem [32, p.75] gives that $\left| \hat{Q}_m \right|$ has a similar decomposition

$$\left| \hat{Q}_m \right| = \hat{k}'_\varepsilon + \hat{q}_\varepsilon$$

with \hat{k}'_ε compact and $\|\hat{q}_\varepsilon\| \leq \sqrt{C^{2m} N \theta e^{2V_{\max}}} + \varepsilon$, with any $\varepsilon > 0$. Since $\|\hat{U}\| = 1$ we deduce a similar decomposition for $\hat{Q}_m = \hat{U} \left| \hat{Q}_m \right| : L^2(I) \rightarrow L^2(I)$. We also use the fact that $C \rightarrow \theta$ as $R \rightarrow \infty$ in (5.6) and we deduce (3.2) and (3.3) for $\hat{F}_\chi : H^{-m}(\mathbb{R}) \rightarrow H^{-m}(\mathbb{R})$. Precisely we obtain that $r_m = \theta^m \sqrt{N \theta e^{2V_{\max}}} = C \theta^m$ with C independent on m .

The fact that the eigenvalues λ_i and their generalized eigenspaces do not depend on the choice of space $H^{-m}(\mathbb{R})$ is due to density of $C_0^\infty(\mathbb{R})$ in Sobolev spaces. We refer to the argument given in the proof of corollary 1 in [12].

Finally, if φ is an eigendistribution of \hat{F}_χ , i.e. $\hat{F}_\chi \varphi = \lambda \varphi$ with $\lambda \neq 0$, we deduce that $\varphi = \frac{1}{\lambda^n} \hat{F}_\chi^n \varphi$ for any $n \geq 1$, and (2.28) implies that $\text{supp}(\varphi) \subset K = \bigcap_{n \in \mathbb{N}} K_n$. On the trapped set we have $\chi = 1$ hence the eigendistribution and eigenvalues of \hat{F}_χ do not depend on χ . This finishes the proof of Theorem 3.1.

6 Dynamics of the canonical map $F : T^*I \rightarrow T^*I$

In this Section we study the trapped set \mathcal{K} and the symbolic dynamics of the canonical map F defined in Proposition 3.2 as associated to the family transfer operator \hat{F}_h . Before we give:

Proof. of Proposition 3.2. This is the same argument as in the proof of Proposition 5.1 except that now the family of operators $\left(e^{\frac{i}{h}\tau(x)} \right)_{h>0}$ which appears in (2.20) is a FIO. As explained in [19, ex.1,p.150], its canonical map is $(x, \xi) \rightarrow (x, \xi + \frac{d\tau}{dx})$. We compose this map with the previous one (5.1) to get (3.6). \square

⁶Actually, we can not apply directly the L^2 -continuity theorem for PDO to \hat{P} because \hat{P} doesn't have a compactly supported Schwartz kernel. However \hat{B} obviously has a compactly supported Schwartz kernel due to the presence of $\hat{\chi}$ in Eq.(5.9). The trick is to approximate \hat{A}_m^{-1} by a properly supported operator Λ_m as it is done in [14, p.45] and then apply the L^2 -continuity theorem to $\hat{\Lambda}_m \hat{B} \hat{\Lambda}_m$.

6.1 The trapped set \mathcal{K} in phase space

We have provided a definition of the trapped set \mathcal{K} in (3.9). We will give now a more precise description of it. Recall that the inverse maps ϕ^{-1} and F^{-1} are uni-valued. For any integer $m \geq 0$, let

$$\tilde{K}_m := F^{-m} (K_m \times [-R, R])$$

where $K_m = \phi^m(I)$ has been defined in (2.8) and R is given by Lemma 3.4. In particular $\tilde{K}_0 = I \times [-R, R]$. Let $\pi : (x, \xi) \in T^*I \rightarrow x \in I$ be the projection map. These sets have the following properties:

$$\begin{aligned} \pi(\tilde{K}_m) &= I, \\ \tilde{K}_{m+1} &\subset \tilde{K}_m \end{aligned} \tag{6.1}$$

Proof. of (6.1). From Lemma 3.4 we have

$$(K_{m+1} \times [-R, R]) \subset F(K_m \times [-R, R])$$

hence

$$\tilde{K}_{m+1} = F^{-m} (F^{-1} (K_{m+1} \times [-R, R])) \subset F^{-m} (K_m \times [-R, R]) = \tilde{K}_m$$

□

Let us define

$$\tilde{K} := \bigcap_m \tilde{K}_m \tag{6.2}$$

Now we combine the sets K_n defined in (2.8) with the sets \tilde{K}_m and define for any integers $a, b \geq 0$

$$\mathcal{K}_{a,b} := \pi^{-1}(K_a) \bigcap \tilde{K}_b \tag{6.3}$$

We have

$$\mathcal{K}_{a+1,b} \subset \mathcal{K}_{a,b}, \quad \mathcal{K}_{a,b+1} \subset \mathcal{K}_{a,b} \tag{6.4}$$

and

$$F^{-1}(\mathcal{K}_{a,b}) = \mathcal{K}_{a-1,b+1}. \tag{6.5}$$

Remark 6.1. We can interpret the trapped set $K \subset I$ with respect to the lifted map $F : T^*I \rightarrow T^*I$, as follows. The trapped set $\pi^{-1}(K) \subset T^*I$ is characterized by

$$\pi^{-1}(K) = \{(x, \xi) \in T^*I, \quad \exists \text{compact } C \Subset T^*I, \forall n \geq 0, F^{-n}(x, \xi) \in C\}$$

i.e. $\pi^{-1}(K)$ can be considered as the “trapped set of the map F in the past”. Similarly $\tilde{K} \subset T^*I$ can be interpreted as the “trapped set of the map F in the future” and $\mathcal{K} \subset T^*I$ as the full trapped set (past and future) since they are characterized by

$$\begin{aligned} \tilde{K} &= \{(x, \xi) \in T^*I, \quad \exists \text{compact } C \Subset T^*I, \forall n \geq 0, F^n(x, \xi) \cap C \neq \emptyset\} \\ \mathcal{K} &= \{(x, \xi) \in T^*I, \quad \exists \text{compact } C \Subset T^*I, \forall n \in \mathbb{Z}, F^n(x, \xi) \cap C \neq \emptyset\} \\ &= \pi^{-1}(K) \cap \tilde{K} \end{aligned} \tag{6.6}$$

From this previous remark, the next definition is equivalent to (3.9).

Definition 6.2. The **trapped set** $\mathcal{K} \subset T^*I$ of the map F is

$$\mathcal{K} := \bigcap_{a=0}^{\infty} \mathcal{K}_{a,a} \quad (6.7)$$

The hypothesis of minimal captivity has been defined in (3.7). The following proposition gives equivalent, stronger and weaker definition of minimal captivity. They are convenient for practical purposes.

Proposition 6.3.

1. The map F is *minimally captive* (i.e. Eq.(3.10) holds true) if and only if the map F satisfies

$$\exists a, \quad \forall (x, \xi) \in \mathcal{K}_{a,a}, \quad \#\left\{F(x, \xi) \cap \mathcal{K}_{a,a}\right\} \leq 1. \quad (6.8)$$

2. If map F is *minimally captive* then

$$\exists a, \exists C, \forall n \text{ s.t. } \forall (x, \xi) \in \mathcal{K}_{a,0}, \quad \#\left\{F^n(x, \xi) \cap \mathcal{K}_{a,0}\right\} \leq C. \quad (6.9)$$

where $\mathcal{K}_{a,0} := (\pi^{-1}(K_a) \cap [-R, R])$ has been defined in (3.7).

Proof. The fact that (6.8) is equivalent to (3.10) is because

$$\forall \varepsilon > 0, \exists a \text{ s.t. } \mathcal{K}_{a,a} \subset \mathcal{K}_\varepsilon$$

$$\forall a, \exists \varepsilon > 0 \text{ s.t. } \mathcal{K}_\varepsilon \subset \mathcal{K}_{a,a}$$

□

6.2 Symbolic dynamics

The purpose of this section is to describe precisely the dynamics of ϕ and F using “symbolic dynamics”. This is very standard for expanding maps [5]. This somehow refines the structure of the sets $\mathcal{K}_{a,b}$ introduced before.

6.2.1 Symbolic dynamics on the trapped set $K \subset I$

Let

$$\mathcal{W}_- := \left\{ (\dots, w_{-2}, w_{-1}, w_0) \in \{1, \dots, N\}^{-\mathbb{N}}, w_{l-1} \rightsquigarrow w_l, \forall l \leq 0 \right\} \quad (6.10)$$

be the set of **admissible left semi-infinite sequences**. For $w \in \mathcal{W}_-$ and $i < j$ we write $w_{i,j} := (w_i, w_{i+1}, \dots, w_j)$ for an extracted sequence. For simplicity we will use the notation

$$\phi_{w_{i,j}} := \phi_{w_{j-1}, w_j} \circ \dots \circ \phi_{w_i, w_{i+1}} : I_{w_i} \rightarrow I_{w_j} \quad (6.11)$$

for the composition of maps. For $n \geq 0$, let

$$I_{w_{-n},0} := \phi_{w_{-n},0} (I_{w_{-n}}) \subset I_{w_0}. \quad (6.12)$$

For any $0 < m < n$ we have the strict inclusions

$$I_{w_{-n},0} \subset I_{w_{-m},0} \subset I_{w_0}.$$

From (2.1), the size of $I_{w_{-n},0}$ is bounded by

$$|I_{w_{-n},0}| \leq \theta^n |I_{w_0}|,$$

hence the sequence of sets $(I_{w_{-n},0})_{n \geq 1}$ is a sequence of non empty and decreasing closed intervals and $\bigcap_{n=1}^{\infty} I_{w_{-n},0}$ is a point in K . We define

Definition 6.4. The “symbolic coding map” is

$$S : \begin{cases} \mathcal{W}_- & \rightarrow K \\ w & \rightarrow S(w) := \bigcap_{n=1}^{\infty} I_{w_{-n},0} \end{cases} \quad (6.13)$$

In some sense we have decomposed the sets K_n , Eq.(2.8), into individual components:

$$K_n = \bigcup_{w_{-n,0} \in \mathcal{W}_-} I_{w_{-n},0} \quad (6.14)$$

$$K = \bigcup_{w \in \mathcal{W}_-} S(w)$$

Let us introduce the **left shift**, a multivalued map, defined by

$$L : \begin{cases} \mathcal{W}_- & \rightarrow \mathcal{W}_- \\ (\dots, w_{-2}, w_{-1}, w_0) & \rightarrow (\dots, w_{-2}, w_{-1}, w_0, w_1) \end{cases}$$

with $w_1 \in \{1, \dots, N\}$ such that $w_0 \rightsquigarrow w_1$. Let the **right shift** be the univalued map defined by

$$R : \begin{cases} \mathcal{W}_- & \rightarrow \mathcal{W}_- \\ (\dots, w_{-2}, w_{-1}, w_0) & \rightarrow (\dots, w_{-2}, w_{-1}) \end{cases}.$$

Proposition 6.5. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{W}_- & \xrightarrow{S} & K \\ R \uparrow \downarrow L & & \phi^{-1} \uparrow \downarrow \phi \\ \mathcal{W}_- & \xrightarrow{S} & K \end{array} \quad (6.15)$$

and the map $S : \mathcal{W}_- \rightarrow K$ is one to one. This means that the dynamics of points on the trapped set K under the maps ϕ^{-1}, ϕ is equivalent to the symbolic dynamics of the shift maps R, L on the set of admissible words \mathcal{W}_- . Notice that the maps R and ϕ^{-1} are univalued, whereas the maps L and ϕ are (in general) multivalued.

Proof. From the definition of S we have

$$\phi_{w_0 w_1} (S(\dots, w_{-2}, w_{-1}, w_0)) = S(\dots, w_{-2}, w_{-1}, w_0, w_1) \quad (6.16)$$

and

$$\phi_{w_{-1} w_0}^{-1} (S(\dots, w_{-2}, w_{-1}, w_0)) = S(\dots, w_{-2}, w_{-1}) \quad (6.17)$$

which gives the diagram (6.15). The map $S : \mathcal{W}_- \rightarrow K$ is surjective by construction. Let us show that the hypothesis (2.2) implies that it is also injective. Let $w, w' \in \mathcal{W}_-$ and suppose that $w \neq w'$, i.e. there exists $k \geq 0$ such that $w_{-k} \neq w'_{-k}$. From (2.2) we have $\phi_{w_{-k}, w_{-k+1}} (I_{w_{-k}}) \cap \phi_{w'_{-k}, w'_{-k+1}} (I_{w'_{-k}}) = \emptyset$. We deduce recursively that $\phi_{w_{-k}, 0} (I_{w_{-k}}) \cap \phi_{w'_{-k}, 0} (I_{w'_{-k}}) = \emptyset$. Since $S(w) \in \phi_{w_{-k}, 0} (I_{w_{-k}})$ and $S(w') \in \phi_{w'_{-k}, 0} (I_{w'_{-k}})$ we deduce that $S(w) \neq S(w')$. Hence S is one to one. \square

6.2.2 The “future trapped set” \tilde{K} in phase space T^*I

Let

$$\mathcal{W}_+ := \left\{ (w_0, w_1, w_2, \dots) \in \{1, \dots, N\}^{\mathbb{N}}, \quad w_l \rightsquigarrow w_{l+1}, \forall l \geq 0 \right\}$$

be the set of admissible right semi-infinite sequences. We still use the notation $w_{i,j} := (w_i, w_{i+1}, \dots, w_j)$ for an extracted sequence. For any $n \geq 0$ let

$$\tilde{I}_{w_0, n} := F^{-n} (I_{w_0, n} \times [-R, R]) \quad (6.18)$$

be the image of the rectangle under the univalued map F^{-n} . Notice that $\pi \left(\tilde{I}_{w_0, n} \right) = I_{w_0}$ where $\pi(x, \xi) = x$ is the canonical projection map. Since the map F^{-1} contracts strictly in variable ξ by the factor $\theta < 1$ then $\left(\tilde{I}_{w_0, n} \right)_{n \in \mathbb{N}}$ is a sequence of decreasing sets: $\tilde{I}_{w_0, n+1} \subset \tilde{I}_{w_0, n}$ and we can define the limit

$$\tilde{S} : w \in \mathcal{W}_+ \rightarrow \tilde{S}(w) := \bigcap_{n \geq 0} \tilde{I}_{w_0, n} \subset \tilde{K}. \quad (6.19)$$

Proposition 6.6. *For every $w \in \mathcal{W}_+$, the set $\tilde{S}(w)$ is a smooth curve given by*

$$\tilde{S}(w) = \{(x, \zeta_w(x)), \quad x \in I_{w_0}, w \in \mathcal{W}_+\}$$

with

$$\zeta_w(x) = - \sum_{k \geq 1} \phi'_{w_0, k}(x) \cdot \tau'(\phi_{w_0, k}(x)), \quad (6.20)$$

We have an estimate of regularity, uniform in w : $\forall \alpha \in \mathbb{N}, \exists C_\alpha > 0, \forall w \in \mathcal{W}_+, \forall x \in I_{w_0},$

$$|(\partial_x^\alpha \zeta_w)(x)| \leq C_\alpha. \quad (6.21)$$

Moreover, with the hypothesis 3.7 of minimal captivity there exists $a \geq 1$ such that these branches do not intersect on $\pi^{-1}(K_a)$,

$$\forall w, w' \in \mathcal{W}_+, \quad w \neq w' \Rightarrow \pi^{-1}(K_a) \cap \tilde{S}(w) \cap \tilde{S}(w') = \emptyset \quad (6.22)$$

The set (6.2) can be expressed as

$$\tilde{K} = \bigcup_{w \in \mathcal{W}_+} \tilde{S}(w).$$

Proof. From (3.6) we get

$$F^{-1}(\phi_{i, j}(x), \xi) = (x, \phi'_{i, j}(x)(\xi - \tau'(\phi_{i, j}(x)))) \quad (6.23)$$

Iterating this equation we get, that

$$\zeta_{w, n}(x) := - \sum_{k=1}^n \phi'_{w_0, k}(x) \cdot \tau'(\phi_{w_0, k}(x)) = F^{-n}(\phi_{w_0, n}(x), 0),$$

thus $\zeta_{w, n}(x) \in \tilde{S}(w)$ for all $n \in \mathbb{N}$ and we get (6.20). In order to prove (6.21) we can check, that the series of $\zeta_{w, n}(x)$ and $\partial_x^\alpha \zeta_{w, n}(x)$ converge with uniform bounds in w which follows after some calculations from (2.1) and the fact that $\phi'_{w_0, k}(x) \leq \theta^k$ independent of w . \square

6.2.3 Symbolic dynamics on the trapped set \mathcal{K} in phase space T^*I

Recall from (6.6) that $\mathcal{K} = \pi^{-1}(K) \cap \tilde{K}$. Let

$$\mathcal{W} := \left\{ (\dots w_{-2}, w_{-1}, w_0, w_1, \dots) \in \{1, \dots, N\}^{\mathbb{Z}}, \quad w_l \rightsquigarrow w_{l+1}, \forall l \in \mathbb{Z} \right\}$$

be the set of bi-infinite admissible sequences. For a given $w \in \mathcal{W}$ and $a, b \in \mathbb{N}$, let

$$\mathcal{I}_{w_{-a},0,w_0,b} := \left(\pi^{-1}(I_{w_{-a},0}) \cap \tilde{I}_{w_0,b} \right) \subset \mathcal{K}_{a,b}$$

where $\mathcal{K}_{a,b}$ has been defined in (6.3).

Definition 6.7. The symbolic coding map is

$$\mathcal{S} : \begin{cases} \mathcal{W} & \rightarrow \mathcal{K} \\ w & \rightarrow \mathcal{S}(w) := \bigcap_{n=1}^{\infty} \mathcal{I}_{w_{-n},0,w_0,n} = \left(\pi^{-1}(S(w_-)) \cap \tilde{S}(w_+) \right) \end{cases} \quad (6.24)$$

with $w_- = (\dots w_{-1}, w_0) \in \mathcal{W}_-$, $w_+ = (w_0, w_1, \dots) \in \mathcal{W}_+$.

More precisely we can express the point $\mathcal{S}(w) \in \mathcal{K}$ as

$$\mathcal{S}(w) = (x_{w_-}, \xi_w), \quad x_{w_-} = S(w_-), \quad \xi_w = \zeta_{w_+}(S(w_-)), \quad (6.25)$$

with ζ_{w_+} given in (6.20). We also have

$$\mathcal{K}_{a,b} = \bigcup_{w \in \mathcal{W}} \mathcal{I}_{w_{-a},0,w_0,b}.$$

Proposition 6.8. The following diagram is commutative

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{S} & \mathcal{K} \\ R \uparrow \downarrow L & & F^{-1} \uparrow \downarrow F \\ \mathcal{W} & \xrightarrow{S} & \mathcal{K}. \end{array} \quad (6.26)$$

If assumption 3.7 of minimal captivity holds true then the map $\mathcal{S} : \mathcal{W} \rightarrow \mathcal{K}$ is one to one. This means that the univalued dynamics of points on the trapped set \mathcal{K} under the maps F^{-1}, F is equivalent to the symbolic dynamics of the full shift maps R, L on the set of words \mathcal{W} .

Proof. Commutativity of the diagram comes from the construction of \mathcal{S} . Also \mathcal{S} is surjective. Let us show that \mathcal{S} is injective. Let $w, w' \in \mathcal{W}$, with $w \neq w'$. There exists $n \geq 0$ such that $(L^n(w))_- \neq (L^n(w'))_-$. So $S((L^n(w))_-) \neq S((L^n(w'))_-)$ because $S : \mathcal{W}_- \rightarrow K$ is one to one from Lemma 6.5. Hence $\mathcal{S}(L^n(w)) \neq \mathcal{S}(L^n(w'))$ and $F^n(\mathcal{S}(w)) \neq F^n(\mathcal{S}(w'))$ from commutativity of the diagram. We apply F^{-n} and deduce that $\mathcal{S}(w) \neq \mathcal{S}(w')$ because F^{-1} and F^{-n} are injective on \mathcal{K} from Assumption 3.7. \square

6.3 Dimension of the trapped set \mathcal{K}

6.3.1 Proof of Theorem 3.10

For $w = (w_k)_{k \in \mathbb{Z}} \in \mathcal{W}$, we note $w_- = (\dots, w_{-2}, w_{-1}, w_0) \in \mathcal{W}_-$ and $w_+ = (w_0, w_1, \dots) \in \mathcal{W}_+$. Let

$$\text{Inv}(w_+) := (\dots w_2, w_1, w_0)$$

be the reversed word. Since the adjacency matrix A is supposed to be symmetric we have that $\text{Inv}(w_+) \in \mathcal{W}_-$. Then, let us consider the following one to one map

$$D : \begin{cases} \mathcal{W} & \rightarrow (\mathcal{W}_- \times \mathcal{W}_-)_l \\ w & \rightarrow (w_-, \text{Inv}(w_+)) \end{cases}$$

where

$$(\mathcal{W}_- \times \mathcal{W}_-)_l := \{(w, w') \in \mathcal{W}_- \times \mathcal{W}_-, \quad w_0 = w'_0\} \quad (6.27)$$

is a subset of $\mathcal{W}_- \times \mathcal{W}_-$. The index l stands for “linked”. Let

$$\Phi := (S \otimes S) \circ D \circ \mathcal{S}^{-1} \quad : \mathcal{K} \rightarrow K \times K$$

where $\mathcal{S} : \mathcal{W} \rightarrow \mathcal{K}$ has been defined in (6.24) and is shown in Proposition 6.8 to be one to one under assumption 3.7. The map $S : \mathcal{W}_+ \rightarrow K$ has been defined in (6.13) and is also one to one. Consider

$$(K \times K)_l := (S \otimes S)((\mathcal{W}_- \times \mathcal{W}_-)_l) \subset K \times K \quad (6.28)$$

the image of (6.27) under the map $S \otimes S$. From the previous remarks, the map $\Phi : \mathcal{K} \rightarrow (K \times K)_l$ is one to one.

Lemma 6.9. *The map $\Phi : \mathcal{K} \rightarrow (K \times K)_l$ is bi-Lipschitz.*

As a consequence of this Lemma, since the Hausdorff and Minkowski dimension is invariant under bi-Lipschitz maps [9, p.24], we deduce that

$$\dim_M(\mathcal{K}) = \dim_M(K \times K)_l \quad (6.29)$$

Before proving Lemma 6.9, let us show how to deduce Theorem 3.10 from it. Let us temporarily write $K_i := K \cap I_i$. From (6.28) we have that

$$(K \times K)_l = \bigcup_i K_i \times K_i$$

hence

$$\dim_M (K \times K)_l = \sup_i (2 \dim_M K_i) = 2 \dim_M K \quad (6.30)$$

Eq.(6.29) and (6.30) give Theorem 3.10.

Proof. of Lemma 6.9. Let $w \in \mathcal{W}$. We write $w = (w_-, w_+)$ as before and $x_{w_-} := S(w_-) \in K$, $\rho = (x_{w_-}, \xi_w) = \mathcal{S}(w) \in \mathcal{K}$. Similarly for another $w' \in \mathcal{W}$ we get another point $\rho' = (x_{w'_-}, \xi_{w'}) \in \mathcal{K}$. We have that

$$\Phi(\rho) = (S(w_-), S(\text{Inv}(w_+))) = (x_{w_-}, x_{\text{Inv}(w_+)}) \in K \times K.$$

That the map Φ is bi-Lipschitz means that

$$|\Phi(\rho) - \Phi(\rho')| \asymp |\rho - \rho'|$$

uniformly⁷ over ρ, ρ' . Equivalently this is

$$\left| x_{w_-} - x_{w'_-} \right| + \left| x_{\text{Inv}(w_+)} - x_{\text{Inv}(w'_+)} \right| \asymp \left| x_{w_-} - x_{w'_-} \right| + |\xi_w - \xi_{w'}| \quad (6.31)$$

uniformly over $w, w' \in \mathcal{W}$. Let us show (6.31). Let $w, w' \in \mathcal{W}$ and let $n \geq 0$ be the integer such that that $(w_+)_j = (w'_+)_j$ for $-n \leq j \leq 0$ but $(w_+)_{-n-1} \neq (w'_+)_{-n-1}$. From the definition (6.12) of the intervals $I_{w_{-n},0}$, we see that the two points $x_{\text{Inv}(w_+)}, x_{\text{Inv}(w'_+)}$ belong both to the interval $I_{(\text{Inv}(w_+))_{-n},0}$ but inside it, they belong to the disjoint sub-intervals $I_{(\text{Inv}(w_+))_{-n-1},0}$ and $I_{(\text{Inv}(w'_+))_{-n-1},0}$ respectively. Hence

$$\left| x_{\text{Inv}(w_+)} - x_{\text{Inv}(w'_+)} \right| \asymp \left| I_{(\text{Inv}(w_+))_{-n},0} \right|$$

uniformly over $w, w' \in \mathcal{W}$, where $|I|$ is the length of the interval I . From the definition (6.18) of the sets $\tilde{I}_{w_0,n}$ we observe that the points $\rho = (x_{w_-}, \xi_w)$ and $\rho' = (x_{w'_-}, \xi_{w'})$ belong respectively to the sets $\tilde{I}_{w_0,n}$ and $\tilde{I}_{w'_0,n}$. Let $\tilde{w}' := (w'_-, w_+)$. We have

$$|\rho - \rho'| = \left| (x_{w_-}, \xi_w) - (x_{w'_-}, \xi_{w'}) \right| \quad (6.32)$$

$$\asymp \left| (x_{w_-}, \xi_w) - (x_{w_-}, \xi_{\tilde{w}'}) \right| + \left| (x_{w_-}, \xi_{\tilde{w}'}) - (x_{w'_-}, \xi_{w'}) \right| \quad (6.33)$$

$$\asymp \left| x_{w_-} - x_{w'_-} \right| + |\xi_w - \xi_{\tilde{w}'}|$$

⁷The notation $|\Phi(\rho) - \Phi(\rho')| \asymp |\rho - \rho'|$ means precisely that there exist $C > 0$ such that for every ρ, ρ' , $C^{-1} |\rho - \rho'| \leq |\Phi(\rho) - \Phi(\rho')| \leq C |\rho - \rho'|$.

The points $\xi_w, \xi_{\tilde{w}'}$ belong to the same set $\tilde{I}_{w_0, n}$. However if assumption of “minimal captivity” holds, they belong to disjoint sub-sets $\tilde{I}_{w_0, n+1}$ and $\tilde{I}'_{w'_0, n+1}$ respectively. Hence

$$|\xi_w - \xi_{\tilde{w}'}| \asymp |J_{w, n}| \quad (6.34)$$

with the interval $J_{w, n} := \tilde{I}_{w_0, n} \cap \pi^{-1}(x_{w_-})$. From the bounded distortion principle [9] we have that

$$\forall x, y \in I_{w_{-n, 0}}, \quad |(D\phi_{w_{-n, 0}})(x)| \asymp |(D\phi_{w_{-n, 0}})(y)| \asymp |I_{w_{-n, 0}}|$$

uniformly with respect to w, n, x, y . From the expression of the canonical map F in (3.6) and the bounded distortion principle, we have that

$$|J_{w, n}| \asymp |(D\phi_{w_{-n, 0}})(x)|, \quad \forall x \in I_{w_0},$$

uniformly with respect to w, n, x . Using the previous results we get

$$\begin{aligned} |x_{w_-} - x_{w'_-}| + |\xi_w - \xi_{w'}| &\asymp |x_{w_-} - x_{w'_-}| + |\xi_w - \xi_{\tilde{w}'}| \\ &\asymp |x_{w_-} - x_{w'_-}| + |J_{w, n}| \\ &\asymp |x_{w_-} - x_{w'_-}| + (D\phi_{w_0, n})(x), \quad \forall x \in I_{w_0}, \\ &\asymp |x_{w_-} - x_{w'_-}| + |I_{\text{Inv}(w_0, n)}| \\ &\asymp |x_{w_-} - x_{w'_-}| + |x_{\text{Inv}(w_+)} - x_{\text{Inv}(w'_+)}|. \end{aligned}$$

We have obtained (6.31) and finished the proof of Lemma 6.9 and Theorem 3.10. \square

7 Proof of Theorem 3.9 for the spectral gap in the semi-classical limit

For the proof of Theorem 3.9, we will follow step by step the same analysis as in Section 5 (and also follow closely the proof of Theorem 2 in [11]). The main difference now is that $\hbar \ll 1$ is a semi-classical parameter (no fixed anymore). In other words, we just perform a linear rescaling in cotangent space: $\xi_h := \hbar\xi$. Our quantization rule for a symbol $A(x, \xi_h) \in S^{-m}(\mathbb{R})$, Eq.(5.7) writes now (see [19] p.22), for $\varphi \in \mathcal{S}(\mathbb{R})$:

$$\left(\hat{A}\varphi\right)(x) := \frac{1}{2\pi\hbar} \int A(x, \xi_h) e^{i(x-y)\xi_h/\hbar} \varphi(y) dy d\xi_h \quad (7.1)$$

For simplicity we will still write ξ instead of ξ_h below.

7.1 The escape function

Let $1 < \kappa < 1/\theta$ and $R > 0$ given in Lemma 3.4. Let $m > 0$, $\eta > 0$ (small) and consider a C^∞ function $A_m(x, \xi)$ on $T^*\mathbb{R}$ so that:

$$\begin{aligned} A_m(x, \xi) &:= \langle \xi \rangle^{-m} && \text{for } |\xi| > R + \eta \\ &:= 1 && \text{for } \xi \leq R \end{aligned}$$

where $\langle \xi \rangle := (1 + \xi^2)^{1/2}$. A_m belongs to the symbol class $S^{-m}(\mathbb{R})$ defined in (5.5).

From Eq. (3.8) we can deduce, similarly to Eq.(5.6) and if η is small enough, that:

$$\forall x \in I, \forall |\xi| > R, \forall i \rightsquigarrow j \quad \frac{A_m(F_{i,j}(x, \xi))}{A_m(x, \xi)} \leq C^m < 1, \quad \text{with } C = \sqrt{\frac{R^2 + 1}{\kappa^2 R^2 + 1}} < 1. \quad (7.2)$$

This means that the function A_m is an **escape function** since it decreases strictly along the trajectories of F outside the zone $\mathcal{Z}_0 := I \times [-R, R]$. For any point $(x, \xi) \in T^*I$ we have the more general bound:

$$\forall x \in I, \forall \xi \in \mathbb{R}, \forall i \rightsquigarrow j \quad \frac{A_m(F_{i,j}(x, \xi))}{A_m(x, \xi)} \leq 1. \quad (7.3)$$

Let $\hbar > 0$. Using the quantization rule (7.1), the symbol A_m can be quantized giving a \hbar -pseudodifferential operator \hat{A}_m which is self-adjoint and invertible on $C^\infty(I)$. In our case \hat{A}_m is simply a multiplication operator by $A_m(\xi)$ in \hbar -Fourier space.

7.2 Using the Egorov Theorem

Let us consider the Sobolev space

$$H^{-m}(\mathbb{R}) := \hat{A}_m^{-1}(L^2(\mathbb{R}))$$

which is the usual Sobolev space as a linear space, except for the norm which depends on \hbar . Then $\hat{F} : H^{-m}(\mathbb{R}) \rightarrow H^{-m}(\mathbb{R})$ is unitary equivalent to

$$\hat{Q} := \hat{A}_m \hat{F} \hat{A}_m^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}).$$

Let $n \in \mathbb{N}^*$, a fixed time which will be made large at the end of the proof, and define

$$\hat{P}^{(n)} := \hat{Q}^{*n} \hat{Q}^n = \hat{A}_m^{-1} \hat{F}^{*n} \hat{A}_m^2 \hat{F}^n \hat{A}_m^{-1}. \quad (7.4)$$

From Egorov Theorem, as in Lemma 5.4), we have that $\hat{B} := \hat{F}^* \hat{A}_m^2 \hat{F}$ is a PDO with principal symbol

$$\begin{aligned} B(x, \xi) &= \chi^2(x) \sum_{j \text{ s.t. } i \rightsquigarrow j} |\phi'_{i,j}(x)| e^{2\text{Re}(V(\phi_{i,j}(x)))} A_m^2(F_{i,j}(x, \xi)), && (x, \xi) \in T^*I \\ &= \chi^2(x) \sum_{j \text{ s.t. } i \rightsquigarrow j} e^{2D((\phi_{i,j}(x)))} A_m^2(F_{i,j}(x, \xi)) \end{aligned}$$

where we have used the “damping function” $D(x) := \operatorname{Re}(V(x)) - \frac{1}{2} \log(|(\phi^{-1})'(x)|)$ already defined in (3.12). Iteratively for every $n \geq 1$, Egorov’s Theorem gives that $\hat{F}^{*n} \hat{A}_m^2 \hat{F}^n$ is a PDO with principal symbol

$$B_n(x, \xi) = \chi^2(x) \sum_{w_{-n,0} \in \mathcal{W}_-} e^{2D_{w_{-n,0}}(x)} A_m^2(F_{w_{-n,0}}(x, \xi))$$

where \mathcal{W}_+ is the set of admissible sequences, defined in (6.10), with the Birkhoff sum $D_{w_{-n,0}}(x) := \sum_{k=1}^n D(\phi_{w_{-n,-k}}(x))$ and

$$F_{w_{-n,0}} := F_{w_{-1}, w_0} \circ \dots \circ F_{w_{-n}, w_{-n+1}}.$$

With the theorem of composition of PDO [34, chap.4] we obtain that $\hat{P}^{(n)}$ is a PDO of order 0 with principal symbol given by

$$P^{(n)}(x, \xi) = \left(\chi^2(x) \sum_{w_{-n,0} \in \mathcal{W}_-} e^{2D_{w_{-n,0}}(x)} \frac{A_m^2(F_{w_{-n,0}}(x, \xi))}{A_m^2(x, \xi)} \right). \quad (7.5)$$

We define

$$\gamma(n) := \sup_{x \in I, w_{-n,0} \in \mathcal{W}_-} \frac{1}{n} D_{w_{-n,0}}(x)$$

hence $e^{2D_{w_{-n,0}}(x)} \leq e^{2n\gamma(n)}$.

From Theorem 3.1, the spectrum of \hat{F}_h does not depend on the choice of χ . Here we take $a \geq 0$ as given in Assumption 3.7 and we choose χ such that $\chi \equiv 1$ on K_{a+1} , $\chi \equiv 0$ on $\mathbb{R} \setminus K_a$. We have $P(x, \xi) = 0$ if $x \in \mathbb{R} \setminus K_a$.

Now we will bound the positive symbol $P^{(n)}(x, \xi)$ from above, considering $x \in K_a$ and different possibilities for the trajectory $F_{w_{-n,0}}(x, \xi)$:

1. If $|\xi| > R$, Eq.(7.2) gives

$$\frac{A_m^2(F_{w_{-n,0}}(x, \xi))}{A_m^2(x, \xi)} = \frac{A_m^2(F_{w_{-n,0}}(x, \xi))}{A_m^2(F_{w_{-n,-1}}(x, \xi))} \frac{A_m^2(F_{w_{-n,-1}}(x, \xi))}{A_m^2(F_{w_{-n,-2}}(x, \xi))} \cdots \frac{A_m^2(F_{w_{-n,-n+1}}(x, \xi))}{A_m^2(x, \xi)} \quad (7.6)$$

$$\leq (C^{2m})^n \quad (7.7)$$

therefore

$$P^{(n)}(x, \xi) \leq (\#\mathcal{W}_n) e^{2n\gamma(n)} (C^{2m})^n \leq (N e^{2\gamma(n)} C^{2m})^n.$$

We have used that $\#\mathcal{W}_n \leq N^n$. Notice that C^{2m} can be made arbitrarily small if m is large.

2. If $|\xi| \leq R$, we have from the hypothesis of minimal captivity 3.7 and Proposition 6.3 that at time $(n-1)$ every point (x', ξ') of the set $F^{n-1}(x, \xi)$ except finitely many points, satisfy $|\xi'| > R$. Using (7.3) and (7.2), for all these points one has $\frac{A_m^2(F_{w-n,0}(x, \xi))}{A_m^2(x, \xi)} \leq C^{2m}$ and for the exceptional point one can only write $\frac{A_m^2(F_{w-n,0}(x, \xi))}{A_m^2(x, \xi)} \leq 1$. This gives

$$P^{(n)}(x, \xi) \leq e^{2n\gamma(n)} ((\#\mathcal{W}_n - 1)C^{2m} + C') \leq \mathcal{B}$$

with the bound

$$\mathcal{B} := e^{2n\gamma(n)} (N^n C^{2m} + C') \quad (7.8)$$

With the L^2 **continuity theorem** for pseudodifferential operators [19, 7] this implies that in the limit $\hbar \rightarrow 0$

$$\|\hat{P}^{(n)}\| \leq \mathcal{B} + \mathcal{O}_n(\hbar) \quad (7.9)$$

Polar decomposition of \hat{Q}^n gives

$$\|\hat{Q}^n\| \leq \|\hat{Q}^n\| = \sqrt{\|\hat{P}^{(n)}\|} \leq (\mathcal{B} + \mathcal{O}_n(\hbar))^{1/2} \quad (7.10)$$

Let $\gamma_+ = \limsup_{n \rightarrow \infty} \gamma(n)$. If we let $\hbar \rightarrow 0$ first, and $m \rightarrow +\infty$ giving $C^{2m} \rightarrow 0$, and $n \rightarrow \infty$, we obtain $(\mathcal{B} + \mathcal{O}_n(\hbar))^{1/(2n)} \rightarrow e^{\gamma_+}$. Therefore for any $\rho > e^{\gamma_+}$, there exists $n_0 \in \mathbb{N}$, $\hbar_0 > 0$, $m_0 > 0$ such that for any $\hbar \leq \hbar_0$, $m > m_0$,

$$\|\hat{F}_\hbar^{n_0}\|_{H^{-m}} = \|\hat{Q}^{n_0}\|_{L^2} \leq \rho^{n_0}. \quad (7.11)$$

Also, there exists $c > 0$ independent of $\hbar \leq \hbar_0$, such that for any r such that $0 \leq r < n_0$ we have $\|\hat{Q}^r\|_{L^2} < c$. As a consequence for any $n \in \mathbb{N}$ we write $n = kn_0 + r$ with $0 \leq r < n_0$ and

$$\|\hat{F}_\hbar^n\|_{H^{-m}} = \|\hat{Q}^n\|_{L^2} \leq \|\hat{Q}^{n_0}\|_{L^2}^k \|\hat{Q}^r\|_{L^2} \leq \rho^n \frac{\|\hat{Q}^r\|_{L^2}}{\rho^r} \leq c\rho^n$$

We have obtained (3.16). Equivalently this gives (3.15).

For any n the spectral radius of \hat{Q} satisfies [28, p.192]

$$r_s(\hat{Q}) \leq \|\hat{Q}^n\|^{1/n} \leq c^{1/n} \rho.$$

So we get that for $\hbar \rightarrow 0$,

$$r_s(\hat{F}_\hbar) = r_s(\hat{Q}) \leq e^{\gamma_+} + o(1) \quad (7.12)$$

which finishes the proof of Theorem 3.9.

8 Proof of Theorem 3.11 about the fractal Weyl law

We will prove this result once more by conjugating the transfer operator by an escape function as in previous Section 7. However we first have to improve the properties of the escape function.

8.1 A refined escape function

8.1.1 Distance function

The escape function A will be constructed from a distance function δ . For $x \in I$, let

$$\tilde{K}(x) := \tilde{K} \cap (\{x\} \times \mathbb{R}) \quad (8.1)$$

where \tilde{K} has been defined in (6.2). With this notation we can define the following distance function.

Definition 8.1. Let $x \in I_{w_0}$ and $\xi \in \mathbb{R}$, we define the distance of (x, ξ) to the set \tilde{K} given in (6.2) by

$$\delta(x, \xi) := \text{dist}(\xi, \tilde{K}(x)) = \min_{w \in \mathcal{W}_+} |\xi - \zeta_w(x)|. \quad (8.2)$$

We will show that the distance function $\delta(x, \xi)$ decreases along the trajectories of F . First, the next Lemma shows how the branches ζ_w are transformed under the canonical map F . This formula follows from straightforward calculations.

Lemma 8.2. For every $w_+ = (w_0, w_1, \dots) \in \mathcal{W}_+$, $x \in I_{w_0}$ we have

$$F_{w_0, w_1}(x, \zeta_{w_+}(x)) = (x', \zeta_{L(w_+)}(x')) \quad (8.3)$$

with $L(w_+) := (w_1, w_2, \dots)$ and $x' = \phi_{w_0, w_1}(x)$.

Lemma 8.3. $\forall i, j, i \rightsquigarrow j, \forall x \in I_i, \forall \xi \in \mathbb{R}$,

$$\delta(F_{i,j}(x, \xi)) \geq \frac{1}{\theta} \delta(x, \xi)$$

where $\theta < 1$ is given by (2.1).

Proof. Let $i = w_0 \rightsquigarrow j = w_1$, $x \in I_{w_0}$. Let $(x', \xi') := F_{w_0, w_1}(x, \xi)$ with $x' \in I_{w_1}$. We use (8.3) and also that F_{w_0, w_1} is expansive in ξ by a factor larger than $\theta^{-1} > 1$ (Eq.(3.6)), and get

$$|\xi' - \zeta_{L(w_+)}(x')| = \left| (F_{w_0, w_1}(x, \xi) - F_{w_0, w_1}(x, \zeta_{w_+}(x))) \right|_{\xi} \geq \frac{1}{\theta} |\xi - \zeta_{w_+}(x)|.$$

Thus

$$\begin{aligned} \delta(F_{w_0, w_1}(x, \xi)) &= \min_{w_+ \in \mathcal{W}_+} |\xi' - \zeta_{w_+}(x')| = \min_{w_+ \in \mathcal{W}_+} |\xi' - \zeta_{L(w_+)}(x')| \\ &\geq \frac{1}{\theta} \min_{w_+ \in \mathcal{W}_+} |\xi - \zeta_{w_+}(x)| = \frac{1}{\theta} \delta(x, \xi). \end{aligned}$$

□

8.1.2 Escape function

The aim of this section is to prove the existence of an escape function with the following properties:

Proposition 8.4. $\forall 1 < \kappa < \theta^{-1}$, $\exists C_0 > 0$, $\forall \mu$, s.t. $0 \leq \mu < \frac{1}{2}$, $\forall m > 0$, there exists an \hbar -dependent order function $A_{m, \mu} \in \mathcal{OF}^{m\mu}(\langle \xi \rangle^{-m})$ (as defined in Definition C.3) which fulfills the following “decay condition”:

$\forall i, j$, s.t. $i \rightsquigarrow j$ and $\forall (x, \xi) \in I_i \times \mathbb{R}$ s.t. $\delta(x, \xi) > C_0 \hbar^\mu$ the following estimate holds:

$$\left(\frac{A_{m, \mu} \circ F_{i, j}}{A_{m, \mu}} \right) (x, \xi) \leq \kappa^{-m}. \quad (8.4)$$

In order to prove the above proposition we first remark that the distance function (8.2) is not differentiable, however Lipschitz.

Lemma 8.5. Let $C_1 := \sup_{x \in I, \omega \in \mathcal{W}_+} |(\partial_x \zeta_\omega)(x)|$. Then $\delta : T^*I \rightarrow \mathbb{R}^+$ is a Lipschitz function with constant $C_1 + 1$

Proof. Let $x, y \in I_i$, then from the fact, that $|(\partial_x \zeta_\omega)(x)|$ is uniformly bounded by C_1 we have

$$|\delta(x, \xi) - \delta(y, \xi)| \leq C_1 |x - y|.$$

On the other hand clearly

$$|\delta(y, \xi) - \delta(y, \zeta)| \leq |\xi - \zeta|$$

thus

$$|\delta(x, \xi) - \delta(y, \zeta)| \leq C_1|x - y| + |\xi - \zeta| \leq (C_1 + 1)\text{dist}((x, \xi), (y, \zeta)).$$

□

Next we choose $0 \leq \mu < 1/2$ and regularize the function δ at the scale \hbar^μ . For this we choose $\chi \in C_0^\infty(\mathbb{R}^2)$ with support in the unit ball $B_1(0)$ of \mathbb{R}^2 and $\chi(x, \xi) > 0$ for $\|(x, \xi)\| < 1$. This function can be rescaled to

$$\chi_{\hbar^\mu}(x, \xi) := \frac{1}{\hbar^{2\mu} \|\chi\|_{L^1}} \chi\left(\frac{x}{\hbar^\mu}, \frac{\xi}{\hbar^\mu}\right)$$

such that $\text{supp}\chi_{\hbar^\mu} \subset B_{\hbar^\mu}(0)$ and $\int \chi_{\hbar^\mu}(x) dx = 1$. Now we can define the regularized distance function by

$$\tilde{\delta}(x, \xi) := \int_{T^*I} \delta(x', \xi') \chi_{\hbar^\mu}(x - x', \xi - \xi') dx' d\xi'.$$

This smoothed distance function $\tilde{\delta}$ differs only at order \hbar^μ from the original one because

$$\begin{aligned} \left| \tilde{\delta}(x, \xi) - \delta(x, \xi) \right| &= \left| \int_{\mathbb{R}^2} (\delta(x, \xi) - \delta(x - x', \xi - \tilde{\xi}')) \chi_{\hbar^\mu}(x', \xi') dx' d\xi' \right| \\ &\leq \sup_{(x', \xi') \in B_{\hbar^\mu}(0)} |\delta(x, \xi) - \delta(x - x', \xi - \xi')| \\ &\leq (C_1 + 1)\hbar^\mu. \end{aligned} \tag{8.5}$$

Furthermore we get the following estimates for its derivatives:

Lemma 8.6. *For all $\alpha, \beta \in \mathbb{N}$ the estimate*

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{\delta}(x, \xi)| \leq C_{\alpha, \beta} \hbar^{-\mu(\alpha + \beta)} (\delta(x, \xi) + C\hbar^\mu)$$

holds

Proof. From the definition of χ_{\hbar^μ} we have $\|\partial_x^\alpha \partial_\xi^\beta \chi_{\hbar^\mu}\|_\infty \leq C_{\alpha, \beta} \hbar^{-2 - (\alpha + \beta)\mu}$ and thus:

$$\begin{aligned} \left| \left(\partial_x^\alpha \partial_\xi^\beta \tilde{\delta}(x, \xi) \right) \right| &= \int_{T^*I} \delta(x', \xi') \partial_x^\alpha \partial_\xi^\beta \chi_{\hbar^\mu}(x - x', \xi - \xi') dx' d\xi' \\ &\leq \pi \hbar^{2\mu} \|\delta\|_{\infty, B_{\hbar^\mu}(x, \xi)} C_{\alpha, \beta} \hbar^{-(2 + \alpha + \beta)\mu} \\ &\leq \pi C_{\alpha, \beta} \hbar^{-(\alpha + \beta)\mu} (\delta(x, \xi) + (C_1 + 1)\hbar^\mu) \end{aligned}$$

where we used the Lipschitz property of δ in the last inequality. □

As $|\delta(x, \xi)| \leq |\xi| + C$ the above lemma gives us directly that $\tilde{\delta} \in S_\mu^1(T^*I)$. Now we define the escape function as:

$$A_{m,\mu}(x, \xi) := \hbar^{m\mu} \left(\hbar^{2\mu} + \left(\tilde{\delta}(x, \xi) \right)^2 \right)^{-\frac{m}{2}}. \quad (8.6)$$

This is obviously a smooth function and it obeys the following estimates:

Lemma 8.7. *The function $A_{m,\mu}$ defined in (8.6) is an \hbar -dependent order function: $A_{m,\mu} \in \mathcal{O}\mathcal{F}^{m\mu}(\langle \xi \rangle^{-m})$.*

Proof. As $\min(0, |\xi| - C) \leq \tilde{\delta}(x, \xi) \leq |\xi| + C$ it follows, that $A_{m,\mu}(x, \xi) \leq \tilde{C} \langle \xi \rangle^{-m}$ and that $A_{m,\mu}(x, \xi) \geq C' \hbar^{m\mu} \langle \xi \rangle^{-m}$. It remains thus to show, that for arbitrary $\alpha, \beta \in \mathbb{N}$ one has:

$$|\partial_x^\alpha \partial_\xi^\beta A_{m,\mu}(x, \xi)| \leq C_{\alpha,\beta} \hbar^{-\mu(\alpha+\beta)} A_{m,\mu}(x, \xi) \quad (8.7)$$

where $C_{\alpha,\beta}$ depends only on α and β . First consider the case $\alpha = 1, \beta = 0$:

$$|\partial_x A_{m,\mu}(x, \xi)| = \left| \hbar^{m\mu} m \frac{(\partial_x \tilde{\delta}(x, \xi)) \tilde{\delta}(x, \xi)}{\left(\hbar^{2\mu} + \left(\tilde{\delta}(x, \xi) \right)^2 \right)^{\frac{m+2}{2}}} \right| \leq C \hbar^{-\mu} A_{m,\mu}(x, \xi)$$

where we used $\tilde{\delta} \leq \sqrt{\hbar^{2\mu} + \tilde{\delta}^2}$ and $|\partial_x \tilde{\delta}| \leq C \hbar^{-\mu} \sqrt{\hbar^{2\mu} + \tilde{\delta}^2}$ which follows from Lemma 8.6 together with (8.5). Inductively one obtains the estimate for arbitrary $\alpha, \beta \in \mathbb{N}$ by repeated use of Lemma 8.6 and (8.5). \square

Finally it remains to show the decay estimates for $\left(\frac{A_{m,\mu} \circ F_{i,j}}{A_{m,\mu}} \right)(x, \xi)$. Combining (8.5) with lemma 8.3 we then get

$$\begin{aligned} \tilde{\delta}(F_{i,j}(x, \xi)) &\geq \delta(F_{i,j}(x, \xi)) - (C_1 + 1)\hbar^\mu \geq \frac{1}{\theta} \delta(x, \xi) - (C_1 + 1)\hbar^\mu \\ &\geq \frac{1}{\theta} \tilde{\delta}(x, \xi) - \left(\frac{1}{\theta} + 1 \right) (C_1 + 1) \hbar^\mu \end{aligned}$$

and thus

$$\frac{A_{m,\mu}(F_{i,j}(x, \xi))}{A_{m,\mu}(x, \xi)} \leq \left(\frac{1 + \left(\frac{1}{\theta} \cdot \frac{\tilde{\delta}(x, \xi)}{\hbar^\mu} - \tilde{C} \right)^2}{1 + \left(\frac{\tilde{\delta}(x, \xi)}{\hbar^\mu} \right)^2} \right)^{\frac{m}{2}} \quad (8.8)$$

where $\tilde{C} = \left(\frac{1}{\theta} + 1 \right) (C_1 + 1)$. Clearly the right side of (8.8) converges to $\left(\frac{1}{\theta} \right)^{-m}$ for $\frac{\tilde{\delta}(x, \xi)}{\hbar^\mu} \rightarrow \infty$ which proves the existence of a desired C_0 and finishes the proof of Proposition 8.4.

8.1.3 Truncation in x

Here we choose a similar truncation operator $\hat{\chi}$ as in Eq.(2.23) but in a finer vicinity of the trapped set K . First notice that $K_{\hbar^\mu} \Subset \phi^{-1}(K_{\hbar^\mu})$ where K_{\hbar^μ} has been defined in Definition 2.2. For \hbar small enough we have $\phi^{-1}(K_{\hbar^\mu}) \Subset I$. Let $\chi \in C_{\phi^{-1}(K_{\hbar^\mu})}^\infty$ such that $\chi(x) = 1$ for $x \in K_{\hbar^\mu}$. χ can be considered as a function $\chi(x, \xi) := \chi(x)$ (independent of ξ) and we have that $\chi_\mu \in S_\mu^0(T^*\mathbb{R})$. As in Eq.(2.23) we define $\hat{\chi} := \text{Op}_\hbar^w(\chi)$ which is the multiplication operator by χ and

$$\hat{F}_{i,j,\chi} := \hat{F}_{i,j}\hat{\chi}, \quad \hat{F}_\chi := \hat{F}\hat{\chi}.$$

We will again omit the χ in the notation and write \hat{F}_\hbar for \hat{F}_χ in the sequel.

8.2 Weyl law

The Weyl law will give an upper bound on the number of eigenvalues of \hat{F}_\hbar in the Sobolev spaces H^m . These estimates will be obtained by conjugating \hat{F}_\hbar with $\text{Op}_\hbar^w(A_{m,\mu})$ in the same way as for the discrete spectrum or the spectral gap. Note that we use the Weyl quantization (see Definition C.2) in this section, because we want to obtain self adjoint operators. In order to be able to conjugate we have to show, that $\text{Op}_\hbar^w(A_{m,\mu}) : H^{-m} \rightarrow L^2$ is an isomorphism. We already know that $\text{Op}_\hbar^w(\langle \xi \rangle^m) : L^2 \rightarrow H^{-m}$ is an isomorphism, thus it suffices to show, that $\hat{B} := \text{Op}_\hbar^w(A_{m,\mu})\text{Op}_\hbar^w(\langle \xi \rangle^m) : L^2 \rightarrow L^2$ is invertible. From the \hbar -local symbol calculus (Theorem C.7) it follows that \hat{B} is an elliptic operator in the \hbar -local symbol class $S_\mu(A_{m,\mu}\langle \xi \rangle^m)$ and thus the invertibility follows from proposition C.10. Note that it is necessary to work in the \hbar -local symbol classes as \hat{B} would not be an elliptic operator in $S_\mu(1)$. Proposition C.10 also gives us the leading order of our inverse \hat{B}^{-1} which is $A_{m,\mu}^{-1}\langle \xi \rangle^{-m}$. So the inverse of $\text{Op}_\hbar^w(A_{m,\mu})$ is again a PDO with leading symbol $A_{m,\mu}^{-1}$.

With the isomorphism $\text{Op}_\hbar^w(A_{m,\mu}) : H^m \rightarrow L^2$ we can thus define a different scalar product on the Sobolev spaces which turns $\text{Op}_\hbar^w(A_{m,\mu})$ into a unitary operator. The Sobolev space equipped with this scalar product will be denoted by $\mathcal{H}_{\hbar,\mu}^m$ and the study of \hat{F}_\hbar is thus unitary equivalent to the study of \hat{Q}_m defined by the following commutative diagram (where we noted $\hat{A}_{m,\mu} := \text{Op}_\hbar^w(A_{m,\mu})$):

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\hat{Q}_m} & L^2(\mathbb{R}) \\ \downarrow \hat{A}_{m,\mu}^{-1} & & \downarrow \hat{A}_{m,\mu}^{-1} \\ \mathcal{H}_{\hbar,\mu}^{-m} & \xrightarrow{\hat{F}_\hbar} & \mathcal{H}_{\hbar,\mu}^{-m} \end{array} \quad (8.9)$$

In the next Lemma, C_0 and κ are as in lemma 8.4.

Lemma 8.8. $\exists C > C_0, \forall \epsilon > 0, \forall \mu$ s.t. $0 \leq \mu < \frac{1}{2}, \forall m > 0$ sufficiently large, as $\hbar \rightarrow 0$ we have:

$$\# \left\{ \lambda_i^{\hbar} \in \sigma \left(\hat{F}_{\hbar} |_{\mathcal{H}_{\hbar, \mu}^m} \right) \mid |\lambda_i^{\hbar}| \geq \epsilon \right\} \leq \frac{1}{2\pi\hbar} \left(\tilde{C}_1 \text{Leb} \{ \mathcal{K}_{C_1 \hbar^{\mu}} \} + \tilde{C}_2 \hbar \right). \quad (8.10)$$

Before proving Lemma 8.8, let us show that it implies Theorem 3.11. From Theorem 3.1, the discrete spectrum of $\hat{F}_{\hbar} |_{\mathcal{H}_{\hbar, \mu}^m}$ is the Ruelle spectrum of resonances $\text{Res} \left(\hat{F}_{\hbar} \right)$, independent of μ and m . With assumption 3.7 we can use Eq.(3.17) and that \mathcal{K} has pure dimension, thus equation (2.13) gives $\text{Leb} \{ \mathcal{K}_{C_1 \hbar^{\mu}} \} = \mathcal{O} \left((\hbar^{\mu})^{\text{codim}_M(\mathcal{K})} \right)$. As $\text{codim}_M(\mathcal{K}) < 2$ and $\mu < \frac{1}{2}$ equation(8.10) and gives

$$\begin{aligned} \# \left\{ \lambda_i^{\hbar} \in \text{Res} \left(\hat{F}_{\hbar} \right) \mid |\lambda_i^{\hbar}| \geq \epsilon \right\} &= \mathcal{O} \left(\hbar^{-1} (\hbar^{\mu})^{\text{codim}_M(\mathcal{K})} \right) \\ &= \mathcal{O} \left(\hbar^{-1} (\hbar^{\mu})^{2-2\dim_H(K)} \right) = \mathcal{O} \left(\hbar^{2\mu-1-2\mu\dim_H(K)} \right) \end{aligned}$$

for any fixed $0 \leq \mu < 1/2$. This gives Theorem 3.11 with $\eta = (1 - 2\mu)(1 - \dim_H(K))$.

Proof. of Lemma 8.8. From (8.9), $\hat{F}_{\chi, \hbar} : \mathcal{H}_{\hbar, \mu}^m \rightarrow \mathcal{H}_{\hbar, \mu}^m$ is unitary equivalent to

$$\hat{Q}_{m, \mu} := \text{Op}_{\hbar}^w (A_{m, \mu}) \hat{F}_{\chi, \hbar} \text{Op}_{\hbar}^w (A_{m, \mu})^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}).$$

Consider

$$\hat{P}_{\mu} := \hat{Q}_{m, \mu}^* \hat{Q}_{m, \mu} = \text{Op}_{\hbar}^w (A_{m, \mu})^{-1} \hat{\chi} \hat{F}_{\hbar}^* \text{Op}_{\hbar}^w (A_{m, \mu})^2 \hat{F}_{\hbar} \hat{\chi} \text{Op}_{\hbar}^w (A_{m, \mu})^{-1}.$$

By the composition Theorem C.7 and the Egorov theorem C.11 for \hbar -local symbols, \hat{P}_{μ} is a PDO with leading symbol $P_{\mu}(x, \xi) \in S_{\mu}(1)$, for $x \in I_i, \xi \in \mathbb{R}$, given by the same expression as in (5.11):⁸

$$P_{\mu}(x, \xi) = \chi^2(x) \sum_{j \text{ s.t. } i \rightsquigarrow j} |\phi'_{i,j}(x)| e^{2\Re(V(\phi_{i,j}(x)))} \frac{A_{m, \mu}^2(F_{i,j}(x, \xi))}{A_{m, \mu}^2(x, \xi)} \text{ mod } \hbar^{1-2\mu} S_{\mu}^{-1}(T^*\mathbb{R}). \quad (8.11)$$

From the definition of χ and Eq.(8.4), the operator \hat{P}_{μ} can be decomposed into self-adjoint operators

$$\hat{P}_{\mu} = \hat{k}_{\mu} + \hat{r}_{\mu}$$

where \hat{k}_{μ} is a PDO with symbol $k_{\mu} \in S_{\mu}^0(T^*\mathbb{R})$ supported on $\mathcal{K}_{C_1 \hbar^{\mu}}$ for some $C_1 > 0$. Hence \hat{k}_{μ} is a trace-class operator. The operator \hat{r}_{μ} is a PDO with symbol $r_{\mu} \in S_{\mu}^0(T^*\mathbb{R})$ such that

$$\|r_{\mu}\|_{\infty} \leq \theta e^{2\|\Re(V)\|_{\infty}} \kappa^{-2m} + \mathcal{O}(\hbar^{1-2\mu}),$$

⁸Also for this calculation it is crucial to work with the \hbar -local calculus in order to obtain sufficient remainder estimates.

hence $\|\hat{r}_\mu\| \leq C\kappa^{-2m} + \mathcal{O}(\hbar^{1-2\mu})$.

Using Lemma A.1 in Appendix A we have that for every $\epsilon > 0$, in the limit $\hbar \rightarrow 0$,

$$\#\left\{\mu_i^\hbar \in \sigma\left(\hat{k}_\mu\right) \mid |\mu_i^\hbar| \geq \epsilon\right\} \leq (2\pi\hbar)^{-1} \left(\tilde{C}_1 \text{Leb}\{\mathcal{K}_{C_1\hbar^\mu}\} + \tilde{C}_2\hbar\right). \quad (8.12)$$

By a standard perturbation argument the same estimates holds for the operator \hat{P}_μ (for m sufficiently large): for every $\epsilon > 0$, in the limit $\hbar \rightarrow 0$,

$$\#\left\{\mu_i^\hbar \in \sigma\left(\hat{P}_\mu\right) \mid |\mu_i^\hbar| \geq \epsilon + \|\hat{r}_\mu\|\right\} \leq (2\pi\hbar)^{-1} \left(\tilde{C}_1 \text{Leb}\{\mathcal{K}_{C_1\hbar^\mu}\} + \tilde{C}_2\hbar\right). \quad (8.13)$$

From the definition $\hat{P}_\mu := \hat{Q}_{m,\mu}^* \hat{Q}_{m,\mu}$, the $\sqrt{\mu_i^\hbar}$ are singular values of $\hat{Q}_{m,\mu}$. Then corollary B.2 from Appendix B shows that the same estimate holds true for the eigenvalues of $\hat{Q}_{m,\mu}$, hence of \hat{F}_\hbar , yielding the result (8.10). \square

A Adapted Weyl type estimates

Lemma A.1. *Let $a_\hbar \in S_\mu(\langle x \rangle^{-2} \langle \xi \rangle^{-2})$ with $0 \leq \mu < \frac{1}{2}$ be a real compactly supported symbol. $\forall \hbar > 0$, $\hat{A} := \text{Op}_\hbar^w(a_\hbar)$ is self-adjoint and trace class on $L^2(\mathbb{R})$ and for any $\epsilon > 0$, as $\hbar \rightarrow 0$:*

$$(2\pi\hbar) \#\left\{\lambda_i^\hbar \in \sigma\left(\hat{A}\right) \mid |\lambda_i^\hbar| \geq \epsilon\right\} \leq C_1 \text{Leb}\{(x, \xi); |a| > 0\} + C_2\hbar \quad (\text{A.1})$$

where C_1 and C_2 depend only on μ and ϵ .

Proof. As a_\hbar is compactly supported \hat{A} is trace class for every \hbar (see theorem C.17 [34]). Consequently also $\frac{1}{\epsilon^2} \hat{A}^2$ is trace class and its trace is given by Lidskii's theorem by $\text{Tr}\left(\frac{1}{\epsilon^2} \hat{A}^2\right) = \sum_i \left(\frac{\lambda_i^\hbar}{\epsilon}\right)^2$. As \hat{A} is self adjoint all λ_i^\hbar are real and one clearly has

$$\#\left\{\lambda_i^\hbar \in \sigma\left(\hat{A}\right) \mid |\lambda_i^\hbar| \geq \epsilon\right\} \leq \text{Tr}\left(\frac{1}{\epsilon^2} \hat{A}^2\right).$$

If we denote by $b_\hbar(x, \xi)$ the complete symbol of \hat{A}^2 we can calculate the trace by the following exact formula

$$\text{Tr}(\hat{A}^2) = \frac{1}{2\pi\hbar} \int b_\hbar(x, \xi) dx d\xi$$

According to the theorem of composition of PDOs b_\hbar can be written as $b_\hbar = b_\hbar^{(1)} + \hbar^1 b_\hbar^{(2)}$ where $\text{supp} b_\hbar^{(1)} = \text{supp} a_\hbar$ and $b_\hbar^{(2)} \in S_\mu(\langle x \rangle^{-2} \langle \xi \rangle^{-2})$ (note that this decomposition depends on μ). Thus

$$\frac{1}{\epsilon^2} \text{Tr}(\hat{A}^2) = \frac{1}{2\pi\hbar\epsilon^2} \left(\int b_\hbar^{(1)}(x, \xi) dx d\xi + \hbar^1 \int b_\hbar^{(2)}(x, \xi) dx d\xi \right) \leq \left(\frac{1}{2\pi\hbar} C_1 \text{Leb}(\text{supp}(a_\hbar)) + C_2\hbar \right)$$

□

B General lemmas on singular values of compact operators

Let $(P_n)_{n \in \mathbb{N}}$ be a family of compact operators on some Hilbert space. For every $n \in \mathbb{N}$ let $(\lambda_{j,n})_{j \in \mathbb{N}} \in \mathbb{C}$ be the sequence of eigenvalues of P_n counted with multiplicity and ordered decreasingly:

$$|\lambda_{0,n}| \geq |\lambda_{1,n}| \geq \dots$$

In the same manner, define $(\mu_{j,n})_{j \in \mathbb{N}} \in \mathbb{R}^+$, the decreasing sequence of singular values of P_n , i.e. the eigenvalues of $\sqrt{P_n^* P_n}$.

Lemma B.1. *Suppose there exists a map $N : \mathbb{N} \rightarrow \mathbb{N}$ s.t. $N(n) \xrightarrow{n \rightarrow \infty} \infty$ and $\mu_{N(n),n} \xrightarrow{n \rightarrow \infty} 0$, then $\forall C > 1$, $|\lambda_{[C \cdot N(n)],n}| \xrightarrow{n \rightarrow \infty} 0$ where $[\cdot]$ stands for the integer part.*

Corollary B.2. *Suppose there exists a map $N : \mathbb{N} \rightarrow \mathbb{N}$ s.t. $\forall \varepsilon > 0$, $\exists A_\varepsilon \geq 0$ s.t. $\forall n \geq A_\varepsilon$,*

$$\#\{j \in \mathbb{N} \text{ s.t. } \mu_{j,n} > \varepsilon\} < N(n),$$

then $\forall C > 1$, $\forall \varepsilon > 0$, $\exists B_{C,\varepsilon} \geq 0$ s.t. $\forall n \geq B_{C,\varepsilon}$

$$\#\{j \in \mathbb{N} \text{ s.t. } |\lambda_{j,n}| > \varepsilon\} \leq C \cdot N(n). \quad (\text{B.1})$$

Proof. (Of corollary B.2). Suppose that for any $\varepsilon > 0$, there exists A_ε s.t. for all $n \geq A_\varepsilon$, $\#\{j \in \mathbb{N} \text{ s.t. } \mu_{j,n} > \varepsilon\} < N(n)$. Then $\mu_{N(n),n} \xrightarrow{n \rightarrow \infty} 0$ and from Lemma B.1, $\forall C > 1$, $|\lambda_{[C \cdot N(n)],n}| \xrightarrow{n \rightarrow \infty} 0$, which can be directly restated as (B.1). □

Proof. (Of lemma B.1). Let $m_{j,n} := -\log \mu_{j,n}$ and $l_{j,n} := -\log |\lambda_{j,n}|$, $M_{k,n} := \sum_{j=0}^k m_{j,n}$ and $L_{k,n} := \sum_{j=0}^k l_{j,n}$. Weyl inequalities relate singular values and eigenvalues by (see [13] p. 50 for a proof) :

$$\prod_{j=1}^k \mu_{j,n} \leq \prod_{j=1}^k |\lambda_{j,n}|, \quad \forall k \geq 1. \quad (\text{B.2})$$

This rewrites:

$$M_{k,n} \leq L_{k,n}, \quad \forall k, n \quad (\text{B.3})$$

The sequence $(l_{j,n})_{j \geq 0}$ is increasing in j so, $\forall n, \forall k$ we have

$$k \cdot l_{k,n} \geq L_{k,n}. \quad (\text{B.4})$$

Suppose that $\mu_{N(n),n} \rightarrow 0$ as $n \rightarrow \infty$ hence

$$m_{N(n),n} \xrightarrow{n \rightarrow \infty} \infty \quad (\text{B.5})$$

Let $C > 1$. The sequence $(m_{j,n})_{j \geq 0}$ is increasing in j hence

$$M_{[C \cdot N(n)],n} \geq ([C \cdot N(n)] - N(n)) \cdot m_{N(n),n}, \quad (\text{B.6})$$

hence

$$\begin{aligned} l_{[C \cdot N(n)],n} &\stackrel{(\text{B.4})}{\geq} \frac{1}{[CN(n)]} \cdot L_{[C \cdot N(n)],n} \stackrel{(\text{B.3})}{\geq} \frac{1}{[C \cdot N(n)]} M_{[C \cdot N(n)],n} \\ &\stackrel{(\text{B.6})}{\geq} \frac{[C \cdot N(n)] - N(n)}{[C \cdot N(n)]} \cdot m_{N(n),n} \stackrel{(\text{B.5})}{\xrightarrow{n \rightarrow \infty}} \infty \end{aligned}$$

Thus $l_{[C \cdot N(n)],n} \xrightarrow{n \rightarrow \infty} \infty$ and $|\lambda_{[C \cdot N(n)],n}| \xrightarrow{n \rightarrow \infty} 0$. \square

C Symbol classes of local \hbar -order

In this Appendix we will first repeat the definitions of the standard symbol classes which are used in this article as well as their well known quantization rules. Then we will introduce a new symbol class which allows \hbar -dependent order functions and will prove some of the classical results which are known in the usual case for these new symbol classes.

C.1 Standard semiclassical Symbol classes and their quantization

The standard symbol classes (see e.g. [34] chapter 4 or [7] ch 7) of \hbar PDO's are defined with respect to an order function $f(x, \xi)$. This order function is required to be a smooth positive valued function on \mathbb{R}^{2n} such that there are constants C_0 and N_0 fulfilling

$$f(x, \xi) \leq C_0 \langle (x, \xi) - (x', \xi') \rangle^{N_0} f(x', \xi'). \quad (\text{C.1})$$

An important example of such an order function is given by $f(x, \xi) = \langle \xi \rangle^m$ with $k \in \mathbb{R}$.

Definition C.1. For $0 \leq \mu \leq \frac{1}{2}$ the symbol classes $\hbar^k S_\mu(m)$ contain all families of functions $a_\hbar(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ parametrized by a parameter $\hbar \in]0, \hbar_0]$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a_\hbar(x, \xi)| \leq C \hbar^{k - \mu(|\alpha| + |\beta|)} f(x, \xi)$$

where C depends only on α and β .

Unless we want to emphasize the dependence of the symbol a_{\hbar} on \hbar we will drop the index in the following. For the special case of order function $f(x, \xi) = \langle \xi \rangle^m$ we also write $S_{\mu}^m = S_{\mu}(\langle \xi \rangle^m)$, if $\mu = 0$ we write $S(f) := S_0(f)$.

As quantization we use two different quantization rules in this article which are called standard quantization respectively Weyl quantization.

Definition C.2. Let $a_{\hbar} \in S_{\mu}(f)$ the Weyl quantization is a family of operators $\text{Op}_{\hbar}^w(a) : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$, defined by

$$(\text{Op}_{\hbar}^w(a_{\hbar})\varphi)(x) = (2\pi\hbar)^{-n} \int e^{\frac{i}{\hbar}\xi(x-y)} a_{\hbar}\left(\frac{x+y}{2}, \xi\right) \varphi(y) dy d\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \quad (\text{C.2})$$

while the standard quantization $\text{Op}_{\hbar}(a) : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is given by

$$(\text{Op}_{\hbar}(a_{\hbar})\varphi)(x) = (2\pi\hbar)^{-n} \int e^{\frac{i}{\hbar}\xi(x-y)} a_{\hbar}(x, \xi) \varphi(y) dy d\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \quad (\text{C.3})$$

Both quantization extend continuously to operators on $\mathcal{S}'(\mathbb{R}^n)$. While the standard quantization is slightly easier to define, the Weyl quantization has the advantage, that real symbols are mapped to formally self adjoint operators.

C.2 Definition of the Symbol classes $S_{\mu}(A_{\hbar})$

In this standard \hbar -PDO calculus the symbols are ordered by there asymptotic behavior for $\hbar \rightarrow 0$. If we take for example a symbol $a \in \hbar^k S_{\mu}(f)$ then $a(x, \xi)$ is of order \hbar^k for all $(x, \xi) \in \mathbb{R}^{2n}$. The symbol classes that we will now introduce will also allow \hbar -dependent order function which will allow to control the \hbar -order of a symbol locally, i.e. in dependence of (x, ξ) . First we define these \hbar -dependent order functions:

Definition C.3. Let f be an order function on \mathbb{R}^{2n} and $0 \leq \mu \leq \frac{1}{2}$. Let $A_{\hbar} \in S_{\mu}(f)$ a (possibly \hbar -dependent) positive symbol such that for some $c \geq 0$ there is a constant C that fulfills

$$A_{\hbar}(x, \xi) \geq C\hbar^c f(x, \xi) \quad (\text{C.4})$$

and that for all multiindices $\alpha, \beta \in \mathbb{N}^n$:

$$\left| \partial_x^{\alpha} \partial_{\xi}^{\beta} A_{\hbar}(x, \xi) \right| \leq C_{\alpha, \beta} \hbar^{-\mu(|\alpha|+|\beta|)} A_{\hbar}(x, \xi) \quad (\text{C.5})$$

holds. Then we call A_{\hbar} an \hbar -dependent order function and say $A_{\hbar} \in \mathcal{OF}^c(f)$

Definition C.4. The symbol class $S_\mu(A_\hbar)$ is then defined to be the space of smooth functions $a_\hbar(x, \xi)$ defined on R^{2n} and parametrized by $\hbar > 0$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta a_\hbar(x, \xi) \right| \leq C_{\alpha, \beta} \hbar^{-\mu(|\alpha|+|\beta|)} A_\hbar(x, \xi) \quad (\text{C.6})$$

By $\hbar^k S_\mu(A_\hbar)$ we will as usual denote the symbols a_\hbar for which $\hbar^{-k} a_\hbar \in S_\mu(A_\hbar)$

As $A_\hbar(x, \xi) \leq C_0 f(x, \xi)$ and from (C.5) it is obvious, that

$$S_\mu(A_\hbar) \subset S_\mu(f) \quad (\text{C.7})$$

and via this inclusion for $a_\hbar \in S_\mu(A_\hbar)$ the standard Quantization $Op_\hbar(a)$ and the Weyl quantization $Op_\hbar^w(a_\hbar)$ are well defined and give continuous operators on $\mathcal{S}(\mathbb{R}^n)$ respectively on $\mathcal{S}'(\mathbb{R}^n)$. Furthermore equation (C.4) gives us a second inclusion

$$S_\mu(f) \subset \hbar^{-c} S_\mu(A_\hbar) \quad (\text{C.8})$$

thus combining these two inclusions we have:

$$\hbar^c S_\mu(f) \subset S_\mu(A_\hbar) \subset S_\mu(f)$$

As for standard $\hbar - PDO$ symbol we can define asymptotic expansions:

Definition C.5. Let $a_j \in S_\mu(A_\hbar)$ for $j = 0, 1, \dots$ then we call $\sum_j \hbar^j a_j$ an asymptotic expansion of $a \in S_\mu(A_\hbar)$ (writing $a \sim \sum_j \hbar^j a_j$) if and only if:

$$a - \sum_{j < N} \hbar^j a_j \in \hbar^N S_\mu(A_\hbar)$$

As in for the standard \hbar -PDOs we have some sort of Borel's theorem also for symbols in $S_\mu(A_\hbar)$

Proposition C.6. Let $a_i \in S_\mu(A_\hbar)$ then there is a symbol $a \in S_\mu(A_\hbar)$ such that

$$a - \sum_{j < k} \hbar^j a_j \in \hbar^k S_\mu(A_\hbar) \quad (\text{C.9})$$

Proof. Once more we can use the inclusion (C.7) into the standard \hbar -PDO classes and obtain the existence of a symbol $a \in S_\mu(f)$ such that

$$a - \sum_{j < k} \hbar^j a_j \in \hbar^k S_\mu(f) \quad (\text{C.10})$$

and we will show that this symbol belongs to $S_\mu(A_\hbar)$ and that (C.9) holds: For the first statement we write

$$a = a - \underbrace{\sum_{j < c} \hbar^j a_j}_{\in \hbar^c S_\mu(f)} + \underbrace{\sum_{j < c} \hbar^j a_j}_{\in S_\mu(A_\hbar)}$$

and use the inverse inclusion (C.8).

In order to prove (C.9) write

$$a - \sum_{j < k} \hbar^j a_j = a - \underbrace{\sum_{j < k+c} \hbar^j a_j}_{\in \hbar^{c+k} S_\mu(f)} + \underbrace{\sum_{j=k}^{k+c-1} \hbar^j a_j}_{\in \hbar^k S_\mu(A_\hbar)}$$

and use once more (C.8). □

The advantage of this new symbol class is, that the order function $A_\hbar(x, \xi)$ itself can depend on \hbar and thus the control in \hbar can be localized. A simple example for such an order function would be $A_\hbar = \hbar^{m\mu} \langle \frac{\xi}{\hbar^\mu} \rangle^m \in \mathcal{O}\mathcal{F}^c(\langle \xi \rangle^m)$. For $\xi \neq 0$ this function is of order \hbar^0 whereas for $\xi = 0$ it is of order $\hbar^{m\mu}$. Thus also all symbols in $S_\mu(A_\hbar)$ have to show this behavior.

C.3 Composition of symbols

By using the inclusion (C.7) we will show a result for the composition of Symbols absolutely analogous to the one in the standard case Theorem 4.18 in [34]. We first note that for $A_\hbar \in \mathcal{O}\mathcal{F}^{c_A}(f_A)$ and $B_\hbar \in \mathcal{O}\mathcal{F}^{c_B}(f_B)$ the product formula for derivative yields that $A_\hbar B_\hbar \in \mathcal{O}\mathcal{F}^{c_A+c_B}(f_A f_B)$ and can now formulate the following theorem:

Theorem C.7. *Let $A_\hbar \in \mathcal{O}\mathcal{F}^{c_A}(f_A)$ and $B_\hbar \in \mathcal{O}\mathcal{F}^{c_B}(f_B)$ be two \hbar -dependent order functions and $a \in S_\mu(A_\hbar)$ and $b \in S_\mu(B_\hbar)$ two \hbar -local symbols. Then there is a symbol*

$$a \# b \in S_\mu(A_\hbar B_\hbar)$$

such that

$$Op_\hbar^w(a) Op_\hbar^w(b) = Op_\hbar^w(a \# b) \quad (\text{C.11})$$

as operators on \mathcal{S} and the at first order we have

$$a \# b - ab \in \hbar^{1-2\mu} S_\mu(A_\hbar B_\hbar) \quad (\text{C.12})$$

Proof. The standard theorem of composition of \hbar -PDOs (see e.g. Th 4.18 in [34]) together with the inclusion of symbol-classes (C.7) provides us a symbol $a\#b \in S_\mu(f_A \cdot f_B)$ that fulfills equation (C.11). Furthermore it provides us with a complete asymptotic expansion for $a\#b$:

$$a\#b - \sum_{k=0}^{N-1} \left(\frac{1}{k!} \left[\frac{i\hbar(\langle D_x, D_\eta \rangle - \langle D_y, D_\xi \rangle)}{2} \right]^k a(x, \xi) b(y, \eta) \right)_{|y=x, \eta=\xi} \in \hbar^{N(1-2\mu)} S_\mu(f_A \cdot f_B) \quad (\text{C.13})$$

In order to prove our theorem it thus only rests to show, that $a\#b \in S_\mu(A_{\hbar}B_{\hbar})$ and that equation (C.12) holds. We start with the second one. First let $N \in \mathbb{N}$ be such that $(N-1)(1-2\mu) \geq c_A + c_B$, then equation (C.13) and inclusion (C.8) assure that the remainder term in (C.13) is in $\hbar^{1-2\mu} S_\mu(A_{\hbar}B_{\hbar})$. For $0 \leq k \leq N-1$ each term in (C.13) can be written as a sum of finitely many terms of the form

$$\frac{(i\hbar)^k}{2^k k!} \left(D_x^\alpha D_\xi^\beta a(x, \xi) \right) \cdot \left(D_x^\gamma D_\xi^\delta b(x, \xi) \right)$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{N}^n$ are multiindices fulfilling $|\alpha| + |\beta| + |\gamma| + |\delta| = 2k$. Via the product formula one easily checks, that these terms are all in $\hbar^{k(1-2\mu)} S_\mu(A_{\hbar}B_{\hbar})$ which proves that $a\#b \in S_\mu(A_{\hbar}B_{\hbar})$. \square

C.4 Ellipticity and inverses

In this section we will define ellipticity for our new symbol classes and will prove a result on L^2 -invertibility.

Definition C.8. We call a symbol $a \in S_\mu(A_{\hbar})$ elliptic if there is a constant C such that:

$$|a(x, \xi)| \geq CA_{\hbar}(x, \xi) \quad (\text{C.14})$$

For an \hbar -dependent order function $A_{\hbar} \in \mathcal{OF}^c(f)$, from (C.5) and (C.4) it follows, that $\hbar^c A_{\hbar}^{-1} \in \mathcal{OF}^c(f^{-1})$ is again a \hbar -dependent order function and we can formulate the following proposition:

Proposition C.9. *If $a \in S_\mu(A_{\hbar})$ is elliptic then $a^{-1} \in \hbar^{-c} S_\mu(\hbar^c A_{\hbar}^{-1})$*

Proof. We have to show, that $|\partial_x^\alpha \partial_\xi^\beta a^{-1}(x, \xi)| \leq C \hbar^{-\mu(|\alpha|+|\beta|)} A_{\hbar}^{-1}(x, \xi)$ uniformly in \hbar, x and ξ . For some first derivative (i.e. for $\alpha \in \mathbb{N}^{2n}, |\alpha| = 1$) we have

$$|\partial_{x,\xi}^\alpha a^{-1}| = \frac{|\partial_{x,\xi}^\alpha a|}{|a^2|} \leq C \frac{\hbar^{-\mu} A_{\hbar}}{A_{\hbar}^2} = C \hbar^{-\mu} A_{\hbar}^{-1}$$

where the inequality is obtained by (C.5) and (C.14). The estimates of higher order derivatives can be obtained by induction. \square

As for standard \hbar -PDOs this notion of ellipticity implies that the corresponding operators are invertible for sufficiently small \hbar .

Proposition C.10. *Let $A_\hbar \in \mathcal{OF}^c(1)$ and $a \in S_\mu(A_\hbar)$ be an elliptic symbol, then $Op_\hbar^w(a) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a bounded operator. Furthermore there exists $\hbar_0 > 0$ such that $Op_\hbar^w(a)$ is invertible for all $\hbar \in]0, \hbar_0]$. Its inverse is again bounded and a pseudodifferential operator $Op_\hbar^w(b)$ with symbol $b \in S_\mu(A_\hbar^{-1})$. At leading order its symbol is given by*

$$b - a^{-1} \in \hbar^{1-2\mu} S_\mu(A_\hbar^{-1})$$

Proof. As $a \in S_\mu(A_\hbar) \subset S_\mu(1)$ the boundedness of $Op_\hbar^w(a)$ follows from theorem 4.23 in [34]. By theorem C.7 we calculate

$$Op_\hbar^w(a)Op_\hbar^w(a^{-1}) = Id + R$$

where $R = Op_\hbar^w(r)$ is a PDO with symbol $r \in \hbar^{1-2\mu} S_\mu(1)$. Again from theorem 4.23 in [34] we obtain $\|R\|_{L^2} \leq C\hbar^{1-2\mu}$ thus there is \hbar_0 such that $\|R\|_{L^2} < 1$ for $\hbar \in]0, \hbar_0]$. According to theorem C.3 in [34] we can conclude that $Op_\hbar^w(a)$ is invertible and that the inverse is given by $Op_\hbar^w(a^{-1})(Id + R)^{-1}$. The semiclassical version of Beals theorem allows us to conclude that $(Id + R)^{-1} = \sum_{k=0}^{\infty} (-R)^k$ is a PDO with symbol in $S_\mu(1)$ (cf. theorem 8.3 and the following remarks in [34]). The representation of $(Id - R)^{-1}$ as a series finally gives us the symbol of the inverse operator at leading order. \square

C.5 Egorov's theorem for diffeomorphisms

In this section we will study the behavior of symbols $a \in S_\mu(A_\hbar)$ under variable changes. Let $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism that equals identity outside some bounded set then the pullback with this coordinate change acts as a continuous operator on $\mathcal{S}(\mathbb{R}^n)$ by:

$$(\gamma^*u)(x) := u(\gamma(x))$$

Which can be extended by its adjoint to a continuous operator $\gamma^* : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$. By a variable change of an operator we understand its conjugation by γ and we are interested for which $a \in S_\mu(A_\hbar)$ the conjugated operator $(\gamma^*)^{-1}Op_\hbar(a)\gamma^*$ is again a \hbar -PDO with symbol a_γ . At leading order this symbol will be the composition of the original symbol with the so called canonical transformation

$$T : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}, (x, \xi) \mapsto (\gamma^{-1}(x), (\partial\gamma(\gamma^{-1}(x)))^T\xi)$$

and the symbol class of a_γ will be $S_\mu(A_{\hbar} \circ T)$. For the $A_{\hbar} \in \mathcal{OF}^c(f)$ defined in Definition C.3 the composition $A_{\hbar} \circ T$ will in general however not be a \hbar -dependent order function itself because the derivatives in x create a supplementary ξ factor which has to be compensated (cf. discussion in chapter 9.3 in [34]). We therefore demand in this section that our order function A_{\hbar} satisfies:

$$\left| \partial_x^\alpha \partial_\xi^\beta A_{\hbar}(x, \xi) \right| \leq C_{\alpha, \beta} \hbar^{\mu(|\alpha|+|\beta|)} \langle \xi \rangle^{-|\beta|} A_{\hbar}(x, \xi)$$

A straightforward calculation shows then, that $A_{\hbar} \circ T \in \mathcal{OF}^c(f \circ T)$ is again a \hbar -dependent order function. The same condition has to be fulfilled by the symbol of the conjugated operator:

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq \hbar^{-\mu(|\alpha|+|\beta|)} \langle \xi \rangle^{-|\beta|} A_{\hbar}(x, \xi) \quad (\text{C.15})$$

Theorem C.11. *Let $a \in S_\mu(A_{\hbar})$ be an symbol which fulfills (C.15) and has compact support in x (i.e. $\overline{\{x \in \mathbb{R}^n | \exists \xi \in \mathbb{R}^n : a(x, \xi) \neq 0\}}$ is compact) and let $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism. Then there is a symbol $a_\gamma \in S_\mu(A_{\hbar} \circ T)$ such that*

$$(Op_{\hbar}(a_\gamma)u)(\gamma(x)) = (Op_{\hbar}(a)(u \circ \gamma))(x) \quad (\text{C.16})$$

for all $u \in \mathcal{S}'(\mathbb{R}^n)$. Furthermore a_γ has the following asymptotic expansion.

$$a_\gamma(\gamma(x), \eta) \sim \sum_{n=0}^{k-n} \frac{1}{n!} \langle i \frac{\hbar}{\langle \eta \rangle} D_y, D_\xi \rangle^n e^{i \frac{\hbar}{\langle \eta \rangle} \rho_x(y, \eta)} a(x, \xi) \Big|_{y=0, \xi=(\partial\gamma(x))^T \eta} \quad (\text{C.17})$$

where $\rho_x(y) = \gamma(y+x) - \gamma(x) - \gamma'(x)y$. The terms of the series are in $\hbar^{\frac{\nu(1-2\mu)}{2}} S_\mu(\langle \eta \rangle^{\frac{\nu}{2}} A_{\hbar} \circ T(\gamma(x), \eta))$.

We will prove this theorem similar to theorem 18.1.17 in [17] by using a parameter dependent stationary phase approximation (Thm7.7.7 in [16]) as well as the following proposition which forms the analog to Proposition 18.1.4 of [17] for our symbol classes and which we will prove first.

Proposition C.12. *Let $a(x, \xi; \hbar) \in C^\infty(\mathbb{R}^{2n})$ a family of smooth functions that fulfills*

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C \hbar^{-l} \langle \xi \rangle^l f(x, \xi) \quad (\text{C.18})$$

where C and l may depend on α and β . Let $a_j \in S_\mu(A_{\hbar})$, $j = 0, 1, \dots$ be a sequence of symbols such that

$$|a(x, \xi) - \sum_{j < k} \hbar^j a_j(x, \xi)| \leq C \hbar^{\tau k} \langle \xi \rangle^{-\tau k} f(x, \xi) \quad (\text{C.19})$$

where $\tau > 0$. Then $a \in S_\mu(A_\hbar)$ and $a \sim \sum \hbar^j a_j$.

Proof. We have to show that for all $k \geq 0$ and $g_k(x, \xi) := a(x, \xi) - \sum_{j < k} \hbar^j a_j(x, \xi)$ we have $|\partial_x^\alpha \partial_\xi^\beta g_k| \leq C \hbar^{k-\mu(|\alpha|+|\beta|)} A_\hbar$. This result can be obtained by iterating the following argument for the first derivative in x_1 :

Let $e_1 \in \mathbb{R}^n$ be the first eigenvector and $0 < \varepsilon < 1$. For arbitrary $j \in \mathbb{N}$ we can write by Taylor's Formula

$$|g_j(x + \varepsilon e_1, \xi) - g_j(x, \xi) - \partial_{x_1} g_j(x, \xi) \varepsilon| \leq C \varepsilon^2 \sup_{t \in [0, \varepsilon]} |\partial_{x_1}^2 g_j(x + t e_1, \xi)|$$

From (C.18) and the property, that all a_j are in $S_\mu(A_\hbar)$ we get

$$\sup_{t \in [0, \varepsilon]} |\partial_{x_1}^2 g_j(x + t e_1, \xi)| \leq C \hbar^{-l} \langle \xi \rangle^l f(x, \xi)$$

for some $l \in \mathbb{R}$ and get

$$|\partial_{x_1} g_j(x, \xi)| \leq C \varepsilon \hbar^{-l} \langle \xi \rangle^l m(x, \xi) + \frac{|g_j(x + \varepsilon e_1, \xi) - g_j(x, \xi)|}{\varepsilon}$$

which turns for $j > \frac{2k+2c+l}{\tau}$ and $\varepsilon = \hbar^{k+l+c} \langle \xi \rangle^{-(k+l+c)}$ into:

$$|\partial_{x_1} g_j(x, \xi)| \leq C \hbar^{c+k} \langle \xi \rangle^{-(c+k)} f(x, \xi) \leq C \hbar^k A_\hbar(x, \xi)$$

where we used (C.8) in the second equation. Thus

$$|\partial_{x_1} g_k(x, \xi)| \leq C \hbar^k A_\hbar(x, \xi) + \left| \sum_{i=k}^j \hbar^i \partial_{x_1} a_i(x, \xi) \right| \leq C \hbar^{k-\mu} A_\hbar(x, \xi)$$

which finishes the proof. □

After having proven this proposition we can start with the proof of theorem C.11:

Proof. If we define

$$a_\gamma(\gamma(x), \eta) := e^{-\frac{i}{\hbar} \gamma(x) \eta} Op_\hbar(a) e^{\frac{i}{\hbar} \gamma(\cdot) \eta} \quad (\text{C.20})$$

then equation (C.16) holds for all $e^{\frac{i}{\hbar} x \eta}$ which form a dense subset of $\mathcal{S}'(\mathbb{R}^n)$. We thus have to show that a_γ defined in (C.20) is in $S_\mu(A_\hbar)$ and that (C.17) holds.

We will first write a_γ as an oscillating integral in order to apply the stationary phase theorem. By definition of $Op_\hbar(a)$ one obtains

$$a_\gamma(\gamma(x), \eta) = \frac{1}{(2\pi\hbar)^n} \iint a(x, \tilde{\xi}) e^{\frac{i}{\hbar} ((x-\tilde{y})\tilde{\xi} + (\gamma(\tilde{y}) - \gamma(x))\eta)} d\tilde{y} d\tilde{\xi}$$

which we can transform by a variable transformation $\tilde{\xi} = \langle \eta \rangle \xi$ and $\tilde{y} = y + x$ into

$$a_\gamma(\gamma(x), \eta) = \frac{1}{(2\pi\tilde{\hbar})^n} \iint a(x, \langle \eta \rangle \xi) e^{\frac{i}{\tilde{\hbar}}(-y\xi + (\gamma(y+x) - \gamma(x))\frac{\eta}{\langle \eta \rangle})} dy d\xi$$

where $\tilde{\hbar} = \frac{\hbar}{\langle \eta \rangle}$.

The critical points of the phase function are given by

$$y = 0 \text{ and } \xi = (\partial\gamma(x))^T \frac{\eta}{\langle \eta \rangle}$$

Let $\chi \in C_c^\infty([-2, 2]^n)$ such that $\chi = 1$ on $[-1, 1]^n$ then we can write

$$a_\gamma(\gamma(x), \eta) = I_1(\tilde{\hbar}) + I_2(\tilde{\hbar})$$

with

$$I_1(\tilde{\hbar}) = \frac{1}{(2\pi\tilde{\hbar})^n} \iint \chi(y) \chi\left(\xi - (\partial\gamma(x))^T \frac{\eta}{\langle \eta \rangle}\right) a(x, \langle \eta \rangle \xi) e^{\frac{i}{\tilde{\hbar}}(-y\xi + (\gamma(y+x) - \gamma(x))\frac{\eta}{\langle \eta \rangle})} dy d\xi$$

and

$$I_2(\tilde{\hbar}) = \frac{1}{(2\pi\tilde{\hbar})^n} \iint \left(1 - \chi(y) \chi\left(\xi - (\partial\gamma(x))^T \frac{\eta}{\langle \eta \rangle}\right)\right) a(x, \langle \eta \rangle \xi) e^{\frac{i}{\tilde{\hbar}}(-y\xi + (\gamma(y+x) - \gamma(x))\frac{\eta}{\langle \eta \rangle})} dy d\xi.$$

While $I_1(\tilde{\hbar})$ still contains critical points, for $I_2(\tilde{\hbar})$ there are no critical points in the support of the integrand anymore.

I_1 is of the form studied in theorem 7.7.7 in [16]. Here the role of x and y is interchanged and there is an additional parameter $\frac{\eta}{\langle \eta \rangle}$. We thus get from this stationary phase theorem

$$\begin{aligned} & \left| I_1(\tilde{\hbar}) - \sum_{\nu=0}^{k-n} \frac{1}{\nu!} \langle i\tilde{\hbar}D_y, D_\xi \rangle^\nu e^{\frac{i}{\tilde{\hbar}}\langle \rho_x(y), \frac{\eta}{\langle \eta \rangle} \rangle} u(x, \xi, y, \eta) \Big|_{y=0, \xi=(\partial\gamma(x))^T \frac{\eta}{\langle \eta \rangle}} \right| \\ & \leq C\tilde{\hbar}^{\frac{k+n}{2}} \sum_{|\alpha| \leq 2k} \sup_{y, \xi} |D_{y, \xi}^\alpha u(x, \xi, y, \eta)| \end{aligned} \quad (\text{C.21})$$

where $u(x, \xi, y, \eta) = \chi(y) \chi\left(\xi - (\partial\gamma(x))^T \frac{\eta}{\langle \eta \rangle}\right) a(x, \langle \eta \rangle \xi)$. Because of (C.15) and (C.1) we can estimate

$$\sup_{y, \xi} |D_{y, \xi}^\alpha u(x, \xi, y, \eta)| \leq C\hbar^{-\mu|\alpha|} f(x, (\partial\gamma(x))^T \eta) = C\hbar^{-\mu|\alpha|} f \circ T(\gamma(x), \eta)$$

Thus transforming the expansion (C.21) back to an expansion in \hbar we get

$$\begin{aligned} & \left| I_1(\hbar) - \sum_{\nu=0}^{k-n} \frac{1}{\nu!} \langle i\frac{\hbar}{\langle \eta \rangle} D_y, D_\xi \rangle^\nu e^{\frac{i}{\hbar}\langle \rho_x(y), \eta \rangle} u(x, \xi, y, \eta) \Big|_{y=0, \xi=(\partial\gamma(x))^T \frac{\eta}{\langle \eta \rangle}} \right| \\ & \leq C\hbar^{\frac{k(1-2\mu)+n}{2}} \langle \eta \rangle^{-\frac{k+n}{2}} f \circ T(\gamma(x), \eta) \end{aligned}$$

As the stationary points for I_2 are not contained in the support of the integrand we get by the non stationary phase theorem:

$$|I_2(\hbar)| \leq C \left(\frac{\hbar}{\langle \eta \rangle} \right)^N f \circ T(\gamma(x), \eta)$$

for all $N \in \mathbb{N}$. Thus we finally get

$$\left| a_\gamma(\gamma(x), \eta) - \sum_{\nu=0}^{k-n} \frac{1}{\nu!} \langle i \frac{\hbar}{\langle \eta \rangle} D_y, D_\xi \rangle^\nu e^{\frac{i}{\hbar} \langle \rho_x(y), \eta \rangle} u(x, \xi, y, \eta) \Big|_{y=0, \xi=(\partial\gamma(x))^T \frac{\eta}{\langle \eta \rangle}} \right| \leq C \hbar^{\frac{k(1-2\mu)+n}{2}} \langle \eta \rangle^{-\frac{k+n}{2}} f \circ T(\gamma(x), \eta) \quad (\text{C.22})$$

If we show that the elements of the series are in $\hbar^{\frac{\nu(1-2\mu)}{2}} S_\mu(\langle \eta \rangle^{\frac{\nu}{2}} A_\hbar \circ T(\gamma(x), \eta))$ then this equation is of the form (C.19). The terms of order ν in the series are of the form

$$\left(\frac{i\hbar}{\langle \eta \rangle} \right)^\nu \partial_y^\alpha e^{\frac{i}{\hbar} \langle \rho_x(y), \eta \rangle} (\partial_\xi^\alpha a)(x, (\partial\gamma(x))^T \eta) \langle \eta \rangle^\nu \Big|_{y=0}$$

Where $\alpha \in \mathbb{N}^n$ with $|\alpha| = \nu$. The second factor $(\partial_\xi^\alpha a)(x, (\partial\gamma(x))^T \eta) \langle \eta \rangle^\nu$ is in $\hbar^{-\mu\nu} S_\mu(A_\hbar \circ T(\gamma(x), \eta))$ as we demanded the condition (C.15) on our symbol a . Thus it remains to show that the other factor is of order $\left(\frac{\hbar}{\langle \eta \rangle} \right)^{\frac{\nu}{2}}$ on the support of a . This is the case because $\rho_x(y)$ vanishes at second order in $y = 0$. Each derivative of $e^{\frac{i}{\hbar} \langle \rho_x(y), \eta \rangle}$ produces a factor $\frac{i}{\hbar} \langle \partial_{y_i} \rho_x(0), \eta \rangle$. But as $\partial_{y_i} \rho_x(0)$ vanishes we need a second derivative, now acting on $\partial_{y_i} \rho_x(y)$, in order to get a contribution. Thus in the worst case $\partial_y^\alpha e^{\frac{i}{\hbar} \langle \rho_x(y), \eta \rangle}$ is of order $\left(\frac{\hbar}{\langle \eta \rangle} \right)^{-\frac{\nu}{2}}$. Thus we have shown that (C.22) is of the form (C.19).

The last thing that we have to show is thus, that a_γ fulfills (C.18). If we consider the definition (C.20) of a_γ we see that $\partial_x^\alpha \partial_\xi^\beta a_\gamma(\gamma(x), \eta)$ can be written as a sum of terms of the form $\frac{P(\eta)}{\hbar^k} e^{-\frac{i}{\hbar} \gamma(x)\eta} O p_\hbar(b) e^{\frac{i}{\hbar} \gamma(\cdot)\eta}$ where $b \in S_\mu(A_\hbar \langle \xi \rangle^j)$ and $P(\eta)$ is a polynomial in η . The constants j, k and the degree of $P(\eta)$ depend on α and β . Thus writing these terms as oscillating integrals and applying the same arguments as above one gets (C.18).

We have thus shown that all the conditions for proposition C.12 are fulfilled and can conclude that a_γ belongs to $S_\mu(A_\hbar)$ and that (C.22) is also an asymptotic expansion w.r.t. the order function A_\hbar . \square

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