Asymptotic spectral gap and Weyl law for Ruelle resonances of open partially expanding maps

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Abstract

We consider a simple model of an open partially expanding map. Its trapped set $\mathcal{K}$ in phase space is a fractal set. We first show that there is a well defined discrete spectrum of Ruelle resonances which describes the asymptotic of correlation functions for large time and which is parametrized by the Fourier component ν on the neutral direction of the dynamics. We introduce a specific hypothesis on the dynamics that we call "minimal captivity". This hypothesis is stable under perturbations and means that the dynamics is univalued on a neighborhood of $\mathcal{K}$. Under this hypothesis we show the existence of an asymptotic spectral gap and a Fractal Weyl law for the upper bound of density of Ruelle resonances in the semiclassical limit $\nu \to \infty$. Some numerical computations with the truncated Gauss map illustrate these results.

[1]

Contents

*Institut Fourier, UMR 5582, 100 rue des Maths, BP74 38402 St Martin d'Hères.

†Institut Fourier, UMR 5582, 100 rue des Maths, BP74 38402 St Martin d'Hères. frederic.faure@ujf-grenoble.fr http://www-fourier.ujf-grenoble.fr/~faure

‡Fachbereich Mathematik, Philipps-Universit¨t Marburg, a Hans-Meerwein-Straße, 35032 Marburg, Germany. weich@mathematik.uni-marburg.de http://www.mathematik.uni-marburg.de/~weich/

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1 Introduction

A "partially expanding map" is a map which is expanding except in some directions which are called "neutral". An "open map" is a map for which the non wandering set (or trapped set) is not the full manifold but a relatively compact subset. The aim of this paper is to study the dynamics of a class of open partially expanding maps from the spectral approach initiated from Ruelle and Bowen. In this approach, the pull back operator by the map, also called transfer operator, is shown to have some discrete spectrum in some specific functional spaces. These eigenvalues called "Ruelle resonances" are very useful to describe the effective long time behavior of the dynamics: to express dynamical correlation functions and deduce statistical properties of the dynamics such as mixing and central limit theorems.

In Section 2 we define the model of expanding maps on some union of intervals $I \subset \mathbb{R}$ precisely called an iterated function scheme (I.F.S.). This is a well studied class of dynamical systems for which the trapped set $K \subset I$ is a Cantor set and has some Hausdorff dimension $\dim_H K \in [0,1[$. In Section 2.4 we extend this model by adding a neutral direction and obtain a "partially expanding map". The transfer operator is defined in Section 2.5. We can decompose the transfer operators into its Fourier components $\nu \in \mathbb{Z}$ with respect to the neutral direction and obtain a family of operators $\hat{F}_\nu$ also written $\hat{F}_\hbar$ with $\hbar := 1/(2\pi\nu)$ (if $\nu \neq 0$).

In Section 3 we present the main new results of this paper. Theorem 12 shows that each transfer operator $\hat{F}_\nu$ has some discrete spectrum of Ruelle resonances in specific Sobolev space. Then Theorem 20 shows that in the limit of large frequencies $|\nu| \to \infty$ the spectral radius of $\hat{F}_\nu$ is bounded by some expression, under some condition that we call "minimal captivity". In order to derive this result we use a semiclassical approach which consists in considering that the operator $\hat{F}_\nu$ has some microlocal properties in phase space T^*I (precisely it is a Fourier integral operator). This allows to consider the associated canonical map F on T^*I. This canonical map has a trapped set $\mathcal{K}$ which is also a Cantor set (which projects on K). We also obtain an upper bound on the number of Ruelle resonances in the limit $|\nu| \to \infty$ in Theorem 22. This upper bound involves the Hausdorff dimension $\dim_H K$ is is usually called fractal Weyl law after the work of J. Sjöstrand in [24]. The "minimal captivity" condition means that the dynamics of the canonical map F restricted to its trapped set $\mathcal{K}$ is one-to-one (whereas the map F on T^*I is multivalued).

In Section 4 we illustrate our results by numerical computations with a particular model: the Gauss map.

The same semiclassical approach has been used before for "closed dynamical systems" in [9] and [16], i.e. for systems in which the trapped set was the full manifold. In these latter papers as well as in [28] a similar result for the asymptotic spectral radius has been obtained. Technically the open aspect here is overcame by using a truncation function χ as explained in Section 2.6. In [16] the author considers models for which the neutral direction is a non commutative compact Lie group and shows discrete spectrum of Ruelle resonances, asymptotic spectral radius and Weyl law. Let us remark that we could extend the present results similarly by considering extensions with compact groups.

As explained in Section 4.1 our results can be applied to "Bowen Series maps" and "Bowen Series transfer operators" associated to the geodesic flow of "convex co-compact hyperbolic surfaces" also called "Shottky surfaces". So our results give some results for the zeroes of the Selberg zeta function and resonances of the Laplacian of these surfaces. In that case the Weyl law of Theorem 22 is in close relation with the results obtained by Lin, Guilloppe and Zworski in [13] where they give an upper bound on the density of resonances for the Laplace-Beltrami operator on open hyperbolic surfaces. We can also apply our results to the quadratic maps and recover results already obtained in [25, 19].

Also let us remark that with the condition of "minimal captivity", the dynamics of the canonical map F in the vicinity of the trapped set $\mathcal{K}$ is univalued and can be identified with the classical dynamics of a "open quantum map". The results of S. Nonnenmacher et M. Zworski [20] about asymptotic spectral radius and Weyl law of these open quantum maps are in that sense very similar to the results presented in this paper.

2 The transfer operator

2.1 Iterated function scheme

The transfer operator studied in this paper is constructed from a simple model of chaotic dynamics called "an iterated function scheme, I.F.S."[8, chap.9]. We give the definition below and refer to Section 4 where many standard examples are presented.

Definition 1. "An iterated function scheme (I.F.S.)". Let $N \in \mathbb{N}$, $N \geq 1$. Let $I_1, \dots I_N \subset \mathbb{R}$ be a finite collection of **disjoint and closed** intervals. Let A be a $N \times N$ matrix, called adjency matrix, with $A_{i,j} \in \{0, 1\}$. We will note $i \rightsquigarrow j$ if $A_{i,j} = 1$. Assume that for each pair $i, j \in \{1, \dots, N\}$ such that $i \rightsquigarrow j$, we have a smooth invertible map $\phi_{i,j} : I_i \to \phi_{i,j}(I_i)$ with $\phi_{i,j}(I_i) \subset \mathrm{Int}(I_j)$. Assume that the map $\phi_{i,j}$ is a **strict contraction**, i.e. there exists $0 < \theta < 1$ such that for every $x \in I_i$,

$$\left|\phi'_{i,j}(x)\right| \leq \theta \tag{1}$$

We suppose that different images of the maps $\phi_{i,j}$ do not intersect (this is the "strong separation condition" in [7, p.35]):

$$\phi_{i,j}(I_i) \cap \phi_{k,l}(I_k) \neq \emptyset \quad \Rightarrow \quad i = k \text{ and } j = l. \tag{2}$$

Remark that the derivative $\phi'_{i,j}(x)$ may be negative. Figure 1 illustrates Definition 1 on a specific example.

2.2 Example with the truncated Gauss map

The Gauss map is

$$G : \begin{cases}]0,1] & \to]0,1[\\ y & \to \left\{\frac{1}{y}\right\} \end{cases} \tag{3}$$

where $\{a\} := a - [a] \in [0, 1[$ denotes the fractional part of $a \in \mathbb{R}$. For $j \in \mathbb{N}\backslash\{0\}$, and $y \in \mathbb{R}$ such that $\frac{1}{j+1} < y \leq \frac{1}{j}$ then $G(y) = G_j(y) := \frac{1}{y} - j$. Notice that $dG/dy < 0$. The inverse map is $y = G_j^{-1}(x) = \frac{1}{x+j}$.

Let $N \geq 1$. We will consider only the first N "branches" $(G_j)_{j=1,\dots N}$. In order to have a well defined I.F.S according to definition 1, for $1 \leq i \leq N$, let $i' = N + 1 - i$ (which just inverses the labels of the Branch's). Let $\alpha_i = G_{i'}^{-1}\left(\frac{1}{N+1}\right)$, $a_i = \frac{1}{1+i'}$, b_i such that $\alpha_i < b_i < \frac{1}{i'}$, and intervals $I_i = [a_i, b_i]$. On these intervals $(I_i)_i$, we define

$$y = \phi_{i,j}(x) = G_{j'}^{-1}(x) = \frac{1}{x + j'}, \quad j' = N + 1 - j'. \tag{4}$$

The adjency matrix is $A = (A_{i,j})_{i,j}$, the full $N \times N$ matrix with all entries $A_{i,j} = 1$.

2.3 The trapped set K

We define

$$I := \bigcup_{i=1}^{N} I_i \tag{5}$$

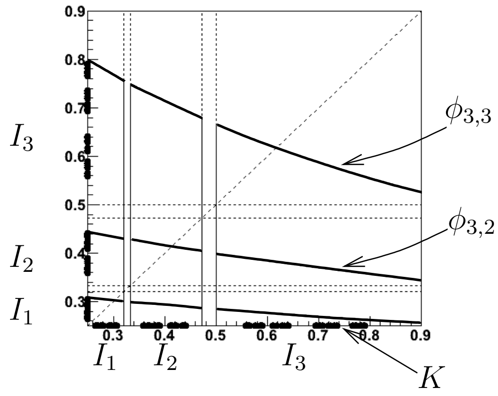

Figure 1: The iterated functions scheme (IFS) defined from the truncated Gauss map (3). Here we have $N = 3$ branches. The maps ϕ: $\phi_{i,j} : I_i \to I_j$, $i, j = 1 \ldots N$ are contractive and given by $\phi_{i,j}(x) = \frac{1}{x+(N+1-j)}$. The trapped set K defined in (9) is a N-adic Cantor set. It is obtained as the limit of the sets $K_0 = (I_1 \cup I_2 \ldots \cup I_N) \supset K_1 = \phi(K_0) \supset K_2 = \phi(K_1) \supset \ldots \supset K$.

and the multivalued map:

$$\phi : I \to I, \qquad \phi = (\phi_{i,j})_{i,j}.$$

ϕ can be iterated and generates a multivalued map $\phi^n : I \to I$ for $n \geq 1$. From hypothesis (2) the inverse map

$$\phi^{-1} : \phi(I) \to I$$

is uni-valued. Throughout the paper we will use the **"unstable Jacobian function"**

$$J(x) := \log \left| \frac{d\phi^{-1}}{dx}(x) \right| \tag{6}$$

defined on $\phi(I)$. From (1), one has

$$\forall x, \quad J(x) > \log \frac{1}{\theta} > 0. \tag{7}$$

Let

$$K_n := \phi^n(I) \tag{8}$$

and $K_0 = I$. Since $K_{n+1} \subset K_n$ we can defined the limit set

$$K := \bigcap_{n \in \mathbb{N}} K_n \tag{9}$$

called the **trapped set**. Then the map

$$\phi^{-1} : K \to K \tag{10}$$

is well defined and uni-valued.

2.3.1 The fractal dimension of the trapped set K and the topological pressure

In this paper we will use the following definition of fractal dimension.

Definition 2. [18, p.76],[7, p.20] If $B \subset \mathbb{R}^d$ is a non empty bounded set, its **upper Minkowski dimension** (or box dimension) is

$$\dim_M B := d - \operatorname{codim}_M B \tag{11}$$

with

$$\operatorname{codim}_M B := \sup \left\{ s \in \mathbb{R} \mid \limsup_{\delta \downarrow 0} \delta^{-s} \cdot \operatorname{Leb}(B_\delta) < +\infty \right\}. \tag{12}$$

where $B_\delta := \left\{ x \in \mathbb{R}^d, \operatorname{dist}(x, B) \leq \delta \right\}$ and $\operatorname{Leb}(.)$ is the Lebesgue measure.

Remark 3. In general

$$\limsup_{\delta \downarrow 0} \delta^{-\operatorname{codim}_M B} \cdot \operatorname{Leb}(B_\delta) < +\infty \tag{13}$$

does not hold, but if it does, B is said to be of **pure dimension**[2]. It is known that the trapped set K defined in (9) has pure dimension and that the above definition of Minkowski dimension coincides with the more usual **Hausdorff dimension** of K [7, p.68]:

$$\dim_M K = \dim_H K \quad \in [0, 1[\tag{14}$$

An efficient way to calculate the fractal dimension $\dim_H K$ is given by the topological pressure. The topological pressure can be defined from the periodic points as follows. A **periodic point** of period $n \geq 1$ is $x \in K$ such that $x = \phi^{-n}(x)$. The topological pressure can be defined in terms of periodic points.

Definition 4. [7, p.72] The **topological pressure** of a function $\varphi \in C(I)$ is

$$\Pr(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{x = \phi^{-n}(x)} e^{\varphi_n(x)} \right)$$

where $\varphi_n(x)$ is the Birkhoff sum of φ along the periodic orbit:

$$\varphi_n(x) := \sum_{k=0}^{n-1} \varphi\left(\phi^{-k}(x)\right)$$

[2] see [24] for comments and further references.

It is interesting to consider the special case of the function $\varphi(x) = -\beta J(x)$ with some $\beta \in \mathbb{R}$ and $J(x)$ defined in (6). This gives the function $\mathrm{P} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathrm{P}(\beta) := \mathrm{Pr}(-\beta J) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{x=\phi^{-n}(x)} \prod_{k=0}^{n-1} \left|\phi'\left(\phi^{-k}(x)\right)\right|^{\beta} \right) \tag{15}$$

The following lemma provides an easy way to compute (numerically) the dimension of K. See figure 2.

Lemma 5. *[7, p.77]* $\mathrm{P}(\beta)$ *is continuous and strictly decreasing in* β *and its unique zero is given by* $\beta = \dim_H K$.

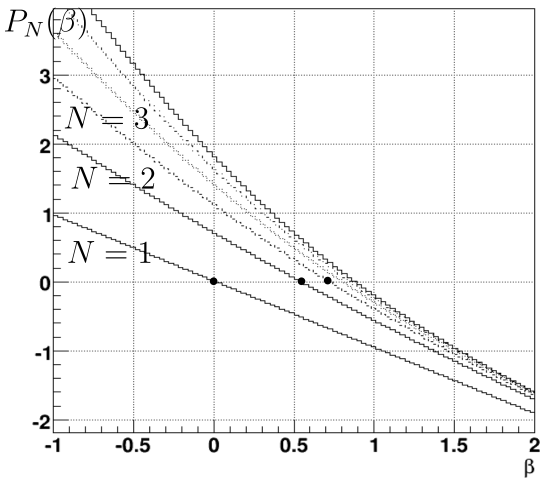

Figure 2: Topological Pressure $\mathrm{P}_N(\beta)$ defined by (15) for the truncated Gauss map example of Section 2.2 for each value of $N = 1, 2, 3\ldots$ being the number of branches. The black points mark the zero of $\mathrm{P}_N(\beta) = 0$ giving the fractal dimension of the trapped set K_N for each value of N: $\dim_H K_1 = 0$, $\dim_H K_2 = 0.531\ldots$, $\dim_H K_3 = 0.705\ldots$, and $\dim_H K_N \underset{N\to\infty}{\to} 1$.

2.4 An extended partially expanding map and reduced transfer operators

The map $\phi^{-1} : \phi(I) \to I$ is univalued and expanding. Let $\tau \in C^{\infty}(I;\mathbb{R})$ be a smooth real valued function called **roof function**. We define the map

$$f : \begin{cases} \phi(I) \times S^1 & \to I \times S^1 \\ (x,y) & \to \left(\phi^{-1}(x), y + \tau(x)\right) \end{cases} \tag{16}$$

with $S^1 := \mathbb{R}/\mathbb{Z}$. Notice that the map f is expanding in the x variable whereas it is neutral in the y variable in the sense that $\frac{\partial f}{\partial y} = 1$. This is called a partially expanding map and is a very simple model of more general partially hyperbolic dynamics [21].

Let $V \in C^\infty(I; \mathbb{C})$ be a smooth complex valued function called a **potential** function.

Definition 6. The transfer operator of the map f with potential V is

$$\hat{\mathcal{F}} : \begin{cases} C^\infty(I \times S^1) & \to C^\infty(\phi(I) \times S^1) \\ \psi(x,y) & \to e^{V(x)}\psi(f(x,y)) \end{cases} \tag{17}$$

Notice that $\psi(x,y)$ can be decomposed into Fourier modes in the y direction. For $\nu \in \mathbb{Z}$, a Fourier mode is

$$\psi_\nu(x,y) = \varphi(x)\, e^{i2\pi\nu y}$$

we have

$$\begin{aligned} \left(\hat{\mathcal{F}}\psi_\nu\right)(x,y) &= e^{V(x)}\psi_\nu(f(x,y)) = e^{V(x)}\varphi\left(\phi^{-1}(x)\right) e^{i2\pi\nu(y+\tau(x))} \\ &= \left(\hat{F}_\nu\varphi\right)(x)\, e^{i2\pi\nu y} \end{aligned}$$

with the reduced transfer operator $\hat{F}_\nu : C^\infty(I) \to C^\infty(\phi(I))$ defined by

$$\left(\hat{F}_\nu\varphi\right)(x) := e^{V(x)} e^{i2\pi\nu\tau(x)}\varphi\left(\phi^{-1}(x)\right) \tag{18}$$

So the operator $\hat{\mathcal{F}}$ is the direct sum of operators $\bigoplus_{\nu\in\mathbb{Z}}\hat{F}_\nu$. From the next Section we will study the individual operator $\hat{F}_\nu$ in (21). Since our main interest is the limit $\nu \to \infty$ of large frequencies in the neutral direction, we will suppose $\nu \neq 0$ and write $\hbar := \frac{1}{2\pi\nu}$. In Section 3.3 we will deduce from our principal results, some asymptotic expansions for time correlation functions of the map (16).

2.5 The transfer operator

Notations: We denote $C_0^\infty(\mathbb{R})$ the space of smooth function on $\mathbb{R}$ with compact support. If $B \subset \mathbb{R}$ is a compact set, we denote $C_B^\infty(\mathbb{R}) \subset C_0^\infty(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ with support included in B. Recall that the inverse map $\phi^{-1} : \phi(I) \to I$ is uni-valued.

Definition 7. Let $\tau \in C^{\infty}(I;\mathbb{R})$ and $V \in C^{\infty}(I;\mathbb{C})$ be smooth functions called respectively **roof function** and **potential** function. Let $\hbar > 0$. We define the **transfer operator**:

$$\hat{F} : \begin{cases} C_I^{\infty}(\mathbb{R}) & \to C_I^{\infty}(\mathbb{R}) \\ \varphi = (\varphi_i)_i & \to \left(\sum_{i=1}^{N} \hat{F}_{i,j}\varphi_i\right)_j \end{cases} \tag{19}$$

with

$$\hat{F}_{i,j} : \begin{cases} C_{I_i}^{\infty}(\mathbb{R}) & \to C_{I_j}^{\infty}(\mathbb{R}) \\ \varphi_i & \to \left(\hat{F}_{i,j}\varphi_i\right)(x) = \begin{cases} e^{V(x)} e^{i\frac{1}{\hbar}\tau(x)} \varphi_i\left(\phi_{i,j}^{-1}(x)\right) & \text{if } i \rightsquigarrow j \text{ and } x \in \phi_{i,j}(I_i) \\ 0 & \text{otherwise} \end{cases} \end{cases} \tag{20}$$

Remark:

- From assumption (2), for any $x \in I$, the sum $\sum_{i=1}^{N}\left(\hat{F}_{i,j}\varphi_i\right)(x)$ which appears on the right hand side of (19) contains at most one non vanishing term. See figure 3.
- For short we can write that

$$\hat{F} : \begin{cases} C_I^{\infty}(\mathbb{R}) & \to C_I^{\infty}(\mathbb{R}) \\ \varphi & \to \begin{cases} e^{V(x)} e^{i\frac{1}{\hbar}\tau(x)} \varphi\left(\phi^{-1}(x)\right) & \text{if } x \in \phi(I) \\ 0 & \text{otherwise} \end{cases} \end{cases} \tag{21}$$

Remark 8.

- For any $\varphi \in C_I^{\infty}(\mathbb{R})$, $n \geq 0$ we have

$$\operatorname{supp}\left(\hat{F}^n \varphi\right) \subset K_n \tag{22}$$

  with K_n defined in (8).

- In the definition (10) we can write $e^{V(x)} e^{i\frac{1}{\hbar}\tau(x)} = \exp\left(i\frac{1}{\hbar}\mathcal{V}(x)\right)$ with $\mathcal{V}(x) := \tau(x) + \hbar(-iV(x))$. More generally we may consider a finite series $\mathcal{V}(x) = \sum_{j=0}^{n} \hbar^j \mathcal{V}_j(x)$ with leading term $\mathcal{V}_0(x) = \tau(x)$ and complex valued sub-leading terms $\mathcal{V}_j : I \to \mathbb{C}$, $j \geq 1$.

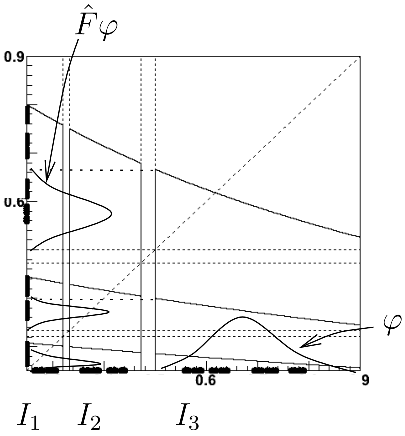

Figure 3: Action of the transfer operator $\hat{F}$ on a function φ as defined in (21). In this schematic figure we have $V = 0$ and $\tau = 0$. In general the factor $e^{V(x)}$ changes the amplitude and $e^{i\frac{1}{\hbar}\tau(x)}$ creates some fast oscillations if $\hbar \ll 1$.

2.6 Extension of the transfer operator to distributions

The transfer operator $\hat{F}$ has been defined on smooth functions $C_I^\infty(\mathbb{R})$ in (21). We will need to extend it to the space of distributions. For that purpose we first introduce a cut-off function $\chi \in C_I^\infty(\mathbb{R})$ such that $\chi(x) = 1$ for every $x \in K_1 = \phi(I)$, i.e. $\chi(\phi_{i,j}(x)) = 1$ for every $x \in I_i$ and j such that $i \rightsquigarrow j$. We denote $\hat{\chi}$ the multiplication operator by the function χ. Let us define:

$$\hat{F}_\chi := \hat{F}\hat{\chi}, \qquad \hat{F}_{i,j,\chi} := \hat{F}_{i,j}\hat{\chi} \tag{23}$$

Note that for any $\varphi \in C_{K_1}^\infty(\mathbb{R})$ we have $\hat{\chi}\varphi = \varphi$ hence $\left(\hat{F}\hat{\chi}\right)\varphi = \hat{F}\varphi$. Also $\hat{\chi}: C_0^\infty(\mathbb{R}) \to C_I^\infty(\mathbb{R})$ hence $\hat{F}_\chi$ is defined on $C_0^\infty(\mathbb{R})$.

The formal adjoint operator $\hat{F}_{i,j,\chi}^*: C_0^\infty(\mathbb{R}) \to C_{I_i}^\infty(\mathbb{R})$ is defined by

$$\langle \varphi_i | \hat{F}_{i,j,\chi}^* \psi_j \rangle = \langle \hat{F}_{i,j,\chi}\varphi_i | \psi_j \rangle, \qquad \forall \varphi_i \in C_0^\infty(\mathbb{R}), \psi_j \in C_0^\infty(\mathbb{R}), \tag{24}$$

with the L^2-scalar product $\langle u|v\rangle := \int \overline{u}(x)\, v(x)\, dx$.

Lemma 9. *For $i \rightsquigarrow j$, the adjoint operator $\hat{F}_{i,j,\chi}^*: C_0^\infty(\mathbb{R}) \to C_{I_i}^\infty(\mathbb{R})$ is given, for $y \in I_i$ by*

$$\left(\hat{F}_{i,j,\chi}^*\psi_j\right)(y) = \chi(y)\left|\phi_{i,j}'(y)\right| e^{\overline{V(\phi_{i,j}(y))}} e^{-\frac{i}{\hbar}\tau(\phi_{i,j}(y))}\psi_j(\phi_{i,j}(y)) \tag{25}$$

The adjoint operator $\hat{F}_\chi^*: C_0^\infty(\mathbb{R}) \to C_I^\infty(\mathbb{R})$ is given by

$$\psi = (\psi_j)_j \to \left(\hat{F}_\chi^*\psi\right)_i(y) = \sum_{j \text{ s.t. } i \rightsquigarrow j} \left(\hat{F}_{i,j,\chi}^*\psi_j\right)(y)$$

Proof. Using the change of variables $x = \phi_{i,j}(y)$ and definition (24), we write

$$\begin{aligned}
\langle \varphi_i | \hat{F}^*_{i,j,\chi} \psi_j \rangle &= \int \overline{\varphi}_i(y) \left(\hat{F}^*_{i,j,\chi} \psi_j \right)(y)\, dy \\
&= \langle \hat{F}_{i,j,\chi} \varphi_i | \psi_j \rangle = \int_{\phi_{i,j}(I_i)} \overline{e^{V(x)} e^{i\frac{1}{\hbar}\tau(x)} \varphi_i\left(\phi^{-1}_{i,j}(x)\right) \chi\left(\phi^{-1}_{i,j}(x)\right)} \psi_j(x)\, dx \\
&= \int_{I_i} \overline{\varphi_i}(y)\, \chi(y) \left|\phi'_{i,j}(y)\right| e^{\overline{V(x)}} e^{-i\frac{1}{\hbar}\tau_j(x)} \psi_j(x)\, dy
\end{aligned}$$

and deduce (25). $\square$

Remark 10.

- Without the cut-off function χ the image of $\hat{F}^*_{i,j}$ may not be continuous on the boundary of I_i.
- An other more general possibility would have been to consider $\chi \in C^\infty_I(\mathbb{R})$ such that $0 < \chi(x)$ for $x \in \mathrm{Int}(I)$ (without assumption that $\chi \equiv 1$ on K_1) and define

  $$\hat{F}_{i,j,\chi} := \hat{\chi}^{-1} \hat{F}_{i,j} \hat{\chi} : C^\infty_{I_i}(\mathbb{R}) \to C^\infty_{I_j}(\mathbb{R}) \tag{26}$$

  which is well defined since $\mathrm{supp}\left(\hat{F}_{i,j}\hat{\chi}\varphi\right) \subset \mathrm{Int}(I_j)$ where χ does not vanish. This more general definition (26) may be more useful in some cases, e.g. we use it in numerical computation. We recover the previous definition (23) if we make the additional assumption that $\chi \equiv 1$ on K_1.

Proposition 11. *By duality the transfer operator (23) extends to distributions:*

$$\hat{F}_\chi : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}) \tag{27}$$

$$\hat{F}^*_\chi : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$$

Similarly to (22) we have that for any $n \geq 1$, any $\alpha \in \mathcal{D}'(\mathbb{R})$,

$$\mathit{supp}\left(\hat{F}^n_\chi \alpha\right) \subset K_n \tag{28}$$

with K_n defined in (8).

Proof. The extension is defined by[3]

$$\hat{F}_{i,j,\chi}(\alpha_i)(\psi_j) = \alpha_i\left(\overline{\hat{F}^*_{i,j,\chi}\overline{\psi_j}}\right), \qquad \alpha_i \in \mathcal{D}'(\mathbb{R}), \psi_j \in C^\infty_0(\mathbb{R}), \tag{29}$$

[3]The complex conjugation appears in (29) because duality is related to scalar product on L^2 by $\alpha(\overline{\varphi}) := \int \overline{\varphi}\alpha = \langle \varphi, \alpha \rangle_{L^2}$.

Then the transfer operator extends to: $\hat{F}_\chi : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$.

If $\psi_j(\phi_{i,j}(y)) = 0, \forall y \in I_i$, Eq.(25) shows that $\hat{F}^*_{i,j,\chi}\psi_j \equiv 0$. More generally let $\psi \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(\psi) \cap K_n = \emptyset$ with $n \geq 1$ and K_n defined in (8). Then

$$\left(\hat{F}^*_\chi\right)^n \psi \equiv 0 \tag{30}$$

For any $\alpha \in \mathcal{D}'(\mathbb{R})$, we deduce that $\left(\hat{F}^n_\chi \alpha\right)(\overline{\psi}) = \alpha\left(\overline{\left(\hat{F}^*_\chi\right)^n \psi}\right) = 0$. By definition, this means that $\mathrm{supp}\left(\hat{F}^n_\chi \alpha\right) \subset K_n$. □

3 Main results

3.1 Discrete spectrum of Ruelle resonances

Theorem 12 below shows that the transfer operator $\hat{F}_\chi$ (for any $\hbar$) has discrete spectrum called "Ruelle resonances" in ordinary Sobolev spaces with negative order and that the spectrum does not depend on the choice of χ. Recall that for $m \in \mathbb{R}$, the **Sobolev space** $H^{-m}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ is defined by ([26] p.271).

$$H^{-m}(\mathbb{R}) := \left\langle \hat{\xi} \right\rangle^m \left(L^2(\mathbb{R})\right) \tag{31}$$

with the differential operator $\hat{\xi} := -i\frac{d}{dx}$ and the notation $\langle x \rangle := (1 + x^2)^{1/2}$.

Theorem 12. ***"Discrete spectrum of resonances".*** *For any fixed $\hbar$, any $m \in \mathbb{R}$, the transfer operator $\hat{F}_\chi$ in (27) is bounded in the Sobolev space $H^{-m}(\mathbb{R})$ and can be written*

$$\hat{F}_\chi = \hat{K} + \hat{R} \tag{32}$$

where $\hat{K}$ is a compact operator and $\hat{R}$ is such that:

$$\left\|\hat{R}\right\|_{H^{-m}(\mathbb{R})} \leq r_m, \qquad r_m := \theta^m \sqrt{N\theta e^{2V_{\max}}} \tag{33}$$

where $0 < \theta < 1$ is given in (1) and $V_{\max} = max_{x\in I}\mathrm{Re}(V(x))$. Notice that $r_m \to 0$ as $m \to +\infty$ and that the operator $\hat{F}_\chi$ has discrete spectrum on the domain $|z| > r_m$. These eigenvalues of $\hat{F}_\chi$ and their eigenspace do not depend on m nor on χ. The support of the eigendistributions is contained in the trapped set K. These discrete eigenvalues are denoted

$$\mathrm{Res}\left(\hat{F}\right) := \{\lambda_i\}_i \subset \mathbb{C}^* \tag{34}$$

*and are called **Ruelle resonances.***

In Section 4 we show the discrete spectrum of Ruelle resonances computed numerically for different examples.

3.2 Asymptotic spectral gap and Fractal Weyl law

We will give some partial description of the discrete spectrum of Ruelle resonances of the operator $\hat{F}_{\chi,\hbar}$, Eq.(27), in the limit $\hbar \to 0$. For brevity we will drop the index χ and simply write $\hat{F}_{\hbar}$. In Theorem 20 below we present a result giving an upper bound for the spectral radius of $\hat{F}_{\hbar}$ in the semiclassical limit $\hbar \to 0$. In Theorem 22 we provide an upper bound for the number of resonances outside any radius $\varepsilon > 0$ as $\hbar \to 0$. This is called "fractal Weyl law". These results rely on the study of the dynamics of a symplectic map or canonical map $F : T^*I \to T^*I$ associated to the family of operators $\left(\hat{F}_{\hbar}\right)_{\hbar}$, that we describe first.

Lemma 13. *The family of operators $\left(\hat{F}_{\hbar}\right)_{\hbar}$ restricted to $C_I^{\infty}(\mathbb{R})$ is a $\hbar$-Fourier integral operator (FIO). Its* ***canonical transform*** *is a multi-valued symplectic map $F : T^*I \to T^*I$ (with $T^*I \cong I \times \mathbb{R}$) given by:*

$$F : \begin{cases} T^*I & \to T^*I \\ (x,\xi) & \to \{F_{i,j}(x,\xi) \qquad \text{with } i,j \text{ s.t. } x \in I_i,\ i \rightsquigarrow j\} \end{cases} \tag{35}$$

with

$$F_{i,j} : \begin{cases} x' & = \phi_{i,j}(x) \\ \xi' & = \frac{1}{\phi'_{i,j}(x)}\xi + \tau'(x') \end{cases} \tag{36}$$

The proof of Lemma 13 will be given in the beginning of Section 6.

Remark 14. For short, we can write

$$F : \begin{cases} T^*I & \to T^*I \\ (x,\xi) & \to \left(\phi(x), \frac{1}{\phi'(x)}\xi + \tau'(\phi(x))\right) \end{cases} \tag{37}$$

We will study the dynamics of F in detail in later Sections, but we can already make some remarks. The term $\frac{d\tau_j}{dx}(x')$ in the expression of ξ', Eq.(36), complicates significantly the dynamics near the zero section $\xi = 0$. However the next Lemma shows that a trajectory from an initial point (x,ξ) with $|\xi|$ large enough, escape towards infinity:

Lemma 15. *For any $1 < \kappa < 1/\theta$, there exists $R \geq 0$ such that for any $|\xi| > R$ and any $i \rightsquigarrow j$,*

$$|\xi'| > \kappa |\xi| \tag{38}$$

where $(x',\xi') = F_{i,j}(x,\xi)$.

Proof. From (36), one has $\xi' = \frac{1}{\phi'_{i,j}(x)}\xi + \tau'(x')$. Also $\left|\frac{1}{\phi'_{i,j}(x)}\right| \geq \theta$ hence

$$\begin{aligned} |\xi'| - \kappa|\xi| &= \left|\frac{1}{\phi'_{i,j}(x)}\xi + \tau'(x')\right| - \kappa|\xi| \geq \left|\frac{1}{\phi'_{i,j}(x)}\xi\right| - |\tau'(x')| - \kappa|\xi| \\ &\geq \left(\frac{1}{\theta} - \kappa\right)|\xi| - \max_x |\tau'(x)| > 0, \end{aligned}$$

the last inequality holds true if $|\xi| > R := \left(\frac{1}{\theta} - \kappa\right)^{-1} \max_x |\tau'|$. $\square$

Definition 16. The **trapped set** in phase space T^*I is defined as

$$\mathcal{K} = \{(x,\xi) \in T^*I, \quad \exists C \Subset T^*I, \ \forall n \in \mathbb{Z}, \ F^n(x,\xi) \cap C \neq \emptyset\} \tag{39}$$

Remark 17. Since the map $F : T^*I \to T^*I$ is a lift of the map $\phi : I \to I$, we have $\mathcal{K} \subset (K \times \mathbb{R})$. For any R given from Lemma 15 we can precise this and obtain:

$$\mathcal{K} \subset (K \times [-R, R])$$

For $\varepsilon > 0$, let $\mathcal{K}_\varepsilon$ denote a $\varepsilon-$neighborhood of the trapped set $\mathcal{K}$, namely

$$\mathcal{K}_\varepsilon := \{(x,\xi) \in T^*I, \quad \exists (x_0,\xi_0) \in \mathcal{K}, \quad \max(|x - x_0|, |\xi - \xi_0|) \leq \varepsilon\}.$$

From now on we will make the following hypothesis on the multi-valued map F.

Assumption 18. *We assume the following property called "**minimal captivity**":*

$$\exists \varepsilon > 0, \quad \forall (x,\xi) \in \mathcal{K}_\varepsilon, \quad \sharp\left\{F(x,\xi) \bigcap \mathcal{K}_\varepsilon\right\} \leq 1. \tag{40}$$

This means that the dynamics of F is univalued on the trapped set $\mathcal{K}$.

Remark 19. In the paper [9] we introduced the property of **"partial captivity"** which is weaker than **"minimal captivity"**: partial captivity roughly states that most of trajectories escape from the trapped set $\mathcal{K}$ whereas minimal captivity states that every trajectory except one, escape from the trapped set $\mathcal{K}$.

In Section 6 we provide more details on the dynamics of the map $F : T^*I \to T^*I$, namely we provide a more precise description of the trapped set $\mathcal{K}$, a detailed symbolic coding for this dynamics and some equivalent statements to the property of minimal captivity.

For the next Theorem, let us define the function $V_0 \in C^\infty(I)$

$$V_0(x) := \frac{1}{2} J(x) \tag{41}$$

called "**potential of reference**" with $J(x)$ defined in (6). Recall from (7) that

$$V_0(x) \geq \frac{1}{2} \log\left(\frac{1}{\theta}\right) > 0.$$

Let us define the function $D \in C^\infty(I)$

$$D(x) := \mathrm{Re}(V(x)) - V_0(x) \tag{42}$$

called "**effective damping function**".

Theorem 20. ***Spectral gap in the semi-classical limit.*** *With assumption 18 of "minimal captivity" (and m sufficiently large so that $r_m \ll 1$ in (33)), the spectral radius of the operators $\hat{F}_\hbar : H^{-m}(\mathbb{R}) \to H^{-m}(\mathbb{R})$ satisfies in the semi-classical limit $\hbar \to 0$:*

$$r_s\left(\hat{F}_\hbar\right) \leq e^{\gamma_+} + o(1) \tag{43}$$

with

$$\gamma_+ := \limsup_{n\to\infty} \left(\sup_{x, w_{0,n}} \frac{1}{n} D_{w_{0,n}}(x) \right)$$

where $D_{w_{0,n}}(x) := \sum_{k=1}^{n} D\left(\phi_{w_{k,n}}(x)\right)$ is the Birkhoff average of the damping function D along a trajectory of length n, starting from the point x. Moreover the norm of the resolvent is controlled uniformly with respect to $\hbar$: for any $\rho > e^{\gamma_+}$, there exist $C_\rho > 0$, $\hbar_\rho > 0$ such that $\forall \hbar < \hbar_\rho$, $\forall |z| > \rho$ then

$$\left\| \left(z - \hat{F}_\hbar\right)^{-1} \right\|_{H^{-m}(\mathbb{R})} \leq C_\rho. \tag{44}$$

Remark

- Notice that Theorem 20 depends on the roof function τ only implicitly through assumption 18. The value of the upper bound (43) does not depend on τ.

- Eq.(44) implies (43) and is equivalent[4] to the following property that the norm $\left\| \hat{F}_\hbar^n \right\|$ is controlled uniformly with respect to $\hbar$: For any $\rho > e^{\gamma_+}$, there exist $c_\rho > 0$, $\hbar_\rho > 0$ such that $\forall \hbar < \hbar_\rho$, for any $n \in \mathbb{N}$,

$$\left\| \hat{F}_\hbar^n \right\|_{H^{-m}(\mathbb{R})} \leq c_\rho . \rho^n \tag{45}$$

  We will use (45) later to compute asymptotic of correlation functions.

Lemma 21. *If assumption 18 holds true and if the adjency matrix A is symmetric then*

$$\dim_M \mathcal{K} = 2\dim_M K = 2\dim_H K \tag{46}$$

where $\dim_M B$ *stands for the Minkowski dimension of a set B as defined in Eq.(11).*

Recall from (14) that $\dim_H K = \dim_M K$.

Theorem 22. ***"Fractal Weyl upper bound"***. *Suppose that the assumption of minimal captivity 18 holds and that the adjency matrix A is symmetric. For any $\varepsilon > 0$, any $\eta > 0$, we have for $\hbar \to 0$*

$$\sharp \left\{ \lambda_i^\hbar \in \mathrm{Res}\left(\hat{F}_\hbar \right) \mid \left| \lambda_i^\hbar \right| \geq \varepsilon \right\} = \mathcal{O}\left(\hbar^{-\dim_H(K)-\eta} \right) \tag{47}$$

3.3 Decay of correlations

In this subsection we present a quite immediate consequence of the existence of an asymptotic spectral radius e^{γ_+} obtained in Theorem 20: we obtain a finite expansion for correlation functions $\langle v | \hat{\mathcal{F}}^n u \rangle$ of the extended transfer operator $\hat{\mathcal{F}}$ defined in (17), with $u, v \in C^\infty(I \times S^1)$.

[4] Let us show the equivalence. In one sense, let $\rho_2 > \rho_1 > e^{\gamma_+}$, suppose that $\left\| \hat{F}_\hbar^n \right\|_{H^{-m}(\mathbb{R})} \leq c_{\rho_1} . \rho_1^n$. Let $|z| > \rho_2$. The relation $\left(z - \hat{F}_\hbar \right)^{-1} = z^{-1} \sum_{n \geq 0} \left(\frac{\hat{F}_\hbar}{z} \right)^n$ gives that

$$\left\| \left(z - \hat{F}_\hbar \right)^{-1} \right\| \leq |z|^{-1} \sum_{n \geq 0} \frac{\left\| \hat{F}_\hbar \right\|^n}{|z|^n} \leq |z|^{-1} c_{\rho_1} \sum_{n \geq 0} \frac{\rho_1^n}{|z|^n} = \frac{c_{\rho_1}}{|z| - \rho_1} \leq \frac{c_{\rho_1}}{\rho_2 - \rho_1} =: C_{\rho_2}$$

For the other sense, suppose that for $|z| > \rho$, $\left\| \left(z - \hat{F}_\hbar \right)^{-1} \right\| \leq C_\rho$. From the Cauchy formula $\hat{F}_\hbar^n = \frac{1}{2\pi i} \oint z^n \left(z - \hat{F}_\hbar \right)^{-1} dz$ where γ is the circle of radius ρ one deduces that $\left\| \hat{F}_\hbar^n \right\| \leq \rho C_\rho \rho^n$.

We first introduce a notation: for a given $\nu \in \mathbb{Z}$, we have seen in Theorem 12 that the transfer operator $\hat{\mathcal{F}}_\nu \equiv \hat{F}_N$ has a discrete spectrum of resonances. For $\rho > 0$ such that there is no eigenvalue on the circle $|z| = \rho$ for any $\nu \in \mathbb{Z}$, we denote by $\Pi_{\rho,\nu}$ the spectral projector of the operator $\hat{F}_\nu$ on the domain $\{z \in \mathbb{C}, |z| > \rho\}$. These projection operators are obviously finite rank and commute with $\hat{F}_\nu$.

Theorem 23. *For any $\rho > e^{\gamma_+}$, there exists $\nu_0 \in \mathbb{N}$ such that for any $u, v \in C^\infty(I \times S^1)$, in the limit $n \to \infty$,*

$$\langle v|\hat{\mathcal{F}}^n u\rangle = \sum_{|\nu|\le\nu_0} \langle v| \left(\hat{F}_\nu \Pi_{\rho,\nu}\right)^n u\rangle + O(\rho^n) \tag{48}$$

Remark 24. In the right hand side of (48) there is a finite sum and each operator $\hat{F}_\nu \Pi_{\rho,\nu}$ is finite rank. Using the spectral decomposition of $\hat{F}_\nu$ we get an expansion of the correlation function $\langle v|\hat{\mathcal{F}}^n u\rangle$ with a finite number of terms which involve the leading Ruelle resonances (i.e. those with modulus greater than ρ) plus the error term $O(\rho^n)$.

Proof. Let $\rho > e^{\gamma_+}$. Recall that $\hbar = \frac{1}{2\pi\nu}$ and that we note $\hat{F}_\hbar = \hat{F}_\nu$. In Theorem 12 we have for $\hbar \to 0$ that $r_s\left(\hat{F}_\hbar\right) \le e^{\gamma_+} + o(1)$. Let the value of ν_0 be such that $r_s\left(\hat{F}_\hbar\right) < \rho$ for every $\nu > \nu_0$. Then

$$\langle v|\hat{\mathcal{F}}^n u\rangle = \sum_{|\nu|\le\nu_0} \langle v| \left(\hat{F}_\nu \Pi_{\rho,\nu}\right)^n u\rangle + O_{\nu_0}(\rho^n) + \sum_{|\nu|>\nu_0} \langle v|\hat{F}_\nu^n u\rangle \tag{49}$$

But $\left|\langle v|\hat{F}_\nu^n u\rangle\right| \le \|u_\nu\|_{H^m} \|v_\nu\|_{H^{-m}} \left\|\hat{F}_\nu^n\right\|_{H^{-m}}$ where $u_\nu, v_\nu \in C^\infty(S^1)$ stand for the Fourier components of the smooth functions $u, v \in C^\infty(I \times S^1)$. On one hand, for smooth functions one has fast decay $\|u_\nu\|, \|v_\nu\| = O(\nu^{-\infty})$. On the other hand from (45), $\left\|\hat{F}_\nu^n\right\|_{H^{-m}} = O(\rho^n)$. So $\left|\sum_{|\nu|>\nu_0} \langle v|\hat{F}_\nu^n u\rangle\right| = O(\rho^n)$. Then (49) gives (48). ☐

We recall the following result called "Perron-Frobenius Lemma":

Lemma 25. *For real potential V, if the map ϕ^{-1} is ergodic then the transfer operator $\hat{\mathcal{F}}$ has a leading and simple eigenvalue $\lambda_0 > 0$ in the Fourier mode $\nu = 0$, i.e.*

$$\hat{\mathcal{F}}_{\nu=0} = \lambda_0 \Pi_{\lambda_0} + \hat{\mathcal{F}}'$$

with Π_{λ_0} being the rank 1 spectral projector associated to λ_0, the remainder operator has $r_s\left(\hat{\mathcal{F}}'\right) < \lambda_0$ and for any $\nu \neq 0$, we also have $r_s\left(\hat{\mathcal{F}}_\nu\right) < \lambda_0$.

For example without potential, i.e. $V = 0$, then $\lambda_0 = \exp(\Pr(-J))$ with J given in (6). As a consequence of Lemma 25 and Theorem 23 we obtain (a result already obtained by Dolgopyat [5]):

Theorem 26. *Let $u, v \in C^{\infty}(I \times S^1)$, then for $n \to \infty$,*

$$\langle v|\hat{\mathcal{F}}^n u\rangle = \lambda_0^n \langle v|\Pi_{\lambda_0} u\rangle + O\left(|\lambda_1|^n\right)$$

where λ_1 is the second eigenvalue with $|\lambda_1| < \lambda_0$. The case $V = 0$ gives that the extended map $f : I \times S^1 \to I \times S^1$ is mixing with exponential decay of correlations.

4 Numerical results for the truncated Gauss map

In this section we still consider the example of the I.F.S. defined from the truncated Gauss map presented in Section 2.2 with N intervals. For the roof function τ and the potential function V which enter in the definition of the transfer operator (20) we will choose:

$$\tau(x) = -J(x), \qquad V(x) = (1-a)J(x), \quad a \in \mathbb{R}. \tag{50}$$

where $J(x) = \log\left(\left|\left(\phi^{-1}\right)'(x)\right|\right) = \log\left(\left|G'(x)\right|\right) = \log\left(\frac{1}{x^2}\right)$ has been defined in 6 and $a \in \mathbb{R}$. Let us write for $\hbar \neq 0$,

$$s = a + ib \in \mathbb{C}, \quad a, b = \frac{1}{\hbar} \in \mathbb{R}.$$

In other words, we consider for every $s \in \mathbb{C}$ the transfer operator $\hat{L}_s := \hat{F}$ as given in (21) and written simply for $s \in \mathbb{C}$ as:

$$\hat{L}_s\varphi = \hat{F}\varphi = e^{V(x)} e^{i\frac{1}{\hbar}\tau(x)} \varphi \circ \phi^{-1} = e^{(1-s)J} \varphi \circ \phi^{-1} \tag{51}$$

As explained in Section 4.1 below, this choice is interesting due its relation with the dynamics on the modular surface. The (adjoint of the) transfer operator $\hat{F}$ constructed in this way is usually called the **Gauss-Kuzmin-Wirsing transfer operator** or "Dieter-Mayer transfer operator" for the truncated Gauss map.

Proposition 27. *For every $N \geq 1$, the minimal captivity assumption 18 holds true for the truncated Gauss transfer operator defined by (50).*

Proof. The canonical map F in (35) is the union of maps from each branch $j = 1 \dots N$ explicitly given by

$$\begin{aligned}\left(x_j', \xi_j'\right) &= F_j(x, \xi) = \left(G_j^{-1}(x), G'\left(x_j'\right)\xi + \tau'(x_j)\right) \\ &= \left(\frac{1}{x+j}, -(x+j)^2 \xi - 2(x+j)\right)\end{aligned}$$

We first observe that the trapped set $\mathcal{K}$ is included in the band $B := \left\{(x, \xi), x \in]0, 1[, -\frac{2}{1+x} < \xi < 0\right\}$ because points outside of it escape towards infinity. Now consider the sub-band

$$B_j := \left\{(x, \xi), x \in]0, 1[, -\frac{2}{j+x} < \xi < -\frac{2}{j+1+x}\right\}$$

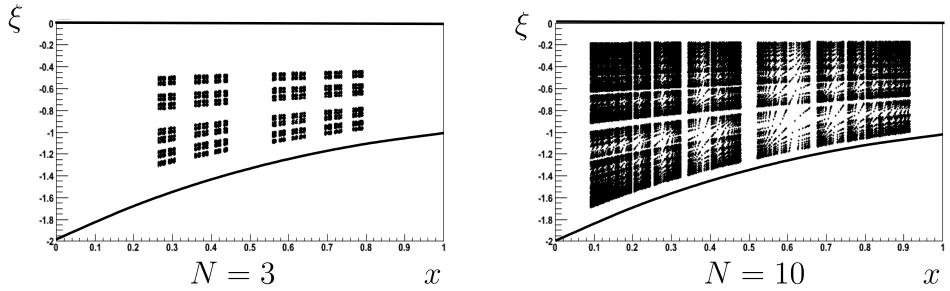

Figure 4: The trapped set $\mathcal{K}_N := \mathcal{K}$ for the truncated Gauss map with functions (50), for the cases of $N = 3$ and $N = 10$ branches. This corresponds to the Gauss-Kuzmin-Wirsing transfer operator (51). We have $\mathcal{K}_N \subset \mathcal{K}_{N+1}$ and for $N \to \infty$, the limit trapped set $\mathcal{K}_\infty = \bigcup_{N\geq 0} \mathcal{K}_N = \left\{(x,\xi), x \in]0,1[, -\frac{2}{1+x} < \xi < 0\right\}$ is the band between the marked black lines. (More precisely, we have represented the periodic points with period $n = 6$. That explains the sparse aspect of the trapped set).

We easily compute that

$$F_j(B_j) = \left\{(x,\xi), x \in]0,1[, -\frac{2}{1+x} < \xi < 0\right\}$$

So the map F^{-1} is injective on B. This implies that the map F is minimally captive. $\square$

4.1 Relation with the zeroes of the Selberg zeta function

For the geodesic flow on the modular surface $\mathrm{SL}_2\mathbb{Z}\backslash\mathrm{SL}_2\mathbb{R}$ it is possible to define the Selberg zeta function:

$$\zeta_{\mathrm{Selberg}}(s) = \prod_{\gamma}\prod_{m\geq 0}\left(1 - e^{-(s+m)|\gamma|}\right), \quad s \in \mathbb{C},$$

where the product is over periodic orbits γ of the geodesic flow and $|\gamma|$ denotes the length of the orbit. Using the Gauss map and continued fractions, C. Series has shown that a periodic orbit γ is in one to one correspondence with a periodic sequence $(w_j)_{j\in\mathbb{Z}} \in (\mathbb{N}\backslash\{0\})^{\mathbb{Z}}$ where $w_j \in \mathbb{N}\backslash\{0\}$ is the number of the branch of the Gauss map $G_{w_j}^{-1}$ in (4). Given $N \geq 1$, we can restrict the product $\prod_\gamma$ over periodic orbits for which $w_j \leq N$, $\forall j \in \mathbb{Z}$ and define a truncated Selberg zeta function:

$$\zeta_{\mathrm{Selberg},N}(s) = \prod_{\gamma,\ w_j\leq N.\forall j,}\ \prod_{m\geq 0}\left(1 - e^{-(s+m)|\gamma|}\right), \quad s \in \mathbb{C},$$

On the other hand, for fixed $s \in \mathbb{C}$, we have from Theorem 12 that the operator $\hat{L}_s$ has discrete spectrum of Ruelle resonances. It is possible to define the dynamical determinant

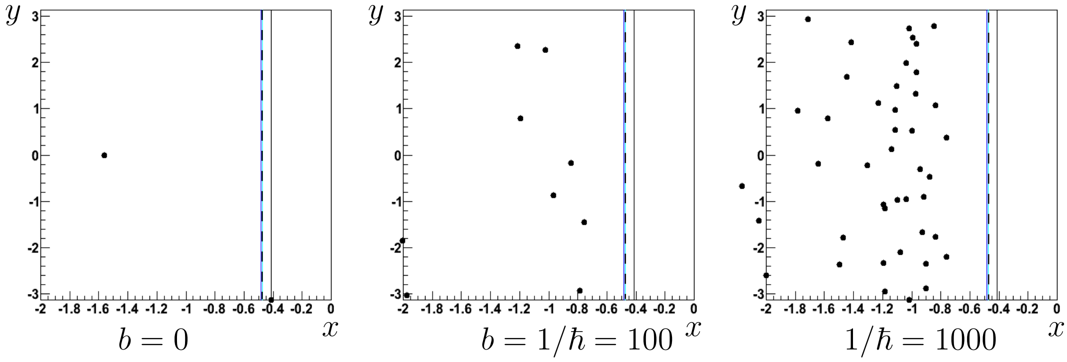

Figure 5: The discrete spectrum of Ruelle resonances λ_j (in log scale writing: $\log\lambda = x+iy$) for the truncated Gauss-Kuzmin-Wirsing transfer operator (51) associated to the Gauss map, for $N=3$ branches and parameters $a=1$, $b=0,100,1000$. For $b=0$ there is the eigenvalue $\lambda = e^{\mathrm{Pr}(-J)}$ at $x = \mathrm{Pr}\,(-J)$, $y=-\pi$ corresponding to the "equilibrium measure". The full vertical line is at $x=\mathrm{Pr}\,(-J)$. The dashed vertical line is at $x=\gamma_+$ which is shown in (43) to be an asymptotic upper bound for $b=1/\hbar\to\infty$.

of $\hat{L}_s$ by

$$d\,(z,s) := \mathrm{Det}\left(1 - z\hat{L}_s\right) := \exp\left(-\sum_{n\geq 1}\frac{z^n}{n}\mathrm{Tr}^\flat\left(\hat{L}_s^n\right)\right),\quad z\in\mathbb{C}$$

where $\mathrm{Tr}^\flat\left(\hat{L}_s^n\right)$ stands for the flat trace of Atiyah-Bott. It is known that the zeroes of $d\,(.,s)$ coincide with multiplies with the Ruelle resonances [1]. In the case $z=1$, we also have that $d\,(1,s)$ coincides with the truncated Selberg zeta function [22][2, p.306]:

$$\mathrm{Det}\left(1-\hat{L}_s\right) = \zeta_{\mathrm{Selberg},N}\,(s) \tag{52}$$

which means that the zeroes of $\zeta_{\mathrm{Selberg},N}\,(.)$ are given (with multiplicity) by the event that 1 is a Ruelle resonance of the transfer operator $\hat{L}_s$. (Remark: in [22][2, p.306] they consider the adjoint operator of $\hat{L}_s$ called the Perron-Frobenius operator).

5 Proof of Theorem 12 about the discrete spectrum

For this proof we follow closely the proof[5] of Theorem 2 in the paper [9] which uses semiclassical analysis.

[5] see also theorem 4 in [10] although we are dealing in this paper with expanding maps instead of hyperbolic maps which simplifies the analysis, since we can work with ordinary Sobolev spaces and not anisotropic Sobolev spaces.

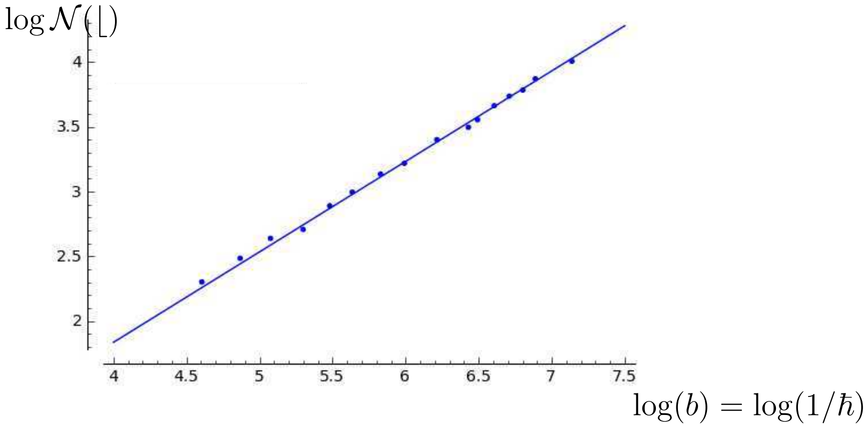

Figure 6: This is the Weyl law for the model of Gauss map with $N=3$ branches. The points represent the number of resonances $\mathcal{N}(b) = \sharp\left\{\lambda_j \in \mathrm{Res}\left(\hat{L}_s\right), |\lambda_j| > e^{-3.5}\right\}$ computed numerically, as a function of the semiclassical parameter $b = 1/\hbar$ in log scale. The linear fit gives $\log \mathcal{N}(b) = -0.70 \cdot \log b - 0.96$ which has to be compared to the fractal Weyl law (47) giving $\log \mathcal{N}(b) \leq -\dim_H(K) \cdot \log b + cste$. From (2) with have $\dim_H K_3 = 0.705$ giving an excellent agreement with the numerical results and suggesting that the upper bound is in fact optimal.

5.1 Dynamics on the cotangent space T^*I

In order to study the spectral properties of the transfer operator, we have first to study the dynamics of the map $\phi : I \to I$ lifted on the cotangent space T^*I.

Proposition 28. *Considering $\hbar > 0$ fixed, the transfer operator $\hat{F}_\chi$ restricted to $C_I^\infty(\mathbb{R})$ is a Fourier integral operator (FIO). Its* ***canonical transform*** *is a multi-valued symplectic map $\mathfrak{F} : T^*I \to T^*I$ on the cotangent space $T^*I \equiv I \times \mathbb{R}$ given by:*

$$\mathfrak{F} : \begin{cases} T^*I & \to T^*I \\ (x,\xi) & \to \{\mathfrak{F}_{i,j}(x,\xi), \quad \text{with } i,j \text{ s.t. } x \in I_i,\ i \rightsquigarrow j\} \end{cases}$$

with

$$\mathfrak{F}_{i,j} : \begin{cases} x' & = \phi_{i,j}(x) \\ \xi' & = \frac{1}{\phi'_{i,j}(x)}\xi \end{cases} \tag{53}$$

Remarks:

- For short, we can write

$$\mathfrak{F}:\begin{cases} T^*I & \to T^*I \\ (x,\xi) & \to \left(\phi\left(x\right), \frac{1}{\phi'(x)}\xi\right)\end{cases} \tag{54}$$

- Notice that the map $\mathfrak{F}$ differs from the canonical map F introduced in (37). The reason is that this latter map is used in the asymptotic limit $\hbar \to 0$, whereas the study in this Section is for fixed $\hbar$.

Proof. of Proposition 54. The operator $\varphi \to \varphi \circ \phi_{i,j}^{-1}$ is a pull back operator, one of the simplest example of Fourier integral operator in the sense of semiclassical homogeneous theory [27], see also [17] example 2 p.150. In that case the canonical map $\mathfrak{F}_{i,j}$ is the map $\phi_{i,j}$ lifted on the cotangent space T^*I in the canonical way. In particular the action on ξ is linear. The term $e^{V(x)}e^{\frac{i}{\hbar}\tau(x)}\chi\left(\phi_{i,j}^{-1}\left(x\right)\right)$ in (20) does not contribute to the expression of $\mathfrak{F}$ because it acts as a pseudodifferential operator, equivalently as a FIO whose canonical map is the identity. □

Remarks

- Observe that the dynamics of the map $\mathfrak{F}$ on T^*I has a quite simple property: the zero section $\{(x,\xi) \in I \times \mathbb{R},\ \xi = 0\}$ is globally invariant and any other point (x,ξ) with $\xi \neq 0$ escapes towards infinity ($\xi \to \pm\infty$) in a controlled manner, because $\left|\phi'_{i,j}\left(x\right)\right| < \theta < 1$, with θ given in (1), hence:

$$|\xi'| \geq \frac{1}{\theta}|\xi| \tag{55}$$

- Due to hypothesis (2) the map $\phi_{i,j}^{-1}$ is uni-valued (when it is defined). Therefore the map $\mathfrak{F}^{-1}$ is also uni-valued and one has

$$\mathfrak{F}^{-1} \circ \mathfrak{F} = \mathrm{Id}_{T^*I} \tag{56}$$

5.2 The escape function

Definition 29. [27, p.2] For $m \in \mathbb{R}$, the **class of symbols** $S^{-m}\left(T^*\mathbb{R}\right)$, with **order** m, is the set of functions on the cotangent space $A \in C^{\infty}\left(T^*\mathbb{R}\right)$ such that for any $\alpha, \beta \in \mathbb{N}$, there exists $C_{\alpha,\beta} > 0$ such that

$$\forall\left(x,\xi\right) \in T^*\mathbb{R}, \quad \left|\partial_x^{\alpha}\partial_{\xi}^{\beta}A\left(x,\xi\right)\right| \leq C_{\alpha,\beta}\left\langle\xi\right\rangle^{-m-|\beta|}, \qquad \text{with } \left\langle\xi\right\rangle = \left(1+\xi^2\right)^{1/2}. \tag{57}$$

Lemma 30. *Let $m > 0$ and let*

$$A_m(x,\xi) := \langle\xi\rangle^{-m} \quad \in S^{-m}(T^*\mathbb{R}).$$

We have

$$\forall R > 0, \forall |\xi| > R, \quad \forall i \rightsquigarrow j, \forall x \in I_i, \qquad \frac{A_m(\mathfrak{F}_{i,j}(x,\xi))}{A_m(x,\xi)} \leq C^m < 1, \tag{58}$$

*with $C = \sqrt{\frac{R^2+1}{R^2/\theta^2+1}} < 1$. We say that A_m is an **escape function**: (58) shows that A_m decreases strictly along the trajectories of $\mathfrak{F}$ outside the zero section.*

Proof. From Eq. (53) and (55) we have

$$\frac{A_m(\mathfrak{F}_{i,j}(x,\xi))}{A_m(x,\xi)} = \frac{(1+\xi^2)^{m/2}}{\left(1+(\xi')^2\right)^{m/2}} \leq \frac{(1+\xi^2)^{m/2}}{(1+\xi^2/\theta^2)^{m/2}} \leq \left(\frac{1+R^2}{1+R^2/\theta^2}\right)^{m/2} = C^m$$

The last inequality is because the function decreases with $|\xi|$. $\square$

Using the standard quantization rule [27, p.2] the symbol A_m can be quantized into a pseudodifferential operator $\hat{A}_m$ (PDO for short) which is self-adjoint and invertible on $C_0^\infty(\mathbb{R})$:

$$\left(\hat{A}_m\varphi\right)(x) = \frac{1}{2\pi}\int A_m(x,\xi)\, e^{i(x-y)\xi}\varphi(y)\, dy d\xi. \tag{59}$$

Conversely A_m is called the symbol of the PDO $\hat{A}_m$. In our simple case, this is very explicit: in Fourier space, $\hat{A}_m$ is simply the multiplication by $\langle\xi\rangle^m$. Its inverse $\hat{A}_m^{-1}$ is the multiplication by $\langle\xi\rangle^{-m}$.

5.3 Use of the Egorov Theorem

Let

$$\hat{Q}_m := \hat{A}_m \hat{F}_\chi \hat{A}_m^{-1} \quad : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

which is unitarily equivalent to $\hat{F}_\chi : H^{-m}(\mathbb{R}) \to H^{-m}(\mathbb{R})$ (from the definition of $H^{-m}(\mathbb{R})$, Eq.(31)). This is expressed by the following commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \overset{\hat{Q}_m}{\to} & L^2(\mathbb{R}) \\ \downarrow \hat{A}_m^{-1} & & \downarrow \hat{A}_m^{-1} \\ H^{-m}(\mathbb{R}) & \overset{\hat{F}_\chi}{\to} & H^{-m}(\mathbb{R}) \end{array}$$

We will therefore study the operator $\hat{Q}_m$ in $L^2(\mathbb{R})$. Notice that $\hat{Q}_m$ is defined a priori on a dense domain $C_0^\infty(\mathbb{R})$. Define

$$\hat{P} := \hat{Q}_m^* \hat{Q}_m = \hat{A}_m^{-1} \left(\hat{F}_\chi^* \hat{A}_m^2 \hat{F}_\chi \right) \hat{A}_m^{-1} = \hat{A}_m^{-1} \hat{B} \hat{A}_m^{-1} \tag{60}$$

with

$$\hat{B} := \hat{F}_\chi^* \hat{A}_m^2 \hat{F}_\chi = \hat{\chi} \hat{F}^* \hat{A}_m^2 \hat{F} \hat{\chi}. \tag{61}$$

Now, the crucial step in the proof is to use Egorov Theorem.

Lemma 31. ***(Egorov theorem)***. *$\hat{B}$ defined in (61) is a pseudo-differential operator with symbol in $S^{-2m}(T^*\mathbb{R})$ given by:*

$$B(x,\xi) = \left(\chi^2(x) \sum_{j\ s.t. i \rightsquigarrow j} \left| \phi'_{i,j}(x) \right| e^{2\mathrm{Re}(V(\phi_{i,j}(x)))} A_m^2 \left(\mathfrak{F}_{i,j}(x,\xi) \right) \right) + R \tag{62}$$

where $R \in S^{-2m-1}(T^*\mathbb{R})$ has a lower order, $x \in I_i$, $\xi \in \mathbb{R}$.

Proof. $\hat{F}$ and $\hat{F}^*$ are Fourier integral operators (FIO) whose canonical maps are respectively $\mathfrak{F}$ and $\mathfrak{F}^{-1}$. The pseudodifferential operator (PDO) $\hat{A}_m$ can also be considered as a FIO whose canonical map is the identity. By composition we deduce that $\hat{B} = \hat{\chi} \hat{F}^* \hat{A}_m^2 \hat{F} \hat{\chi}$ is a FIO whose canonical map is the identity since $\mathfrak{F}^{-1} \circ \mathfrak{F} = I$ from (56). Therefore $\hat{B}$ is a PDO. Using (20), (25) we obtain that the principal symbol of $\hat{B}$ is the first term of (62). ☐

Remark: contrary to (61), $\hat{F} \hat{A}_m \hat{F}^*$ is not a PDO, but a FIO whose canonical map $\mathfrak{F} \circ \mathfrak{F}^{-1}$ is multivalued.

Now by **theorem of composition of PDO** [27, p.11], Eq.(60) and Eq.(62) imply that $\hat{P}$ is a PDO with symbol in $S^0(\mathbb{R})$ and principal symbol:

$$P(x,\xi) = \frac{B(x,\xi)}{A_m^2(x,\xi)} = \left(\chi^2(x) \sum_{j \text{ s.t. } i \rightsquigarrow j} \left| \phi'_{i,j}(x) \right| e^{2\mathrm{Re}(V(\phi_{i,j}(x)))} \frac{A_m^2 \left(\mathfrak{F}_{i,j}(x,\xi) \right)}{A_m^2(x,\xi)} \right), x \in I_i, \xi \in \mathbb{R}. \tag{63}$$

The estimate (58) gives the following upper bound for any $R > 0$, $x \in I$ and $|\xi| > R$:

$$|P(x,\xi)| \leq \chi^2(x) C^{2m} \sum_{j, i \rightsquigarrow j} \left| \phi'_{i,j}(x) \right| e^{2\mathrm{Re}(V(\phi_{i,j}(x)))} \leq C^{2m} N \theta e^{2V_{\max}}$$

with $V_{\max} = \max_{x \in I} \mathrm{Re}(V(x))$.

We apply[6] the L^2**-continuity theorem for PDO** to $\hat{P}$ as given in [12, th 4.5 p.42]. The result is that for any $\varepsilon > 0$,

$$\hat{P} = \hat{k}_\varepsilon + \hat{p}_\varepsilon$$

with $\hat{k}_\varepsilon$ a smoothing operator (hence compact) and $\|\hat{p}_\varepsilon\| \leq C^{2m} N\theta e^{2V_{\max}} + \varepsilon$.

If $\hat{Q}_m = \hat{U} \left|\hat{Q}_m\right|$ is the polar decomposition of $\hat{Q}_m$, with $\hat{U}$ unitary, then from (60), $\hat{P} = \left|\hat{Q}_m\right|^2$, hence $\left|\hat{Q}_m\right| = \sqrt{\hat{P}}$ and the spectral theorem ([27] p.75) gives that $\left|\hat{Q}_m\right|$ has a similar decomposition

$$\left|\hat{Q}_m\right| = \hat{k}'_\varepsilon + \hat{q}_\varepsilon$$

with $\hat{k}'_\varepsilon$ compact and $\|\hat{q}_\varepsilon\| \leq \sqrt{C^{2m} N\theta e^{2V_{\max}}} + \varepsilon$, with any $\varepsilon > 0$. Since $\left\|\hat{U}\right\| = 1$ we deduce a similar decomposition for $\hat{Q}_m = \hat{U} \left|\hat{Q}_m\right| : L^2(I) \to L^2(I)$. We also use the fact that $C \to \theta$ as $R \to \infty$ in (58) and we deduce (32) and (33) for $\hat{F}_\chi : H^{-m}(\mathbb{R}) \to H^{-m}(\mathbb{R})$.

The fact that the eigenvalues λ_i and their generalized eigenspaces do not depend on the choice of space $H^{-m}(\mathbb{R})$ is due to density of $C_0^\infty(\mathbb{R})$ in Sobolev spaces. We refer to the argument given in the proof of corollary 1 in [10].

Finally, if φ is an eigendistribution of $\hat{F}_\chi$, i.e. $\hat{F}_\chi \varphi = \lambda\varphi$ with $\lambda \neq 0$, we deduce that $\varphi = \frac{1}{\lambda^n} \hat{F}_\chi^n \varphi$ for any $n \geq 1$, and (28) implies that $\operatorname{supp}(\varphi) \subset K = \bigcap_{n\in\mathbb{N}} K_n$. On the trapped set we have $\chi = 1$ hence the eigendistribution and eigenvalues of $\hat{F}_\chi$ do not depend on χ. This finishes the proof of Theorem 12.

6 Dynamics of the canonical map $F : T^*I \to T^*I$

The map F appears in Proposition 13. In this Section we study its trapped set and its symbolic dynamics. Before we give:

Proof. of Proposition 13. This is the same argument as in the proof of Proposition 28 except that now the family of operators $\left(e^{\frac{i}{\hbar}\tau(x)}\right)_{\hbar>0}$ which appears in (20) is a FIO. As explained in [17] example 1 p.150, its canonical map is $(x,\xi) \to \left(x, \xi + \frac{d\tau}{dx}\right)$. We compose with the previous canonical map (53) to get (36). □

6.1 The trapped set $\mathcal{K}$ in phase space

We have provided a definition of the trapped set $\mathcal{K}$ in (39). We will give now a more precise description of it. Recall that the inverse maps ϕ^{-1} and F^{-1} are uni-valued. For any integer

[6]Actually, we can not apply directly the L^2-continuity theorem for PDO to $\hat{P}$ because $\hat{P}$ doesn't have a compactly supported Schwartz kernel. However $\hat{B}$ obviously has a compactly supported Schwartz kernel due to the presence of $\hat{\chi}$ in Eq.(61). The trick is to approximate $\hat{A}_m^{-1}$ by a properly supported operator Λ_m as it is done in [12, p.45] and then apply the L^2-continuity theorem to $\hat{\Lambda}_m \hat{B} \hat{\Lambda}_m$.

$m \geq 0$, let

$$\tilde{K}_m := F^{-m}\left(K_m \times [-R, R]\right)$$

where $K_m = \phi^m(I)$ has been defined in (8) and R is given by Lemma 15. In particular $\tilde{K}_0 = I \times [-R, R]$. Let $\pi : (x, \xi) \in T^*I \to x \in I$ be the projection map. These sets have the following properties:

$$\pi\left(\tilde{K}_m\right) = I,$$

$$\tilde{K}_{m+1} \subset \tilde{K}_m \tag{64}$$

Proof. of (64). From Lemma 15 we have

$$\left(K_{m+1} \times [-R, R]\right) \subset F\left(K_m \times [-R, R]\right)$$

hence

$$\tilde{K}_{m+1} = F^{-m}\left(F^{-1}\left(K_{m+1} \times [-R, R]\right)\right) \subset F^{-m}\left(K_m \times [-R, R]\right) = \tilde{K}_m$$

□

Let us define

$$\tilde{K} := \bigcap_m \tilde{K}_m \tag{65}$$

Now we combine the sets K_n defined in (8) with the sets $\tilde{K}_m$ and define for any integers $a, b \geq 0$

$$\mathcal{K}_{a,b} := \pi^{-1}\left(K_a\right) \bigcap \tilde{K}_b \tag{66}$$

We have

$$\mathcal{K}_{a+1,b} \subset \mathcal{K}_{a,b}, \quad \mathcal{K}_{a,b+1} \subset \mathcal{K}_{a,b} \tag{67}$$

and

$$F^{-1}\left(\mathcal{K}_{a,b}\right) = \mathcal{K}_{a-1,b+1}. \tag{68}$$

Remark 32. We can interpret the trapped set $K \subset I$ with respect to the lifted map $F : T^*I \to T^*I$, as follows. The trapped set $\pi^{-1}(K) \subset T^*I$ is characterized by

$$\pi^{-1}(K) = \left\{(x, \xi) \in T^*I, \quad \exists \text{compact } C \Subset T^*I, \forall n \geq 0, F^{-n}(x, \xi) \in C\right\}$$

i.e. $\pi^{-1}(K)$ can be considered as the "trapped set of the map F in the past". Similarly $\tilde{K} \subset T^*I$ can be interpreted as the "trapped set of the map F in the future"and $\mathcal{K} \subset T^*I$ as the full trapped set (past and future) since they are characterized by

$$\tilde{K} = \{(x, \xi) \in T^*I, \quad \exists \text{compact } C \Subset T^*I, \forall n \geq 0, F^n(x, \xi) \cap C \neq \emptyset\}$$

$$\begin{aligned} \mathcal{K} &= \{(x, \xi) \in T^*I, \quad \exists \text{compact } C \Subset T^*I, \forall n \in \mathbb{Z}, F^n(x, \xi) \cap C \neq \emptyset\} \\ &= \pi^{-1}(K) \cap \tilde{K} \end{aligned} \tag{69}$$

From this previous remark, the next definition is equivalent to (39).

Definition 33. The **trapped set** $\mathcal{K} \subset T^*I$ of the map F is

$$\mathcal{K} := \bigcap_{a=0}^{\infty} \mathcal{K}_{a,a} \tag{70}$$

The hypothesis of minimal captivity has been defined in (18). The following proposition gives equivalent, stronger and weaker definition of minimal captivity. They are convenient for practical purposes.

Proposition 34.

1. *The map F is minimally captive (i.e. Eq.(40) holds true) if and only if the map F satisfies*

$$\exists a, \quad \forall (x,\xi) \in \mathcal{K}_{a,a}, \quad \sharp \left\{ F(x,\xi) \bigcap \mathcal{K}_{a,a} \right\} \leq 1. \tag{71}$$

2. *If map F is minimally captive then*

$$\exists a, \exists C, \forall n \text{ s.t. } \forall (x,\xi) \in \mathcal{K}_{a,0}, \quad \sharp \left\{ F^n(x,\xi) \bigcap \mathcal{K}_{a,0} \right\} \leq C. \tag{72}$$

   where $\mathcal{K}_{a,0} := \left(\pi^{-1}(K_a) \bigcap [-R,R]\right)$ has been defined in (18).

3. *If there exists two smooth functions $u_1, u_2 : I \to \mathbb{R}$ with $u_1 < u_2$ such that for $U := \{(x,\xi), u_1(x) < \xi < u_2(x)\}$ one has $\mathcal{K} \subset U$ and*

$$\forall (x,\xi) \in U, \quad \sharp \{F(x,\xi) \cap U\} \leq 1.$$

   then the map F is minimally captive.

Proof. The fact that (71) is equivalent to (40) is because

$$\forall \varepsilon > 0, \exists a \quad \text{s.t. } \mathcal{K}_{a,a} \subset \mathcal{K}_{\varepsilon}$$
$$\forall a, \exists \varepsilon > 0 \text{ s.t. } \mathcal{K}_{\varepsilon} \subset \mathcal{K}_{a,a}$$

□

6.2 Symbolic dynamics

The purpose of this Section is to describe precisely the dynamics of ϕ and F using "symbolic dynamics". This is very standard for expanding maps [3]. This somehow refines the structure of the sets $\mathcal{K}_{a,b}$ introduced before.

6.2.1 Symbolic dynamics on the trapped set $K \subset I$

Let

$$\mathcal{W}_- := \left\{ (\ldots, w_{-2}, w_{-1}, w_0) \in \{1, \ldots, N\}^{-\mathbb{N}}, w_{l-1} \rightsquigarrow w_l, \forall l \leq 0 \right\} \tag{73}$$

be the set of **admissible left semi-infinite sequences**. For $w \in \mathcal{W}_-$ and $i < j$ we write $w_{i,j} := (w_i, w_{i+1}, \ldots w_j)$ for an extracted sequence. For simplicity we will use the notation

$$\phi_{w_{i,j}} := \phi_{w_{j-1},w_j} \circ \ldots \circ \phi_{w_i,w_{i+1}} : I_{w_i} \to I_{w_j} \tag{74}$$

for the composition of maps. For $n \geq 0$, let

$$I_{w_{-n,0}} := \phi_{w_{-n,0}} \left(I_{w_{-n}} \right) \subset I_{w_0} \tag{75}$$

For any $0 < m < n$ we have the strict inclusions

$$I_{w_{-n,0}} \subset I_{w_{-m,0}} \subset I_{w_0}$$

From (1), the size of $I_{w_{-n,0}}$ is bounded by:

$$\left| I_{w_{-n,0}} \right| \leq \theta^n \left| I_{w_0} \right|$$

hence the sequence of sets $\left(I_{w_{-n,0}} \right)_{n \geq 1}$ is a sequence of non empty and decreasing closed intervals and $\bigcap_{n=1}^{\infty} I_{w_{-n,0}}$ is a point in K. We define

Definition 35. The "**symbolic coding map**" is

$$S : \begin{cases} \mathcal{W}_- & \to K \\ w & \to S(w) := \bigcap_{n=1}^{\infty} I_{w_{-n,0}} \end{cases} \tag{76}$$

In some sense we have decomposed the sets K_n, Eq.(8), into individual components:

$$K_n = \bigcup_{w_{-n,0} \in \mathcal{W}_-} I_{w_{-n,0}} \tag{77}$$

$$K = \bigcup_{w \in \mathcal{W}_-} S(w)$$

Let us introduce the **left shift,** a multivalued map, defined by

$$L : \begin{cases} \mathcal{W}_- & \to \mathcal{W}_- \\ (\ldots, w_{-2}, w_{-1}, w_0) & \to (\ldots, w_{-2}, w_{-1}, w_0, w_1) \end{cases}$$

with $w_1 \in \{1,\ldots,N\}$ such that $w_0 \rightsquigarrow w_1$. Let the **right shift** be the univalued map defined by

$$R:\begin{cases}\mathcal{W}_- & \to \mathcal{W}_-\\ (\ldots,w_{-2},w_{-1},w_0) & \to (\ldots,w_{-2},w_{-1})\end{cases}$$

Proposition 36. *The following diagram is commutative*

$$\begin{array}{ccc}\mathcal{W}_- & \xrightarrow{S} & K\\ R\uparrow\!\downarrow L & & \phi^{-1}\uparrow\!\downarrow\phi\\ \mathcal{W}_- & \xrightarrow{S} & K\end{array} \tag{78}$$

and the map $S:\mathcal{W}_-\to K$ is one to one. This means that the dynamics of points on the trapped set K under the maps ϕ^{-1},ϕ is equivalent to the symbolic dynamics of the shift maps R,L on the set of admissible words $\mathcal{W}_-$. Notice that the maps R and ϕ^{-1} are univalued, whereas the maps L and ϕ are multivalued (in general).

Proof. From the definition of S we have

$$\phi_{w_0w_1}\left(S\left(\ldots,w_{-2},w_{-1},w_0\right)\right)=S\left(\ldots,w_{-2},w_{-1},w_0,w_1\right) \tag{79}$$

and

$$\phi^{-1}_{w_{-1}w_0}\left(S\left(\ldots,w_{-2},w_{-1},w_0\right)\right)=S\left(\ldots,w_{-2},w_{-1}\right) \tag{80}$$

which gives the diagram (78). The map $S:\mathcal{W}_-\to K$ is surjective by construction. Let us show that the hypothesis (2) implies that it is also injective. Let $w,w'\in\mathcal{W}_-$ and suppose that $w\neq w'$, i.e. there exists $k\geq 0$ such that $w_{-k}\neq w'_{-k}$. From (2) we have $\phi_{w_{-k},w_{-k+1}},\left(I_{w_{-k}}\right)\cap\phi_{w'_{-k},w'_{-k+1}}\left(I_{w'_{-k}}\right)=\emptyset$. We deduce recursively that $\phi_{w_{-k,0}}\left(I_{w_{-k}}\right)\cap\phi_{w'_{-k,0}}\left(I_{w'_{-k}}\right)=\emptyset$. Since $S\left(w\right)\in\phi_{w_{-k,0}}\left(I_{w_{-k}}\right)$ and $S\left(w'\right)\in\phi_{w'_{-k,0}}\left(I_{w'_{-k}}\right)$ we deduce that $S\left(w\right)\neq S\left(w'\right)$. So S is one to one. $\square$

6.2.2 The "future trapped set" $\tilde{K}$ in phase space T^*I

Let us consider

$$\mathcal{W}_+:=\left\{(w_0,w_1,w_2\ldots)\in\{1,\ldots,N\}^{\mathbb{N}},\quad w_l\rightsquigarrow w_{l+1},\forall l\geq 0\right\}$$

be the set of admissible right semi-infinite sequences. We still use the notation $w_{i,j}:=(w_i,w_{i+1},\ldots w_j)$ for an extracted sequence. For any $n\geq 0$ let:

$$\tilde{I}_{w_{0,n}}:=F^{-n}\left(I_{w_{0,n}}\times[-R,R]\right) \tag{81}$$

be the image of the rectangle under the univalued map F^{-n}. Notice that $\pi\left(\tilde{I}_{w_{0,n}}\right)=I_{w_0}$ where $\pi(x,\xi)=x$ is the canonical projection map. Since the map F^{-1} contracts strictly in variable ξ by the factor $\theta<1$ then $\left(\tilde{I}_{w_{0,n}}\right)_{n\in\mathbb{N}}$ is a sequence of decreasing sets: $\tilde{I}_{w_{0,n+1}}\subset\tilde{I}_{w_{0,n}}$ and we can define the limit

$$\tilde{S}: w\in\mathcal{W}_+\to\tilde{S}(w):=\bigcap_{n\geq0}\tilde{I}_{w_{0,n}}\subset\tilde{K} \tag{82}$$

Proposition 37. *For every $w\in\mathcal{W}_+$, the set $\tilde{S}(w)$ is a smooth curve given by*

$$\tilde{S}(w)=\{(x,\zeta_w(x)),\quad x\in I_{w_0}, w\in\mathcal{W}_+\}$$

with

$$\zeta_w(x)=-\sum_{k\geq1}\phi'_{w_{0,k}}(x)\cdot\tau'\left(\phi_{w_{0,k}}(x)\right), \tag{83}$$

We have an estimate of regularity, uniform in w: $\forall\alpha\in\mathbb{N}$, $\exists C_\alpha>0$, $\forall w\in\mathcal{W}_+$, $\forall x\in I_{w_0}$,

$$|(\partial_x^\alpha\zeta_w)(x)|\leq C_\alpha \tag{84}$$

Moreover, with the hypothesis 18 of minimal captivity there exists $a\geq1$ such that these branches do not intersect on $\pi^{-1}(K_a)$,

$$\forall w,w'\in\mathcal{W}_+,\quad w\neq w'\Rightarrow\pi^{-1}(K_a)\cap\tilde{S}(w)\cap\tilde{S}(w')=\emptyset \tag{85}$$

The set (65) can be expressed as

$$\tilde{K}=\bigcup_{w\in\mathcal{W}_+}\tilde{S}(w)$$

Proof. From (36) we get

$$F^{-1}(\phi_{i,j}(x),\xi)=\left(x,\phi'_{i,j}(x)(\xi-\tau'(\phi_{i,j}(x)))\right). \tag{86}$$

Iterating this equation we get, that

$$\zeta_{w,n}(x):=-\sum_{k=1}^{n}\phi'_{w_{0,k}}(x)\cdot\tau'\left(\phi_{w_{0,k}}(x)\right)=F^{-n}\left(\phi_{w_{0,n}}(x),0\right)$$

thus $\zeta_{w,n}(x)\in\tilde{S}(w)$ for all $n\in\mathbb{N}$ and we get (83). In order to prove (84) we can check, that the series of $\zeta_{w,n}(x)$ and $\partial_x^\alpha\zeta_{w,n}(x)$ converge with uniform bounds in w which follows after some calculations from (1) and the fact that $\phi'_{w_{0,k}}(x)\leq\theta^k$ independent of w. ☐

6.2.3 Symbolic dynamics on the trapped set $\mathcal{K}$ in phase space T^*I

Recall from (69) that $\mathcal{K} = \pi^{-1}(K) \cap \tilde{K}$. Let

$$\mathcal{W} := \left\{ (\dots w_{-2}, w_{-1}, w_0, w_1, \dots) \in \{1, \dots, N\}^{\mathbb{Z}}, \quad w_l \rightsquigarrow w_{l+1}, \forall l \in \mathbb{Z} \right\}$$

be infinite admissible sequences. For a given $w \in \mathcal{W}$ and$a, b \in \mathbb{N}$, let

$$\mathcal{I}_{w_{-a,0}, w_{0,b}} := \left(\pi^{-1}\left(I_{w_{-a,0}}\right) \cap \tilde{I}_{w_{0,b}} \right) \quad \subset \mathcal{K}_{a,b}$$

where $\mathcal{K}_{a,b}$ has been defined in (66).

Definition 38. The **symbolic coding map** is

$$\mathcal{S} : \begin{cases} \mathcal{W} & \rightarrow \mathcal{K} \\ w & \rightarrow \mathcal{S}(w) := \bigcap_{n=1}^{\infty} \mathcal{I}_{w_{-n,0}, w_{0,n}} = \left(\pi^{-1}\left(S(w_-)\right) \cap \tilde{S}(w_+) \right) \end{cases} \tag{87}$$

with $w_- = (\dots w_{-1}, w_0) \in \mathcal{W}_-$, $w_+ = (w_0, w_1, \dots) \in \mathcal{W}_+$.

More precisely we can express the point $\mathcal{S}(w) \in \mathcal{K}$ as

$$\mathcal{S}(w) = \left(x_{w_-}, \xi_w\right), \quad x_{w_-} = S(w_-), \quad \xi_w = \zeta_{w_+}\left(S(w_-)\right), \tag{88}$$

withζ_{w_+} given in (83). We also have

$$\mathcal{K}_{a,b} = \bigcup_{w \in \mathcal{W}} \mathcal{I}_{w_{-a,0}, w_{0,b}}$$

Proposition 39. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\mathcal{S}} & \mathcal{K} \\ R \uparrow\downarrow L & & F^{-1} \uparrow\downarrow F \\ \mathcal{W} & \xrightarrow{\mathcal{S}} & \mathcal{K} \end{array} \tag{89}$$

If assumption 18 of minimal captivity holds true then the map $\mathcal{S} : \mathcal{W} \rightarrow \mathcal{K}$ is one to one. This means that the univalued dynamics of points on the trapped set $\mathcal{K}$ under the maps F^{-1}, F is equivalent to the symbolic dynamics of the full shift maps R, L on the set of words $\mathcal{W}$.

Proof. Commutativity of the diagram comes from the construction of $\mathcal{S}$. Also $\mathcal{S}$ is surjective. Let us show that $\mathcal{S}$ is injective. Let $w, w' \in \mathcal{W}$, with $w \neq w'$. There exists $n \geq 0$ such that $(L^n(w))_- \neq (L^n(w'))_-$. So $S\left((L^n(w))_-\right) \neq S\left((L^n(w'))_-\right)$ because $S : \mathcal{W}_- \to K$ is one to one from Lemma 36. Hence $\mathcal{S}(L^n(w)) \neq \mathcal{S}(L^n(w'))$ and $F^n(\mathcal{S}(w)) \neq F^n(\mathcal{S}(w'))$ from commutativity of the diagram. We apply F^{-n} and deduce that $\mathcal{S}(w) \neq \mathcal{S}(w')$ because F^{-1} and F^{-n} are injective on $\mathcal{K}$ from assumption 18. $\square$

6.3 Dimension of the trapped set $\mathcal{K}$

6.3.1 Proof of Theorem 21

For $w = (w_k)_{k\in\mathbb{Z}} \in \mathcal{W}$, we note $w_- = (\ldots, w_{-2}, w_{-1}, w_0) \in \mathcal{W}_-$ and $w_+ = (w_0, w_1, \ldots) \in \mathcal{W}_+$. Let

$$\mathrm{Inv}(w_+) := (\ldots w_2, w_1, w_0)$$

be the reversed word. Since the adjency matrix A is supposed to be symmetric we have that $\mathrm{Inv}(w_+) \in \mathcal{W}_-$. Then, let us consider the following one to one map

$$D : \begin{cases} \mathcal{W} & \to (\mathcal{W}_- \times \mathcal{W}_-)_l \\ w & \to (w_-, \mathrm{Inv}(w_+)) \end{cases}$$

where

$$(\mathcal{W}_- \times \mathcal{W}_-)_l := \{(w, w') \in \mathcal{W}_- \times \mathcal{W}_-, \quad w_0 = w_0'\} \tag{90}$$

is a subset of $\mathcal{W}_- \times \mathcal{W}_-$. The index l stands for "linked". Let

$$\Phi := (S \otimes S) \circ D \circ \mathcal{S}^{-1} \quad : \mathcal{K} \to K \times K$$

where $\mathcal{S} : \mathcal{W} \to \mathcal{K}$ has been defined in (87) and is shown in Proposition 39 to be one to one under assumption 18. The map $S : \mathcal{W}_+ \to K$ has been defined in (76) and is also one to one. Consider

$$(K \times K)_l := (S \otimes S)\left((\mathcal{W}_- \times \mathcal{W}_-)_l\right) \quad \subset K \times K \tag{91}$$

the image of (90) under the map $S \otimes S$. From the previous remarks, the map $\Phi : \mathcal{K} \to (K \times K)_l$ is one to one.

Lemma 40. *The map $\Phi : \mathcal{K} \to (K \times K)_l$ is bi-Lipchitz.*

As a consequence of this Lemma, since the Hausdorff and Minkowski dimension is invariant under bi-Lipchitz maps [7, p.24], we deduce that

$$\dim_M(\mathcal{K}) = \dim_M (K \times K)_l \tag{92}$$

Before proving Lemma 40, let us show how to deduce Theorem 21 from it. Let us temporally write $K_i := K \cap I_i$. From (91) we have that

$$(K \times K)_l = \bigcup_i K_i \times K_i$$

hence

$$\dim_M (K \times K)_l == \sup_i (2 \dim_M K_i) = 2 \dim_M K \tag{93}$$

Eq.(92) and (93) give Theorem 21.

Proof. of Lemma 40. Let $w \in \mathcal{W}$. We write $w = (w_-, w_+)$ as before and $x_{w_-} := S(w_-) \in K$, $\rho = (x_{w_-}, \xi_w) = \mathcal{S}(w) \in \mathcal{K}$. Similarly for another $w' \in \mathcal{W}$ we get another point $\rho' = \left(x_{w'_-}, \xi_{w'}\right) \in \mathcal{K}$. We have that

$$\Phi(\rho) = (S(w_-), S(\mathrm{Inv}(w_+))) = \left(x_{w_-}, x_{\mathrm{Inv}(w_+)}\right) \in K \times K.$$

That the map Φ is bi-Lipchitz means that

$$|\Phi(\rho) - \Phi(\rho')| \asymp |\rho - \rho'|$$

uniformly[7] over ρ, ρ'. Equivalently this is

$$\left|x_{w_-} - x_{w'_-}\right| + \left|x_{\mathrm{Inv}(w_+)} - x_{\mathrm{Inv}\left(w'_+\right)}\right| \asymp \left|x_{w_-} - x_{w'_-}\right| + |\xi_w - \xi_{w'}| \tag{94}$$

uniformly over $w, w' \in \mathcal{W}$. Let us show (94). Let $w, w' \in \mathcal{W}$ and let $n \geq 0$ be the integer such that that $(w_+)_j = \left(w'_+\right)_j$ for $-n \leq j \leq 0$ but $(w_+)_{-n-1} \neq \left(w'_+\right)_{-n-1}$. From the definition (75) of the intervals $I_{w_{-n,0}}$, we see that the two points $x_{\mathrm{Inv}(w_+)}, x_{\mathrm{Inv}\left(w'_+\right)}$ belong both to the interval $I_{(\mathrm{Inv}(w_+))_{-n,0}}$ but inside it, they belong to the disjoint sub-intervals $I_{(\mathrm{Inv}(w_+))_{-n-1,0}}$ and $I_{\left(\mathrm{Inv}\left(w'_+\right)\right)_{-n-1,0}}$ respectively. Hence

$$\left|x_{\mathrm{Inv}(w_+)} - x_{\mathrm{Inv}\left(w'_+\right)}\right| \asymp \left|I_{(\mathrm{Inv}(w_+))_{-n,0}}\right|$$

uniformly over $w, w' \in \mathcal{W}$, where $|I|$ is the length of the interval I. From the definition (81) of the sets $\tilde{I}_{w_{0,n}}$ we observe that the points $\rho = (x_{w_-}, \xi_w)$ and $\rho' = \left(x_{w'_-}, \xi_{w'}\right)$ belong respectively to the sets $\tilde{I}_{w_{0,n}}$ and $\tilde{I}_{w'_{0,n}}$. Let $\tilde{w}' := \left(w'_-, w_+\right)$ .We have

$$\begin{aligned} |\rho - \rho'| &= \left|(x_{w_-}, \xi_w) - \left(x_{w'_-}, \xi_{w'}\right)\right| && (95) \\ &\asymp \left|(x_{w_-}, \xi_w) - (x_{w_-}, \xi_{\tilde{w}'})\right| + \left|(x_{w_-}, \xi_{\tilde{w}'}) - \left(x_{w'_-}, \xi_{w'}\right)\right| && (96) \\ &\asymp \left|x_{w_-} - x_{w'_-}\right| + |\xi_w - \xi_{\tilde{w}'}| \end{aligned}$$

[7]The notation $|\Phi(\rho) - \Phi(\rho')| \asymp |\rho - \rho'|$ means precisely that there exist $C > 0$ such that for every ρ, ρ', $C^{-1} |\rho - \rho'| \leq |\Phi(\rho) - \Phi(\rho')| \leq C |\rho - \rho'|$.

The points $\xi_w, \xi_{\tilde{w}'}$ belong to the same set $\tilde{I}_{w_{0,n}}$. However if assumption of "minimal captivity" holds, they belong to disjoint sub-sets $\tilde{I}_{w_{0,n+1}}$ and $\tilde{I}_{w'_{0,n+1}}$ respectively. Hence

$$|\xi_w - \xi_{\tilde{w}'}| \asymp |J_{w,n}| \tag{97}$$

with the interval $J_{w,n} := \tilde{I}_{w_{0,n}} \cap \pi^{-1}\left(x_{w_-}\right)$. From the bounded distortion principle [7] we have that

$$\forall x, y \in I_{w_{-n,0}}, \quad \left|\left(D\phi_{w_{-n,0}}\right)(x)\right| \asymp \left|\left(D\phi_{w_{-n,0}}\right)(y)\right| \asymp \left|I_{w_{-n,0}}\right|$$

uniformly with respect to w, n, x, y. From the expression of the canonical map F in (36) and the bounded distortion principle, we have that

$$|J_{w,n}| \asymp \left|\left(D\phi_{w_{-n,0}}\right)(x)\right|, \quad \forall x \in I_{w_0},$$

uniformly with respect to w, n, x. Using the previous results we get

$$\begin{aligned}
\left|x_{w_-} - x_{w'_-}\right| + |\xi_w - \xi_{w'}| &\asymp \left|x_{w_-} - x_{w'_-}\right| + |\xi_w - \xi_{\tilde{w}'}| \\
&\asymp \left|x_{w_-} - x_{w'_-}\right| + |J_{w,n}| \\
&\asymp \left|x_{w_-} - x_{w'_-}\right| + \left(D\phi_{w_{0,n}}\right)(x), \quad \forall x \in I_{w_0}, \\
&\asymp \left|x_{w_-} - x_{w'_-}\right| + \left|I_{\mathrm{Inv}(w_{0,n})}\right|, \\
&\asymp \left|x_{w_-} - x_{w'_-}\right| + \left|x_{\mathrm{Inv}(w_+)} - x_{\mathrm{Inv}\left(w'_+\right)}\right|.
\end{aligned}$$

We have obtained (94) and finished the proof of Lemma 40 and Theorem 21. □

7 Proof of Theorem 20 for the spectral gap in the semi-classical limit

For the proof of Theorem 20, we will follow step by step the same analysis as in Section 5 (and also follow closely the proof of Theorem 2 in [9]). The main difference now is that $\hbar \ll 1$ is a semi-classical parameter (no more fixed). In other words, we just perform a linear rescaling in cotangent space: $\xi_h := \hbar\xi$. Our quantization rule for a symbol $A(x, \xi_h) \in S^{-m}(\mathbb{R})$, Eq.(59) writes now (see [17] p.22), for $\varphi \in \mathcal{S}(\mathbb{R})$:

$$\left(\hat{A}\varphi\right)(x) := \frac{1}{2\pi\hbar}\int A(x, \xi_h)\, e^{i(x-y)\xi_h/\hbar}\varphi(y)\, dy d\xi_h \tag{98}$$

For simplicity we will still write ξ instead of ξ_h below.

7.1 The escape function

Let $1 < \kappa < 1/\theta$ and $R > 0$ given in Lemma 15. Let $m > 0$, $\eta > 0$ (small) and consider a C^∞ function $A_m(x,\xi)$ on $T^*\mathbb{R}$ so that:

$$\begin{aligned} A_m(x,\xi) &:= \langle\xi\rangle^{-m} \qquad \text{for } |\xi| > R+\eta \\ &:= 1 \qquad \text{for } \xi \le R \end{aligned}$$

where $\langle\xi\rangle := (1+\xi^2)^{1/2}$. A_m belongs to the symbol class $S^{-m}(\mathbb{R})$ defined in (57).

From Eq. (38) we can deduce, similarly to Eq.(58) and if η is small enough, that:

$$\forall x \in I, \forall |\xi| > R, \forall i \rightsquigarrow j \qquad \frac{A_m(F_{i,j}(x,\xi))}{A_m(x,\xi)} \le C^m < 1, \quad \text{with } C = \sqrt{\frac{R^2+1}{\kappa^2R^2+1}} < 1 \tag{99}$$

This means that the function A_m is an **escape function** since it decreases strictly along the trajectories of F outside the zone $\mathcal{Z}_0 := I \times [-R,R]$. For any point $(x,\xi) \in T^*I$ we have the more general bound:

$$\forall x \in I, \forall \xi \in \mathbb{R}, \forall i \rightsquigarrow j \qquad \frac{A_m(F_{i,j}(x,\xi))}{A_m(x,\xi)} \le 1. \tag{100}$$

Let $\hbar > 0$. Using the quantization rule (98), the symbol A_m can be quantized giving a $\hbar$-pseudodifferential operator $\hat{A}_m$ which is self-adjoint and invertible on $C^\infty(I)$. In our case $\hat{A}_m$ is simply a multiplication operator by $A_m(\xi)$ in $\hbar-$Fourier space.

7.2 Using the Egorov Theorem

Let us consider the Sobolev space

$$H^{-m}(\mathbb{R}) := \hat{A}_m^{-1}\left(L^2(\mathbb{R})\right)$$

which is the usual Sobolev space as a linear space, except for the norm which depends on $\hbar$. Then $\hat{F} : H^{-m}(\mathbb{R}) \to H^{-m}(\mathbb{R})$ is unitary equivalent to

$$\hat{Q} := \hat{A}_m\hat{F}\hat{A}_m^{-1} \quad : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

Let $n \in \mathbb{N}^*$, a fixed time which will be made large at the end of the proof, and define

$$\hat{P}^{(n)} := \hat{Q}^{*n}\hat{Q}^n = \hat{A}_m^{-1}\hat{F}^{*n}\hat{A}_m^2\hat{F}^n\hat{A}_m^{-1} \tag{101}$$

From Egorov Theorem, as in Lemma 31), we have that $\hat{B} := \hat{F}^*\hat{A}_m^2\hat{F}$ is a PDO with principal symbol

$$\begin{aligned} B(x,\xi) &= \chi^2(x) \sum_{j \text{ s.t.} i\rightsquigarrow j} \left|\phi'_{i,j}(x)\right| e^{2\mathrm{Re}(V(\phi_{i,j}(x)))} A_m^2(F_{i,j}(x,\xi)), \qquad (x,\xi) \in T^*I \\ &= \chi^2(x) \sum_{j \text{ s.t.} i\rightsquigarrow j} e^{2D((\phi_{i,j}(x)))} A_m^2(F_{i,j}(x,\xi)) \end{aligned}$$

where we have used the "damping function" $D(x) := \mathrm{Re}(V(x)) - \frac{1}{2}\log\left(\left|\left(\phi^{-1}\right)'(x)\right|\right)$ already defined in (42). Iteratively for every $n \geq 1$, Egorov Theorem gives that $\hat{F}^{*n}\hat{A}_m^2\hat{F}^n$ is a PDO with principal symbol

$$B_n(x,\xi) = \chi^2(x)\sum_{w_{-n,0}\in\mathcal{W}_-} e^{2D_{w_{-n,0}}(x)}A_m^2\left(F_{w_{-n,0}}(x,\xi)\right)$$

where $\mathcal{W}_+$ is the set of admissible sequences, defined in (73), with the Birkhoff sum $D_{w_{-n,0}}(x) := \sum_{k=1}^{n} D\left(\phi_{w_{-n,-k}}(x)\right)$ and

$$F_{w_{-n,0}} := F_{w_{-1},w_0}\circ\ldots\circ F_{w_{-n},w_{-n+1}}.$$

With the Theorem of composition of PDO ([6, chap.4]), we obtain that $\hat{P}^{(n)}$ is a PDO of order 0 with principal symbol given by

$$P^{(n)}(x,\xi) = \left(\chi^2(x)\sum_{w_{-n,0}\in\mathcal{W}_-} e^{2D_{w_{-n,0}}(x)}\frac{A_m^2\left(F_{w_{-n,0}}(x,\xi)\right)}{A_m^2(x,\xi)}\right) \tag{102}$$

We define

$$\gamma_{(n)} := \sup_{x\in I, w_{-n,0}\in\mathcal{W}_-}\frac{1}{n}D_{w_{-n,0}}(x)$$

hence $e^{2D_{w_{-n,0}}(x)} \leq e^{2n\gamma_{(n)}}$.

From Theorem 12, the spectrum of $\hat{F}_\hbar$ does not depend on the choice of χ. Here we take $a \geq 0$ as given in Assumption 18 and we choose χ such that $\chi \equiv 1$ on K_{a+1}, $\chi \equiv 0$ on $\mathbb{R}\backslash K_a$. We have $P(x,\xi) = 0$ if $x \in \mathbb{R}\backslash K_a$.

Now we will bound the positive symbol $P^{(n)}(x,\xi)$ from above, considering $x \in K_a$ and different possibilities for the trajectory $F_{w_{-n,0}}(x,\xi)$:

1. If $|\xi| > R$, Eq.(99) gives

$$\frac{A_m^2\left(F_{w_{-n,0}}(x,\xi)\right)}{A_m^2(x,\xi)} = \frac{A_m^2\left(F_{w_{-n,0}}(x,\xi)\right)}{A_m^2\left(F_{w_{-n,-1}}(x,\xi)\right)}\frac{A_m^2\left(F_{w_{-n,-1}}(x,\xi)\right)}{A_m^2\left(F_{w_{-n,-2}}(x,\xi)\right)}\cdots\frac{A^2\left(F_{w_{-n,-n+1}}(x,\xi)\right)}{A^2(x,\xi)} \leq \left(C^{2m}\right)^n \tag{103}$$

therefore

$$P^{(n)}(x,\xi) \leq (\sharp\mathcal{W}_n)\,e^{2n\gamma_{(n)}}\left(C^{2m}\right)^n \leq \left(Ne^{2\gamma_{(n)}}C^{2m}\right)^n$$

We have used that $\sharp\mathcal{W}_n \leq N^n$. Notice that C^{2m} can be made arbitrarily small if m is large.

2. If $|\xi| \leq R$, we have from the hypothesis of minimal captivity 18 that at time $(n-1)$ every point (x',ξ') of the set $F^{n-1}(x,\xi)$ except at most one point satisfy $|\xi'| > R$.

Using (100) and (99), for all these points one has $\frac{A_m^2\left(F_{w_{-n,0}}(x,\xi)\right)}{A_m^2(x,\xi)} \leq C^{2m}$ and for the exceptional point one can only write $\frac{A_m^2\left(F_{w_{-n,0}}(x,\xi)\right)}{A_m^2(x,\xi)} \leq 1$. This gives

$$P^{(n)}(x,\xi) \leq e^{2n\gamma_{(n)}}\left((\sharp\mathcal{W}_n - 1)\,C^{2m} + 1\right) \leq \mathcal{B}$$

with the bound

$$\mathcal{B} := e^{2n\gamma_{(n)}}\left(N^n C^{2m} + 1\right) \tag{104}$$

With L^2 **continuity theorem** for pseudodifferential operators [17, 4] this implies that in the limit $\hbar \to 0$

$$\left\|\hat{P}^{(n)}\right\| \leq \mathcal{B} + \mathcal{O}_n(\hbar) \tag{105}$$

Polar decomposition of $\hat{Q}^n$ gives

$$\left\|\hat{Q}^n\right\| \leq \left\|\left|\hat{Q}^n\right|\right\| = \sqrt{\left\|\hat{P}^{(n)}\right\|} \leq \left(\mathcal{B} + \mathcal{O}_n(\hbar)\right)^{1/2} \tag{106}$$

Let $\gamma_+ = \limsup_{n\to\infty}\gamma_{(n)}$. Let $\rho > e^{\gamma_+}$. We let $\hbar \to 0$ first, after $m \to +\infty$ giving $C^{2m} \to 0$ and then $n \to \infty$ so that $\left(\mathcal{B} + \mathcal{O}_n(\hbar)\right)^{1/(2n)} \to e^{\gamma_+}$. We have obtained that for any $\rho > e^{\gamma_+}$, there exists $n_0 \in \mathbb{N}$, $\hbar_0 > 0$, $m_0 > 0$ such that for any $\hbar \leq \hbar_0$, $m > m_0$,

$$\left\|\hat{F}_\hbar^{n_0}\right\|_{H^{-m}} = \left\|\hat{Q}^{n_0}\right\|_{L^2} \leq \rho^{n_0}. \tag{107}$$

Also, there exists $c > 0$ independent of $\hbar \leq \hbar_0$, such that for any r such that $0 \leq r < n_0$ we have $\left\|\hat{Q}^r\right\|_{L^2} < c$. As a consequence for any $n \in \mathbb{N}$ we write $n = kn_0 + r$ with $0 \leq r < n_0$ and

$$\left\|\hat{F}_\hbar^{n}\right\|_{H^{-m}} = \left\|\hat{Q}^{n}\right\|_{L^2} \leq \left\|\hat{Q}^{n_0}\right\|_{L^2}^k \left\|\hat{Q}^{r}\right\|_{L^2} \leq \rho^n \frac{\left\|\hat{Q}^{r}\right\|_{L^2}}{\rho^r} \leq c\rho^n$$

We have obtained (45). Equivalently this gives (44).

For any n the spectral radius of $\hat{Q}$ satisfies [23, p.192]

$$r_s\left(\hat{Q}\right) \leq \left\|\hat{Q}^n\right\|^{1/n} \leq c^{1/n}\rho$$

So we get that for $\hbar \to 0$,

$$r_s\left(\hat{F}_\hbar\right) = r_s\left(\hat{Q}\right) \leq e^{\gamma_+} + o(1). \tag{108}$$

which finishes the proof of Theorem 20.

8 Proof of Theorem 22 about the Fractal Weyl law.

8.1 A refined escape function

8.1.1 Distance function

The escape function A will be constructed from a distance function δ. For $x \in I$, let

$$\tilde{K}(x) := \tilde{K} \bigcap (\{x\} \times \mathbb{R}) \tag{109}$$

where $\tilde{K}$ has been defined in (65). With this notation we can define the following distance function.

Definition 41. Let $x \in I_{w_0}$ and $\xi \in \mathbb{R}$, we define the distance of (x, ξ) to the set $\tilde{K}$ given in (65) by

$$\delta(x, \xi) := \operatorname{dist}\left(\xi, \tilde{K}(x)\right) = \min_{w \in \mathcal{W}_+} |\xi - \zeta_w(x)| \tag{110}$$

We will show that the distance function $\delta(x, \xi)$ decreases along the trajectories of F. First, the next Lemma shows how the branches ζ_w are transformed under the canonical map F and follows from straightforward calculations.

Lemma 42. *For every $w_+ = (w_0, w_1, \ldots) \in \mathcal{W}_+$, $x \in I_{w_0}$ we have*

$$F_{w_0, w_1}\left(x, \zeta_{w_+}(x)\right) = \left(x', \zeta_{L(w_+)}(x')\right) \tag{111}$$

with $L(w_+) := (w_1, w_2, \ldots)$ and $x' = \phi_{w_0, w_1}(x)$.

Lemma 43. $\forall i, j, i \rightsquigarrow j, \forall x \in I_i, \forall \xi \in \mathbb{R}$,

$$\delta\left(F_{i,j}(x, \xi)\right) \geq \frac{1}{\theta} \delta(x, \xi)$$

where $\theta < 1$ is given by (1).

Proof. Let $i = w_0 \rightsquigarrow j = w_1$, $x \in I_{w_0}$. Let $(x', \xi') := F_{w_0, w_1}(x, \xi)$ with $x' \in I_{w_1}$. We use (111) and also that F_{w_0, w_1} is expansive in ξ by a factor larger than $\theta^{-1} > 1$ (Eq.(36)), and get

$$\left|\xi' - \zeta_{L(w_+)}(x')\right| = \left|\left(F_{w_0, w_1}(x, \xi) - F_{w_0, w_1}\left(x, \zeta_{w_+}(x)\right)\right)_\xi\right| \geq \frac{1}{\theta}\left|\xi - \zeta_{w_+}(x)\right|$$

Thus

$$\begin{aligned}\delta\left(F_{w_0,w_1}(x,\xi)\right) &= \min_{w\in\mathcal{W}_+}\left|\xi'-\zeta_{w_+}(x')\right| = \min_{w_+\in\mathcal{W}_+}\left|\xi'-\zeta_{L(w_+)}(x')\right| \\ &\geq \frac{1}{\theta}\min_{w_+\in\mathcal{W}_+}\left|\xi-\zeta_{w_+}(x)\right| = \frac{1}{\theta}\delta(x,\xi)\end{aligned}$$

□

8.1.2 Escape function

The aim of this section is to prove the existence of an escape function with the following properties:

Proposition 44. *$\forall 1<\kappa<\theta^{-1}$,$\exists \mathsf{C}_0>0$,$\forall\mu$, s.t. $0\leq\mu<\frac{1}{2}$, $\forall m>0$, there exists an $\hbar$-dependent order function $A_{m,\mu}\in\mathcal{OF}^{m\mu}(\langle\xi\rangle^{-m})$(as defined in Definition 54) which fulfills the following 'decay-condition':*
$\forall i,j,\quad$ s.t. $i\rightsquigarrow j$ and$\forall(x,\xi)\in I_i\times\mathbb{R}$ s.t. $\delta(x,\xi)>\mathsf{C}_0\hbar^{\mu}$ the following estimate holds:

$$\left(\frac{A_{m,\mu}\circ F_{i,j}}{A_{m,\mu}}\right)(x,\xi)\leq\kappa^{-m}, \tag{112}$$

In order to prove the above proposition we first remark that the distance function (110) is not differentiable, however Lipschitz.

Lemma 45. *Let $C_1:=\sup_{x\in I,\omega\in\mathcal{W}_+}|(\partial_x\zeta_\omega)(x)|$. Then $\delta:T^*I\to\mathbb{R}^+$is a Lipschitz function with constant C_1+1*

Proof. Let $x,y\in I_i$, then from the fact, that $|(\partial_x\zeta_\omega)(x)|$ is uniformly bounded by C_1 we have

$$|\delta(x,\xi)-\delta(y,\xi)|\leq C_1|x-y|.$$

On the other hand clearly

$$|\delta(y,\xi)-\delta(y,\zeta)|\leq|\xi-\zeta|$$

thus

$$|\delta(x,\xi)-\delta(y,\zeta)|\leq C_1|x-y|+|\xi-\zeta|\leq(C_1+1)\mathrm{dist}((x,\xi),(y,\zeta))$$

□

Next we choose $0 \leq \mu < 1/2$ and regularize the function δ at the scale $\hbar^{\mu}$. For this we choose $\chi \in C_0^{\infty}(\mathbb{R}^2)$ with support in the unit ball $B_1(0)$ of $\mathbb{R}^2$ and $\chi(x,\xi) > 0$ for $\|(x,\xi)\| < 1$. This function can be rescaled to

$$\chi_{\hbar^{\mu}}(x,\xi) := \frac{1}{\hbar^{2\mu}\|\chi\|_{L^1}}\chi\left(\frac{x}{\hbar^{\mu}},\frac{\xi}{\hbar^{\mu}}\right)$$

such that $\mathrm{supp}\chi_{\hbar^{\mu}} \subset B_{\hbar^{\mu}}(0)$ and $\int \chi_{\hbar^{\mu}}(x)\,dx = 1$. Now we can define the regularized distance function by

$$\tilde{\delta}(x,\xi) := \int_{T^*I} \delta(x',\xi')\,\chi_{\hbar^{\mu}}(x-x',\xi-\xi')\,dx'd\xi'.$$

This smoothed distance function $\tilde{\delta}$ differs only at order $\hbar^{\mu}$ from the original one because

$$\begin{aligned}
\left|\tilde{\delta}(x,\xi)-\delta(x,\xi)\right| &= |\int_{\mathbb{R}^2}(\delta(x,\xi)-\delta(x-x',\xi-\tilde{\xi'}))\chi_{\hbar^{\mu}}(x',\xi')dx'd\xi'| \\
&\leq \sup_{(x',\xi')\in B_{\hbar^{\mu}}(0)} |(\delta(x,\xi)-\delta(x-x',\xi-\xi')| \\
&\leq (C_1+1)\hbar^{\mu}. \qquad (113)
\end{aligned}$$

Furthermore we get the following estimates for its derivatives:

Lemma 46. *For all $\alpha,\beta \in \mathbb{N}$ the estimate*

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}\tilde{\delta}(x,\xi)| \leq C_{\alpha,\beta}\hbar^{-\mu(\alpha+\beta)}(\delta(x,\xi)+C\hbar^{\mu})$$

holds

Proof. From the definition of $\chi_{\hbar^{\mu}}$ we have $\|\partial_x^{\alpha}\partial_{\xi}^{\beta}\chi_{\hbar^{\mu}}\|_{\infty} \leq C_{\alpha,\beta}\hbar^{-2-(\alpha+\beta)\mu}$ and thus:

$$\begin{aligned}
\left|\left(\partial_x^{\alpha}\partial_{\xi}^{\beta}\tilde{\delta}(x,\xi)\right)\right| &= \int_{T^*I}\delta(x',\xi')\,\partial_x^{\alpha}\partial_{\xi}^{\beta}\chi_{\hbar^{\mu}}(x-x',\xi-\xi')dx'd\xi' \\
&\leq \pi\hbar^{2\mu}\|\delta\|_{\infty,B_{\hbar^{\mu}(x,\xi)}}C_{\alpha,\beta}\hbar^{-(2+\alpha+\beta)\mu} \\
&\leq \pi C_{\alpha,\beta}\hbar^{-(\alpha+\beta)\mu}(\delta(x,\xi)+(C_1+1)\hbar^{\mu})
\end{aligned}$$

where we used the Lipschitz property of δ in the last inequality. ☐

As $|\delta(x,\xi)| \leq |\xi|+\mathrm{C}$ the above lemma gives us directly that $\tilde{\delta} \in S_{\mu}^{1}(T^*I)$. Now we define the escape function as:

$$A_{m,\mu}(x,\xi) := \hbar^{m\mu}\left(\hbar^{2\mu}+\left(\tilde{\delta}(x,\xi)\right)^2\right)^{-\frac{m}{2}} \qquad (114)$$

This is obviously a smooth function and it obeys the following estimates:

Lemma 47. *The function $A_{m,\mu}$ defined in (114) is an $\hbar$-dependent order function: $A_{m,\mu} \in \mathcal{OF}^{m\mu}(\langle\xi\rangle^{-m})$.*

Proof. As $\min(0,|\xi|-C) \leq \tilde{\delta}(x,\xi) \leq |\xi|+C$ it follows, that $A_{m,\mu}(x,\xi) \leq \tilde{C}\langle\xi\rangle^{-m}$ and that $A_{m,\mu}(x,\xi) \geq C'\hbar^{m\mu}\langle\xi\rangle^{-m}$. It remains thus to show, that for arbitrary $\alpha,\beta \in \mathbb{N}$ one has:

$$|\partial_x^\alpha \partial_\xi^\beta A_{m,\mu}(x,\xi)| \leq C_{\alpha,\beta}\hbar^{-\mu(\alpha+\beta)}A_{m,\mu}(x,\xi) \tag{115}$$

where $C_{\alpha,\beta}$ depends only on α and β

First consider the case $\alpha = 1, \beta = 0$

$$|\partial_x A_{m,\mu}(x,\xi)| = |\hbar^{m\mu} m \frac{(\partial_x\tilde{\delta}(x,\xi))\tilde{\delta}(x,\xi)}{\left(\hbar^{2\mu} + \left(\tilde{\delta}\left(x,\xi\right)\right)^2\right)^{\frac{m+2}{2}}}| \leq C\hbar^{-\mu}A_{m,\mu}\left(x,\xi\right)$$

where we used $\tilde{\delta} \leq \sqrt{\hbar^{2\mu}+\tilde{\delta}^2}$ and $|\partial_x\tilde{\delta}| \leq C\hbar^{-\mu}\sqrt{\hbar^{2\mu}+\tilde{\delta}^2}$ which follows from lemma 46 together with (113). Inductively one obtains the estimate for arbitrary $\alpha,\beta \in \mathbb{N}$ by repeated use of lemma 46 and (113). □

Finally it remains to show the decay estimates for $\left(\frac{A_{m,\mu}\circ F_{i,j}}{A_{m,\mu}}\right)(x,\xi)$.

Combining (113) with lemma 43 we then get

$$\tilde{\delta}(F_{i,j}(x,\xi)) \geq \delta(F_{i,j}(x,\xi))-(C_1+1)\hbar^\mu \geq \frac{1}{\theta}\delta(x,\xi)-(C_1+1)\hbar^\mu \geq \frac{1}{\theta}\tilde{\delta}(x,\xi)-(\frac{1}{\theta}+1)(C_1+1)\hbar^\mu$$

and thus

$$\frac{A_{m,\mu}(F_{i,j}(x,\xi))}{A_{m,\mu}(x,\xi)} \leq \left(\frac{1+(\frac{1}{\theta}\cdot\frac{\tilde{\delta}(x,\xi)}{\hbar^\mu}-\tilde{C})^2}{1+(\frac{\tilde{\delta}(x,\xi)}{\hbar^\mu})^2}\right)^{\frac{m}{2}} \tag{116}$$

where $\tilde{C} = (\frac{1}{\theta}+1)(C_1+1)$. Clearly the right side of (116) converges to $(\frac{1}{\theta})^{-m}$ for $\frac{\tilde{\delta}(x,\xi)}{\hbar^\mu} \to \infty$ which proves the existence of a desired C_0 and finishes the proof of proposition 44.

8.1.3 Truncation in x

Here we choose a similar truncation operator $\hat{\chi}$ as in Eq.(23) but in a finer vicinity of the trapped set K. First notice that $K_{\hbar^\mu} \Subset \phi^{-1}\left(K_{\hbar^\mu}\right)$ where $K_{\hbar^\mu}$ has been defined in definition 2. For $\hbar$ small enough we have $\phi^{-1}\left(K_{\hbar^\mu}\right) \Subset I$. Let $\chi \in C^\infty_{\phi^{-1}(K_{\hbar^\mu})}$ such that $\chi\left(x\right) = 1$ for $x \in K_{\hbar^\mu}$. χ can be considered as a function $\chi\left(x,\xi\right) := \chi\left(x\right)$ (independent of ξ) and we have that $\chi_\mu \in S^0_\mu\left(T^*\mathbb{R}\right)$. As in Eq.(23) we define $\hat{\chi} := \mathrm{Op}^w_\hbar(\chi)$ which is the multiplication operator by χ,

$$\hat{F}_{i,j,\chi} := \hat{F}_{i,j}\hat{\chi}, \qquad \hat{F}_\chi := \hat{F}\hat{\chi}.$$

We will again omit the χ in the notation and write $\hat{F}_\hbar$ for $\hat{F}_\chi$ in the sequel.

8.2 Weyl law

The Weyl law will give an upper bound on the number of eigenvalues of $\hat{F}_\hbar$ on the Sobolev spaces H^m. These estimates will be obtained by conjugating $\hat{F}_\hbar$ with $Op_\hbar^w(A_{m,\mu})$ in the same way as for the discrete spectrum or the spectral gap. Note that we use the Weyl quantization (see Definition 53) in this section, because we want to obtain self adjoint operators. In order to be able to conjugate we have to show, that $Op_\hbar^w(A_{m,\mu}) : H^{-m} \to L^2$ is an isomorphism. We already know that $Op_\hbar^w(\langle\xi\rangle^m) : L^2 \to H^{-m}$ is an isomorphism, thus it suffices to show, that $\hat{B} := Op_\hbar^w(A_{m,\mu})Op_\hbar^w(\langle\xi\rangle^m) : L^2 \to L^2$ is invertible. From the $\hbar$-local symbol calculus (Theorem 58) it follows that $\hat{B}$ is an elliptic operator in the $\hbar$-local symbol class $S_\mu(A_{m,\mu}\langle\xi\rangle^m)$ and thus the invertibility follows from proposition 61. Note that it is necessary to work in the $\hbar$-local symbol classes as $\hat{B}$ would not be an elliptic operator in $S_\mu(1)$. Proposition 61 also gives us the leading order of our inverse $\hat{B}^{-1}$ which is $A_{m,\mu}^{-1}\langle\xi\rangle^{-m}$. So the inverse of $Op_\hbar^w(A_{m,\mu})$ is again a PDO with leading symbol $A_{m,\mu}^{-1}$.

With the isomorphism $Op_\hbar^w(A_{m,\mu}) : H^m \to L^2$ we can thus define a different scalar product on the Sobolev spaces which turns $Op_\hbar^w(A_{m,\mu})$ into a unitary operator. The Sobolev space equipped with this scalar product will be denoted by $\mathcal{H}_{\hbar,\mu}^m$ and the study of $\hat{F}_\hbar$ is thus unitary equivalent to the study of $\hat{Q}_m$ defined by the following commutative diagram (where we noted $\hat{A}_{m,\mu} := Op_\hbar^w(A_{m,\mu})$):

$$\begin{array}{ccc} L^2(\mathbb{R}) & \overset{\hat{Q}_m}{\to} & L^2(\mathbb{R}) \\ \downarrow \hat{A}_{m,\mu}^{-1} & & \downarrow \hat{A}_{m,\mu}^{-1} \\ \mathcal{H}_{\hbar,\mu}^{-m} & \overset{\hat{F}_\hbar}{\to} & \mathcal{H}_{\hbar,\mu}^{-m} \end{array} \tag{117}$$

In the next Lemma, C_0 and κ are as in lemma 44.

Lemma 48. $\exists \mathsf{C} > \mathsf{C}_0$, $\forall \epsilon > 0$, $\forall \mu$ *s.t.* $0 \le \mu < \frac{1}{2}$, $\forall m > 0$ *sufficiently large, as* $\hbar \to 0$ *we have:*

$$\sharp\left\{\lambda_i^\hbar \in \sigma\left(\hat{F}_\hbar|_{\mathcal{H}_{\hbar,\mu}^m}\right) \mid \left|\lambda_i^\hbar\right| \ge \epsilon\right\} \le \frac{1}{2\pi\hbar}\left(\tilde{C}_1 \mathrm{Leb}\left\{\mathcal{K}_{C_1\hbar^\mu}\right\} + \tilde{C}_2\hbar\right). \tag{118}$$

Before proving Lemma 48, let us show that it implies Theorem 22. From Theorem 12, the discrete spectrum of $\hat{F}_\hbar|_{\mathcal{H}_{\hbar,\mu}^m}$ *is* the Ruelle spectrum of resonances $\mathrm{Res}\left(\hat{F}_\hbar\right)$, independent of μ and m. With assumption 18 we can use Eq.(46) and that $\mathcal{K}$ has pure dimension thus equation (13) gives $\mathrm{Leb}\left\{\mathcal{K}_{C_1\hbar^\mu}\right\} = \mathcal{O}\left((\hbar^\mu)^{\mathrm{codim}_M(\tilde{K})}\right)$. As $\mathrm{codim}_M(\mathcal{K}) < 2$ and $\mu < \frac{1}{2}$ equation(118) and gives

$$\begin{aligned} \sharp\left\{\lambda_i^\hbar \in \mathrm{Res}\left(\hat{F}_\hbar\right) \mid \left|\lambda_i^\hbar\right| \ge \epsilon\right\} &= \mathcal{O}\left(\hbar^{-1}(\hbar^\mu)^{\mathrm{codim}_M(\tilde{K})}\right) \\ &= \mathcal{O}\left(\hbar^{-1}(\hbar^\mu)^{2-2\dim_H(K)}\right) = \mathcal{O}\left(\hbar^{2\mu-1-2\mu\dim_H(K)}\right) \end{aligned}$$

for any fixed $0 \leq \mu < 1/2$. This gives Theorem 22 with $\eta = (1-2\mu)(1-\dim_{\mathrm{H}}(K))$.

Proof. of Lemma 48. From (117), $\hat{F}_{\chi,\hbar} : \mathcal{H}^m_{\hbar,\mu} \to \mathcal{H}^m_{\hbar,\mu}$ is unitary equivalent to

$$\hat{Q}_{m,\mu} := \mathrm{Op}^w_\hbar\left(A_{m,\mu}\right)\hat{F}_\hbar\hat{\chi}\mathrm{Op}^w_\hbar\left(A_{m,\mu}\right)^{-1} : L^2\left(\mathbb{R}\right) \to L^2\left(\mathbb{R}\right).$$

Consider

$$\hat{P}_\mu := \hat{Q}^*_{m,\mu}\hat{Q}_{m,\mu} = \mathrm{Op}^w_\hbar\left(A_{m,\mu}\right)^{-1}\hat{\chi}\hat{F}^*_\hbar\mathrm{Op}^w_\hbar\left(A_{m,\mu}\right)^2\hat{F}_\hbar\hat{\chi}\mathrm{Op}^w_\hbar\left(A_{m,\mu}\right)^{-1}.$$

By the composition Theorem (Th 58) and the Egorov theorem (Th 62) for $\hbar$-local symbols, $\hat{P}_\mu$ is a PDO with leading symbol $P_\mu\left(x,\xi\right) \in S_\mu\left(1\right)$, for $x \in I_i$, $\xi \in \mathbb{R}$, given by the same expression as in (63):[8]

$$P_\mu\left(x,\xi\right) = \chi^2\left(x\right)\sum_{j \text{ s.t. } i\rightsquigarrow j}\left|\phi'_{i,j}\left(x\right)\right|e^{2\Re\left(V\left(\phi_{i,j}(x)\right)\right)}\frac{A^2_{m,\mu}\left(F_{i,j}\left(x,\xi\right)\right)}{A^2_{m,\mu}\left(x,\xi\right)} \mod \hbar^{1-2\mu}S^{-1}_\mu\left(T^*\mathbb{R}\right) \tag{119}$$

From the definition of χ and Eq.(112), the operator $\hat{P}_\mu$ can be decomposed into self-adjoint operators

$$\hat{P}_\mu = \hat{k}_\mu + \hat{r}_\mu$$

where $\hat{k}_\mu$ is a PDO with symbol $k_\mu \in S^0_\mu\left(T^*\mathbb{R}\right)$ supported on $\mathcal{K}_{C_1\hbar^\mu}$ for some $C_1 > 0$. Hence $\hat{k}_\mu$ is a trace-class operator. The operator $\hat{r}_\mu$ is a PDO with symbol $r_\mu \in S^0_\mu\left(T^*\mathbb{R}\right)$ such that

$$\left\|r_\mu\right\|_\infty \leq \theta e^{2\|\Re(V)\|_\infty}\kappa^{-2m} + \mathcal{O}\left(\hbar^{1-2\mu}\right),$$

hence $\|\hat{r}_\mu\| \leq C\kappa^{-2m} + \mathcal{O}(\hbar^{1-2\mu})$.

Using lemma 49 in appendix A we have that for every $\epsilon > 0$, in the limit $\hbar \to 0$,

$$\sharp\left\{\mu^\hbar_i \in \sigma\left(\hat{k}_\mu\right) \mid \left|\mu^\hbar_i\right| \geq \epsilon\right\} \leq \left(2\pi\hbar\right)^{-1}\left(\tilde{C}_1\mathrm{Leb}\left\{\mathcal{K}_{C_1\hbar^\mu}\right\} + \tilde{C}_2\hbar\right). \tag{120}$$

By a standard perturbation argument the same estimates holds for the operator $\hat{P}_\mu$(for m sufficiently large): for every $\epsilon > 0$, in the limit $\hbar \to 0$,

$$\sharp\left\{\mu^\hbar_i \in \sigma\left(\hat{P}_\mu\right) \mid \left|\mu^\hbar_i\right| \geq \epsilon + \left\|\hat{r}_\mu\right\|\right\} \leq \left(2\pi\hbar\right)^{-1}\left(\tilde{C}_1\mathrm{Leb}\left\{\mathcal{K}_{C_1\hbar^\mu}\right\} + \tilde{C}_2\hbar\right). \tag{121}$$

From the definition $\hat{P}_\mu := \hat{Q}^*_{m,\mu}\hat{Q}_{m,\mu}$, the $\sqrt{\mu^\hbar_i}$ are singular values of $\hat{Q}_{m,\mu}$. Then corollary 51 from appendix B shows that the same estimate holds true for *the eigenvalues* of $\hat{Q}_{m,\mu}$, hence of $\hat{F}_\hbar$, yielding the result (118). $\square$

A Adapted Weyl type estimates

[8] Also for this calculation it is crucial to work with the $\hbar$-local calculus in order to obtain sufficient remainder estimates.

Lemma 49. *Let $a_\hbar \in S_\mu(\langle x\rangle^{-2}\langle\xi\rangle^{-2})$ with $0 \le \mu < \frac{1}{2}$ be a real compactly supported symbol. $\forall\hbar > 0$, $\hat{A} := \mathrm{Op}_\hbar^w(a_\hbar)$ is self-adjoint and trace class on $L^2(\mathbb{R})$ and for any $\epsilon > 0$, as $\hbar \to 0$:*

$$(2\pi\hbar)\sharp\left\{\lambda_i^\hbar \in \sigma\left(\hat{A}\right) \mid \left|\lambda_i^\hbar\right| \ge \epsilon\right\} \le C_1 \mathrm{Leb}\left\{(x,\xi)\,;\,|a| > 0\right\} + C_2\hbar \tag{122}$$

where C_1 and C_2 depend only on μ and ϵ.

Proof. As $a_\hbar$ is compactly supported $\hat{A}$ is trace class for every $\hbar$ (see theorem C.17 [6]). Consequently also $\frac{1}{\epsilon^2}\hat{A}^2$ is trace class and its trace is given by Lidskii's theorem by $Tr(\frac{1}{\epsilon^2}\hat{A}^2) = \sum_i \left(\frac{\lambda_i^\hbar}{\epsilon}\right)^2$. As $\hat{A}$ is self adjoint all $\lambda_i^\hbar$ are real and one clearly has

$$\sharp\left\{\lambda_i^\hbar \in \sigma\left(\hat{A}\right) \mid \left|\lambda_i^\hbar\right| \ge \epsilon\right\} \le Tr\left(\frac{1}{\epsilon^2}\hat{A}^2\right).$$

If we denote by $b_\hbar(x,\xi)$ the complete symbol of $\hat{A}^2$ we can calculate the trace by the following exact formula

$$Tr(\hat{A}^2) = \frac{1}{2\pi\hbar}\int b_\hbar(x,\xi)dxd\xi$$

According to the theorem of composition of PDO's $b_\hbar$can be written as $b_\hbar = b_\hbar^{(1)} + \hbar^1 b_\hbar^{(2)}$ where $\mathrm{supp} b_\hbar^{(1)} = \mathrm{supp} a_\hbar$ and $b_\hbar^{(2)} \in S_\mu(\langle x\rangle^{-2}\langle\xi\rangle^{-2})$(note that this decomposition depends on μ. Thus

$$\frac{1}{\epsilon^{\mathbf{2}}}Tr(\hat{A}^2) = \frac{1}{2\pi\hbar\epsilon^2}\left(\int b_\hbar^{(1)}(x,\xi)dxd\xi + \hbar^1\int b_\hbar^{(1)}(x,\xi)dxd\xi\right) \le \left(\frac{1}{2\pi\hbar}C_1\mathrm{Leb}(\mathrm{supp}(a_\hbar)) + C_2\hbar\right)$$

□

B General lemmas on singular values of compact operators

Let $(P_\nu)_{\nu\in\mathbb{N}}$ be a family of compact operators on some Hilbert space. Consider any P_ν and let $(\lambda_{j,\nu})_{j\in\mathbb{N}^*} \in \mathbb{C}$ be the sequence of its eigenvalues ordered decreasingly according to multiplicity:

$$|\lambda_{1,\nu}| \ge |\lambda_{2,\nu}| \ge ...$$

In the same manner, define $(\mu_{j,\nu})_{j\in\mathbb{N}^*} \in \mathbb{R}^+$, the decreasing sequence of singular values of P_ν -i.e. the eigenvalues of $\sqrt{P_\nu^* P_\nu}$.

Lemma 50. *Suppose there exits a map $N : \mathbb{N} \to \mathbb{N}$ s.t. $N(\nu) \to \infty$ and $\mu_{N(\nu),\nu} \to 0$* as ν grows. *Then $\forall C > 1$, $|\lambda_{[C \cdot N(\nu)],\nu}| \to_{\nu \to \infty} 0$.*

Corollary 51. *Let $N : \mathbb{N} \to \mathbb{N}$ be as in lemma 50. Suppose that $\forall \epsilon > 0$, $\exists A_\epsilon \geq 0$ s.t. $\forall \nu \geq A_\epsilon$; $\#\{ j \in \mathbb{N}^* \mid \mu_{j,\nu} > \epsilon\} < N(\nu)$. Then for any $C > 1$, $\epsilon > 0$ there exists $B_{C,\epsilon} \geq 0$ such that:*

$$\forall \nu \geq B_{C,\epsilon};\ \#\{j \in \mathbb{N}^* \mid |\lambda_{j,\nu}| > \epsilon\} \leq [C \cdot N(\nu)]. \tag{123}$$

Proof. (Of corollary 51). Suppose that for any $\epsilon > 0$, there exists a rank A_ϵ s.t. for all $\nu \geq A_\epsilon$ $\#\{j \in \mathbb{N}^* \mid \mu_{j,\nu} > \epsilon\} < N(\nu)$, which means that $\mu_{N(\nu),\nu} \to_{\nu \to \infty} 0$ and from Lemma 50, $\forall C > 1$, $|\lambda_{[C \cdot N(\nu)],\nu}| \to_{\nu \to \infty} 0$, which can be directly restated as (123). □

Proof. (Of lemma 50) The main relation between singular and eigenvalues is given by the Weyl inequalities (see [11] p. 50 for a proof):

$$\prod_{j=1}^{k} \mu_{j,\nu} \leq \prod_{j=1}^{k} |\lambda_{j,\nu}|\,;\ \forall k \in \mathbb{N}^*. \tag{124}$$

Let $m_{j,\nu} := -log\,(\mu_{j,\nu})$, $l_{j,\nu} := -log\,(|\lambda_{j,\nu}|)$ to define $M_{k,\nu} := \sum_{j=1}^{k} m_{j,\nu}$, and $L_{k,\nu} := \sum_{j=1}^{k} l_{j,\nu}$. The Weyl inequalities (124) thus read : $M_{k,\nu} \leq L_{k,\nu}$, $\forall k \in \mathbb{N}^*$. Both sequences $(l_{j,\nu})_{j\geq 1}$ and $(m_{j,\nu})_{j\geq 1}$ are increasing so, $\forall k \in \mathbb{N}^*$, $k \cdot l_{k,\nu} \geq L_{k,\nu}$, and for any $k, K \in \mathbb{N}^*$,

$$M_{k+K,\nu} \geq K \cdot m_{k,\nu}. \tag{125}$$

Suppose that $\mu_{N(\nu),\nu} \to 0$ (hence $m_{N(\nu),\nu} \to \infty$) as $\nu \to \infty$ and choose some constant $C > 1$, By (125) we have that

$$M_{[C \cdot N(\nu)],\nu} \geq ([C \cdot N(\nu)] - N(\nu)) \cdot m_{N(\nu),\nu}, \tag{126}$$

and therefore, since $l_{[C \cdot N(\nu)],\nu} \geq \frac{1}{[CN(\nu)]} \cdot L_{[C \cdot N(\nu)],\nu} \geq \frac{1}{[C \cdot N(\nu)]} M_{[C \cdot N(\nu)],\nu}$, from (126) we get

$$l_{[CN(\nu)],\nu} \geq \frac{[C \cdot N(\nu)] - N(\nu)}{[C \cdot N(\nu)]} \cdot m_{[CN(\nu)],\nu}. \tag{127}$$

Notice that $[C \cdot N(\nu)] - N(\nu) > 0$ for ν large enough. Therefore (127) gives the result. □

C Symbol classes of local $\hbar$-order

In this Appendix we will first repeat the definitions of the standard symbol classes which are used in this article as well as their well known quantization rules. Then we will introduce a new symbol class which allows $\hbar$-dependent order functions and will prove some of the classical results which are known in the usual case for these new symbol classes.

C.1 Standard semiclassical Symbol classes and their quantization

The standard symbol classes (see e.g. [6] chapter 4 or [4] ch 7) of $\hbar$PDO's are defined with respect to an order function $f(x,\xi)$. This order function is required to be a smooth positive valued function on $\mathbb{R}^{2n}$ such that there are constants C_0 and N_0 fulfilling

$$f(x,\xi) \leq C_0 \langle (x,\xi) - (x',\xi') \rangle^{N_0} f(x',\xi'). \tag{128}$$

An important example of such an order function is given by $f(x,\xi) = \langle \xi \rangle^m$ with $k \in \mathbb{R}$.

Definition 52. For $0 \leq \mu \leq \frac{1}{2}$ the symbol classes $\hbar^k S_\mu(m)$ contain all families of functions $a_\hbar(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ parametrized by a parameter $\hbar \in]0,\hbar_0]$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a_\hbar(x,\xi)| \leq C \hbar^{k-\mu(|\alpha|+|\beta|)} f(x,\xi)$$

where C depends only on α and β.

Unless we want to emphasize the dependence of the symbol $a_\hbar$ on $\hbar$ we will drop the index in the following. For the special case of order function $f(x,\xi) = \langle \xi \rangle^m$ we also write $S_\mu^m = S_\mu(\langle \xi \rangle^m)$, if $\mu = 0$ we write $S(f) := S_0(f)$.

As quantization we use two different quantization rules in this article which are called standard quantization respectively Weyl quantization.

Definition 53. Let $a_\hbar \in S_\mu(f)$ the Weyl quantization is a family of operators $\mathrm{Op}_\hbar^w(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, defined by

$$\left(\mathrm{Op}_\hbar^w(a_\hbar)\varphi\right)(x) = (2\pi\hbar)^{-n} \int e^{\frac{i}{\hbar}\xi(x-y)} a_\hbar\left(\frac{x+y}{2}, \xi\right) \varphi(y) dy d\xi, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{129}$$

while the standard quantization $\mathrm{Op}_\hbar(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is given by

$$\left(\mathrm{Op}_\hbar(a_\hbar)\varphi\right)(x) = (2\pi\hbar)^{-n} \int e^{\frac{i}{\hbar}\xi(x-y)} a_\hbar(x,\xi) \varphi(y) dy d\xi, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{130}$$

Both quantization extend continuously to operators on $\mathcal{S}'(\mathbb{R}^n)$. While the standard quantization is slightly easier to define, the Weyl quantization has the advantage, that real symbols are mapped to formally self adjoint operators.

C.2 Definition of the Symbol classes $S_\mu(A_\hbar)$

In this standard $\hbar$-PDO calculus the symbols are ordered by there asymptotic behavior for $\hbar \to 0$. If we take for example a symbol $a \in \hbar^k S_\mu(f)$ then $a(x,\xi)$ is of order $\hbar^k$ for all $(x,\xi) \in \mathbb{R}^{2n}$. The symbol classes that we will now introduce will also allow $\hbar-$dependent order function which will allow to control the $\hbar$-order of a symbol locally, i.e. in dependence of (x,ξ). First we define these $\hbar$-dependent order functions:

Definition 54. Let f be an order function on $\mathbb{R}^{2n}$ and $0 \leq \mu \leq \frac{1}{2}$. Let $A_\hbar \in S_\mu(f)$ a (possibly $\hbar$-dependent) positive symbol such that for some $c \geq 0$ there is a constant C that fulfills

$$A_\hbar(x,\xi) \geq C\hbar^c f(x,\xi) \tag{131}$$

and that for all multiinidices $\alpha, \beta \in \mathbb{N}^n$:

$$\left|\partial_x^\alpha \partial_\xi^\beta A_\hbar(x,\xi)\right| \leq C_{\alpha,\beta}\hbar^{-\mu(|\alpha|+|\beta|)} A_\hbar(x,\xi) \tag{132}$$

holds. Then we call $A_\hbar$an $\hbar$-dependent order function and say $A_\hbar \in \mathcal{OF}^c(f)$

Definition 55. The symbol class $S_\mu(A_\hbar)$ is then defined to be the space of smooth functions $a_\hbar(x,\xi)$ defined on R^{2n} and parametrized by $\hbar > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a_\hbar(x,\xi)\right| \leq C_{\alpha,\beta}\hbar^{-\mu(|\alpha|+|\beta|)} A_\hbar(x,\xi) \tag{133}$$

By $\hbar^k S_\mu(A_\hbar)$ we will as usual denote the symbols $a_\hbar$ for which $\hbar^{-k} a_\hbar \in S_\mu(A_\hbar)$

As $A_h(x,\xi) \leq C_0 f(x,\xi)$ and from (132) it is obvious, that

$$S_\mu(A_\hbar) \subset S_\mu(f) \tag{134}$$

and via this inclusion for $a_\hbar \in S_\mu(A_\hbar)$ the standard Quantization $Op_\hbar(a)$ and the Weyl quantization $Op_\hbar^w(a_\hbar)$ are well defined and give continuous operators on $\mathcal{S}(\mathbb{R}^n)$respectively on $\mathcal{S}'(\mathbb{R}^n)$. Furthermore equation (131) gives us a second inclusion

$$S_\mu(f) \subset \hbar^{-c} S_\mu(A_\hbar) \tag{135}$$

thus combining these two inclusions we have:

$$\hbar^c S_\mu(f) \subset S_\mu(A_\hbar) \subset S_\mu(f)$$

As for standard $\hbar - PDO$ symbol we can define asymptotic expansions:

Definition 56. Let $a_j \in S_\mu(A_\hbar)$ for $j = 0, 1, \dots$ then we call $\sum_j \hbar^j a_j$ an asymptotic expansion of $a \in S_\mu(A_\hbar)$ (writing $a \sim \sum_j \hbar^j a_j$) if and only if:

$$a - \sum_{j<N} \hbar^j a_j \in \hbar^N S_\mu(A_\hbar)$$

As in for the standard $\hbar$-PDOs we have some sort of Borel's theorem also for symbols in $S_\mu(A_\hbar)$

Proposition 57. *Let $a_i \in S_\mu(A_\hbar)$ then there is a symbol $a \in S_\mu(A_\hbar)$ such that*

$$a - \sum_{j<k} \hbar^j a_j \in \hbar^k S_\mu(A_\hbar) \tag{136}$$

Proof. Once more we can use the inclusion (134) into the standard $h - PDO$ classes and obtain the existence of a symbol $a \in S_\mu(f)$ such that

$$a - \sum_{j<k} \hbar^j a_j \in \hbar^k S_\mu(f) \tag{137}$$

and we will show that this symbol belongs to $S_\mu(A_\hbar)$ and that (136) holds: For the first statement we write

$$a = \underbrace{a - \sum_{j<c} \hbar^j a_j}_{\in \hbar^c S_\mu(f)} + \underbrace{\sum_{j<c} \hbar^j a_j}_{\in S_\mu(A_\hbar)}$$

and use the inverse inclusion (135).

In order to prove (136) write

$$a - \sum_{j<k} \hbar^j a_j = \underbrace{a - \sum_{j<k+c} \hbar^j a_j}_{\in \hbar^{c+k} S_\mu(f)} + \underbrace{\sum_{j=k}^{k+c-1} \hbar^j a_j}_{\in \hbar^k S_\mu(A_\hbar)}$$

and use once more (135). ☐

The advantage of this new symbol class is, that the order function $A_\hbar(x, \xi)$ itself can depend on $\hbar$ and thus the control in $\hbar$ can be localized. A simple example for such an order function would be $A_\hbar = \hbar^{m\mu} \langle \frac{\xi}{\hbar^\mu} \rangle^m \in \mathcal{OF}^c(\langle \xi \rangle^m)$. For $\xi \neq 0$ this function is of order $\hbar^0$ whereas for $\xi = 0$ it is of order $\hbar^{m\mu}$. Thus also all symbols in $S_\mu(A_\hbar)$ have to show this behavior.

C.3 Composition of symbols

By using the inclusion (134) we will show a result for the composition of Symbols absolutely analogous to the one in the standard case Th 4.18 in [6]. We first note that for $A_\hbar \in \mathcal{OF}^{c_A}(f_A)$ and $B_\hbar \in \mathcal{OF}^{c_B}(f_B)$ the product formula for derivative yields that $A_\hbar B_\hbar \in \mathcal{OF}^{c_A+c_B}(f_A f_B)$ and can now formulate the following theorem:

Theorem 58. *Let $A_\hbar \in \mathcal{OF}^{c_A}(f_A)$ and $B_\hbar \in \mathcal{OF}^{c_B}(f_B)$ be two $\hbar$-dependent order functions and $a \in S_\mu(A_\hbar)$ and $b \in S_\mu(B_\hbar)$ two $\hbar$-local symbols. Then there is a symbol*

$$a\#b \in S_\mu(A_\hbar B_\hbar)$$

such that

$$Op_\hbar^w(a)Op_\hbar^w(b) = Op_\hbar^w(a\#b) \tag{138}$$

as operators on $\mathcal{S}$ and the at first order we have

$$a\#b - ab \in \hbar^{1-2\mu} S_\mu(A_\hbar B_\hbar) \tag{139}$$

Proof. The standard theorem of composition of $\hbar$-PDOs (see e.g. Th 4.18 in [6]) together with the inclusion of symbol-classes (134) provides us a symbol $a\#b \subset S_\mu(f_A \cdot f_B)$ that fulfills equation (138). Furthermore it provides us with a complete asymptotic expansion for $a\#b$:

$$a\#b - \sum_{k=0}^{N-1} \left(\frac{1}{k!} \left[\frac{i\hbar(\langle D_x, D_\eta\rangle - \langle D_y, D_\xi\rangle)}{2} \right]^k a(x,\xi)b(y,\eta) \right)_{|y=x,\eta=\xi} \in \hbar^{N(1-2\mu)} S_\mu(f_A \cdot f_B) \tag{140}$$

In order to prove our theorem it thus only rests to show, that $a\#b \in S_\mu(A_\hbar B_\hbar)$ and that equation (139) holds. We start with the second one. First let $N \in \mathbb{N}$ be such that $(N-1)(1-2\mu) \geq c_A + c_B$, then equation (140) and inclusion (135) assure that the remainder term in (140) is in $\hbar^{1-2\mu} S_\mu(A_\hbar B_\hbar)$. For $0 \leq k \leq N-1$ each term in (140) can be written as a sum of finitely many terms of the form

$$\frac{(i\hbar)^k}{2^k k!} \left(D_x^\alpha D_\xi^\beta a(x,\xi) \right) \cdot \left(D_x^\gamma D_\xi^\delta b(x,\xi) \right)$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{N}^n$ are multiindices fulfilling $|\alpha| + |\beta| + |\gamma| + |\delta| = 2k$. Via the product formula one easily checks, that these terms are all in $\hbar^{k(1-2\mu)} S_\mu(A_\hbar B_\hbar)$ which proves that $a\#b \in S_\mu(A_\hbar B_\hbar)$. ☐

C.4 Ellipticity and inverses

In this section we will define ellipticity for our new symbol classes and will prove a result on L^2-invertibility.

Definition 59. We call a symbol $a \in S_\mu(A_\hbar)$ elliptic if there is a constant C such that:

$$|a(x,\xi)| \geq CA_\hbar(x,\xi) \tag{141}$$

For an $\hbar$-dependent order function $A_\hbar \in \mathcal{OF}^c(f)$From (132) and (131) it follows, that $\hbar^c A_\hbar^{-1} \in \mathcal{OF}^c(f^{-1})$ is again a $\hbar$dependent order function and we can formulate the following proposition

Proposition 60. *If $a \in S_\mu(A_\hbar)$ is elliptic then $a^{-1} \in \hbar^{-c} S_\mu(\hbar^c A_\hbar^{-1})$*

Proof. We have to show, that $|\partial_x^\alpha \partial_\xi^\beta a^{-1}(x,\xi)| \leq C\hbar^{-\mu(|\alpha|+|\beta|)} A_\hbar^{-1}(x,\xi)$ uniformly in $\hbar, x$ and ξ. For some first derivative (i.e. for $\alpha \in \mathbb{N}^{2n}$, $|\alpha| = 1$) we have

$$|\partial_{x,\xi}^\alpha a^{-1}| = \frac{|\partial_{x,\xi}^\alpha a|}{|a^2|} \leq C\frac{\hbar^{-\mu} A_\hbar}{A_\hbar^2} = C\hbar^{-\mu} A_\hbar^{-1}$$

where the inequality is obtained by (132) and (141). The estimates of higher order derivatives can be obtained by induction. □

As for standard $\hbar$-PDOs this notion of ellipticity implies that the corresponding operators are invertible for sufficiently small $\hbar$.

Proposition 61. *Let $A_\hbar \in \mathcal{OF}^c(1)$ and $a \in S_\mu(A_\hbar)$ be an elliptic symbol, then $Op_\hbar^w(a) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded operator. Furthermore there exists $\hbar_0 > 0$ such that $Op_\hbar^w(a)$ is invertible for all $\hbar \in]0, \hbar_0]$. Its inverse is again bounded and a pseudodifferential operator $Op_\hbar^w(b)$ with symbol $b \in S_\mu(A_\hbar^{-1})$. At leading order its symbol is given by*

$$b - a^{-1} \in \hbar^{1-2\mu} S_\mu(A_\hbar^{-1})$$

Proof. As $a \in S_\mu(A_\hbar) \subset S_\mu(1)$ the boundedness of $Op_\hbar^w(a)$ follows from theorem 4.23 in [6]. By theorem 58 we calculate

$$Op_\hbar^w(a) Op_\hbar^w(a^{-1}) = Id + R$$

where $R = Op_\hbar^w(r)$ is a PDO with symbol $r \in \hbar^{1-2\mu} S_\mu(1)$. Again from theorem 4.23 in [6] we obtain $\|R\|_{L^2} \leq C\hbar^{1-2\mu}$ thus there is $\hbar_0$ such that $\|R\|_{L^2} < 1$ for $\hbar \in]0, \hbar_0]$. According to theorem C.3 in [6] we can conclude that $Op_\hbar^w(a)$ is invertible and that the inverse is given by $Op_\hbar^w(a^{-1})(Id+R)^{-1}$. The semiclassical version of Beals theorem allows us to conclude that $(Id+R)^{-1} = \sum_{k=0}^{\infty}(-R)^k$ is a PDO with symbol in $S_\mu(1)$ (cf. theorem 8.3 and the following remarks in [6]). The representation of $(Id-R)^{-1}$ as a series finally gives us the symbol of the inverse operator at leading order. $\square$

C.5 Egorov's theorem for Diffeomorphisms

In this section we will study the behavior of symbols $a \in S_\mu(A_\hbar)$ under variable changes. Let $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism that equals identity outside some bounded set then the pullback with this coordinate change acts as a continuous operator on $\mathcal{S}(\mathbb{R}^n)$ by:

$$(\gamma^* u)(x) := u(\gamma(x))$$

Which can be extended by its adjoint to a continuous operator $\gamma^* : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. By a variable change of an operator we understand its conjugation by γ and we are interested for which $a \in S_\mu(A_\hbar)$ the conjugated operator $(\gamma^*)^{-1} Op_\hbar(a)\gamma^*$ is again a $\hbar$-PDO with symbol a_γ. At leading order this symbol will be the composition of the original symbol with the so called canonical transformation

$$T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, (x,\xi) \mapsto (\gamma^{-1}(x), (\partial\gamma(\gamma^{-1}(x)))^T \xi)$$

and the symbol class of a_γ will be $S_\mu(A_\hbar \circ T)$. For the $A_\hbar \in \mathcal{OF}^c(f)$ defined in Definition 54 the composition $A_\hbar \circ T$ will in general however not be a $\hbar$-dependent order function itself because the derivatives in x create a supplementary ξ factor which has to be compensated (cf. discussion in chapter 9.3 in [6]). We therefore demand in this section that our order function $A_\hbar$ satisfies:

$$\left|\partial_x^\alpha \partial_\xi^\beta A_\hbar(x,\xi)\right| \le C_{\alpha,\beta} \hbar^{\mu(|\alpha|+|\beta|)} \langle\xi\rangle^{-|\beta|} A_\hbar(x,\xi)$$

A straightforward calculation shows then, that $A_\hbar \circ T \in \mathcal{OF}^c(f \circ T)$ is again a $\hbar$dependent order function. The same condition has to be fulfilled by the symbol of the conjugated operator:

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le \hbar^{-\mu(|\alpha|+|\beta|)} \langle\xi\rangle^{-|\beta|} A_\hbar(x,\xi) \tag{142}$$

Theorem 62. *Let $a \in S_\mu(A_\hbar)$ be an symbol which fulfills (142) and has compact support in x (i.e. $\overline{\{x \in \mathbb{R}^n | \exists \xi \in \mathbb{R}^n : a(x,\xi) \neq 0\}}$ is compact) and let $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then there is a symbol $a_\gamma \in S_\mu(A_\hbar \circ T)$ such that*

$$(Op_\hbar(a_\gamma)u)(\gamma(x)) = (Op_\hbar(a)(u \circ \gamma))(x) \tag{143}$$

for all $u \in \mathcal{S}'(\mathbb{R}^n)$. Furthermore a_γ has the following asymptotic expansion.

$$a_\gamma(\gamma(x),\eta) \sim \sum_{\nu=0}^{k-n} \frac{1}{\nu!}\langle i\frac{\hbar}{\langle\eta\rangle}D_y, D_\xi\rangle^\nu e^{\frac{i}{\hbar}\langle\rho_x(y),\eta\rangle} a(x,\xi)\Big|_{y=0,\xi=(\partial\gamma(x))^T\eta} \tag{144}$$

where $\rho_x(y) = \gamma(y+x)-\gamma(x)-\gamma'(x)y$. The terms of the series are in $\hbar^{\frac{\nu(1-2\mu)}{2}} S_\mu(\langle\eta\rangle^{\frac{\nu}{2}} A_\hbar \circ T(\gamma(x),\eta))$.

We will prove this theorem similar to theorem 18.1.17 in [15] by using a parameter dependent stationary phase approximation (Thm7.7.7 in [14]) as well as the following proposition which forms the analog to Proposition 18.1.4 of [15] for our symbol classes and which we will prove first.

Proposition 63. *Let $a(x,\xi;\hbar) \in C^\infty(\mathbb{R}^{2n})$ a family of smooth functions that fulfills*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C\hbar^{-l}\langle\xi\rangle^l f(x,\xi) \tag{145}$$

where C and l may depend on α and β. Furthermore let And let $a_j \in S_\mu(A_\hbar)$, $j = 0,1,\dots$ be a sequence of symbols such that

$$|a(x,\xi) - \sum_{j<k}\hbar^j a_j(x,\xi)| \le C\hbar^{\tau k}\langle\xi\rangle^{-\tau k} f(x,\xi) \tag{146}$$

where $\tau > 0$. Then $a \in S_\mu(A_\hbar)$ and $a \sim \sum \hbar^j a_j$.

Proof. We have to show that for all $k \ge 0$ and $g_k(x,\xi) := a(x,\xi) - \sum_{j<k}\hbar^j a_j(x,\xi)$ we have $|\partial_x^\alpha\partial_\xi^\beta g_k| \le C\hbar^{k-\mu(|\alpha|+|\beta|)}A_\hbar$. This result can be obtained by iterating the following argument for the first derivative in x_1:

Let $e_1 \in \mathbb{R}^n$ be the first eigenvector and $0 < \epsilon < 1$. For arbitrary $j \in \mathbb{N}$ we can write by Tailor's Formula

$$|g_j(x+\epsilon e_1,\xi) - g_j(x,\xi) - \partial_{x_1}g_j(x,\xi)\epsilon)| \le C\epsilon^2 \sup_{t\in[0,\epsilon]} |\partial_{x_1}^2 g_j(x+te_1,\xi)|$$

From (145) and the property, that all a_j are in $S_\mu(A_\hbar)$ we get

$$\sup_{t\in[0,\epsilon]} |\partial_{x_1}^2 g_j(x+te_1,\xi)| \le C\hbar^{-l}\langle\xi\rangle^l f(x,\xi)$$

for some $l \in \mathbb{R}$ and get

$$|\partial_{x_1} g_j(x,\xi)| \leq C\epsilon\hbar^{-l}\langle\xi\rangle^l m(x,\xi) + \frac{|g_j(x+\epsilon e_1,\xi) - g_j(x,\xi)|}{\epsilon}$$

which turns for $j > \frac{2k+2c+l}{\tau}$ and $\epsilon = \hbar^{k+l+c}\langle\xi\rangle^{-(k+l+c)}$ into:

$$|\partial_{x_1} g_j(x,\xi)| \leq C\hbar^{c+k}\langle\xi\rangle^{-(c+k)} f(x,\xi) \leq C\hbar^k A_\hbar(x,\xi)$$

where we used (135) in the second equation. Thus

$$|\partial_{x_1} g_k(x,\xi)| \leq C\hbar^k A_\hbar(x,\xi) + |\sum_{i=k}^{j} \hbar^i \partial_{x_1} a_i(x,\xi)| \leq C\hbar^{k-\mu} A_\hbar(x,\xi)$$

which finishes the proof. □

After having proven this proposition we can start with the proof of theorem 62:

Proof. If we define

$$a_\gamma(\gamma(x),\eta) := e^{-\frac{i}{\hbar}\gamma(x)\eta} Op_\hbar(a) e^{\frac{i}{\hbar}\gamma(\cdot)\eta} \tag{147}$$

then equation (143) holds for all $e^{\frac{i}{\hbar}x\eta}$ which form a dense subset of $\mathcal{S}'(\mathbb{R}^n)$. We thus have to show that a_γ defined in (147) is in $S_\mu(A_\hbar)$ and that (144) holds.

We will first write a_γ as an oscillating integral in order to apply the stationary phase theorem. By definition of $Op_\hbar(a)$ one obtains

$$a_\gamma(\gamma(x),\eta) = \frac{1}{(2\pi\hbar)^n} \iint a(x,\tilde{\xi}) e^{\frac{i}{\hbar}((x-\tilde{y})\tilde{\xi} + (\gamma(\tilde{y})-\gamma(x))\eta)} d\tilde{y} d\tilde{\xi}$$

which we can transform by a variable transformation $\tilde{\xi} = \langle\eta\rangle\xi$ and $\tilde{y} = y + x$ into

$$a_\gamma(\gamma(x),\eta) = \frac{1}{(2\pi\tilde{\hbar})^n} \iint a(x,\langle\eta\rangle\xi) e^{\frac{i}{\tilde{\hbar}}(-y\xi + (\gamma(y+x)-\gamma(x))\frac{\eta}{\langle\eta\rangle})} dy d\xi$$

where $\tilde{\hbar} = \frac{\hbar}{\langle\eta\rangle}$.

The critical points of the phase function are given by

$$y = 0 \text{ and } \xi = (\partial\gamma(x))^T \frac{\eta}{\langle\eta\rangle}$$

Let $\chi \in C_c^\infty([-2,2]^n)$ such that $\chi = 1$ on $[-1,1]^n$ then we can write

$$a_\gamma(\gamma(x),\eta) = I_1(\tilde{\hbar}) + I_2(\tilde{\hbar})$$

with

$$I_1(\tilde{\hbar}) = \frac{1}{(2\pi\tilde{\hbar})^n} \iint \chi(y)\, \chi\left(\xi - (\partial\gamma(x))^T \frac{\eta}{\langle\eta\rangle}\right) a(x,\langle\eta\rangle\xi) e^{\frac{i}{\tilde{\hbar}}(-y\xi + (\gamma(y+x)-\gamma(x))\frac{\eta}{\langle\eta\rangle})} dy d\xi$$

and

$$I_2(\tilde{\hbar}) = \frac{1}{(2\pi\tilde{\hbar})^n}\iint \left(1 - \chi(y)\,\chi\left(\xi - (\partial\gamma(x))^T \frac{\eta}{\langle\eta\rangle}\right)\right) a(x, \langle\eta\rangle\xi) e^{\frac{i}{\hbar}(-y\xi + (\gamma(y+x)-\gamma(x))\frac{\eta}{\langle\eta\rangle})} dy d\xi.$$

While $I_1(\hbar)$ still contains critical points, for $I_2(\hbar)$ there are no critical points in the support of the integrand anymore.

I_1 is of the form studied in theorem 7.7.7 in [14]. Here the role of x and y is interchanged and there is an additional parameter $\frac{\eta}{\langle\eta\rangle}$. We thus get from this stationary phase theorem

$$\begin{aligned}&\left|I_1(\tilde{\hbar}) - \sum_{\nu=0}^{k-n} \tfrac{1}{\nu!}\langle i\tilde{\hbar}D_y, D_\xi\rangle^\nu e^{\frac{i}{\hbar}\langle\rho_x(y), \frac{\eta}{\langle\eta\rangle}\rangle} u(x,\xi,y,\eta)\big|_{y=0,\xi=(\partial\gamma(x))^T\frac{\eta}{\langle\eta\rangle}}\right| \\ &\le C\tilde{\hbar}^{\frac{k+n}{2}} \sum_{|\alpha|\le 2k} \sup_{y,\xi} |D^\alpha_{y,\xi} u(x,\xi,y,\eta)|\end{aligned} \tag{148}$$

where $u(x,\xi,y,\eta) = \chi(y)\,\chi\left(\xi - (\partial\gamma(x))^T\frac{\eta}{\langle\eta\rangle}\right) a(x,\langle\eta\rangle\xi)$. Because of (142) and (128) we can estimate

$$\sup_{y,\xi} |D^\alpha_{y,\xi} u(x,\xi,y,\eta)| \le C\hbar^{-\mu|\alpha|} f(x, (\partial\gamma(x))^T\eta) = C\hbar^{-\mu|\alpha|} f\circ T(\gamma(x),\eta)$$

Thus transforming the expansion (148) back to an expansion in $\hbar$ we get

$$\begin{aligned}&\left|I_1(\hbar) - \sum_{\nu=0}^{k-n} \tfrac{1}{\nu!}\langle i\tfrac{\hbar}{\langle\eta\rangle}D_y, D_\xi\rangle^\nu e^{\frac{i}{\hbar}\langle\rho_x(y),\eta\rangle} u(x,\xi,y,\eta)\big|_{y=0,\xi=(\partial\gamma(x))^T\frac{\eta}{\langle\eta\rangle}}\right| \\ &\le C\hbar^{\frac{k(1-2\mu)+n}{2}}\langle\eta\rangle^{-\frac{k+n}{2}} f\circ T(\gamma(x),\eta)\end{aligned}$$

As the stationary points for I_2 are not contained in the support of the integrand we get by the non stationary phase theorem:

$$|I_2(\hbar)| \le C\left(\frac{\hbar}{\langle\eta\rangle}\right)^N f\circ T(\gamma(x),\eta)$$

for all $N\in\mathbb{N}$. Thus we finally get

$$\begin{aligned}&\left|a_\gamma(\gamma(x),\eta) - \sum_{\nu=0}^{k-n} \tfrac{1}{\nu!}\langle i\tfrac{\hbar}{\langle\eta\rangle}D_y, D_\xi\rangle^\nu e^{\frac{i}{\hbar}\langle\rho_x(y),\eta\rangle} u(x,\xi,y,\eta)\big|_{y=0,\xi=(\partial\gamma(x))^T\frac{\eta}{\langle\eta\rangle}}\right| \\ &\le C\hbar^{\frac{k(1-2\mu)+n}{2}}\langle\eta\rangle^{-\frac{k+n}{2}} f\circ T(\gamma(x),\eta)\end{aligned} \tag{149}$$

If we show that the elements of the series are in $\hbar^{\frac{\nu(1-2\mu)}{2}} S_\mu(\langle\eta\rangle^{\frac{\nu}{2}} A_\hbar \circ T(\gamma(x),\eta))$ then this equation is of the form (146). The terms of order ν in the series are of the form

$$\left(\frac{i\hbar}{\langle\eta\rangle}\right)^\nu \partial^\alpha_y e^{\frac{i}{\hbar}\langle\rho_x(y),\eta\rangle} (\partial^\alpha_\xi a)(x, (\partial\gamma(x))^T\eta)\langle\eta\rangle^\nu{}_{\big|y=0}$$

Where $\alpha \in \mathbb{N}^n$ with $|\alpha| = \nu$. The second factor $(\partial_\xi^\alpha a)(x, (\partial\gamma(x))^T\eta)\langle\eta\rangle^\nu$ is in $\hbar^{-\mu\nu}S_\mu(A_\hbar \circ T(\gamma(x),\eta))$ as we demanded the condition (142) on our symbol a. Thus it remains to show that the other factor is of order $\left(\frac{\hbar}{\langle\eta\rangle}\right)^{\frac{\nu}{2}}$ on the support of a. This is the case because $\rho_x(y)$ vanishes at second order in $y = 0$. Each derivative of $e^{\frac{i}{\hbar}\langle\rho_x(y),\eta\rangle}$ produces a factor $\frac{i}{\hbar}\langle\partial_{y_i}\rho_x(0),\eta\rangle$. But as $\partial_{y_i}\rho_x(0)$ vanishes we need a second derivative, now acting on $\partial_{y_i}\rho_x(y)$, in order to get a contribution. Thus in the worst case $\partial_y^\alpha e^{\frac{i}{\hbar}\langle\rho_x(y),\eta\rangle}$ is of order $\left(\frac{\hbar}{\langle\eta\rangle}\right)^{-\frac{\nu}{2}}$. Thus we have shown that (149) is of the form (146).

The last thing that we have to show is thus, that a_γ fulfills (145). If we consider the definition (147)of a_γ we see that $\partial_x^\alpha\partial_\xi^\beta a_\gamma(\gamma(x),\eta)$ can be written as a sum of terms of the form $\frac{P(\eta)}{\hbar^k}e^{-\frac{i}{\hbar}\gamma(x)\eta}Op_\hbar(b)e^{\frac{i}{\hbar}\gamma(\cdot)\eta}$ where $b \in S_\mu(A_\hbar\langle\xi\rangle^j)$ and $P(\eta)$ is a polynomial in η. The constants j, k and the degree of $P(\eta)$ depend on α and β. Thus writing these terms as oscillating integrals and applying the same arguments as above one gets (145).

We have thus shown that all the conditions for proposition 63 are fulfilled and can conclude that a_γ belongs to $S_\mu(A_\hbar)$ and that (149) is also an asymptotic expansion w.r.t. the order function $A_\hbar$. $\square$

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