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On the poset of computation rules for nonassociative calculus

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Abstract

The symmetric maximum, denoted by \oplus , is an extension of the usual maximum \vee operation so that 0 is the neutral element, and $-x$ is the symmetric (or inverse) of x , i.e., $x \oplus (-x) = 0$. However, such an extension does not preserve the associativity of \vee . This fact asks for systematic ways of parenthesing (or bracketing) terms of a sequence (with more than two arguments) when using such an extended maximum. We refer to such systematic (predefined) ways of parenthesing as computation rules.

As it turns out there are infinitely many computation rules, each of which corresponds to a systematic way of bracketing arguments of sequences. Essentially, computation rules reduce to deleting terms of sequences based on the condition $x \oplus (-x) = 0$. This observation gives raise to a quasi-order on the set of such computation rules: say that rule 1 is below rule 2 if for all sequences of numbers, rule 1 deletes more terms of the sequence than rule 2.

In this paper we present a study of this quasi-ordering of computation rules. In particular, we show that the induced poset of all equivalence classes of computation rules is uncountably infinite, has infinitely many maximal elements, has infinitely many atoms, and it embeds the powerset of natural numbers ordered by inclusion.

Keywords: symmetric maximum, nonassociative algebra, computation rule, partially ordered set

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1 Introduction

Among the wide variety of algebraic structures sofar studied in the realm of aggregation theory, only a few have been considered with nonassociative fundamental operations; see e.g. [2, 5, 6, 7, 9], see also [8] for a recent reference. If commutativity, distributivity, and existence of neutral element and of symmetric element, etc., are often abandoned, associativity remains a desirable property in order to avoid ambiguities when assessing the outcome of composed computations within the algebraic structure. However, in certain situations such nonassociative operations are both natural and necessary: this is the case of the symmetric maximum [5, 6].

For a preliminary discussion, consider the set \mathbb{N} of nonnegative integers and the maximum operation \vee defined on it. Let us try to build on \mathbb{Z} an operation \otimes behaving like a group addition but coinciding with \vee on the positive side, that is, for every $a, b \in \mathbb{Z}$, $a \otimes a = a$ (idempotency), $a \otimes 0 = 0 \otimes a = a$ (neutral element), $a \otimes (-a) = 0$ (symmetry), $a \otimes b = a \vee b$ if $a, b \geq 0$. If such an operation exists, it is necessarily nonassociative as shown below:

$$-3 \otimes (3 \otimes 2) = -3 \otimes 3 = 0 \quad (1)$$

$$(-3 \otimes 3) \otimes 2 = 0 \otimes 2 = 2. \quad (2)$$

One can show [5] that the best definition (in the sense that it fails associativity on the smallest possible domain) of \otimes is given by:

$$a \otimes b = \begin{cases} -(|a| \vee |b|) & \text{if } b \neq -a \text{ and } |a| \vee |b| = -a \text{ or } = -b \\ 0 & \text{if } b = -a \\ |a| \vee |b| & \text{otherwise.} \end{cases} \quad (3)$$

Except for the case $b = -a$, $a \otimes b$ equals the element among the two that has the greatest absolute value.

The main problem is how to interpret this nonassociative operation when evaluating expressions like $\otimes_{i=1}^n a_i$, as it was the case in [6]. The solution proposed in [5, 6] was to define *computation rules*, that is, to define systematic ways of putting parentheses so that no ambiguity occurs. Since we deal with commutative operations, a simple example of a computation rule is the following: put parentheses around each pair of maximal symmetric terms. If we apply this to our example above, this rule corresponds to (2). Another one is to make the computation separately on positive and on negative terms, and to aggregate the result: $(\otimes_i a_i^+) \otimes (\otimes_i a_i^-)$. This corresponds to (1).

It is easy to see that there are many possible computation rules, but to study them, one needs to formalize the intuitive idea of a computation rule. The aim of this paper is twofold: to propose a formal definition of a computation rule, which was lacking in [5], and to study the set of all computation rules endowed with a very natural ordering. As we will see, the poset of computation rules induced by this ordering is uncountable; in fact, from Corollary 24 below, it follows that this poset is equimorphic (equivalent with respect to embeddability) to the power set of positive integers ordered by inclusion. Moreover, we show that the poset of computation rules has infinitely many atoms and has infinitely many maximal elements; these are completely described in Subsections 4.3 and 4.4.

Throughout the paper, we adopt the following notation: if Z is a set of symbols, then $\mathcal{L}(Z)$ denotes the language (set of words, including the empty word ε) built on the alphabet Z .

2 The symmetric maximum

In this section we recall basic concepts and preliminary results needed hereinafter (for further developments see [5, 6] and [8, §9.3]). However, we assume that the reader is familiar with elementary notions in the theory of ordered sets, and refer the reader, e.g., to [1, 3, 4] for basic background.

Let C be a chain endowed with an order \leq and least element 0, and let $C^- = \{-c : c \in C\}$ be its dually isomorphic copy, which we refer to as its symmetric counterpart.

We define $\tilde{C} = C \cup C^-$, and set $-0 = 0$. Since we will only consider countable sequences of elements of \tilde{C} , without loss of generality, we may assume that $\tilde{C} = \mathbb{Z}$, or a finite symmetric interval of it.

Let us introduce a binary operation \otimes on \tilde{C} fulfilling the following independent conditions:

- (I) \otimes coincides with \vee on C^2 .
- (II) $-a$ is the symmetric of a , i.e., $a \otimes (-a) = 0$.
- (III) $-(a \otimes b) = (-a) \otimes (-b)$ for all $a, b \in C$.

As observed in Section 1, (I) and (II) imply that \otimes is not associative. Note also that from (III), it follows that \otimes coincides with the minimum on C^- . The following results are not difficult to verify.

Proposition 1. Under the conditions (I), (II) and (III) above, no operation is associative on a larger domain than that on which the symmetric maximum defined by (3) is associative.

Proposition 2. The symmetric maximum has the following properties:

- (i) \otimes is commutative on \tilde{C} .
- (ii) 0 is the neutral element of \otimes .
- (iii) \otimes is associative on an expression involving $a_1, \dots, a_n \in \tilde{C}$, with $|\{i : a_i \neq 0\}| > 2$, if and only if $\bigvee_{i=1}^n a_i \neq -\bigwedge_{i=1}^n a_i$.
- (iv) \otimes is nondecreasing in each argument on \tilde{C} .

See [6] for a proof. For the sake of completeness, and since this result is central in the paper, we give an alternative proof of (iii).

Proof. We show that (iii) holds. To show that the condition is necessary, suppose that $\bigvee_{i=1}^n a_i = -\bigwedge_{i=1}^n a_i$, and let $1 \leq j, k \leq n$ be such that $a_j = \bigvee_{i=1}^n a_i$ and $a_k = \bigwedge_{i=1}^n a_i$.

If there exists $1 \leq t \leq n$ such that $0 < a_t < a_j$, then

$$\left(\bigvee_{i:0 < a_i \leq a_t} a_i \right) \otimes \left(\left(\bigvee_{i:a_i > a_t} a_i \right) \otimes \left(\bigwedge_{i:a_i \leq 0} a_i \right) \right) = \bigvee_{i:0 < a_i \leq a_t} a_i > 0.$$

However, $(\bigvee_{i:a_i > 0} a_i) \otimes (\bigwedge_{i:a_i \leq 0} a_i) = 0$, and hence \otimes is not associative. The case when there exists $1 \leq t \leq n$ such that $a_k < a_t < 0$ follows similarly.

Therefore we assume that for every $1 \leq i \leq n$, if $a_i > 0$, then $a_i = a_j$, and if $a_i < 0$, then $a_i = a_k$. Since $|\{i : a_i \neq 0\}| > 2$, we have $|\{i : a_i = a_j\}| > 1$ or $|\{i : a_i = a_k\}| > 1$. Without loss of generality, suppose that the former holds, and let $1 \leq t \leq n$, $t \neq j$, such that $a_t = a_j$. In this case, we have

$$a_t \otimes \left(\left(\bigvee_{\substack{i:i \neq t \\ a_i > 0}} a_i \right) \otimes \left(\bigwedge_{i:a_i \leq 0} a_i \right) \right) = a_t > 0,$$

but $(\bigvee_{i:a_i > 0} a_i) \otimes (\bigwedge_{i:a_i \leq 0} a_i) = 0$, and hence \otimes is not associative.

To show that the condition is sufficient, suppose that $\bigvee_{i=1}^n a_i \neq -\bigwedge_{i=1}^n a_i$. We consider the case $a_j = \bigvee_{i=1}^n a_i > -\bigwedge_{i=1}^n a_i$; the other case follows similarly. By (3), $a_j \otimes a_i = a_j$ for every $1 \leq i \leq n$, and thus no matter how we parenthesize a_1, \dots, a_n using \otimes , we always obtain a_j . Hence, \otimes is associative. \square

Property (iii) of Proposition 2 will be the basis for defining computation rules.

3 Computation rules

The lack of associativity of \otimes induces ambiguity when evaluating expressions like $\otimes_{i=1}^n a_i$. To overcome this difficulty, computation rules were proposed in [5, 6], and that amount to eliminating situations where nonassociativity occurs, as characterized by property (iii) in Proposition 2.

Given a sequence $(a_i)_{i \in I}$ with $I \subseteq \mathbb{N}$, we say that it *fulfills associativity* if either $|I| \leq 2$ or $\bigvee_{i \in I} a_i \neq -\bigwedge_{i \in I} a_i$. Hence $\otimes_{i \in I} a_i$ is well-defined if and only if $(a_i)_{i \in I}$ fulfills associativity. Informally speaking, a computation rule is a systematic (predefined) way to delete symbols in a sequence in order to make it associative, provided that this corresponds to some arrangement of parentheses.

Example 3. Consider the following sequence in \mathbb{Z} : 3, 2, 1, 0, -2, -3, -3. A possible way to make the sequence associative is to delete 3, -3, which corresponds to the arrangement

$$(3 \otimes -3) \otimes (-3 \otimes 2 \otimes -2 \otimes 1 \otimes 0) = -3.$$

Another possibility is to delete all occurrences of maximal symmetric symbols, that is, first 3, -3 then 2, -2, which corresponds to:

$$(3 \otimes (-3 \otimes -3)) \otimes (2 \otimes -2) \otimes 1 \otimes 0 = 1.$$

Even though deleting the 3 makes this sequence associative, it does not correspond to any arrangement of parentheses.

In this section we reassemble these ideas and propose a formalism where the intuitive idea of a computation rule is made precise, and show that our formalization fulfills our initial requirements.

Since 0 is the neutral element of \mathbb{O} , we deal with sequences (words) built on $Z = \tilde{C} \setminus \{0\}$, including the empty sequence ε . Hence, we consider the language $\mathcal{L}(Z)$. Nonempty words are denoted by $\sigma = (a_i)_{i \in I}$, where I is a finite index set.

We are interested in computing expressions $\mathbb{O}_{i \in I} a_i$ unambiguously. Since \mathbb{O} is commutative, the order of symbols in the word does not matter, and we can consider any particular ordering of the word, like the decreasing order of the absolute values of the elements in the sequence:

$$(1, 3, -2, -3, 3, 1, 2) \rightarrow (3, 3, -3, 2, -2, 1, 1).$$

Hence, we do not deal with words, but with such ordered sequences. We denote by \mathfrak{S} the set of all such sequences. We introduce a convenient and unambiguous encoding of sequences, based on two mappings. The mapping θ assigns to every $\sigma \in \mathfrak{S}$, the list of the absolute values in σ in decreasing order:

$$\theta(\sigma) = (n_1, \dots, n_q).$$

We assume that $\theta(\sigma)$ is always a finite sequence of arbitrary length. The mapping $\psi : \mathfrak{S} \rightarrow \cup_{i \in \mathbb{N}} (\mathbb{N}_0^2)^i$ is defined by:

$$\psi((a_i)_{i \in I}) = ((p_1, m_1), \dots, (p_q, m_q))$$

where p_k, m_k are the numbers of occurrences of the k -th greatest absolute value of elements in the sequence, p_k being for the positive element, and m_k for the negative one. In other words, for $\theta(\sigma) = (n_1, \dots, n_q)$, the sequence σ can be rewritten after reordering as:

$$\sigma = \underbrace{(n_1, \dots, n_1)}_{p_1 \text{ times}} \underbrace{(-n_1, \dots, -n_1)}_{m_1 \text{ times}}, \dots, \underbrace{(n_q, \dots, n_q)}_{p_q \text{ times}} \underbrace{(-n_q, \dots, -n_q)}_{m_q \text{ times}}.$$

Note that no pair in $\psi(\sigma)$ can be $(0, 0)$.

Example 4. Consider the sequence $\sigma = (1, 3, -3, 2, -2, -2, 3, 1, 1, 1)$. Then

$$\begin{aligned} \theta(\sigma) &= (3, 2, 1) \\ \psi(\sigma) &= ((2, 1), (1, 2), (4, 0)). \end{aligned}$$

Note that $\theta(\sigma)$ and $\psi(\sigma)$ uniquely determine σ . Also, saying that σ fulfills associativity means that either p_1 or m_1 is 0. We denote by \mathfrak{S}_0 the set of sequences which do *not* fulfill associativity.

Definition 5. There exist five elementary rules $\rho_i : \mathfrak{S} \rightarrow \mathfrak{S}$, defined as follows. For any sequence σ with $\psi(\sigma) = ((p_k, m_k)_{k=1, \dots, q})$:

- (i) elementary rule ρ_1 : if $p_1 > 1$ and $m_1 > 0$, the number p_1 is changed into $p_1 = 1$;
- (ii) elementary rule ρ_2 : if $m_1 > 1$ and $p_1 > 0$, the number m_1 is changed into $m_1 = 1$;

- (iii) elementary rule ρ_3 : if $p_1 > 0$, $m_1 > 0$, the pair (p_1, m_1) is changed into $(p_1 - c, m_1 - c)$, where $c = p_1 \wedge m_1$. If this results in the pair $(0,0)$, then this pair is deleted, and all subsequent pairs (p_k, m_k) , $k = 2, 3, \dots$, are renumbered as (p_{k-1}, m_{k-1}) .
- (iv) elementary rule ρ_4 : if $p_1 > 0$, $m_1 > 0$, and if $p_2 > 0$, the number p_2 is changed into $p_2 = 0$. If this results in the pair $(0,0)$, then this pair is deleted, and all subsequent pairs (p_k, m_k) , $k = 3, 4, \dots$, are renumbered as (p_{k-1}, m_{k-1}) .
- (v) elementary rule ρ_5 : if $p_1 > 0$, $m_1 > 0$, and if $m_2 > 0$, the number m_2 is changed into $m_2 = 0$. If this results in the pair $(0,0)$, then this pair is deleted, and all subsequent pairs (p_k, m_k) , $k = 3, 4, \dots$, are renumbered as (p_{k-1}, m_{k-1}) .

Rules ρ_1, \dots, ρ_5 have no action (i.e., $\rho_i(\sigma) = \sigma$) if the conditions of application are not satisfied.

We define the (computation) alphabet as $\Psi = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$.

Definition 6. A *computation rule* R is any word built on Ψ , i.e., $R \in \mathcal{L}(\Psi)$. We say that R is a *well-formed computation rule* (w.f.c.r.) if for any sequence $\sigma \in \mathfrak{S}$ we have $R(\sigma) \in \mathfrak{S} \setminus \mathfrak{S}_0$. We denote by \mathfrak{R} the set of well-formed computation rules.

For example, $\rho_2\rho_3\rho_1$, $\rho_4^*\rho_1$, $(\rho_1\rho_3)^*(\rho_4\rho_5)^*$ are computation rules, where as usual w^* denotes the infinite concatenation $www\cdots$ of the word w (we recall that words are read from left to right). Observe that only the two latter rules are well-formed.

Note that from Definition 5, we have $R(\sigma) = \sigma$ for any rule R and any sequence σ in $\mathfrak{S} \setminus \mathfrak{S}_0$. We give examples of w.f.c.r.'s that include those already proposed in [5] (we leave to the reader the proof that they are well-formed):

- (i) $\langle \cdot \rangle_0 = \rho_3^*$,
- (ii) $\langle \cdot \rangle_- = (\rho_1\rho_2\rho_3)^*$,
- (iii) $\langle \cdot \rangle_-^+ = (\rho_4\rho_5)^*\langle \cdot \rangle_- = (\rho_4\rho_5)^*\rho_1\rho_2\rho_3$,
- (iv) $\langle \cdot \rangle_{\text{pess}} = (\rho_4\rho_5)^*\rho_1\rho_3$,
- (v) $\langle \cdot \rangle_{\text{opt}} = (\rho_4\rho_5)^*\rho_2\rho_3$,
- (vi) $\langle \cdot \rangle_L = (\rho_1\rho_3)^*$,
- (vii) $\langle \cdot \rangle_R = (\rho_2\rho_3)^*$.

Note that $\langle \sigma \rangle_-^+ = \varepsilon$ for all $\sigma \in \mathfrak{S}_0$.

We use $\mathbb{Q}(R(\sigma))$ to denote the value of $\mathbb{Q}_{i \in I} a_i$ after applying the computation rule $R \in \mathfrak{R}$ to $\psi(\sigma) = \psi((a_i)_{i \in I})$. To compute $\mathbb{Q}(R(\sigma))$, one needs to delete symbols in the sequence $\theta(\sigma)$ exactly as they are deleted in $\psi(\sigma)$. We say that $R, R' \in \mathfrak{R}$ are *equivalent*, denoted by $R \sim R'$, if for any sequence $\sigma \in \mathfrak{S}$ we have $\mathbb{Q}(R(\sigma)) = \mathbb{Q}(R'(\sigma))$.

The next fundamental theorem shows that our setting covers all possible ways of putting parentheses on words in $\mathcal{L}(Z)$ in order to make them associative¹.

¹It is noteworthy to observe that this framework is suitable for any nonassociative operation that satisfies (ii) and (iii) of Proposition 2.

Theorem 7. Any computation rule applied to some $\sigma \in \mathfrak{S}$ corresponds to an arrangement of parentheses and a permutation on σ . Conversely, any arrangement of parentheses and permutation on some $\sigma \in \mathfrak{S}$ making the sequence associative is equivalent to a computation rule applied to σ .

Proof. Let us define 5 basic rules applied on any sequence σ with $\psi(\sigma) = (p_k, m_k)_{k \in K}$ as follows:

- (i) basic rule $\rho'_1{}^k$, for a given $k \in K$: if $p_k > 1$, the number p_k is changed into $p_k - 1$;
- (ii) basic rule $\rho'_2{}^k$, for a given $k \in K$: if $m_k > 1$, the number m_k is changed into $m_k - 1$;
- (iii) basic rule $\rho'_3{}^k$, for a given $k \in K$: if $p_k > 0, m_k > 0$, the pair (p_k, m_k) is changed into $(p_k - 1, m_k - 1)$;
- (iv) basic rule $\rho'_4{}^k$, for $k > 1$: if $p_k > 0$, the number p_k is changed into $p_k - 1$;
- (v) basic rule $\rho'_5{}^k$, for $k > 1$: if $m_k > 0$, the number m_k is changed into $m_k - 1$,

For all these rules, if a pair (0,0) appears, it is immediately deleted. Observe that the elementary rules are concatenations of the above basic rules. Indeed, we have:

$$\rho_1 = (\rho'_1{}^1)^*, \quad \rho_2 = (\rho'_2{}^1)^*, \quad \rho_3 = (\rho'_3{}^1)^*, \quad \rho_4 = (\rho'_4{}^2)^*, \quad \rho_5 = (\rho'_5{}^2)^*.$$

Claim 1. Any way of parenthesing a word in $\mathcal{L}(Z)$ corresponds to a word (rule) in $\mathcal{L}(\{\rho'_1{}^k, \dots, \rho'_5{}^k\}_{k \in \mathbb{N}})$, and conversely.

Proof of Claim 1. Indeed, consider a word $w \in \mathcal{L}(Z)$: parentheses are put around 2 consecutive elements, like $(a \otimes b)$, where a or b can be the result of a pair of parentheses too. Only three cases can occur:

- (i) either $a = b$, then $(a \otimes b) = a = b$. This corresponds to basic rules $\rho'_1{}^k$ (if $a > 0$) or $\rho'_2{}^k$ (if $a < 0$) for a suitable k ;
- (ii) or $a = -b$, then $(a \otimes b) = 0$. This corresponds to the basic rule $\rho'_3{}^k$ for a suitable k ;
- (iii) otherwise $|a| < |b|$ (or $|a| > |b|$). Then $(a \otimes b) = b$ and this corresponds to the basic rules $\rho'_4{}^k$ (if $a > 0$) or $\rho'_5{}^k$ (if $a < 0$) for a suitable k . \square

Claim 2. Given a sequence $\sigma \in \mathfrak{S}_0$, for any rule ρ in $\mathcal{L}(\{\rho'_1{}^k, \dots, \rho'_5{}^k\}_{k \in \mathbb{N}})$ making σ associative, there exists a computation rule R in $\mathcal{L}(\Psi)$ such that $\otimes(\rho(\sigma)) = \otimes(R(\sigma))$.

Proof of Claim 2. We have already established that any elementary rule is a particular rule in $\mathcal{L}(\{\rho'_1{}^k, \dots, \rho'_5{}^k\}_{k \in \mathbb{N}})$, and therefore this is true also for any computation rule in $\mathcal{L}(\Psi)$.

Take then any rule ρ in $\mathcal{L}(\{\rho'_1{}^k, \dots, \rho'_5{}^k\}_{k \in \mathbb{N}})$ making σ associative. The result $\otimes(\rho(\sigma))$ is some number in σ , say δn_k , with $\delta = 1$ or -1 (i.e., the k th positive or negative symbol in $\theta(\sigma)$). Let us construct a computation rule R such that $\otimes(R(\sigma)) = \delta n_k$ as follows:

- Suppose $k = 1, \delta = 1$ (provided $p_1 > 1$). Then $R = \rho_2 \rho_3$. For the case $\delta = -1$, we find $R = \rho_1 \rho_3$.

- Suppose $k > 1$, $\delta = 1$ (provided $p_k > 0$) or $\delta = -1$ (provided $m_k > 0$). Apply the following algorithm:

- Initialization: $R \leftarrow \varepsilon$
- For $i = 2$ to $k - 1$, Do:
 - * If $p_i = 0$ put $R \leftarrow R\rho_5$
 - * If $m_i = 0$ put $R \leftarrow R\rho_4$
 - * Otherwise put $R \leftarrow R\rho_4\rho_5$
- Case $\delta = 1$: if $m_k > 0$, $R \leftarrow R\rho_5$.
- Case $\delta = -1$: if $p_k > 0$, $R \leftarrow R\rho_4$.
- $R \leftarrow R\rho_1\rho_2\rho_3$

By construction, R is equivalent to ρ on σ , and the proof of the claim is now complete. \square

Theorem 7 now follows from Claims 1 and 2. \square

Remark 8. Note that a well-formed rule in $\mathcal{L}(\{\rho_1^k, \dots, \rho_5^k\}_{k \in \mathbb{N}})$ (i.e., making any σ associative) is not necessarily equivalent to a w.f.c.r. in \mathfrak{R} . For instance, consider the well-formed rule $\rho = \rho_5^3((\rho_1^1)^*(\rho_2^1)^*\rho_3^1)^*$, and apply it on the sequences:

$$\sigma = (2, 3)(1, 0)(0, 1)(2, 1), \quad \sigma' = (2, 3)(1, 1)(0, 1)(2, 0).$$

Then $\mathcal{O}(\rho(\sigma)) = n_2$ and $\mathcal{O}(\rho(\sigma')) = n_4$. Let us try to build an equivalent w.f.c.r. $R \in \mathfrak{R}$. Since the second pair in σ is the final result, one cannot touch it. Therefore, R contains only ρ_1, ρ_2, ρ_3 , and thus one finds $-n_3$ on σ' . Hence, compositions of basic rules may result in rules more general than our computation rules. However, those rules that are not computation rules are rather artificial.

Hereinafter, we will make use of the following “factorization scheme” for computation rules.

Lemma 9. Let R be a w.f.c.r. in \mathfrak{R} .

- (i) **Factorization:** Rule R can be factorized into a composition

$$R = T_1 T_2 \cdots T_i \cdots \quad (4)$$

where each term has the form $T_i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$, with $\omega_i \in \mathcal{L}(\{\rho_4, \rho_5\})$ (possibly empty), and $a_i, b_i \in \{0, 1\}$.

- (ii) **Simplification:** Suppose that in (4) there exists $j \in \mathbb{N}$ such that $\omega_j = \omega\rho_4^*$ or $\omega\rho_5^*$ for some $\omega \in \mathcal{L}(\{\rho_4, \rho_5\})$, or that ρ_4 and ρ_5 alternate infinitely many times in ω_j . Let

$$k_1 = \min\{j : \omega_j = \omega\rho_4^* \text{ or } \omega\rho_5^*\}, \text{ and}$$

$$k_2 = \min\{j : \rho_4 \text{ and } \rho_5 \text{ alternate infinitely many times in } \omega_j\}.$$

- If $k_1 < k_2$, then $R \sim T_1 \cdots T_{k_1}$.
- Otherwise, $k_2 \leq k_1$, and $R \sim T_1 \cdots T'_{k_2}$, where $T'_{k_2} = (\rho_4\rho_5)^* \rho_1^{a_{k_2}} \rho_2^{b_{k_2}} \rho_3$.

Proof. Let R be a w.f.c.r. Then R is necessarily infinite, otherwise one can always construct a sequence σ such that $R(\sigma) \notin \mathfrak{S} \setminus \mathfrak{S}_0$. Also, ρ_3 necessarily belongs to R , otherwise the sequence σ with $\psi(\sigma) = (2, 1)$ would not be made associative by R . Therefore, the word R can be cut into terms where ρ_3 acts as a separator, i.e., $R = R_1\rho_3R_2\rho_3\cdots$, with $R_i \in \mathcal{L}(\{\rho_1, \rho_2, \rho_4, \rho_5\})$. Now observe that ρ_1 and ρ_1^k are equivalent for any $k > 1$, and the same holds for ρ_2 . Moreover, the order between ρ_1, ρ_2 and ω_i is unimportant because none of these symbols can make the sequence σ associative (i.e., the rule will not stop after applying these elementary rules), and each of them applies on a different symbol of σ . This proves that each term T_i can be written in form (4).

Observe that since R is infinite, there can be infinitely many factors T_i or finitely many, provided one factor T_i has an infinite ω_i . In the first case, there is no last factor and the proof of (i) is complete. In the second case, it remains to prove that the last factor T_l has the same form, i.e., it ends with ρ_3 . Suppose on the contrary that there are elementary rules ρ_4, ρ_5 after ρ_3 . If σ is made associative after applying ρ_3 , then the rule stops and the remaining ρ_4, ρ_5 are useless. If not, it is because ρ_3 has acted on a pair (p, p) with $p > 0$. But if the next pair is, say, $(1, 1)$, σ will not be made associative by the remaining ρ_4, ρ_5 , contradicting the fact that R is well-formed.

Let us prove (ii). Suppose first that $k_2 \leq k_1$. Observe that any ω_i where ρ_4, ρ_5 alternate infinitely many times is equivalent to $(\rho_4\rho_5)^*$. Moreover, $(\rho_4\rho_5)^*$ deletes all pairs after the current one. Therefore, it remains only the current pair, and $\rho_1^{a_{k_2}}\rho_2^{b_{k_2}}\rho_3$ necessarily stops on it, for any value of a_{k_2}, b_{k_2} .

Suppose now that $k_1 < k_2$, and $\omega_{k_1} = \omega\rho_4^*$ (the other case is similar). Then ρ_4^* deletes all pairs after the current pair of the form $(p', 0)$, and stops at the first pair of the form (p', m') with $m' > 0$, which is transformed into $(0, m')$. Then $\rho_1^{a_{k_1}}\rho_2^{b_{k_1}}\rho_3$ makes the current pair either of the form $(0, 0)$, or $(p, 0)$ or $(0, m)$. In the two last cases, R stops. In the first case, the current term is deleted, and the next pair encountered is $(0, m')$, where the rule stops. The proof of (ii) is now complete. \square

Remark 10. Note that (ii) of Lemma 9 does not refer to every ω containing a ρ_4^* or a ρ_5^* . For instance, if $\omega = \rho_5\rho_4^*\rho_5$, then the subsequent terms of R are relevant.

Remark 11. If $T_i = \omega_i\rho_1^{a_i}\rho_2^{b_i}\rho_3$, where $\omega_i \neq (\rho_4\rho_5)^*, \omega\rho_4^*, \omega\rho_5^*$, for any $\omega \in \mathcal{L}(\{\rho_4, \rho_5\})$, then there is σ_i such that $T_i(\sigma_i) = \varepsilon$ and $T_i(\sigma_i\sigma) = \sigma$ for every $\sigma \in \mathfrak{S}$.

We refer to the compositions given in (ii) as *factorized irredundant forms* of computation rules. For instance, $\langle \cdot \rangle_-^+$ can be factorized into two equivalent compositions

$$\langle \cdot \rangle_-^+ = (\rho_4\rho_5)^*(\rho_1\rho_2\rho_3)^* = (\rho_4\rho_5)^*\rho_1\rho_2\rho_3.$$

but only the second is a factorized irredundant form. Note that our previous examples of w.f.c.r.'s are given in factorized irredundant forms.

Now it is natural to ask whether two equivalent rules have necessarily the same factorized irredundant form. The next proposition shows that there is a unique factorized irredundant form for each equivalence class of computation rules.

Proposition 12. Let $T = T_1 \cdots T_n$ and $T' = T'_1 \cdots T'_m$ be two rules in factorized irredundant form, where n, m may be infinite. Then $T \sim T'$ if and only if $n = m$ and for every $1 \leq i \leq n$, $T_i = T'_i$.

Proof. Clearly, the conditions are sufficient. So let us prove that they are also necessary.

First, we show that $n = m$. For a contradiction, suppose that $n \neq m$, say $n < m$. In particular, for every $j < m$, ω'_j is not of the $\omega\rho_4^*$ nor $\omega\rho_5^*$ form for any $\omega \in \mathcal{L}(\{\rho_4, \rho_5\})$, and $\omega'_j \neq (\rho_4\rho_5)^*$.

Note that $(a_1, b_1) = (a'_1, b'_1)$, otherwise $T \not\sim T'$ (just consider $(2, 1)$, $(2, 2)$, or $(1, 2)$). Thus, to verify that $T_1 = T'_1$, it suffices to show that $\omega_1 = \omega'_1$. Let p and p' be the number of times that ρ_4 and ρ_5 alternate in ω_1 and ω'_1 , respectively. It is easy to see that $p = p'$ (just consider sequences of the form $(1, 1)(1, 0)^a(0, 1)[(1, 0)(0, 1)]^q(1, 0)(0, 1)^b$, for suitable $a, b \in \{0, 1\}$ and $q \in \mathbb{N}$). Moreover, either both start with ρ_4 or both start with ρ_5 (just consider strings of the form $(1, 1)[(1, 0)(0, 1)]^p$).

So suppose both start with ρ_4 and $p = 2t - 1$ (the case $p = 2t$ is similar), say

$$\omega_1 = \rho_4^{l_1} \rho_5^{r_1} \cdots \rho_4^{l_t} \rho_5^{r_t} \quad \text{and} \quad \omega'_1 = \rho_4^{l'_1} \rho_5^{r'_1} \cdots \rho_4^{l'_t} \rho_5^{r'_t}$$

where $r_t, r'_t \neq 0$, and let $k = \min\{j : l_j \neq l'_j \text{ or } r_j \neq r'_j\}$, say $l_k < l'_k$ (the other cases are dealt with similarly). Then, for

$$\sigma = (1, 1)[(1, 0)(0, 1)]^{k-1}(1, 0)^{l_k+1}(0, 1)[(1, 0)(0, 1)]^{t-k}$$

$\mathbb{Q}(T(\sigma)) > \mathbb{Q}(T'(\sigma))$, which contradicts $T \sim T'$. Hence, $\omega_1 = \omega'_1$, and we conclude $T_1 = T'_1$. In fact, following exactly the same steps, one can verify that $T_i = T'_i$, for every $i < n$.

Now, as in the case above $(a_n, b_n) = (a'_n, b'_n)$, otherwise $T \not\sim T'$. Moreover, by assumption, we have that $\omega_n = \omega\rho_4^*$ or $\omega\rho_5^*$ for some $\omega \in \mathcal{L}(\{\rho_4, \rho_5\}) \setminus \{(\rho_4\rho_5)^*\}$, or that $\omega_n = (\rho_4\rho_5)^*$. Since $\omega'_n \neq (\rho_4\rho_5)^*$, $\omega_n \neq (\rho_4\rho_5)^*$. Hence, ρ_4 and ρ_5 must alternate the same number of times, say

$$\omega_n = \rho_4^{l_1} \rho_5^{r_1} \cdots \rho_4^{l_t} \rho_5^{r_t} \quad \text{and} \quad \omega'_n = \rho_4^{l'_1} \rho_5^{r'_1} \cdots \rho_4^{l'_t} \rho_5^{r'_t},$$

where either $l_t = * \neq l'_t$ and $r_t = r'_t = 0$, or $r_t = * \neq r'_t$ and $l_t, l'_t > 0$. Without loss of generality, suppose that the latter holds. Then, for

$$\sigma = (1, 1)[(1, 0)(0, 1)]^{n-1}(1, 0)(0, 1)^{r'_t+1}$$

$\mathbb{Q}(T(\sigma)) < \mathbb{Q}(T'(\sigma))$, again a contradiction. Using Lemma 9 (ii), we see that all possible cases have been considered and, since each leads to a contradiction, we have $n = m$.

Now, by making use of (concatenations of) sequences of the form

$$(1, 1)(1, 0)^a(0, 1)[(1, 0)(0, 1)]^q(1, 0)(0, 1)^b(2, 1)^c(2, 2)^d(1, 2)^e,$$

if both have infinitely many terms T_i, T'_i , then $T_i = T'_i$ for every $i \in \mathbb{N}$, and if T and T' have the same (finite) number of terms, say n , then $T_i = T'_i$ for every $i < n$, and $(a_n, b_n) = (a'_n, b'_n)$.

Thus, to complete the proof it remains to show that in the latter case, we have $\omega_n = \omega'_n$; in fact, both ω_n and ω'_n are $(\rho_4\rho_5)^*$, or $\omega\rho_4^*$ or $\omega\rho_5^*$ for some $\omega \in \mathcal{L}(\{\rho_4, \rho_5\}) \setminus \{(\rho_4\rho_5)^*\}$.

For the sake of a contradiction, suppose first that $\omega_n = (\rho_4\rho_5)^*$ but $\omega'_n = \omega\rho_4^*$ or $\omega'_n = \omega\rho_5^*$ where ρ_4 and ρ_5 alternate finitely many times in ω , say p times. Then

$$\mathbb{Q}(T(\sigma(1, 1)[(1, 0)(0, 1)]^{p+2})) < \mathbb{Q}(T'(\sigma(1, 1)[(1, 0)(0, 1)]^{p+2})),$$

where σ is the concatenation of the sequences σ_i , $1 \leq i < n$, given in Remark 11.

Now suppose that $\omega_n = \omega\rho_4^*$ and $\omega'_n = \omega'\rho_5^*$ where ρ_4 and ρ_5 alternate finitely many times in ω and ω' , say p and p' times, respectively. (The remaining cases can be dealt with similarly.) Without loss of generality, suppose that $p \leq p'$, then taking a as the ceiling of $\frac{p}{2}$ we have

- $\mathfrak{O}(T(\sigma(1, 1)[(1, 0)(0, 1)]^a(0, 1))) > \mathfrak{O}(T(\sigma(1, 1)[(1, 0)(0, 1)]^a(0, 1)))$, if ω and ω' both start with ρ_4 or ρ_5 , or ω and ω' start with ρ_5 and ρ_4 , respectively, and
- $\mathfrak{O}(T(\sigma(1, 1)(0, 1)[(1, 0)(0, 1)]^a(0, 1))) > \mathfrak{O}(T(\sigma(1, 1)(0, 1)[(1, 0)(0, 1)]^a(0, 1)))$, if ω and ω' start with ρ_4 and ρ_5 , respectively,

where again σ is the concatenation of the sequences σ_i , $1 \leq i < n$, given in Remark 11.

Since both cases yield the desired contradiction, the proof is now complete. \square

4 The poset $(\mathfrak{R}/\sim, \leq)$ of computation rules

The above considerations allow us to focus on the quotient \mathfrak{R}/\sim of equivalence classes rather than on the whole set of w.f.c.r.'s. Moreover, by making use of Lemma 9, we can focus on factorized irredundant forms.

We consider the following order \leq on \mathfrak{R}/\sim , which was introduced in [5]. Let R, R' be two computation rules in \mathfrak{R}/\sim and, for each sequence $\sigma = (a_i)_{i \in I}$, let J_σ and J'_σ , $J_\sigma, J'_\sigma \subseteq I$, be the sets of indices of the terms in σ deleted by R and R' , respectively. Then, we write $R \leq R'$ if for all sequences $\sigma \in \mathfrak{S}$ we have $J_\sigma \supseteq J'_\sigma$. To simplify our exposition, we use $R(\sigma) \sqsubseteq R'(\sigma)$ to denote the fact that $J_\sigma \supseteq J'_\sigma$. If $J_\sigma = J'_\sigma$, then we simply write $R(\sigma) = R'(\sigma)$. Moreover, we may adopt the same notation to arbitrary substrings of w.f.c.r.'s.

It is easy to verify that \leq is reflexive and transitive (but, as we will see, not linear). Also, it is antisymmetric: if two rules R, R' delete exactly the same terms, i.e., $R \leq R'$ and $R' \leq R$, then they are equivalent. Conversely, it follows from Proposition 12 that if two rules are equivalent, then they have the same factorized irredundant form, therefore $R \leq R'$ and $R' \leq R$. Thus, $(\mathfrak{R}/\sim, \leq)$ is a poset (partially ordered set). In what follows, we make no distinction between w.f.c.r.'s and the elements of \mathfrak{R}/\sim , which will be always written in the factorized irredundant form.

4.1 Preliminary results

In the following, let $\omega, \omega' \in \mathcal{L}(\{\rho_4, \rho_5\})$, and $a, b, c, d \in \{0, 1\}$.

Lemma 13. Let $T, T' \in \mathfrak{R}/\sim$. If $T \geq T'$, then $\omega\rho_1^a\rho_2^b\rho_3T \geq \omega\rho_1^a\rho_2^b\rho_3T'$. Moreover, if ρ_4 and ρ_5 alternate finitely many times in ω , then $\omega\rho_1^a\rho_2^b\rho_3T > \omega\rho_1^a\rho_2^b\rho_3T'$ (resp. $\omega\rho_1^a\rho_2^b\rho_3T \parallel \omega\rho_1^a\rho_2^b\rho_3T'$) if and only if $T > T'$ (resp. $T \parallel T'$).

Proof. From Lemma 9 (ii), if ρ_4 and ρ_5 alternate infinitely many times in ω , then

$$\omega\rho_1^a\rho_2^b\rho_3T = (\rho_4\rho_5)^*\rho_1^a\rho_2^b\rho_3 = \omega\rho_1^a\rho_2^b\rho_3T'.$$

So we may assume that $\omega = \rho_4^{a_1} \rho_5^{b_1} \cdots \rho_4^{a_n} \rho_5^{b_n}$, with $a_i, b_i \in \mathbb{N} \cup \{*\}$. We assume also that $a_i \neq 0$ for $2 \leq i \leq n$, and $b_i \neq 0$ for $1 \leq i \leq n-1$. We treat the case $a_1 \neq 0, b_n \neq 0$, the remaining cases follow similarly.

To see that $\omega \rho_1^a \rho_2^b \rho_3 T \geq \omega \rho_1^a \rho_2^b \rho_3 T'$, just note that for every string $\gamma = (p_1, m_1) \cdots (p_k, m_k)$, $(p_1, m_1) \geq (1, 1)$, we have

$$\begin{aligned} \omega \rho_1^a \rho_2^b \rho_3 T(\gamma) &= (\rho_1^a \rho_2^b \rho_3)(p_1, m_1) T((p'_1, m'_1) \cdots (p'_{k'}, m'_{k'})) \\ &\sqsupseteq (\rho_1^a \rho_2^b \rho_3)(p_1, m_1) T'((p'_1, m'_1) \cdots (p'_{k'}, m'_{k'})) = \omega \rho_1^a \rho_2^b \rho_3 T'(\gamma). \end{aligned}$$

Hence, $\omega \rho_1^a \rho_2^b \rho_3 T \geq \omega \rho_1^a \rho_2^b \rho_3 T'$, and by antisymmetry, the strict inequality occurs if and only if $T > T'$. Similarly, if $T \parallel T'$, then by taking

- $\sigma_>$ and $\sigma_<$ such that $T(\sigma_>) \sqsupset T'(\sigma_>)$ and $T(\sigma_<) \sqsubset T'(\sigma_<)$, respectively, and
- $\gamma_> = \sigma \sigma_>$ and $\gamma_< = \sigma \sigma_<$, where σ is given in Remark 11 (for $T_i = \omega \rho_1^a \rho_2^b \rho_3$),

we can verify that $\omega \rho_1^a \rho_2^b \rho_3 T \parallel \omega \rho_1^a \rho_2^b \rho_3 T'$. This completes the proof of the lemma. \square

By repeated applications of Lemma 13, we have the following corollary.

Corollary 14. Let $T, T' \in \mathfrak{A}/\sim$, and let $R = T_1 T_2 \cdots T_m \in \mathcal{L}(\Psi)$, where $T_i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$. If $T \geq T'$, then $RT \geq RT'$. Furthermore, if ρ_4 and ρ_5 alternate finitely many times in each ω_i , then $RT > RT'$ (resp. $RT \parallel RT'$) if and only if $T > T'$ (resp. $T \parallel T'$).

Remark 15. In fact, by Corollary 14 it follows that if ρ_4 and ρ_5 alternate finitely many times in each ω_i , then $T \geq T'$ (resp. $T \parallel T'$) if and only if $RT \geq RT'$ (resp. $RT \parallel RT'$).

Lemma 16. Let $T, T' \in \mathcal{L}(\Psi)$ such that $T \geq T'$. Then $\omega \rho_1^a \rho_2^b \rho_3 T \geq \omega \rho_1^c \rho_2^d \rho_3 T'$ if and only if $(a, b) = (c, d)$ or $(c, d) = (1, 1)$. Moreover, $\omega \rho_1^a \rho_2^b \rho_3 T > \omega \rho_1^c \rho_2^d \rho_3 T'$ if and only if $(a, b) \neq (c, d) = (1, 1)$.

Proof. To see that the condition in the first claim is sufficient, observe that if $(a, b) = (c, d)$, then by Lemma 13 $\omega \rho_1^a \rho_2^b \rho_3 T \geq \omega \rho_1^c \rho_2^d \rho_3 T'$. If $(c, d) = (1, 1)$, then for every nonassociative string $\sigma = (p_1, m_1) \cdots (p_k, m_k)$ (i.e., $(p_1, m_1) \geq (1, 1)$) we have

$$\begin{aligned} \omega \rho_1^a \rho_2^b \rho_3 T(\sigma) &= (\rho_1^a \rho_2^b \rho_3)(p_1, m_1) T((p'_1, m'_1) \cdots (p'_{k'}, m'_{k'})) \\ &\sqsupseteq T'((p'_1, m'_1) \cdots (p'_{k'}, m'_{k'})) = \omega \rho_1^c \rho_2^d \rho_3 T'(\sigma), \end{aligned}$$

and hence $\omega \rho_1^a \rho_2^b \rho_3 T \geq \omega \rho_1^c \rho_2^d \rho_3 T'$. Moreover, if $(a, b) \neq (c, d) = (1, 1)$, then by considering $(2, 1)$ if (a, b) equals $(0, 1)$ or $(0, 0)$, and $(1, 2)$ if (a, b) equals $(1, 0)$, one can easily verify that $\omega \rho_1^a \rho_2^b \rho_3 T > \omega \rho_1^c \rho_2^d \rho_3 T'$, thus showing that the condition of the second claim is also sufficient.

To verify that the conditions in the first and second claims are also necessary, it suffices to show that if $(a, b), (c, d) \neq (1, 1)$ and $(a, b) \neq (c, d)$, then $\omega \rho_1^a \rho_2^b \rho_3 T \parallel \omega \rho_1^c \rho_2^d \rho_3 T'$. But this fact can be easily verified by making use of the strings $(2, 1), (1, 2)$ or $(2, 2)$. \square

Lemma 17. If ρ_4 and ρ_5 alternate infinitely many times in ω but not in ω' , then $\omega \rho_1^a \rho_2^b \rho_3 \sim \omega \rho_1^a \rho_2^b \rho_3 T < \omega' \rho_1^a \rho_2^b \rho_3 T'$, for every $T, T' \in \mathcal{L}(\Psi)$.

Proof. Let $\omega = (\rho_4\rho_5)^*$ and $\omega' = \rho_4^{a_1}\rho_5^{b_1}\cdots\rho_4^{a_n}\rho_5^{b_n}$, with $a_i, b_i \in \mathbb{N} \cup \{*\}$. (By Lemma 9, $\omega\rho_1^a\rho_2^b\rho_3 \sim \omega\rho_1^a\rho_2^b\rho_3T$, for every $T \in \mathcal{L}(\Psi)$.) Then for every string $\gamma = (p_1, m_1) \cdots (p_k, m_k)$, $(p_1, m_1) \geq (1, 1)$,

$$\omega\rho_1^a\rho_2^b\rho_3(\gamma) = (\rho_1^a\rho_2^b\rho_3)(p_1, m_1) \sqsubseteq (\rho_1^a\rho_2^b\rho_3)(p_1, m_1)T'((p'_1, m'_1) \cdots (p'_{k'}, m'_{k'})) = \omega'\rho_1^a\rho_2^b\rho_3T'(\gamma).$$

Hence, $\omega\rho_1^a\rho_2^b\rho_3 \leq \omega'\rho_1^a\rho_2^b\rho_3T'$. For $1 \leq i \leq n$, let $\alpha_i = 0$ (resp. $\beta_i = 0$) if $a_i = 0$ (resp. $b_i = 0$) and $\alpha_i = 1$ (resp. $\beta_i = 1$) otherwise. By considering

$$\sigma = (1, 1)(1, 0)^{\alpha_1}(0, 1)^{\beta_1} \cdots (1, 0)^{\alpha_n}(0, 1)^{\beta_n},$$

one can easily verify that $\omega\rho_1^a\rho_2^b\rho_3 < \omega'\rho_1^a\rho_2^b\rho_3T'$. \square

Lemma 18. Let $\omega = \rho_4^{a_1}\rho_5^{b_1}\cdots\rho_4^{a_n}\rho_5^{b_n}$, $n \geq 0$, and let $\omega' = \rho_4^{a'_1}\rho_5^{b'_1}\cdots\rho_4^{a'_m}\rho_5^{b'_m}$, $m \geq 0$. For $T = \omega\rho_1^a\rho_2^b\rho_3\langle \cdot \rangle_-^+$ and $T' = \omega'\rho_1^a\rho_2^b\rho_3\langle \cdot \rangle_-^+$, the following assertions hold:

- (i) If $n = m = 1$, then $T \parallel T'$ if and only if $b_1 \neq b'_1$, or $[b_1 = b'_1 = 0 \text{ and } a_1 \neq a'_1]$.
- (ii) If $n = m > 1$, then $T \parallel T'$ if and only if
 - a) $b_n \neq b'_n$, or
 - b) $b_n = b'_n = 0$ and $a_n \neq a'_n$, or
 - c) $b_n = b'_n \neq 0$ and there exists $1 \leq j < n$ such that $(a_j, b_j) \neq (a'_j, b'_j)$, or
 - d) $b_n = b'_n = 0$, $a_n = a'_n \neq 0$, and $a_{n-1} \neq a'_{n-1}$ or there exists $1 \leq j < n - 1$ such that $(a_j, b_j) \neq (a'_j, b'_j)$.
- (iii) If $n \neq m$, then $T \parallel T'$.

Proof. We may assume that $a_i \neq 0$ for $2 \leq i \leq n$ and $b_i \neq 0$ for $1 \leq i \leq n - 1$, and that $a'_j \neq 0$ for $2 \leq j \leq m$ and $b'_j \neq 0$ for $1 \leq j \leq m - 1$.

(i): To prove sufficiency, suppose first that $b_1 \neq b'_1$, say $b_1 > b'_1$. Then, by considering $\sigma = (1, 1)(0, 1)^{b_1}(1, 1)$ and $\sigma' = (1, 1)(0, 1)^{b'_1+1}$, we see that $T \not\leq T'$ and $T \not\geq T'$, respectively.

So suppose that $b_1 = b'_1 = 0$ and $a_1 \neq a'_1$, say $a_1 > a'_1$. Then, by considering $\sigma = (1, 1)(1, 0)^{a_1}(1, 1)$ and $\sigma' = (1, 1)(1, 0)^{a'_1+1}$, we see that $T \not\leq T'$ and $T \not\geq T'$, respectively.

We prove necessity by counterposition. Observe first that if $b_1 = b'_1$ and $(a_1, b_1) = (a'_1, b'_1)$, then $T = T'$. So suppose that $b_1 = b'_1 \neq 0$ and $a_1 \neq a'_1$, say $a_1 > a'_1$. We claim that $T < T'$. By making use of $\sigma = (1, 1)(1, 0)^{a_1+1}$, we see that $T(\sigma) \sqsubset T'(\sigma)$. Thus, we only have to show that $T \leq T'$.

Let $\sigma = (p_1, m_1)(p_2, m_2) \cdots (p_k, m_k)$. If the action of $\rho_1^a\rho_2^b\rho_3$ does not delete all terms of (p_1, m_1) , then $T(\sigma) \sqsubseteq T'(\sigma)$ since the ‘‘subrule’’ $\langle \cdot \rangle_-^+$ in T and T' does not act on σ ; hence, without loss of generality, we may further assume that $(p_1, m_1) = (1, 1)$.

Now, if $m_2 \neq 0$, then $T(\sigma) \sqsubseteq T'(\sigma)$, and if $p_2 = 0$, then $T(\sigma) = T'(\sigma)$; hence, we may assume $m_2 = 0$ and $p_2 \neq 0$. In fact, we may suppose that $p_2 \leq a'_1$ since, otherwise, $T(\sigma) \sqsubset T'(\sigma)$.

Under the assumption that $m_2 = 0$ and $p_2 \leq a'_1$, and applying the same reasoning to (p_3, m_3) , we again derive that the only case to consider is when $m_3 = 0$ and $0 < p_3 \leq$

$a'_1 - p_2$. Proceeding in this way, we may eventually arrive at (p_j, m_j) with $m_i = 0$ and $p_i > 0$ for $i = 2, \dots, j-1$, $m_j = 0$ and

$$a'_1 - \sum_{i=2}^{j-1} p_i \leq 0.$$

The only case to consider reduces then to $p_j = 0$, hence $(p_j, m_j) = (0, 0)$, so this term disappears, and similarly all remaining terms until (p_k, m_k) . Otherwise, if

$$a'_1 - \sum_{i=2}^{k-1} p_i > 0$$

we have to consider the case $(p_k, m_k) = (p_k, 0)$ with $0 < p_k \leq a'_1 - \sum_{i=2}^{k-1} p_i$. But then clearly $T(\sigma) \sqsubset T'(\sigma)$. In any case we have $T \leq T'$ (hence, $T \not\parallel T'$).

(ii): The proof of sufficiency in the case when $(a_j, b_j) = (a'_j, b'_j)$, for $1 \leq j < n$, follows exactly the same steps as in the proof of (i), by adding $(0, 1)[(1, 0)(0, 1)]^{n-2}$ (after the first $(1, 1)$) to the strings used above. We consider the case when $b_n = b'_n \neq 0$ and there exists $1 \leq j < n$ such that $(a_j, b_j) \neq (a'_j, b'_j)$, say $a_j > a'_j$. The case $b_n = b'_n = 0$, $a_{n-1} = a'_{n-1} \neq 0$, and $(a_j, b_j) \neq (a'_j, b'_j)$ (say $b_j > b'_j$) for some $1 \leq j < n-1$, follows similarly by interchanging the roles of ρ_4 and ρ_5 , and those of $(1, 0)$ and $(0, 1)$ (and a'_j and b'_j) in the strings below.

So let σ be given by

- $\sigma = (1, 1)(0, 1)(1, 1)^{j-2}(1, 0)^{a'_j+1}(0, 1)(1, 1)^{n-j-1}(1, 0)$ if $1 < j$, and
- $\sigma = (1, 1)(1, 0)^{a'_j+1}(0, 1)(1, 1)^{n-1}(1, 0)$ otherwise.

Then $T(\sigma) \neq \varepsilon = T'(\sigma)$ and thus $T \not\leq T'$. Now let σ' be given by

- $\sigma' = (1, 1)(0, 1)(1, 1)^{j-2}(1, 0)^{a'_j+1}(0, 1)(1, 1)^{n-j-2}(1, 0)$ if $1 < j$ (with $(1, 1)^{n-j-2} = (0, 0)$ whenever $n-j-2 \leq 0$), and
- $\sigma' = (1, 1)(1, 0)^{a'_j+1}(0, 1)(1, 1)^{n-2}(1, 0)$ otherwise.

Then $T(\sigma') = \varepsilon \neq T'(\sigma')$ and thus $T \not\geq T'$.

The proof of necessity is similar to case (i). If none of the conditions of (ii) is satisfied, then we may assume that $b_n = b'_n \neq 0$, $(a_j, b_j) = (a'_j, b'_j)$ for every $1 \leq j < n$, and focus on the case $a_n \neq a'_n$ (for otherwise $T = T'$). (The case when $b_n = b'_n = 0$, $a_n = a'_n$, $b_{n-1} = b'_{n-1}$, and $(a_j, b_j) = (a'_j, b'_j)$ for every $1 \leq j < n-1$ follows similarly by the above mentioned substitutions.)

So suppose without loss of generality that $a_n > a'_n = t$. As in case (i), we show that $T < T'$. By making use of $\sigma = (1, 1)(0, 1)(1, 1)^{n-2}(1, 0)^{a'_n+1}$, we see that $T(\sigma) \sqsubset T'(\sigma)$.

So let $\sigma = (p_1, m_1)(p_2, m_2) \cdots (p_k, m_k)$. By reasoning as in (i), we may assume that $k > t + n$ and, since $(a_j, b_j) = (a'_j, b'_j)$ for every $1 \leq j < n$, that $(p_2, m_2) = (0, 1)$, $(p_j, m_j) = (1, 1)$ for $3 \leq j < n+1$, and that $p_j = 0$ for each $n+1 \leq j \leq t+n$; for otherwise we reach the same conclusion $T(\sigma) \sqsubseteq T'(\sigma)$.

If $p_{t+n+1} = 0$ or $m_{t+n+1} = 0$, then $T(\sigma) = T'(\sigma)$ or $T(\sigma) \sqsubset T'(\sigma)$, respectively. If $p_{t+n+1} \neq 0$ and $m_{t+n+1} \neq 0$, then $T(\sigma) \sqsubseteq T'(\sigma)$.

(iii): Suppose that $n \neq m$, say $1 \leq n < m$. First we consider the case $n = 1$. Let $\sigma = (1, 1)(0, 1)(1, 0)$. Then $T(\sigma) \neq \varepsilon = T'(\sigma)$ and thus $T \not\leq T'$. Let $\sigma' = (1, 1)(0, 1)^\alpha(1, 1)^{m-1}(1, 0)$ where $\alpha = 0$ if $b_1 = 0$, and $\alpha = 1$ otherwise. Then $T(\sigma') = \varepsilon \neq T'(\sigma')$ and thus $T \not\geq T'$.

Suppose now that $n > 1$. Then, for $\sigma = (1, 1)(0, 1)(1, 1)^{m-2}(1, 0)$, we have $T(\sigma) \neq \varepsilon = T'(\sigma)$ and thus $T \not\leq T'$.

To show that $T \not\geq T'$, let σ' be given by

- $\sigma' = (1, 1)(0, 1)(1, 1)^{n-1}(1, 1)(1, 1)^{m-n-1}(1, 0)$ if $b_n, b'_m \neq 0$,
- $\sigma' = (1, 1)(0, 1)(1, 1)^{n-2}(1, 0)(1, 1)(0, 1)(1, 1)^{m-n-1}(1, 0)$ if $b_n = 0$ and $b'_m \neq 0$,
- $\sigma' = (1, 1)(0, 1)(1, 1)^{n-1}(1, 1)(1, 1)^{m-n-2}(1, 0)(0, 1)$ if $b_n \neq 0$ and $b'_m = 0$, and
- $\sigma' = (1, 1)(0, 1)(1, 1)^{n-2}(1, 0)(1, 1)(0, 1)(1, 1)^{m-n-1}(1, 0)$ if $b_n, b'_m = 0$.

In each case we get $T(\sigma') = \varepsilon \neq T'(\sigma)$, and thus $T \not\geq T'$. \square

Remark 19. Note that the proofs of (i) and (ii) of Lemma 18 show that if $b_n = b'_n \neq 0$ and $(a_j, b_j) = (a'_j, b'_j)$ for $1 \leq j < n$, then $T < T'$ if and only if $a_n > a'_n$. Similarly, if $b_n = b'_n = 0$, $a_n = a'_n \neq 0$, $a_{n-1} = a'_{n-1}$, and $(a_j, b_j) = (a'_j, b'_j)$ for $1 \leq j < n - 1$, then $T < T'$ if and only if $b_{n-1} > b'_{n-1}$.

Moreover, if $b_n = b'_n$, $a_n = a'_n$, and there exists $1 \leq j < n$ such that $a_j > a'_j$ or $b_j > b'_j$, then we have that $\omega' \rho_1 \rho_2 \rho_3 \langle \cdot \rangle_-^+ \not\leq \omega \rho_1^a \rho_2^b \rho_3 T$ for every $T \in \mathfrak{R}$ and any $a, b \in \{0, 1\}$. To illustrate, suppose $a_j > a'_j$. Then consider σ given by

- $\sigma = (1, 1)(0, 1)(1, 1)^{j-2}(1, 0)^{a'_j+1}(0, 1)[(1, 0)(0, 1)]^{n-j-1}(1, 0)$ if $1 < j$, and
- $\sigma = (1, 1)(1, 0)^{a'_j+1}(0, 1)[(1, 0)(0, 1)]^{n-1}(1, 0)$ otherwise.

Remark 20. By reasoning as in the proof of (iii) of Lemma 18 and taking ω and ω' as above with $m < n$, one can show that $\omega' \rho_1 \rho_2 \rho_3 \langle \cdot \rangle_-^+ \not\leq \omega \rho_1^a \rho_2^b \rho_3 T$ for every $T \in \mathfrak{R}$ and any $a, b \in \{0, 1\}$.

4.2 The subposet \mathfrak{R}_{123} .

Let $\mathfrak{R}_{123} = \{R \in \mathfrak{R}/\sim : R \in \mathcal{L}(\{\rho_1, \rho_2, \rho_3\})\}$. Writing these rules in factorized irredundant form, they read $R = T_1 T_2 \cdots$ where, for each $i \in \mathbb{N}$, $T_i = \rho_1^a \rho_2^b \rho_3$ for some $a, b \in \{0, 1\}$. Note that from Proposition 12 it follows that each such expression is necessarily in \mathfrak{R}/\sim , is in the factorized irredundant form, and has infinite length.

For $T \in \mathfrak{R}_{123}$, and $a, b \in \{0, 1\}$, set $\mathcal{I}_{ab}^T = \{i \in \mathbb{N} : T_i = \rho_1^a \rho_2^b \rho_3\}$. Since T is of infinite length, $(\mathcal{I}_{ab}^T)_{a,b \in \{0,1\}}$ is a partition of \mathbb{N} . Moreover, $(\mathcal{I}_{ab}^T)_{a,b \in \{0,1\}}$ uniquely determines T .

Proposition 21. Let $T, T' \in \mathfrak{R}_{123}$. Then

- (i) $T \leq T'$ if and only if $\mathcal{I}_{11}^T \supseteq \mathcal{I}_{11}^{T'}$ and $\mathcal{I}_{ab}^T \subseteq \mathcal{I}_{ab}^{T'}$, for any $(a, b) \neq (1, 1)$.

In particular,

- (ii) $T < T'$ whenever $\mathcal{I}_{11}^T \supset \mathcal{I}_{11}^{T'}$ and $\mathcal{I}_{ab}^T \subseteq \mathcal{I}_{ab}^{T'}$, for any $(a, b) \neq (1, 1)$;

(iii) $T \parallel T'$ whenever $\mathcal{I}_{ab}^T \parallel \mathcal{I}_{ab}^{T'}$ for some $a, b \in \{0, 1\}$.

Proof. Clearly, (iii) follows from (i). Since $(\mathcal{I}_{ab}^T)_{a,b \in \{0,1\}}$ is a partition of \mathbb{N} , (ii) also follows from (i).

To see that the conditions in (i) are sufficient, note that, if T acts on a string $\sigma = (p_1, m_1) \cdots (p_k, m_k)$, then its factor T_i acts on the term (p_i, m_i) . Suppose that for some σ we have $T(\sigma) \sqsupset T'(\sigma)$. Then, using the above remark, for some i we have $T_i(p_i, m_i) \sqsupset T'_i(p_i, m_i)$, which means that $T'_i = \rho_1^a \rho_2^b \rho_3$, $T_i = \rho_1^c \rho_2^d \rho_3$ with $(c, d) \leq (a, b)$ pointwise and $(c, d) \neq (a, b)$. There are two possibilities:

- $(a, b) = (1, 1)$, and hence $\mathcal{I}_{11}^T \not\supseteq \mathcal{I}_{11}^{T'}$, or
- $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$, in which case $(c, d) = (0, 0)$, and thus $\mathcal{I}_{00}^T \not\supseteq \mathcal{I}_{00}^{T'}$.

To show that the conditions in (i) are necessary, suppose first that $\mathcal{I}_{11}^T \not\supseteq \mathcal{I}_{11}^{T'}$. Let $i = \min\{j \in \mathcal{I}_{11}^{T'} : j \notin \mathcal{I}_{11}^T\}$. If $i \in \mathcal{I}_{ab}^T$ for $ab = 00$ or 01 , consider $\sigma = (1, 1)^{i-1}(2, 1)$. Then $T(\sigma) = (1, 0) \neq \varepsilon = T'(\sigma)$. If $i \in \mathcal{I}_{10}^T$, consider $\sigma = (1, 1)^{i-1}(1, 2)$. Then $T(\sigma) = (0, 1) \neq \varepsilon = T'(\sigma)$. Thus $T \not\leq T'$.

So we may assume that $\mathcal{I}_{11}^T \supseteq \mathcal{I}_{11}^{T'}$. We treat the case $\mathcal{I}_{01}^T \not\supseteq \mathcal{I}_{01}^{T'}$; the remaining cases follow similarly. Let $i = \min\{j \in \mathcal{I}_{01}^{T'} : j \notin \mathcal{I}_{01}^T\}$. If $i \in \mathcal{I}_{10}^T$, consider $\sigma = (1, 1)^{i-1}(2, 1)$. Then $T(\sigma) = (1, 0) \neq \varepsilon = T'(\sigma)$. If $i \in \mathcal{I}_{00}^T$, consider $\sigma = (1, 1)^{i-1}(2, 2)$. Then $T(\sigma) = (1, 0) \neq \varepsilon = T'(\sigma)$. Thus $T \not\leq T'$. \square

As immediate corollaries we have the following results.

Corollary 22. Let $T \in \mathfrak{R}_{123}$.

- T is the least rule if and only if $\mathcal{I}_{11}^T = \mathbb{N}$, i.e., $T = \langle \cdot \rangle_-$.
- T is an atom if and only if $\mathcal{I}_{11}^T = \mathbb{N} \setminus \{i\}$ for some $i \in \mathbb{N}$.
- T is a maximal element if and only if $\mathcal{I}_{11}^T = \emptyset$.

Corollary 23. Let $T, T' \in \mathfrak{R}_{123}$ where $T = T_1 T_2 \cdots$ and $T' = T'_1 T'_2 \cdots$. Then $T \wedge T' = S$ where $\mathcal{I}_{11}^S = \mathcal{I}_{11}^T \cup \mathcal{I}_{11}^{T'} \cup \bigcup_{(a,b) \neq (1,1)} \mathcal{I}_{ab}^T \oplus \mathcal{I}_{ab}^{T'}$, where \oplus denotes the symmetric difference, and $\mathcal{I}_{ab}^S = \mathcal{I}_{ab}^T \cap \mathcal{I}_{ab}^{T'}$ for every $(a, b) \neq (1, 1)$.

In other words, \mathfrak{R}_{123} constitutes a \wedge -semilattice. Now, by Proposition 21, if $T, T' \leq R \in \mathfrak{R}_{123}$, then $\mathcal{I}_{11}^T, \mathcal{I}_{11}^{T'} \supseteq \mathcal{I}_{11}^R$ and $\mathcal{I}_{ab}^T, \mathcal{I}_{ab}^{T'} \subseteq \mathcal{I}_{ab}^R$ for every $(a, b) \neq (1, 1)$. Hence, Corollary 23 can be refined by considering intervals of the form $[\langle \cdot \rangle_-, R]$ for some $R \in \mathfrak{R}_{123}$.

Corollary 24. Let $R \in \mathfrak{R}_{123}$. Then $([\langle \cdot \rangle_-, R], \leq)$ constitutes a lattice under \wedge and \vee defined by

- $T \wedge T' = S$ where $\mathcal{I}_{11}^S = \mathcal{I}_{11}^T \cup \mathcal{I}_{11}^{T'} \cup \bigcup_{(a,b) \neq (1,1)} \mathcal{I}_{ab}^T \oplus \mathcal{I}_{ab}^{T'}$, and $\mathcal{I}_{ab}^S = \mathcal{I}_{ab}^T \cap \mathcal{I}_{ab}^{T'}$ for every $(a, b) \neq (1, 1)$;
- $T \vee T' = S$ where $\mathcal{I}_{11}^S = \mathcal{I}_{11}^T \cap \mathcal{I}_{11}^{T'}$, and $\mathcal{I}_{ab}^S = \mathcal{I}_{ab}^T \cup \mathcal{I}_{ab}^{T'}$ for every $(a, b) \neq (1, 1)$,

for every $T, T' \in [\langle \cdot \rangle_-, R]$, with $T = T_1 T_2 \cdots$ and $T' = T'_1 T'_2 \cdots$. Moreover, $([\langle \cdot \rangle_-, R], \leq)$ is order-isomorphic to $(\mathcal{P}(\bigcup_{(a,b) \neq (1,1)} \mathcal{I}_{ab}^R), \subseteq)$.

From Corollary 24, it follows that $(\mathfrak{R}_{123}, \leq)$ embeds the power set of integers ordered by inclusion. Furthermore, for $R \in \mathfrak{R}_{123}$, if $|\bigcup_{(a,b) \neq (1,1)} \mathcal{I}_{ab}^R| = n$ is finite, then $|\langle \cdot \rangle_-, R| = 2^n$.

4.3 Least element and atoms

We turn to the study of the atoms of \mathfrak{R}/\sim . The next proposition was presented in [5].

Proposition 25. The rule $\langle \cdot \rangle_{-}^{\pm}$ is the least element of \mathfrak{R}/\sim .

Proof. It follows immediately from the fact that $\langle \cdot \rangle_{-}^{\pm}$ deletes every term of a nonassociative string. \square

Proposition 26. Let $T = \omega \rho_1^a \rho_2^b \rho_3 T'$ be an element of \mathfrak{R}/\sim , where $(a, b) \neq (1, 1)$. Then T is an atom if and only if ρ_4 and ρ_5 alternate infinitely many times in ω (and therefore $T' = \varepsilon$).

Proof. Note that if ρ_4 and ρ_5 alternate infinitely many times in ω , then $T = \omega \rho_1^a \rho_2^b \rho_3$. By Lemma 16, the condition is sufficient.

To show that it is also necessary, let T be an atom, and for the sake of a contradiction suppose that ρ_4 and ρ_5 alternate finitely many times in ω . Let $T'' = (\rho_4 \rho_5)^* T$. Then $T'' = (\rho_4 \rho_5)^* \rho_1^a \rho_2^b \rho_3 > \langle \cdot \rangle_{-}^{\pm}$. Moreover, by Lemma 17 we have $T > T''$, which contradicts the fact that T is an atom. \square

Consequently, for $(a, b) < (1, 1)$, we have only 3 atoms, namely $(\rho_4 \rho_5)^* \rho_3$, $(\rho_4 \rho_5)^* \rho_1 \rho_3$ and $(\rho_4 \rho_5)^* \rho_2 \rho_3$.

Proposition 27. If $T = \omega \rho_1 \rho_2 \rho_3 T'$ is an atom, then ρ_4 and ρ_5 alternate finitely many times in ω .

Proof. If ρ_4 and ρ_5 alternate infinitely many times in ω , then $T = \langle \cdot \rangle_{-}^{\pm}$. \square

Proposition 28. Let $T = \omega \rho_1 \rho_2 \rho_3 T'$ such that ρ_4 and ρ_5 alternate finitely many times in ω . If T is an atom, then $T' = \langle \cdot \rangle_{-}^{\pm}$.

Proof. Let $T = \omega \rho_1 \rho_2 \rho_3 T'$ be an atom. Suppose for the sake of a contradiction that $T' \neq \langle \cdot \rangle_{-}^{\pm}$. By Proposition 25, $T' > \langle \cdot \rangle_{-}^{\pm}$, and by Lemma 17, $T'' = \omega \rho_1 \rho_2 \rho_3 \langle \cdot \rangle_{-}^{\pm} > \langle \cdot \rangle_{-}^{\pm}$. Moreover, by Lemma 13, $T > T''$, which contradicts the fact that T is an atom. \square

Proposition 29. Let $T = \omega \rho_1 \rho_2 \rho_3 \langle \cdot \rangle_{-}^{\pm}$. Then T is an atom if and only if $\omega = \rho_4^{a_1} \rho_5^{b_1} \cdots \rho_4^{a_n} \rho_5^{b_n}$ with $a_i \neq 0$ for $2 \leq i \leq n$ and $b_i \neq 0$ for $1 \leq i \leq n-1$, and such that

- (i) $b_n \neq 0$ and a_n is infinite, or
- (ii) $b_n = 0$, $a_n \neq 0$ and b_{n-1} is infinite.

Proof. Necessity follows from Propositions 27 and 28, and Lemma 18 and Remark 19. Sufficiency follows from Lemma 18 and Remark 19. \square

We can now explicitly describe the atoms of \mathfrak{R}/\sim .

Theorem 30. A w.f.c.r. T is an atom of \mathfrak{R}/\sim if and only if $T = (\rho_4 \rho_5)^* \rho_1^a \rho_2^b \rho_3$, for $(a, b) \neq (1, 1)$, or $T = \omega \rho_1 \rho_2 \rho_3 \langle \cdot \rangle_{-}^{\pm}$ where $\omega = \rho_4^{a_1} \rho_5^{b_1} \cdots \rho_4^{a_n} \rho_5^{b_n}$ with $a_i \neq 0$ for $2 \leq i \leq n$ and $b_i \neq 0$ for $1 \leq i \leq n-1$, and such that

- (i) $b_n \neq 0$ and a_n is infinite, or
- (ii) $b_n = 0$, $a_n \neq 0$ and b_{n-1} is infinite.

4.4 Maximal elements

We now focus on the maximal elements of \mathfrak{R}/\sim . In [5], it was proved that $\langle \cdot \rangle_0$ is a maximal element of the set of well-formed computation rules.

Proposition 31. Let $T = \omega \rho_1^a \rho_2^b \rho_3 T'$. If T is maximal, then

- (i) ρ_4 and ρ_5 alternate finitely many times in ω ,
- (ii) $(a, b) < (1, 1)$, and
- (iii) T' is maximal.

Proof. Conditions (i) and (ii) follow from Lemmas 17 and 16, respectively. Condition (iii) follows from Lemma 13. \square

As it turns out, every maximal element of \mathfrak{R}_{123} is also maximal in \mathfrak{R}/\sim .

Proposition 32. Let $T \in \mathfrak{R}_{123}$. If $\mathcal{I}_{11}^T = \emptyset$, then T is a maximal element of \mathfrak{R}/\sim .

Proof. It suffices to show that for every $T' = T'_1 T'_2 \cdots$ such that $T' \geq T = T_1 T_2 \cdots$, we have $T' \in \mathcal{R}_{123}$.

For the sake of a contradiction, suppose $T' \notin \mathcal{R}_{123}$, and let $i = \min\{j \in \mathbb{N} : T'_j \notin \mathcal{L}(\{\rho_1, \rho_2, \rho_3\})\}$. Note that $T_i = \rho_1^a \rho_2^b \rho_3$ for $(a, b) < (1, 1)$, for every $i \in \mathbb{N}$. Since $T' \geq T$, $T'_i = \omega \rho_1^a \rho_2^b \rho_3$ where ρ_4 and ρ_5 alternate finitely many times in ω . Without loss of generality, suppose $\omega = \rho_4 \omega'$. Consider $\sigma = (1, 1)^{i-1} (1, 0)^2$. Then $T(\sigma) = (1, 0)^2 > (1, 0) = T'(\sigma)$, thus yielding the desired contradiction.

Hence, $T' \in \mathcal{R}_{123}$ and, by Corollary 22, $T' = T$. Thus T is maximal in \mathcal{R} . \square

Now we consider the maximal elements $T \in (\mathfrak{R}/\sim) \setminus \mathfrak{R}_{123}$.

Proposition 33. Let $T = \omega \rho_1^a \rho_2^b \rho_3 R$. If $\omega \notin \mathcal{L}(\rho_4) \cup \mathcal{L}(\rho_5)$, then T is not maximal.

Proof. Let $\omega = \rho_4^{a_1} \rho_5^{b_1} \cdots \rho_4^{a_n} \rho_5^{b_n}$, $n \geq 1$. Without loss of generality, suppose that $a_i \neq 0$, $b_i \neq 0$, for every $1 \leq i \leq n$. Assume also that $a, b = 0$; the other cases $(a, b) < (1, 1)$ follow similarly.

Let $R' = (\rho_1 \rho_2 \rho_3)^n R$, and set $T' = \rho_1^a \rho_2^b \rho_3 R'$. Let $\gamma = (1, 1)(1, 0)(0, 1)$. Then $T(\gamma) = \varepsilon < (1, 0)(0, 1) = T'(\sigma)$, and thus $T < T'$. Hence, T is not maximal. \square

As an immediate corollary we get the following necessary condition for maximality.

Corollary 34. Let $T = T_1 T_2 \cdots$ with $T_i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$. If T is maximal, then for every $i \in \mathbb{N}$, $\omega_i \in \mathcal{L}(\rho_4) \cup \mathcal{L}(\rho_5)$.

Remark 35. Let $T = T_1 T_2 \cdots$ where $T_i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$, with $\omega_i \in \mathcal{L}(\rho_4) \cup \mathcal{L}(\rho_5)$. If for some $i \in \mathbb{N}$, ω_i is ρ_4^* , then $T = T_1 \cdots T_i$. Otherwise, $T = T_1 T_2 \cdots$, and for each i there is a string γ such that T_i acts on γ .

Proposition 36. Let $T = T_1 T_2 \cdots$ where $T_i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$, and $T' = T'_1 T'_2 \cdots$ where $T'_i = \omega'_i \rho_1^{a'_i} \rho_2^{b'_i} \rho_3$, with $\omega_i, \omega'_i \in \mathcal{L}(\rho_4) \cup \mathcal{L}(\rho_5)$. Then $T \parallel T'$ if and only if one of the following holds:

- (i) each ω_i, ω'_i has finite length, and $\omega_i \neq \omega'_i$, for some $i \in \mathbb{N}$,
- (ii) $T = T_1 \cdots T_i$ and neither ρ_4^* nor ρ_5^* occur in T_j , $1 \leq j \leq i-1$, nor in T' .
- (iii) $T = T_1 \cdots T_i$ and $T' = T'_1 \cdots T'_j$ where neither ρ_4^* nor ρ_5^* occur in T_l , $1 \leq l \leq i-1$, nor in T'_k , $1 \leq k \leq j-1$, or $\omega_t \neq \omega'_t$, for some $1 \leq t \leq i \wedge j$.

Proof. To see that the conditions in (i)-(iii) are necessary, observe that if $T \parallel T'$ (T and T' in factorized irredundant forms), then we must have $T \neq T'$. Since each ω_i and each ω'_i is in $\mathcal{L}(\rho_4) \cup \mathcal{L}(\rho_5)$, one of (i)-(iii) must occur.

To show that (i) is sufficient, assume that each ω_i, ω'_i has finite length and, without loss of generality, suppose that $\omega_1 \neq \omega'_1$. We consider 3 representative cases; the remaining cases follow similarly.

Suppose that $\omega_1 \in \mathcal{L}(\{\rho_4\})$ and $\omega'_1 \in \mathcal{L}(\{\rho_5\})$. Take $\sigma = (1, 1)(0, 1)$ and $\sigma' = (1, 1)(1, 0)$. Then $T(\sigma) = (0, 1) \neq \varepsilon = T'(\sigma)$, but $T(\sigma') = \varepsilon \neq (1, 0) = T'(\sigma')$.

Suppose that $\omega_1 \in \mathcal{L}(\{\rho_4\})$ and $\omega'_1 = \varepsilon$. Take $\sigma = (1, 1)(1, 0)$ and $\sigma' = (1, 1)(1, 1)$. Then $T(\sigma) = \varepsilon \neq (1, 0) = T'(\sigma)$, but $T(\sigma') = (0, 1) \neq \varepsilon = T'(\sigma')$.

Suppose now that $\omega_1 \in \rho_4^n$ and $\omega'_1 = \rho_4^m$, say, $n < m$. Take $\sigma = (1, 1)(1, 0)^{n+1}$ and $\sigma' = (1, 1)(1, 0)^n(1, 1)$. Then $T(\sigma) = (1, 0) \neq \varepsilon = T'(\sigma)$, but $T(\sigma') = \varepsilon \neq (0, 1) = T'(\sigma')$.

In all representative cases we conclude that $T \parallel T'$.

To show that (ii) is sufficient, suppose that $T = T_1 \cdots T_i$ and neither ρ_4^* nor ρ_5^* occur in T_l , $1 \leq l \leq i-1$, nor in T' . Let $k = \min\{j : \omega_j \neq \omega'_j\}$. If $k \leq i-1$, then the proof of (i) can be used to show that $T \parallel T'$.

So suppose that $k = i$ and, without loss of generality, suppose that $\omega_i = \rho_4^*$ and $\omega'_i = \rho_4^m$, $m > 0$. Take $\sigma = (1, 1)^i(1, 0)^{m+1}$ and $\sigma' = (1, 1)(1, 0)^m(1, 1)$. Then $T(\sigma) = \varepsilon \neq (1, 0) = T'(\sigma)$, but $T(\sigma') = (0, 1) \neq \varepsilon = T'(\sigma')$, and again we have that $T \parallel T'$.

Finally, to show that (iii) is sufficient, suppose that $T = T_1 \cdots T_i$ and $T' = T'_1 \cdots T'_j$ where neither ρ_4^* nor ρ_5^* occur in T_l , $1 \leq l \leq i-1$, nor in T'_k , $1 \leq k \leq j-1$, or $\omega_t \neq \omega'_t$, for some $1 \leq t \leq i \wedge j$.

Now, as case (i), we may assume that $i < j$ (the case $i > j$ is similar), and that $\omega_i \in \mathcal{L}(\{\rho_4\})$ and $\omega'_i = \rho_4^m$, $m > 0$. But then, as in case (ii), we again have $T \parallel T'$, and thus the proof is now complete. \square

From Lemma 16, the above necessary condition and Propositions 32 and 36, we obtain the following explicit description of the maximal elements of \mathfrak{R}/\sim .

Theorem 37. Let $T \in \mathfrak{R}/\sim$. Then T is maximal if and only if

- (i) T is a maximal element of \mathfrak{R}_{123} , or
- (ii) $T = T_1 T_2 \cdots$ where $T_i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$ with $\omega_i \in \mathcal{L}(\rho_4) \cup \mathcal{L}(\rho_5)$ and $(a_i, b_i) < (1, 1)$.

5 Concluding remarks: An alternative ordering of \mathfrak{R}/\sim

An alternative ordering of \mathfrak{R}/\sim was proposed in [5], and which is defined as follows. Given $R \in \mathfrak{R}/\sim$, let $\text{Ker}(R) = \{\sigma : R(\sigma) = \varepsilon\}$. For $R, R' \in \mathfrak{R}/\sim$, we write $R \leq_{\text{Ker}} R'$ if $\text{Ker}(R) \supseteq \text{Ker}(R')$. Clearly, \leq_{Ker} is a partial ordering of \mathfrak{R}/\sim , and if $R \leq R'$, then $R \leq_{\text{Ker}} R'$; see [5]. As it turns out, the converse is also true.

Proposition 38. Let $R, R' \in \mathfrak{R}/\sim$. Then $R \leq R'$ if and only if $R \leq_{\text{Ker}} R'$.

Proof. To prove Proposition 38 it remains to show that if $R \parallel R'$, then $R \parallel_{\text{Ker}} R'$, i.e., $R \not\leq_{\text{Ker}} R'$ and $R' \not\leq_{\text{Ker}} R$.

So suppose that $R \parallel R'$, that is, there exist σ_1 and σ_2 such that $R(\sigma_1) \sqsubset R'(\sigma_1)$ and $R(\sigma_2) \supset R'(\sigma_2)$.

Let σ'_1 the string be obtained from σ_1 by removing the indices in $R(\sigma_1)$ such that $R(\sigma'_1) = \varepsilon \neq R'(\sigma'_1)$. Hence, $R' \not\leq_{\text{Ker}} R$.

Similarly, let σ'_2 the string be obtained from σ_2 by removing the indices in $R'(\sigma_2)$ such that $R(\sigma'_2) \neq \varepsilon = R'(\sigma'_2)$. Hence, $R \not\leq_{\text{Ker}} R'$, and thus $R \parallel_{\text{Ker}} R'$. \square

We have presented a partial description of the poset \mathfrak{R}/\sim ; being uncountable, there is little hope of obtaining an explicit description as it was the case of the subposet \mathcal{R}_{123} , which was shown to be isomorphic to the power set of natural numbers.

Looking at directions of further research, we are inevitably drawn to the question in determining whether \mathfrak{R}/\sim constitutes a \wedge -semilattice and, if that is the case, whether its closed intervals constitute lattices, as it was the case of the subposet \mathcal{R}_{123} .

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