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LOCAL AND GLOBAL TAMENESS IN KRULL MONOIDS

WEIDONG GAO, ALFRED GEROLDINGER, AND WOLFGANG A. SCHMID

ABSTRACT. Let H be a Krull monoid with finite class group G such that every class contains a prime divisor. Then the global tame degree $\mathfrak{t}(H)$ equals zero if and only if H is factorial (equivalently, $|G| = 1$). If $|G| > 1$, then $\mathfrak{D}(G) \leq \mathfrak{t}(H) \leq 1 + \mathfrak{D}(G)(\mathfrak{D}(G) - 1)/2$, where $\mathfrak{D}(G)$ is the Davenport constant of G . We analyze the case when $\mathfrak{t}(H)$ equals the lower bound, and we show that $\mathfrak{t}(H)$ grows asymptotically as the upper bound, when both terms are considered as functions of the rank of G . We provide more precise results if G is either cyclic or an elementary 2-group.

Dedicated to Marco Fontana on the occasion of his 65th birthday

1. INTRODUCTION

In an atomic monoid, every non-unit can be written as a finite product of atoms (irreducible elements). The multiplicative monoid of non-zero elements from a noetherian domain is such an atomic monoid, and furthermore it is a Krull monoid if and only if the domain is integrally closed. In a given monoid H , all factorizations into atoms are unique (in other words, H is factorial) if and only if H is a Krull monoid with trivial class group. Otherwise, the non-uniqueness of factorizations is described by arithmetical invariants, such as sets of lengths, catenary and tame degrees.

The concepts of local and global tameness have found some attention in recent literature, and they were studied in settings ranging from numerical monoids to noetherian domains (confer [5, 14, 4, 15, 11, 23, 24, 2, 21, 18, 22]). We recall their definitions. Let H be a monoid and $u \in H$ an atom. Then the local tame degree $\mathfrak{t}(H, u)$ is the smallest N with the following property: for any multiple a of u and any factorization $a = v_1 \cdot \dots \cdot v_n$ of a , which does not contain the u , there is a short subproduct which is a multiple of u , say $v_1 \cdot \dots \cdot v_m$, and a refactorization of this subproduct which contains u , say $v_1 \cdot \dots \cdot v_m = uu_2 \cdot \dots \cdot u_\ell$, such that $\max\{\ell, m\} \leq N$. Thus the local tame degree $\mathfrak{t}(H, u)$ measures the distance between any factorization of a multiple a of u and a factorization of a which contains the u . The global tame degree $\mathfrak{t}(H)$ is the supremum of the local tame degrees over all atoms $u \in H$, and H is called (globally) tame if the global tame degree $\mathfrak{t}(H)$ is finite.

Local tameness is a basic finiteness property in the theory of non-unique factorizations in the sense that in many settings where an arithmetical finiteness property has to be derived, local tameness has to be proved first (confer the proof of the Structure Theorem for sets of lengths in [12, Section 4.3]). Krull monoids with finite class group are globally tame. But if the domain fails to be integrally closed, this does not remain true any more, not even for non-principal orders in number fields. Indeed, a non-principal order \mathfrak{o} in an algebraic number field is always locally tame, but it is globally tame if and only if for every prime ideal \mathfrak{p} containing the conductor there is precisely one prime ideal $\bar{\mathfrak{p}}$ in the principal order $\bar{\mathfrak{o}}$ such that $\bar{\mathfrak{p}} \cap \mathfrak{o} = \mathfrak{p}$ (equivalently, if and only if its elasticity is finite). Higher dimensional analogs will be mentioned after Definition 3.1.

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The focus of the present paper is on Krull monoids H with finite class group G such that every class contains a prime divisor, and for simplicity suppose now that $|G| > 2$. There is the straightforward inequality

$$D(G) \leq \mathfrak{t}(H) \leq 1 + \frac{D(G)(D(G) - 1)}{2},$$

where $D(G)$ is the Davenport constant of G . We analyze the case when $\mathfrak{t}(H)$ equals the lower bound, and we show that $\mathfrak{t}(H)$ grows asymptotically as the upper bound, when both terms are considered as functions of the rank of G (Theorem 4.12). This result, which indicates the general behavior of the tame degree, will be complemented by more precise results if G is either cyclic or an elementary 2-group (Theorems 5.1 and 5.2). Arithmetical invariants (such as sets of lengths, sets of distances, the elasticity, the catenary degree, or the monotone catenary degree) of a Krull monoid as above depend only on the class group G but not on the number of prime divisors in the classes, and therefore all investigations can be carried through in the associated monoid of zero-sum sequences instead of doing them in H . In general, this is not the case for the tame degree, and we provide the first example revealing this fact (see Theorem 5.1, but also Proposition 3.3.1 and Remark 3.4.1). Moreover, note that the existing computational methods (as first presented in [5]) cannot be applied to obtain this or many other examples given in the present paper (the problem is the large number of variables involved in the systems of linear Diophantine equations to be solved).

2. PRELIMINARIES

Let \mathbb{N} denote the set of positive integers, $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, let $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ an interval of integers. By a monoid, we mean a commutative semigroup with unit element which satisfies the cancellation laws. All our concepts will be formulated in the language of monoids. The monoids we have in mind are multiplicative monoids of nonzero elements of noetherian or Mori domains, monoids of ideals (with suitable multiplication), and additive monoids of certain classes of modules ([17, 12, 1, 8]).

Arithmetic of monoids. Let H be a monoid. We denote by $\mathfrak{q}(H)$ a quotient group of H with $H \subset \mathfrak{q}(H)$, by H^\times the group of invertible elements, and by $H_{\text{red}} = \{aH^\times \mid a \in H\}$ the associated reduced monoid. We say that H is reduced if $H^\times = \{1\}$. Furthermore, let $\mathcal{A}(H)$ be the set of atoms (irreducible elements) of H . For a set P , we denote by $\mathcal{F}(P)$ the free (abelian) monoid with basis P . Then every $a \in \mathcal{F}(P) = F$ has a unique representation in the form

$$a = \prod_{p \in P} p^{\nu_p(a)} \quad \text{with} \quad \nu_p(a) \in \mathbb{N}_0 \quad \text{and} \quad \nu_p(a) = 0 \quad \text{for almost all } p \in P.$$

We call $\text{supp}_P(a) = \text{supp}(a) = \{p \in P \mid \nu_p(a) > 0\} \subset P$ the *support* of a , and $|a|_F = |a| = \sum_{p \in P} \nu_p(a) \in \mathbb{N}_0$ the *length* of a . We will often consider submonoids of free abelian monoids, and in all these situations the length $|\cdot|$ refers to the largest free abelian monoid under consideration. The free monoid $Z(H) = \mathcal{F}(\mathcal{A}(H_{\text{red}}))$ is called the *factorization monoid* of H , and the unique homomorphism

$$\pi: Z(H) \rightarrow H_{\text{red}} \quad \text{satisfying} \quad \pi(u) = u \quad \text{for all } u \in \mathcal{A}(H_{\text{red}})$$

is called the *factorization homomorphism* of H . For $a \in H$, the set

$$Z(a) = \pi^{-1}(aH^\times) \subset Z(H) \quad \text{is the set of factorizations of } a, \quad \text{and}$$

$$L(a) = \{|z| \mid z \in Z(a)\} \subset \mathbb{N}_0 \quad \text{is the set of lengths of } a.$$

By definition, we have $Z(a) = \{1\}$ and $L(a) = \{0\}$ for all $a \in H^\times$. The monoid H is called

- *atomic* if $Z(a) \neq \emptyset$ for all $a \in H$;
- *factorial* if $|Z(a)| = 1$ for all $a \in H$ (equivalently, H is atomic and every atom is a prime).

Let $z, z' \in Z(H)$. Then we can write

$$z = u_1 \cdots u_\ell v_1 \cdots v_m \quad \text{and} \quad z' = u_1 \cdots u_\ell w_1 \cdots w_n,$$

where $\ell, m, n \in \mathbb{N}_0$, $u_1, \dots, u_\ell, v_1, \dots, v_m, w_1, \dots, w_n \in \mathcal{A}(H_{\text{red}})$ are such that $\{v_1, \dots, v_m\} \cap \{w_1, \dots, w_n\} = \emptyset$. The *distance* between z and z' is defined by

$$d(z, z') = \max\{m, n\} = \max\{|z \gcd(z, z')^{-1}|, |z' \gcd(z, z')^{-1}|\} \in \mathbb{N}_0.$$

Krull monoids. A monoid homomorphism $\varphi: H \rightarrow D$ is called

- a *divisor homomorphism* if $\varphi(a) \mid \varphi(b)$ implies that $a \mid b$ for all $a, b \in H$.
- *cofinal* if for every $a \in D$ there exists some $u \in H$ such that $a \mid \varphi(u)$.
- a *divisor theory* (for H) if $D = \mathcal{F}(P)$ for some set P , φ is a divisor homomorphism, and for every $p \in P$ (equivalently for every $p \in \mathcal{F}(P)$), there exists a finite subset $\emptyset \neq X \subset H$ satisfying $p = \gcd(\varphi(X))$.

The quotient group $\mathcal{C}(\varphi) = \mathfrak{q}(D)/\mathfrak{q}(\varphi(H))$ is called the *class group* of φ . For $a \in \mathfrak{q}(D)$, we denote by $[a] = [a]_\varphi = a \mathfrak{q}(\varphi(H)) \in \mathfrak{q}(D)/\mathfrak{q}(\varphi(H))$ the class containing a . If $\varphi: H \rightarrow \mathcal{F}(P)$ is a cofinal divisor homomorphism, then

$$G_P = \{[p] = p \mathfrak{q}(\varphi(H)) \mid p \in P\} \subset \mathcal{C}(\varphi)$$

is called the *set of classes containing prime divisors*. By its very definition, every class $g \in \mathcal{C}(\varphi)$ is a subset of $\mathfrak{q}(D)$ and $P \cap g$ is the set of prime divisors lying in g . The monoid H is called a *Krull monoid* if it satisfies one of the following equivalent properties ([12, Theorem 2.4.8] or [17, Chapter 22]):

- (a) H is v -noetherian and completely integrally closed,
- (b) H has a divisor theory,
- (c) H has a divisor homomorphism into a free monoid.

If H is a Krull monoid, then a divisor theory is essentially unique and the associated class group depends only on H (it is called the class group of H). An integral domain R is a Krull domain if and only if its multiplicative monoid $R \setminus \{0\}$ is a Krull monoid, and thus Property (a) shows that a noetherian domain is Krull if and only if it is integrally closed.

The main examples of Krull monoids which we have in mind are those stemming from number theory: rings of integers in algebraic number fields, holomorphy rings in algebraic function fields and regular congruence monoids in these domains are Krull monoids with finite class group such that every class contains infinitely many prime divisors ([12, Section 2.11]). If R is an integral separable finitely generated algebra over an infinite field k such that $\dim_k(R) \geq 2$, then R is noetherian and every class contains infinitely many prime divisors ([19]). Monoid domains and power series domains that are Krull are discussed in [20, 3]. For the role of Krull monoids in module theory we refer to [6, 7, 1]. Module theory provides natural examples of Krull monoids where $G_P \subsetneq G$ but $G_P = -G_P$ holds true.

Monoids of zero-sum sequences. Let G be an additive abelian group, $G_0 \subset G$ a subset and $\mathcal{F}(G_0)$ the free monoid with basis G_0 . According to the tradition of Combinatorial Number Theory, the elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0 . For a sequence

$$S = g_1 \cdots g_\ell = \prod_{g \in G_0} g^{v_g(S)} \in \mathcal{F}(G_0),$$

we call

$$\sigma(S) = \sum_{i=1}^{\ell} g_i \quad \text{the sum of } S \quad \text{and} \quad \Sigma(S) = \left\{ \sum_{i \in I} g_i \mid \emptyset \neq I \subset [1, \ell] \right\} \quad \text{the set of subsums of } S.$$

Furthermore, S is called *zero-sum free* if $0 \notin \Sigma(S)$, and it is a *minimal zero-sum sequence* if $|S| \geq 1$, $\sigma(S) = 0$ and $\sum_{i \in I} g_i \neq 0$ for all $\emptyset \neq I \subsetneq [1, \ell]$. The monoid

$$\mathcal{B}(G_0) = \{U \in \mathcal{F}(G_0) \mid \sigma(U) = 0\}$$

is called the *monoid of zero-sum sequences* over G_0 . Since the embedding $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is a divisor homomorphism, $\mathcal{B}(G_0)$ is a Krull monoid by Property (c). The monoid $\mathcal{B}(G)$ is factorial if and only if $|G| \leq 2$. For every arithmetical invariant $*$ (H) defined for a monoid H , it is usual to write $*$ (G_0) instead of $*$ ($\mathcal{B}(G_0)$) (whenever the meaning is clear from the context). In particular, we set $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$, $\mathcal{Z}(G_0) = \mathcal{Z}(\mathcal{B}(G_0))$, and $\mathfrak{t}(G_0) = \mathfrak{t}(\mathcal{B}(G_0))$. The atoms of $\mathcal{B}(G_0)$ are precisely the minimal zero-sum sequences over G_0 , and

$$D(G_0) = \sup\{|U| \mid U \in \mathcal{A}(G_0)\} \in \mathbb{N} \cup \{\infty\}$$

is the *Davenport constant* of G_0 . Suppose that G is finite with $|G| > 1$, say

$$G \cong C_{n_1} \oplus \cdots \oplus C_{n_r} \cong C_{q_1} \oplus \cdots \oplus C_{q_s},$$

where $r, s \in \mathbb{N}$, $n_1, \dots, n_r \in \mathbb{N}$, $1 < n_1 \mid \dots \mid n_r$, and q_1, \dots, q_s are prime powers (not equal to 1). Then $r = r(G)$ is the *rank* of G , $s = r^*(G)$ is the *total rank* of G , $d(G) = D(G) - 1$ is the maximal length of a zero-sum free sequence over G , and we define

$$d^*(G) = \sum_{i=1}^{r(G)} (n_i - 1), \quad D^*(G) = d^*(G) + 1, \quad \text{and} \quad k^*(G) = \sum_{i=1}^{r^*(G)} \frac{q_i - 1}{q_i}.$$

Furthermore, we set $d^*(\{0\}) = k^*(\{0\}) = 0$. A straightforward example shows that $D^*(G) \leq D(G)$. Moreover, equality holds for groups of rank $r(G) \leq 2$, for p -groups, and some other types of groups but not in general ([9, 16]). If $t \in \mathbb{N}$ and $(e_1, \dots, e_t) \in G^t$, then (e_1, \dots, e_t) is said to be *independent* if e_1, \dots, e_t are all nonzero and if, for every $(m_1, \dots, m_t) \in \mathbb{Z}^t$, the equation $\sum_{i=1}^t m_i e_i = 0$ implies that $m_i e_i = 0$ for all $i \in [1, t]$. Furthermore, (e_1, \dots, e_t) is said to be a *basis* of G if it is independent and $G = \langle e_1, \dots, e_t \rangle$.

3. TAMENESS AND TRANSFER HOMOMORPHISMS

In this section we introduce the concepts of tameness and that of transfer homomorphisms. Our main reference is Section 3.2 in [12]. We present the material in a way suitable for our applications in the following sections. Among others we will show that a Krull monoid is locally tame if and only if the associated block monoid is locally tame, a fact which has not been observed so far. Furthermore, we establish a purely combinatorial characterization of the tame degree of a Krull monoid provided that every class contains sufficiently many prime divisors (Proposition 3.5).

Definition 3.1. Let H be an atomic monoid.

1. For $b \in H$, let $\omega(H, b)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in H$, if $b \mid a_1 \cdots a_n$, then there is a subset $\Omega \subset [1, n]$ such that

$$|\Omega| \leq N \quad \text{and} \quad b \mid \prod_{\nu \in \Omega} a_\nu.$$

2. For $a \in H$ and $x \in \mathcal{Z}(H)$, let $\mathfrak{t}(a, x) \in \mathbb{N}_0 \cup \{\infty\}$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

If $\mathcal{Z}(a) \cap x\mathcal{Z}(H) \neq \emptyset$ and $z \in \mathcal{Z}(a)$, then there exists $z' \in \mathcal{Z}(a) \cap x\mathcal{Z}(H)$ such that $d(z, z') \leq N$. For subsets $H' \subset H$ and $X \subset \mathcal{Z}(H)$, we define

$$\mathfrak{t}(H', X) = \sup\{\mathfrak{t}(a, x) \mid a \in H', x \in X\} \in \mathbb{N}_0 \cup \{\infty\}.$$

H is said to be *locally tame* if $\mathfrak{t}(H, u) < \infty$ for all $u \in \mathcal{A}(H_{\text{red}})$.

3. We set

$$\omega(H) = \sup\{\omega(H, u) \mid u \in \mathcal{A}(H)\} \in \mathbb{N}_0 \cup \{\infty\},$$

and we call

$$\mathfrak{t}(H) = \mathfrak{t}(H, \mathcal{A}(H_{\text{red}})) = \sup\{\mathfrak{t}(H, u) \mid u \in \mathcal{A}(H_{\text{red}})\} \in \mathbb{N}_0 \cup \{\infty\}$$

the *tame degree* of H . The monoid H is said to be (*globally*) *tame* if $\mathfrak{t}(H) < \infty$.

To analyze the above terminology, suppose that H is reduced. By definition, an atom $u \in H$ is prime if and only if $\omega(H, u) = 1$. Thus $\omega(H) = 1$ if and only if H is factorial, and the $\omega(H, u)$ values measure how far away an atom is from being a prime. Let $a \in H$ and $u \in \mathcal{A}(H)$. If $u \nmid a$, then $\mathfrak{t}(a, u) = 0$. Otherwise, $\mathfrak{t}(a, u)$ is the smallest N with the following property: if $z = v_1 \cdots v_n$ is any factorization of a into atoms v_1, \dots, v_n , then there is a subset $\Omega \subset [1, n]$, say $\Omega = [1, m]$, and a factorization $z' = uu_2 \cdots u_\ell v_{m+1} \cdots v_n$ of a with atoms u_2, \dots, u_ℓ such that $\max\{\ell, m\} \leq N$. If u is a prime, then every factorization of a contains u ; thus we can choose $z' = z$ in the above definition, hence $\mathfrak{d}(z, z') = 0$ and $\mathfrak{t}(H, u) = 0$. If u is not a prime, then $\omega(H, u) \leq \mathfrak{t}(H, u)$, and throughout this paper we will use the following characterization of $\mathfrak{t}(H, u)$; it is the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

If $m \in \mathbb{N}$ and $v_1, \dots, v_m \in \mathcal{A}(H)$ are such that $u \mid v_1 \cdots v_m$, but u divides no proper subproduct of $v_1 \cdots v_m$, then there exist $\ell \in \mathbb{N}$ and $u_2, \dots, u_\ell \in \mathcal{A}(H)$ such that $v_1 \cdots v_m = uu_2 \cdots u_\ell$ and $\max\{\ell, m\} \leq N$ (in other words, $\max\{1 + \min L(u^{-1}v_1 \cdots v_m), m\} \leq N$).

Globally, we have that H is factorial if and only if $\mathfrak{t}(H) = 0$, and in the non-factorial case we have $\omega(H) \leq \mathfrak{t}(H)$. Moreover, it is not difficult to show that H is tame if and only if $\omega(H) < \infty$ ([14]).

If H is v -noetherian, then $\omega(H, b) < \infty$ for all $b \in H$, but this need not be true for the $\mathfrak{t}(H, u)$ values. In other words, a v -noetherian monoid is not necessarily locally tame. Apart from Krull monoids which will be discussed below, main examples of locally tame monoids are C-monoids: if R is a noetherian domain with integral closure \overline{R} , non-zero conductor \mathfrak{f} , finite residue field R/\mathfrak{f} and finite class group $\mathcal{C}(\overline{R})$, then R is a C-monoid, and there is an explicit characterization when R is globally tame (see [12, Theorem 2.11.9] and [13, 18, 25]).

A central method to investigate arithmetical phenomena in a given class of monoids H (such as noetherian domains) is to construct a simpler auxiliary monoid B and a homomorphism $\theta: H \rightarrow B$ which will be called a transfer homomorphism and which allows to shift arithmetical properties from B to H . The machinery of transfer homomorphisms is most highly developed for Krull monoids but not restricted to them. The auxiliary monoids associated to Krull monoids are monoids of zero-sum sequences over their respective class groups. We start with the necessary definitions.

A monoid homomorphism $\theta: H \rightarrow B$ is called a *transfer homomorphism* if the following holds:

(T1) $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.

(T2) If $u \in H$, $b, c \in B$ and $\theta(u) = bc$, then there exist $v, w \in H$ such that $u = vw$, $\theta(v) \simeq b$ and $\theta(w) \simeq c$.

A transfer homomorphism $\theta: H \rightarrow B$ between atomic monoids allows a unique extension $\overline{\theta}: Z(H) \rightarrow Z(B)$ to the factorization monoids satisfying $\overline{\theta}(uH^\times) = \theta(u)B^\times$ for all $u \in \mathcal{A}(H)$.

For $a \in H$ and $x \in Z(H)$, we denote by $\mathfrak{t}(a, x, \theta)$ the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

If $Z(a) \cap xZ(H) \neq \emptyset$, $z \in Z(a)$ and $\overline{\theta}(z) \in \overline{\theta}(x)Z(B)$, then there exists some $z' \in Z(a) \cap xZ(H)$ such that $\overline{\theta}(z') = \overline{\theta}(z)$ and $\mathfrak{d}(z, z') \leq N$.

Then

$$\mathfrak{t}(H, x, \theta) = \sup\{\mathfrak{t}(a, x, \theta) \mid a \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

is called the *tame degree in the fibres*. We will make substantial use of this concept in Section 5.

Lemma 3.2. *Let H be a reduced Krull monoid, $H \hookrightarrow F = \mathcal{F}(P)$ a cofinal divisor homomorphism, and let $G_P \subset G = F/H$ be the set of all classes containing prime divisors. Let $\tilde{\beta}: F \rightarrow \mathcal{F}(G_P)$ denote the unique homomorphism defined by $\tilde{\beta}(p) = [p]$ for all $p \in P$. Further, let $u \in \mathcal{A}(H)$ and $U = \tilde{\beta}(u)$.*

1. The homomorphism $\beta = \tilde{\beta} | H: H \rightarrow \mathcal{B}(G_P)$ is a transfer homomorphism, and hence

$$\mathfrak{t}(G_P, U) \leq \mathfrak{t}(H, u) \leq \mathfrak{t}(G_P, U) + \mathfrak{t}(H, u, \beta).$$

2. $\mathfrak{t}(H, u, \beta) \leq 1 + |u| \leq 1 + \mathsf{D}(G_P)$ for all $u \in \mathcal{A}(H)$.
3. Suppose that $G_P = -G_P$ and that every nontrivial class contains at least two distinct prime divisors. Then $1 + |u| = \mathfrak{t}(H, u, \beta)$ for all $u \in \mathcal{A}(H)$ with $|u| \geq 3$. In particular, if $\mathsf{D}(G_P) \geq 3$, then $\mathsf{D}(G_P) + 1 = \max\{\mathfrak{t}(H, u, \beta) \mid u \in \mathcal{A}(H)\}$.

Proof. See [12, Theorem 3.2.5 and Proposition 3.4.8] for 1., and [10, Proposition 4.2] for 2. and 3. \square

For any Krull monoid H , we denote by $\tilde{\beta}: F \rightarrow \mathcal{F}(G_P)$, by $\beta: H \rightarrow \mathcal{B}(G_P)$, and by $\bar{\beta}: \mathcal{Z}(H) \rightarrow \mathcal{Z}(G_P)$ the homomorphisms as defined above.

Proposition 3.3. *Let H be a reduced Krull monoid, $H \hookrightarrow F = \mathcal{F}(P)$ a cofinal divisor homomorphism, $G = F/H$, and $G_P \subset G$ the set of all classes containing prime divisors such that $\mathsf{D}(G_P) \geq 2$.*

1. H is locally tame if and only if $\mathcal{B}(G_P)$ is locally tame. More precisely, we have

$$\mathfrak{t}(G_P, U) \leq \mathfrak{t}(H, u) \leq \mathfrak{t}(G_P, U) + |U| + 1 \quad \text{for every } u \in \mathcal{A}(H).$$

2. $\mathfrak{t}(H, u) \leq \max\left\{\mathfrak{t}(G_P, U), \frac{3+(|u|-1)(\mathsf{D}(G_P)-1)}{2}\right\} \leq 1 + \frac{|u|(\mathsf{D}(G_P)-1)}{2}$ for every $u \in \mathcal{A}(H)$.
3. $\mathfrak{t}(G_P) \leq \mathfrak{t}(H) \leq \max\left\{\mathfrak{t}(G_P), \frac{3+(\mathsf{D}(G_P)-1)^2}{2}\right\} \leq 1 + \frac{\mathsf{D}(G_P)(\mathsf{D}(G_P)-1)}{2}$, and if $\mathfrak{t}(H) > \mathsf{D}(G_P)$, then

$$\mathfrak{t}(H) \leq \max\{1 + \min \mathsf{L}(W) \mid W \in \mathcal{B}(G_P \setminus \{0\}), |W| \leq \mathsf{D}(G_P)(\mathsf{D}(G_P) - 1)\}.$$

4. Suppose that $G_P = -G_P$. If $U \in \mathcal{A}(G_P)$ with $|U| \geq 3$, then $\mathfrak{t}(G_P, U) \geq |U|$. In particular, if $\mathsf{D}(G_P) \geq 3$, then $\mathfrak{t}(G_P) \geq \mathsf{D}(G_P)$.

Proof. 1. The inequalities follow immediately from Items 1. and 2. of Lemma 3.2, and by the very definition the inequalities show that H is locally tame if and only if $\mathcal{B}(G_P)$ is locally tame.

2. See [12, Theorem 3.4.10.6].

3. The first inequalities follow immediately from 1. and 2. Suppose that $\mathfrak{t}(H) > \mathsf{D}(G_P)$. If $\mathfrak{t}(H)$ is infinite, then $\mathsf{D}(G_P)$ is necessarily also infinite and the last inequality is clearly true. So, suppose $\mathfrak{t}(H)$ is finite. There are atoms $u, u_2, \dots, u_\ell, v_1, \dots, v_m$ such that $u \mid v_1 \cdots v_m$, but u divides no proper subproduct, and $\mathfrak{t}(H) = \mathfrak{t}(H, u) = \max\{\ell, m\}$. Since $m \leq |u| \leq \mathsf{D}(G_P)$, it follows that $\mathfrak{t}(H, u) = \ell = 1 + \min \mathsf{L}(w)$ with $w = u^{-1}v_1 \cdots v_m$. Since

$$|w| = |v_1 \cdots v_m| - |u| \leq |u|\mathsf{D}(G_P) - |u| \leq \mathsf{D}(G_P)(\mathsf{D}(G_P) - 1),$$

the assertion follows.

4. Suppose that $U = g_1 \cdots g_m$ with $m \geq 3$, and set $V_i = (-g_i)g_i$ for all $i \in [1, m]$. Then $U \mid V_1 \cdots V_m$, but U divides no proper subproduct. Since $U(-U) = V_1 \cdots V_m$, it follows that $\mathfrak{t}(G_P, U) \geq \max\{2, m\} = m = |U|$. The statement on $\mathfrak{t}(G_P)$ is an immediate consequence. \square

Remarks 3.4. 1. Proposition 3.3 shows that the property whether H is locally tame or not depends only on G_P . This is not true for global tameness. We argue as follows. By [15, Example 4.13], there is a tame Krull monoid H' with class group G' , set of prime divisors $G'_P \subset G'$ such that $\mathsf{D}(G'_P) = \infty$. Since H' is tame, $\mathcal{B}(G'_P)$ is tame by Proposition 3.3.3. By a Realization Theorem for Krull monoids ([12, Theorem 2.5.4]), there is a Krull monoid H with class group G , set of prime divisors $G_P \subset G$, and an isomorphism $\Phi: G \rightarrow G'$ with $\Phi(G_P) = G'_P$ such that every class in G_P contains at least two distinct prime divisors. Then [15, Theorem 4.2] implies that H is not tame, but $\mathcal{B}(G_P) \cong \mathcal{B}(G'_P)$ is tame.

2. Statement 3 of Proposition 3.3 shows that the finiteness of the Davenport constant implies that H is globally tame and hence locally tame. Note, if G_P is finite, then $\mathsf{D}(G_P)$ is finite, and the converse

holds if G has finite total rank ([12, Theorem 3.4.2]). Moreover, if $G_P = G$, then G is finite if and only if $D(G)$ is finite if and only if H is locally tame if and only if H is tame ([14, Theorem 4.4]).

As mentioned above, we can have that $\mathfrak{t}(G_P) < \infty = \mathfrak{t}(H)$. We will also give an example of a finite abelian class group such that $\mathfrak{t}(G) < \mathfrak{t}(H)$ (see Theorem 5.1). Thus in general the tame degree $\mathfrak{t}(H)$ does not coincide with the tame degree of the associated monoid of zero-sum sequences. However, if every class contains sufficiently many prime divisors, then the following proposition offers a characterization of $\mathfrak{t}(H)$ in terms of zero-sum theory. This opens the door to study the arithmetical invariant $\mathfrak{t}(H)$ with methods from Combinatorial and Additive Number Theory.

Proposition 3.5. *Let H be a Krull monoid with class group G and let $G_P \subset G$ denote the set of classes containing prime divisors. Suppose that $G_P = -G_P$ and that $2 < D(G_P) < \infty$.*

1. *Let $u \in \mathcal{A}(H)$ and $U = \beta(u)$ with $|U| \geq 3$. If every nontrivial class contains at least $|U| + 1$ distinct prime divisors, then*

$$\mathfrak{t}(H, u) = \max\{|U|, 1 + \min \mathfrak{L}(A_1 \cdots A_m) \mid m \in \mathbb{N}, U = S_1 \cdots S_m \text{ and, for all } i \in [1, m], \\ S_i, A_i \in \mathcal{F}(G_P) \setminus \{1\} \text{ with } S_i A_i \in \mathcal{A}(G_P)\}.$$

2. *If every nontrivial class contains at least $D(G_P) + 1$ distinct prime divisors, then*

$$\mathfrak{t}(H) = \max\{D(G_P), 1 + \min \mathfrak{L}(A_1 \cdots A_m) \mid m \in \mathbb{N}, A_1, \dots, A_m \in \mathcal{F}(G_P) \setminus \{1\} \\ \text{are zero-sum free such that } \sigma(A_1) \cdots \sigma(A_m) \in \mathcal{A}(G_P)\}.$$

Proof. We may suppose that H is reduced, and we consider a divisor theory $H \hookrightarrow F = \mathcal{F}(P)$.

1. First note that

$$\sigma(A_1 \cdots A_m) = \sigma(A_1) + \dots + \sigma(A_m) = -\sigma(U) = 0,$$

hence $A_1 \cdots A_m \in \mathcal{B}(G_P)$. Since $S_i A_i \in \mathcal{A}(G_P)$ and $S_i \neq 1$, it follows that $|A_i| \leq D(G_P) - 1$ for all $i \in [1, m]$, and hence $|A_1 \cdots A_m| \leq m(D(G_P) - 1) \leq |U|(D(G_P) - 1)$. Thus we get

$$\min \mathfrak{L}(A_1 \cdots A_m) \leq |A_1 \cdots A_m|/2 \leq (D(G_P) - 1)D(G_P)/2.$$

Thus the set $\{|U|, 1 + \min \mathfrak{L}(\cdot) \mid \dots\}$ is finite, and we denote by t' its maximum.

First we show that $\mathfrak{t}(H, u) \leq t'$. Let $a \in H$ with $\mathfrak{t}(H, u) = \mathfrak{t}(a, u)$. Let $\ell, m \in \mathbb{N}$, $v_1, \dots, v_m, u_2, \dots, u_\ell \in \mathcal{A}(H)$ such that $u \mid v_1 \cdots v_m$, but u divides no proper subproduct, $v_1 \cdots v_m = uu_2 \cdots u_\ell$ and $\max\{\ell, m\} = \mathfrak{t}(a, u)$. If $\ell \leq m$, then $\mathfrak{t}(a, u) = m \leq |U| \leq t'$. Suppose that $\ell > m$. Then $\mathfrak{t}(a, u) = \ell$ and $\ell - 1 = \min \mathfrak{L}(u^{-1}a)$. Since u divides $v_1 \cdots v_m$ but no proper subproduct, there are $s_1, \dots, s_m, a_1, \dots, a_m \in F \setminus \{1\}$ such that $u = s_1 \cdots s_m$, and $v_i = s_i a_i$ for all $i \in [1, m]$. Setting $S_i = \beta(s_i)$ and $A_i = \beta(a_i)$ for all $i \in [1, m]$ we obtain that

$$\mathfrak{t}(H, u) = \ell = 1 + \min \mathfrak{L}(u^{-1}a) = 1 + \min \mathfrak{L}(\beta(u^{-1}a)) = 1 + \min \mathfrak{L}(A_1 \cdots A_m) \leq t'.$$

Next we show that $t' \leq \mathfrak{t}(H, u)$. If $t' = |U|$, then the statement follows from Proposition 3.3. Suppose that $t' > |U|$, and let $S_1, A_1, \dots, S_m, A_m$ be as in the definition of t' such that $t' = 1 + \min \mathfrak{L}(A_1 \cdots A_m)$. There are $s_1, \dots, s_m \in F \setminus \{1\}$ such that $u = s_1 \cdots s_m$ and $S_i = \beta(s_i)$ for all $i \in [1, m]$. Set $\gamma = |A_1 \cdots A_m|$. Since every class contains at least $|U| + 1$ distinct prime divisors, there are primes $p_1, \dots, p_\gamma \in P \setminus \text{supp}_P(u)$ and elements $a_1, \dots, a_m \in F$ such that $a_1 \cdots a_m = p_1 \cdots p_\gamma$, $\gcd_F(u, a_1 \cdots a_m) = 1$, and $\beta(a_i) = A_i$ for all $i \in [1, m]$. Now we define $v_i = s_i a_i$ for all $i \in [1, m]$, and observe that $v_1, \dots, v_m \in \mathcal{A}(H)$. By construction, $u \mid v_1 \cdots v_m$, but u does not divide any proper subproduct. Let $u_2, \dots, u_\ell \in \mathcal{A}(H)$ such that $\max\{\ell, m\} \leq \mathfrak{t}(a, u)$. Then

$$\ell \geq 1 + \min \mathfrak{L}(u^{-1}a) = 1 + \min \mathfrak{L}(A_1 \cdots A_m) = t' > |U| \geq m,$$

and hence

$$\mathfrak{t}(H, u) \geq \mathfrak{t}(a, u) = \max\{\ell, m\} = \ell \geq t'.$$

2. Let t' denote the maximum on the right hand side. First we show that for all $u \in \mathcal{A}(H)$, we have $\mathfrak{t}(H, u) \leq t'$. We choose $u \in \mathcal{A}(H)$ and use the equation for $\mathfrak{t}(H, u)$ derived in 1. Clearly, we have $|u| \leq D(G_P) \leq t'$. Let $S_1, A_1, \dots, S_m, A_m$ be as in 1. Since $U = S_1 \cdot \dots \cdot S_m \in \mathcal{A}(G_P)$, it follows that $U' = \sigma(S_1) \cdot \dots \cdot \sigma(S_m) \in \mathcal{A}(G_P)$ and hence $-U' = \sigma(A_1) \cdot \dots \cdot \sigma(A_m) \in \mathcal{A}(G_P)$. Thus all the assumptions on A_1, \dots, A_m of 2. are satisfied, and thus $\mathfrak{t}(H, u) \leq t'$.

Conversely, we show that $t' \leq \mathfrak{t}(H)$. If $t' = D(G_P)$, then $t' = D(G_P) \leq \mathfrak{t}(H, u)$ for some $u \in \mathcal{A}(H)$ with $|u| = D(G_P)$. Suppose that $t' = 1 + \min L(A_1 \cdot \dots \cdot A_m)$ with A_1, \dots, A_m be as in 2. For $i \in [1, m]$, we define $S_i = -\sigma(A_i)$, and we set $U = S_1 \cdot \dots \cdot S_m$. Then $U, S_1 A_1, \dots, S_m A_m \in \mathcal{A}(G_P)$, and for any $u \in \beta^{-1}(U)$, we have $t' = 1 + \min L(A_1 \cdot \dots \cdot A_m) \leq \mathfrak{t}(H, u)$ by 1. \square

4. KRULL MONOIDS WITH SMALL AND WITH LARGE GLOBAL TAME DEGREE

Let H be a Krull monoid with class group G such that every class contains a prime divisor, and suppose that $|G| \geq 3$. Then, by Proposition 3.3, we have

$$D^*(G) \leq D(G) \leq \mathfrak{t}(G) \leq \mathfrak{t}(H) \leq 1 + \frac{D(G)(D(G) - 1)}{2}.$$

The main result in this section is Theorem 4.12. It characterizes when the equality $D^*(G) = \mathfrak{t}(G)$ and when the equality $D^*(G) = \mathfrak{t}(H)$ do hold. These characterizations reveal the first example showing that $\mathfrak{t}(G) < \mathfrak{t}(H)$ may happen. On the other hand, if we consider $\mathfrak{t}(H)$ and $D(G)$ as functions of the rank r of G (with fixed exponent), then $\mathfrak{t}(H)$ is growing as the upper bound given above.

We start with two lemmas providing lower bounds for the global tame degree.

Lemma 4.1. *Let G be a finite abelian group with $|G| > 1$.*

1. *Then $\mathfrak{t}(G) \geq 1 + \exp(G)k^*(G)$.*
2. *If G is cyclic of order $|G| = n \geq 25$, then $\mathfrak{t}(G) > 2n - 7\sqrt{n} + 10$.*

Proof. 1. See [12, Proposition 6.5.2].

2. Let G be cyclic of order $|G| = n \geq 25$. We start with a special construction (which is very similar to [12, Proposition 6.5.2]). We set $n = qm + j$, where $q \in [2, n - 2]$ with $\gcd(q, n) = 1$, and $m, j \in \mathbb{N}$, and choose a non-zero element $g \in G$. The atoms

$$U = (qg)^n, U_0 = g^n, U_1 = (-g)g, V = (qg)g^{n-q}, V' = (qg)(-g)^q$$

are pairwise distinct, and we have

$$A = V^{n-m}V'^m = UU_1^{qm}U_0^{n-q-m} \in \mathcal{B}(G).$$

Since $|\mathcal{Z}(UU_1^{qm}U_0^{n-q-m})| = 1$, A has precisely one factorization which is divisible by U . Therefore we obtain that

$$(4.1) \quad \begin{aligned} \mathfrak{t}(G) &\geq \mathfrak{t}(A, U) \geq d(V^{n-m}V'^m, UU_1^{qm}U_0^{n-q-m}) \\ &= \max\{n, 1 + qm + n - q - m\} = n + (q - 1)(m - 1). \end{aligned}$$

Thus, it would remain to find q, m and j fulfilling the relevant conditions such that $n + (q - 1)(m - 1)$ is greater than $2n - 7\sqrt{n} + 10$. The main obstacle here is that q needs to be co-prime to n . To side-step this problem for the most part, we first apply the first part of this result.

It is easy to see that if n is divisible by at least three distinct primes then $k(G) \geq 2 - 1/n$, and the result follows directly from the first part.

So, we may assume that n is divisible by at most two distinct primes. For such an n it is well-known and not hard to see that among each four consecutive integers there is at least one co-prime to n . Indeed, consider two distinct primes p, q , and assume for a contradiction $a, a + 1, a + 2, a + 3$ are all divisible by p or q , say $p \mid a$, then $p \nmid (a + 1)$ so $q \mid a + 1$ implying that p needs to divide $a + 2$ and thus $p = 2$ and $q \neq 2$, and $a + 3$ is divisible neither by p nor by q , a contradiction.

Thus, we can choose some q that is co-prime to n from the set $\{\lfloor \sqrt{n} \rfloor - 3, \dots, \lfloor \sqrt{n} \rfloor\}$; note that here we use the condition $n \geq 25$ to ensure that these elements are at least 2.

We then set $m = \lfloor n/q \rfloor$, the quotient of the Euclidean division of n by q , and j the rest (note that j is non-zero as q does not divide n). It follows that $m \geq \lfloor \sqrt{n} \rfloor$.

From this we get that $(m-1)(q-1) > (\sqrt{n}-2)(\sqrt{n}-5)$, and the claim follows. \square

It is apparent from the proof that also for $n < 25$ non-trivial bounds can be obtained using the same method. However, if one wishes to have a bound for some specific (small) values of n , one should in any case rather use the bound $n + (q-1)(m-1)$ directly for an in this case suitable choice of q , or at least not use the crude estimate $\lfloor \sqrt{n} \rfloor > \sqrt{n} - 1$ so that we made no effort to avoid the condition $n \geq 25$. Moreover, for n a prime or also a prime-power one can get somewhat better bounds in essentially the same way, using $q = \lfloor \sqrt{n} \rfloor$ or $q \in \{\lfloor \sqrt{n} \rfloor - 1, \lfloor \sqrt{n} \rfloor\}$, respectively.

Remark 4.2.

1. If G is cyclic of order $|G| = n \geq 5$ and n is a prime number, then $\mathfrak{t}(G) \geq 2n - 4\sqrt{n} + 4$.
2. If G is cyclic of order $|G| = n \geq 9$ and n is a prime-power, then $\mathfrak{t}(G) \geq 2n - 5\sqrt{n} + 6$.

Lemma 4.3. *Let G be a finite abelian group, and let $r \in \mathbb{N}$ be even.*

1. Let $(e_1, \dots, e_r) \in G^r$ be independent such that $\gcd(\text{ord}(e_i), \text{ord}(e_j)) > 1$ for all $i, j \in [1, r]$,
 $e_0 = -e_1 - \dots - e_r$, $\bar{e} = \sum_{i=1}^r (-1)^{i+1} e_i$,

$$U = \bar{e} \prod_{i \in [1, r] \text{ odd}} (-e_0 - e_i) \prod_{i \in [1, r] \text{ even}} (e_0 + e_i),$$

and

$$\frac{r}{2} = \text{ord}(e_i)k_i + j_i \quad \text{with } j_i \in [0, \text{ord}(e_i) - 1] \quad \text{and } k_i = \left\lfloor \frac{r}{2 \text{ord}(e_i)} \right\rfloor \quad \text{for all } i \in [1, r].$$

Then, U is a minimal zero-sum sequence and

$$\mathfrak{t}(G, U) \geq 1 + \sum_{i=1}^r \left(2 \left\lfloor \frac{r}{2 \text{ord}(e_i)} \right\rfloor + j_i \right).$$

2. If $G = C_n^r$ with $n \geq 2$ and $\gcd(r-1, n) = 1$, then $\mathfrak{t}(G) \geq 1 + 2r \lfloor \frac{r}{2n} \rfloor$ and $\mathfrak{t}(C_2^r) \geq 1 + \frac{r^2}{2}$.

Proof. 1. We set $S = \bar{e}^{-1}U$. Since $\sigma(S) = -\bar{e}$, it follows that $U = \bar{e}S$ is a zero-sum sequence. To show that U is a minimal zero-sum sequence it remains to verify that S is zero-sumfree. Assume to the contrary that S has a non-empty zero-sum subsequence

$$T = \prod_{i \in I_o} (-e_0 - e_i) \prod_{i \in I_e} (e_0 + e_i), \quad \text{where } I_o \text{ and } I_e \text{ are subsets of } [1, r] \text{ of odd and even numbers, resp.}$$

For $i \in [1, r]$ let $a_i \in [0, \text{ord}(e_i) - 1]$ such that $\sigma(T) = \sum_{i=1}^r a_i e_i$. Recalling the definition of e_0 we infer that

$$a_j e_j = (|I_o| - |I_e| + \delta_j) e_j \quad \text{where } \delta_j = \begin{cases} -1 & \text{for } j \in I_o, \\ 1 & \text{for } j \in I_e, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sigma(T) = 0$ and (e_1, \dots, e_r) is independent, it follows that $a_j e_j = 0$ for each $j \in [1, r]$, that is $\text{ord}(e_j) \mid (|I_o| - |I_e| + \delta_j)$.

Now, since T is non-empty not all δ_j equal 0. However, this implies that no δ_j equals 0. Indeed, if $\delta_k = 0$ for some k , then $\text{ord}(e_k) \mid |I_o| - |I_e|$, and considering some k' such that $\delta_{k'} \neq 0$ we infer that $\text{ord}(e_{k'})$ divides $|I_o| - |I_e| + 1$ or $|I_o| - |I_e| - 1$. This entails that $\gcd(\text{ord}(e_k), \text{ord}(e_{k'}))$ divides two consecutive integers, a contradiction. Consequently, $I_o \cup I_e = [1, r]$, $T = S$, and thus $\sigma(T) = \sigma(S) = \bar{e} \neq 0$, a contradiction.

We set $V_0 = \bar{e} \prod_{\nu=1}^r (-1)^\nu e_\nu$, and we define

$$V_i = \begin{cases} (-e_0 - e_i) \prod_{\nu \in [1, r] \setminus \{i\}} (-e_\nu), & \text{if } i \text{ is odd,} \\ (e_0 + e_i) \prod_{\nu \in [1, r] \setminus \{i\}} e_\nu, & \text{if } i \text{ is even.} \end{cases}$$

By construction, we have $U \mid \prod_{\nu=0}^r V_\nu$, but U does not divide any proper subproduct. Furthermore, we have

$$W = U^{-1} \prod_{\nu=0}^r V_\nu = \prod_{\nu=1}^r ((-e_\nu) e_\nu)^{r/2}.$$

Next we study $L(W)$. For any nonzero $g \in G$, $k \in \mathbb{N}_0$, and $j \in [0, \text{ord}(g) - 1]$, we have

$$L\left(\left((-g)g\right)^{k \text{ord}(g)+j}\right) = \{2k + \nu(\text{ord}(g) - 2) + j \mid \nu \in [0, k]\},$$

and the minimum of this set equals $2k + j$. Thus, for every $i \in [1, r]$, we obtain that

$$L\left(\left((-e_i)e_i\right)^{r/2}\right) = \{2k_i + \nu(\text{ord}(e_i) - 2) + j_i \mid \nu \in [0, k_i]\},$$

with

$$\min L\left(\left((-e_i)e_i\right)^{r/2}\right) = 2 \left\lfloor \frac{r}{2 \text{ord}(e_i)} \right\rfloor + j_i.$$

Since

$$L(W) = \sum_{i=1}^r L\left(\left((-e_i)e_i\right)^{r/2}\right),$$

$\min L(W)$ is the sum of the minima, and it follows that

$$\mathfrak{t}(G, U) \geq 1 + \min L(W) = 1 + \sum_{i=1}^r \left(2 \left\lfloor \frac{r}{2 \text{ord}(e_i)} \right\rfloor + j_i\right).$$

2. Suppose that $G = C_n^r$ with r and n as above. After choosing a basis (e_1, \dots, e_r) of G with $\text{ord}(e_1) = \dots = \text{ord}(e_r) = n$, the first inequality follows immediately from 1. Now suppose that $n = 2$. If $r \equiv 0 \pmod{4}$, then the statement on $\mathfrak{t}(C_2^r)$ follows from the first statement. If $r = 4k + 2$ with $k \in \mathbb{N}_0$, then $r/2 = 2k + 1$, and 1. implies that

$$\mathfrak{t}(G) \geq 1 + r(2k + 1) = 1 + \frac{r^2}{2}. \quad \square$$

Lemma 4.4. *Let G be a finite abelian group. Then $\exp(G)\mathfrak{k}^*(G) \geq \mathfrak{d}^*(G)$, and equality holds if and only if G is a p -group of the form $G = C_n^r$ where $n, r \in \mathbb{N}$.*

Proof. By definition, the statement holds if $|G| = 1$. Suppose that $|G| > 1$, say

$$G \cong C_{n_1} \oplus \dots \oplus C_{n_r} \cong C_{q_1} \oplus \dots \oplus C_{q_s},$$

where $r, s \in \mathbb{N}$, $n_1, \dots, n_r \in \mathbb{N}$, $1 < n_1 \mid \dots \mid n_r$, and q_1, \dots, q_s are prime powers. Note that $\exp(G) = n_r = \text{lcm}(q_1, \dots, q_s)$. Obviously, the statement holds for cyclic groups of prime power order. Suppose that G is cyclic but not a p -group. Then $r = 1$ and $s \geq 2$. Since $\frac{q_j - 1}{q_j} \geq \frac{1}{2}$ for all $j \in [1, s]$, it follows that

$$\sum_{j=1}^s \frac{q_j - 1}{q_j} \geq 1 \quad \text{and hence} \quad n_r \sum_{j=1}^s \frac{q_j - 1}{q_j} \geq n_r > \sum_{i=1}^r (n_i - 1).$$

Thus the statement holds for cyclic groups. If G is not cyclic, then

$$\exp(G)\mathfrak{k}^*(G) = \exp(G) \sum_{i=1}^r \mathfrak{k}^*(C_{n_i}) \stackrel{(1)}{\geq} \sum_{i=1}^r n_i \mathfrak{k}^*(C_{n_i}) \stackrel{(2)}{\geq} \sum_{i=1}^r \mathfrak{d}^*(C_{n_i}) = \mathfrak{d}^*(G),$$

where equality in (1) holds if and only if $n_1 = \dots = n_r$ and equality in (2) holds if and only if n_1, \dots, n_r are prime powers. \square

Proposition 4.5. *Let G_1, G_2 be finite abelian groups with $\mathfrak{t}(G_1) > \mathfrak{D}(G_1)$.*

1. *If $\mathfrak{t}(G_2) > \mathfrak{D}(G_2)$ or $\mathfrak{D}(G_2) = \mathfrak{D}^*(G_2)$, then $\mathfrak{t}(G_1 \oplus G_2) \geq \mathfrak{t}(G_1) + \mathfrak{t}(G_2) - 1$.*
2. *If $\mathfrak{d}(G_1 \oplus G_2) = \mathfrak{d}^*(G_1) + \mathfrak{d}^*(G_2)$, then $\mathfrak{t}(G_1 \oplus G_2) > \mathfrak{D}(G_1 \oplus G_2)$.*

Proof. 1. Let $i \in [1, 2]$. By definition of $\mathfrak{t}(G_i)$, there exists an $U^{(i)} \in \mathcal{A}(G_i)$ with $\mathfrak{t}(G_i) = \mathfrak{t}(G_i, U^{(i)})$. By definition of $\mathfrak{t}(G_i, U^{(i)})$, there are $\ell_i, m_i \in \mathbb{N}$, $U_2^{(i)}, \dots, U_{\ell_i}^{(i)}, V_1^{(i)}, \dots, V_{m_i}^{(i)} \in \mathcal{A}(G_i)$ such that $U^{(i)} \mid V_1^{(i)} \cdot \dots \cdot V_{m_i}^{(i)}$, but $U^{(i)}$ divides no proper subproduct, $U^{(i)} U_2^{(i)} \cdot \dots \cdot U_{\ell_i}^{(i)} = V_1^{(i)} \cdot \dots \cdot V_{m_i}^{(i)}$, and $\mathfrak{t}(G_i, U^{(i)}) = \max\{\ell_i, m_i\}$. If $\mathfrak{t}(G_i, U^{(i)}) = \mathfrak{t}(G_i) > \mathfrak{D}(G_i)$, then $m_i \leq |U^{(i)}| \leq \mathfrak{D}(G_i)$ implies that $m_i < \ell_i = \mathfrak{t}(G_i, U^{(i)})$ and

$$\ell_i - 1 = \min \mathfrak{L}(U_2^{(i)} \cdot \dots \cdot U_{\ell_i}^{(i)}).$$

Now suppose that $\mathfrak{t}(G_2) = \mathfrak{D}(G_2) = \mathfrak{D}^*(G_2)$. Then we provide a new construction of the above type where we have $\ell_2 \geq m_2$ and $\ell_2 - 1 = \min \mathfrak{L}(U_2^{(2)} \cdot \dots \cdot U_{\ell_2}^{(2)})$. For simplicity we use the same notation as above. Let (e_1, \dots, e_r) be a basis of G such that $\text{ord}(e_i) = n_i$ for all $i \in [1, r]$, $1 < n_1 \mid \dots \mid n_r$ and $\mathfrak{d}^*(G) = \sum_{i=1}^r (n_i - 1)$. We set $e_0 = e_1 + \dots + e_r$,

$$U^{(2)} = (-e_0)e_0, \quad V_1^{(2)} = e_0 \prod_{i=1}^r e_i^{n_i-1}, \quad \text{and} \quad V_2^{(2)} = -V_1^{(2)}.$$

Then we get $U^{(2)} \prod_{i=1}^r ((-e_i)e_i)^{n_i-1} = V_1^{(2)} V_2^{(2)}$, $\mathfrak{t}(G_2, U^{(2)}) = \mathfrak{t}(G_2) = 1 + \mathfrak{d}^*(G)$, $\ell_2 = \mathfrak{d}^*(G) + 1 \geq 2 = m_2$ and

$$\ell_2 - 1 = \min \mathfrak{L}(U_2^{(2)} \cdot \dots \cdot U_{\ell_2}^{(2)}),$$

with the obvious definition of $U_2^{(2)}, \dots, U_{\ell_2}^{(2)}$.

We continue simultaneously for both cases. For $\nu \in [1, m_i]$, we set $V_\nu^{(i)} = S_\nu^{(i)} A_\nu^{(i)}$, with $S_\nu^{(i)}, A_\nu^{(i)} \in \mathcal{F}(G_i)$ such that $U^{(i)} = S_1^{(i)} \cdot \dots \cdot S_{m_i}^{(i)}$. We choose an element $g_i \in G_i$ with $g_i \mid S_1^{(i)}$, and define

$$U = g_1^{-1} g_2^{-1} (g_1 + g_2) U^{(1)} U^{(2)} \quad \text{and} \quad V_1 = g_1^{-1} g_2^{-1} (g_1 + g_2) V_1^{(1)} V_1^{(2)}.$$

Then $U, V_1 \in \mathcal{A}(G_1 \oplus G_2)$ and $U \mid V_1 V_2^{(1)} \cdot \dots \cdot V_{m_1}^{(1)} V_2^{(2)} \cdot \dots \cdot V_{m_2}^{(2)}$, but U divides no proper subproduct. For $W = U^{-1} V_1 V_2^{(1)} \cdot \dots \cdot V_{m_1}^{(1)} V_2^{(2)} \cdot \dots \cdot V_{m_2}^{(2)}$ we get

$$W = U_2^{(1)} \cdot \dots \cdot U_{\ell_1}^{(1)} U_2^{(2)} \cdot \dots \cdot U_{\ell_2}^{(2)}$$

and

$$\mathfrak{L}(W) = \mathfrak{L}(U_2^{(1)} \cdot \dots \cdot U_{\ell_1}^{(1)}) + \mathfrak{L}(U_2^{(2)} \cdot \dots \cdot U_{\ell_2}^{(2)}).$$

This show that $\min \mathfrak{L}(W) = (\ell_1 - 1) + (\ell_2 - 1)$. Summing up we obtain that

$$\mathfrak{t}(G_1 \oplus G_2) \geq \mathfrak{t}(G_1 \oplus G_2, U) \geq \max\{m_1 + m_2, \ell_1 + \ell_2 - 1\} = \ell_1 + \ell_2 - 1 = \mathfrak{t}(G_1) + \mathfrak{t}(G_2) - 1.$$

2. Using 1. we infer that

$$\begin{aligned} \mathfrak{t}(G_1 \oplus G_2) &\geq \mathfrak{t}(G_1) + \mathfrak{t}(G_2) - 1 \geq \mathfrak{D}(G_1) + \mathfrak{D}(G_2) \\ &\geq \mathfrak{d}^*(G_1) + \mathfrak{d}^*(G_2) + 2 = \mathfrak{d}(G_1 \oplus G_2) + 2 = \mathfrak{D}(G_1 \oplus G_2) + 1. \end{aligned} \quad \square$$

For the rest of this section, let H be a reduced Krull monoid, $H \hookrightarrow \mathcal{F}(P)$ a divisor theory with class group G and suppose that every class contains a prime divisor.

Lemma 4.6. *Let $G = C_2^r$ with $r \geq 3$ and $\ell \in [1, r + 1]$. Let A_1, \dots, A_ℓ be pairwise distinct zero-sum free sequences with $|A_i| = r$. Then there exist some $k \in [0, \ell - 1]$ and $U_1, \dots, U_k \in \mathcal{A}(G)$ with $U_1 \cdots U_k \mid A_1 \cdots A_\ell$ such that*

$$|U_1 \cdots U_k| \geq 3(\ell - 1).$$

Proof. See [12, Lemma 6.6.5]. □

Lemma 4.7. *Let $G = C_2^r$ with $r \geq 3$, $n \geq 3$, and $u, v_1, \dots, v_n \in \mathcal{A}(H)$ such that $u \mid v_1 \cdots v_n$ and does not divide any proper subproduct. Furthermore, for every $i \in [1, n]$, suppose that $v_i = s_i a_i$ with $a_i, s_i \in \mathcal{F}(P) \setminus \{1\}$ such that $u = \prod_{i=1}^n s_i$ and set $w = u^{-1} v_1 \cdots v_n$. Then $\min \mathbf{L}(w) \leq \frac{n(r-1)+1}{2}$.*

Proof. We set $U = \beta(u)$, $W = \beta(w)$, $V_i = \beta(v_i)$, $S_i = \tilde{\beta}(s_i)$ and $A_i = \tilde{\beta}(a_i)$ for all $i \in [1, n]$. It is sufficient to verify the upper bound for $\min \mathbf{L}(W)$. After renumbering if necessary there is some $\ell \in [0, n]$ such that $|A_1| = \dots = |A_\ell| = r$ and $|A_i| \leq r - 1$ for all $i \in [\ell + 1, n]$. If $\ell = 0$, then $\min \mathbf{L}(W) \leq |W|/2 = n(r - 1)/2$. Let $\ell \geq 1$, and assume to the contrary that there are distinct $i, j \in [1, \ell]$ such that $A_i = A_j$, say $i = 1$ and $j = 2$. Then

$$\sigma(S_1) = \sigma(A_1) = \sigma(A_2) = \sigma(S_2),$$

hence $u = s_1 s_2$ and $n \leq 2$, a contradiction. Thus A_1, \dots, A_ℓ are pairwise distinct zero-sum free sequences. By Lemma 4.6 there exist some $k \in [0, \ell - 1]$ and $U_1, \dots, U_k \in \mathcal{A}(G)$ such that

$$U_1 \cdots U_k \mid A_1 \cdots A_\ell \quad \text{and} \quad |U_1 \cdots U_k| \geq 3(\ell - 1).$$

Setting $W' = (U_1 \cdots U_k)^{-1} W$ we infer that

$$|W'| \leq |W| - 3(\ell - 1) \leq \ell r + (n - \ell)(r - 1) - 3(\ell - 1) = n(r - 1) - 2(\ell - 1) + 1.$$

Thus W has a factorization of length at most

$$k + \frac{|W'|}{2} \leq \frac{1}{2} \left(n(r - 1) + 2(k - (\ell - 1)) + 1 \right) \leq \frac{n(r - 1) + 1}{2}. \quad \square$$

Lemma 4.8. *Let $G = C_2^r$ with $r \geq 3$, and suppose that $\mathbf{t}(G) \geq 2 + \frac{r(r-1)}{2}$. Then there are $U, V_1, \dots, V_{r+1} \in \mathcal{A}(G)$, where $U \mid V_1 \cdots V_{r+1}$ but U divides no proper subproduct, such that the following properties are satisfied:*

- (a) $U = e_1 \cdots e_{r+1}$, and $V_i = e_i A_i$ where $A_i \in \mathcal{F}(G)$ and $e_i = \gcd(U, V_i)$ for all $i \in [1, r + 1]$.
- (b) $A_i A_j$ is not zero-sum free for all $i, j \in [1, r + 1]$ distinct.
- (c) For $W = U^{-1} V_1 \cdots V_{r+1}$ we have $\gcd(U, W) = 1$ and $\mathbf{t}(G) = \mathbf{t}(G, U) = 1 + \min \mathbf{L}(W)$.

Proof. Let $U \in \mathcal{A}(G)$ with $\mathbf{t}(G) = \mathbf{t}(G, U) \geq 2 + \frac{r(r-1)}{2}$. Then there are $V_1, \dots, V_m \in \mathcal{A}(G)$ with $U \mid V_1 \cdots V_m$, $U \nmid \prod_{i \in I} V_i$ for any $I \subsetneq [1, m]$, and such that

$$\mathbf{t}(G, U) = 1 + \min \mathbf{L}(W).$$

Then Lemma 4.7 (applied with $H = \mathcal{B}(G)$) implies that

$$1 + \frac{r(r-1)}{2} \leq \min \mathbf{L}(W) \leq \frac{m(r-1) + 1}{2},$$

and hence $m = r + 1 = |U|$. Therefore we may assume that Property (a) holds.

Assume to the contrary that Property (b) fails. Then there exist some $i, j \in [1, r + 1]$ distinct such that $A_i A_j$ is zero-sum free, say $i = 1$ and $j = 2$. We set

$$\overline{U} = (e_2 + e_1) \prod_{i=3}^{r+1} e_i \quad \text{and} \quad \overline{V}_2 = (e_2 + e_1) A_2 A_1.$$

Then $\overline{U}, \overline{V}_2 \in \mathcal{A}(G)$, $\overline{U} \mid \overline{V}_2 V_3 \cdots V_{r+1}$ and

$$\overline{U}^{-1} \overline{V}_2 V_3 \cdots V_{r+1} = W = U^{-1} V_1 \cdots V_{r+1}.$$

Thus Lemma 4.7 implies that $\min \mathbf{L}(W) \leq \frac{r(r-1)}{2}$, a contradiction. Finally we assume to the contrary that Property (c) fails. This means that the set $I \subset [1, r+1]$, defined as

$$\prod_{i \in I} e_i = \gcd(U, W),$$

is nonempty. Let $i \in I$. Then e_i divides $W = A_1 \cdots A_{r+1}$. If $e_i \mid A_i^{-1} W$, then $U \mid \prod_{j \in [1, r+1] \setminus \{i\}} V_j$, a contradiction. Thus $e_i \mid A_i$ whence $V_i = e_i^2$. If there would exist $i, j \in I$ distinct, then $A_i A_j = e_i e_j$ would be zero-sum free. This implies that $|I| \leq 1$.

Since we assumed I to be nonempty, we get that $|I| = 1$, say $I = \{r+1\}$. Then for $i \in [1, r]$ we have

$$V_i = e_i A_i \quad \text{and} \quad V_{r+1} = e_{r+1}^2.$$

After renumbering if necessary, we may suppose that for some $\ell \in [0, r]$ we have $|A_1| = \cdots = |A_\ell| = r$ and $|A_i| \leq r-1$ for all $i \in [\ell+1, r]$. If $\ell = 0$, then $\min \mathbf{L}(W) \leq |W|/2 \leq (1+r(r-1))/2$, a contradiction. Suppose that $\ell \geq 1$. By Lemma 4.6, there exists some $k \in [1, \ell-1]$ and $U_1, \dots, U_k \in \mathcal{A}(G)$ such that

$$U_1 \cdots U_k \mid A_1 \cdots A_\ell \quad \text{and} \quad |U_1 \cdots U_k| \geq 3(\ell-1).$$

Setting $W' = (U_1 \cdots U_k)^{-1} W$ we infer that

$$|W'| \leq |W| - 3(\ell-1) \leq \ell r + (r-\ell)(r-1) + 1 - 3(\ell-1) = r(r-1) - 2\ell + 4.$$

Let $W' = U_{k+1} \cdot W''$ with $U_{k+1} \in \mathcal{A}(G)$ and $e_{r+1} \mid U_{k+1}$. Since $U \nmid \prod_{i=1}^r V_i$, e_{r+1} occurs exactly once in W' which implies that $|U_{k+1}| \geq 3$ and $|W''| = |W'| - |U_{k+1}| \leq r(r-1) - 2\ell + 1$. Thus W has a factorization with length at most

$$k+1 + \frac{|W''|}{2} = \frac{1}{2}(r(r-1) + 2(k+1-\ell) + 1) \leq \frac{r(r-1)}{2} + \frac{1}{2}$$

whence $\min \mathbf{L}(W) \leq \frac{r(r-1)}{2}$, a contradiction. \square

Lemma 4.9. *Let $G = C_2^r$ with $r \in \mathbb{N}$.*

1. $\mathfrak{t}(G) = \mathbf{D}(G)$ if and only if $r \in [2, 3]$.
2. If $r = 2$, then $\mathfrak{t}(H) = \mathbf{D}(G)$.
3. If $r = 3$ and if there is a nontrivial class containing at least two distinct prime divisors, then $\mathfrak{t}(H) = \mathbf{D}(G) + 1$.

Proof. We proceed in four steps, distinguishing the cases $r = 1$, $r = 2$, $r = 3$, and $r \geq 4$.

(i) If $r = 1$, then $\mathcal{B}(G)$ is factorial and hence $\mathfrak{t}(G) = 0 < \mathbf{D}(G) = 2$.

(ii) Suppose that $r = 2$. Then $\mathbf{D}(G) = 3$, and by Proposition 3.3 we have

$$3 \leq \mathfrak{t}(H) \leq \max \left\{ \mathfrak{t}(G), \left\lfloor \frac{3 + (\mathbf{D}(G_P) - 1)^2}{2} \right\rfloor \right\} = \max\{\mathfrak{t}(G), 3\},$$

and hence it suffices to verify that $\mathfrak{t}(G) \leq 3$. This can be done by a quick direct check.

(iii) Suppose that $r = 3$. To show the statement on $\mathfrak{t}(G)$, we assume to the contrary that $\mathfrak{t}(G) \geq \mathbf{D}(G) + 1 = r + 2 = 5 = 2 + \frac{3 \cdot 2}{2}$. Then let U, V_1, \dots, V_5 have all the properties of Lemma 4.8, and use all the notations of that lemma. In particular, we have $U = e_1 \cdots e_4$. Then (e_1, e_2, e_3) is a basis of G , $e_4 = e_1 + e_2 + e_3$, and $G = \{0, e_1, e_2, e_3, e_1 + e_2 + e_3, e_2 + e_3, e_1 + e_3, e_1 + e_2\}$. Since $\gcd(U, W) = 1$, it follows that

$$\text{supp}(A_1) \subset \text{supp}(W) \subset \{e_2 + e_3, e_1 + e_3, e_1 + e_2\}.$$

On the other hand, A_1 is zero-sum free with $\sigma(A_1) = e_1$ and with $|A_1| \in [2, 3]$, a contradiction.

Now suppose that there is a nontrivial class containing at least two distinct prime divisors. First we show that $\mathfrak{t}(H) \geq 5$. Let (e_1, e_2, e_3) be a basis of G and let $p_i \in P \cap e_i$ for all $i \in [1, 3]$, and let $p'_3 \in P \cap e_3$ with $p'_3 \neq p_3$. Let $u = q_1 q_2 q_3 p_3 \in \mathcal{A}(H)$ with $\beta(u) = (e_1 + e_2 + e_3)(e_1 + e_3)(e_2 + e_3)e_3$ such that $q_1 \in P \cap (e_1 + e_2 + e_3)$, $q_2 \in P \cap (e_1 + e_3)$, and $q_3 \in P \cap (e_2 + e_3)$. Now we define

$$v_1 = q_1 p_1 p_2 p'_3, \quad v_2 = q_2 p_1 p'_3, \quad v_3 = q_3 p_2 p'_3, \quad \text{and} \quad v_4 = p_3 p_3.$$

Then $v_1, v_2, v_3, v_4 \in \mathcal{A}(H)$, $u \mid v_1 v_2 v_3 v_4$, but u does not divide any proper subproduct. Since we have $\mathfrak{L}(u^{-1} v_1 v_2 v_3 v_4) = \{4\}$, it follows that

$$\mathfrak{t}(H, u) \geq \mathfrak{t}(a, u) \geq \max\{4, 1 + \min \mathfrak{L}(u^{-1} a)\} = 5 = \mathfrak{D}(G) + 1.$$

Assume to the contrary that $\mathfrak{t}(H) > 5$. We choose $a \in H$ and $u \in \mathcal{A}(H)$ such that $\mathfrak{t}(H) = \mathfrak{t}(a, u) \geq 6$. Then there are $u_2, \dots, u_\ell, v_1, \dots, v_m \in \mathcal{A}(H)$ such that $u \mid v_1 \dots v_m$, but u divides no proper subproduct, and $\max\{\ell, m\} = \mathfrak{t}(a, u)$. We set $U = \beta(u)$, $w = u^{-1} a$, and $W = \beta(w)$. Since $m \leq |U| \leq \mathfrak{D}(G) = 4$, it follows that $\mathfrak{t}(a, u) = \ell = 1 + \min \mathfrak{L}(W) \geq 6$. From this we get that $|U| = 4$ and $|W| \in [10, 12]$. Then for every $i \in [1, 4]$, there are $p_i \in P$ and $a_i \in F \setminus \{1\}$ such that $v_i = p_i a_i$ and $u = p_1 p_2 p_3 p_4$. We set $A_i = \beta(a_i)$ for all $i \in [1, 4]$, and after renumbering if necessary there is an $s \in [0, 4]$ such that $|A_1| = \dots = |A_s| = 3$, and $3 > |A_{s+1}| \geq \dots \geq |A_4|$. Note that $W = A_1 \dots A_4$, and since $\sigma(A_1), \dots, \sigma(A_4)$ are pairwise distinct, the sequences A_1, \dots, A_4 are pairwise distinct. Since $|A_1 \dots A_4| = |W| \in [10, 12]$, $|W| = 10$ implies $s \geq 2$, $|W| = 11$ implies $s \geq 3$, and $|W| = 12$ implies $s = 4$. By Lemma 4.6 there exist $k \in [0, s-1]$ and $W_1, \dots, W_k \in \mathcal{A}(G)$ such that $W_1 \dots W_k \mid A_1 \dots A_s$ and $|W_1 \dots W_k| \geq 3(s-1)$. This implies that

$$5 \leq \min \mathfrak{L}(W) \leq k + \frac{|W| - |W_1 \dots W_k|}{2} \leq (s-1) + \frac{|W| - 3(s-1)}{2} = \frac{|W| - (s-1)}{2},$$

a contradiction.

(iv) Suppose that $r \geq 4$. If $r \geq 4$ is even, then Lemma 4.3.2 shows that

$$\mathfrak{t}(G) \geq 1 + \frac{r^2}{2} > r + 1 = \mathfrak{D}(G).$$

If $r \geq 5$ is odd, then again by Lemma 4.3.2 we get that

$$\mathfrak{t}(G) \geq \mathfrak{t}(C_2^{r-1}) \geq 1 + \frac{(r-1)^2}{2} > r + 1 = \mathfrak{D}(G). \quad \square$$

Lemma 4.10. *Let $G = C_3^r$ with $r \in \mathbb{N}$. Then $\mathfrak{t}(H) = \mathfrak{D}(G)$ if and only if $r = 1$.*

Proof. Let $r = 1$. Then $\mathfrak{D}(G) = 3$, and by Proposition 3.3 we have

$$3 \leq \mathfrak{t}(H) \leq \max \left\{ \mathfrak{t}(G), \left\lfloor \frac{3 + (\mathfrak{D}(G_P) - 1)^2}{2} \right\rfloor \right\} = \max\{\mathfrak{t}(G), 3\},$$

and hence it suffices to check that $\mathfrak{t}(G) \leq 3$. Let $U \in \mathcal{A}(G)$. If $|U| = 2$, then $\mathfrak{t}(G, U) \leq 1 + \frac{|U|(\mathfrak{D}(G)-1)}{2} = 3$. If $|U| = 3$, then $U = g^3$, $V_1 = V_2 = V_3 = (-g)g$ for some nonzero $g \in G$, and hence $\mathfrak{t}(G, U) = 2$.

Note that $\mathfrak{D}(C_3^r) = \mathfrak{D}^*(C_3^r) = 2r + 1$. Since $\mathfrak{t}(H) \geq \mathfrak{t}(G)$, Proposition 4.5 implies that it is sufficient to show that $\mathfrak{t}(C_3 \oplus C_3) > 5 = \mathfrak{D}(C_3 \oplus C_3)$.

Let $G = C_3 \oplus C_3$, and let (e_1, e_2) be a basis of G . We define

$$V_1 = V_2 = e_1(-e_1 - e_2)^2(e_2 - e_1)^2, \quad V_3 = V_4 = e_2(-e_1 - e_2)^2(e_1 - e_2)^2, \quad V_5 = (e_1 + e_2)(-e_1)^2(e_1 - e_2),$$

and

$$U = e_1^2 e_2^2 (e_1 + e_2).$$

Then $U, V_1, \dots, V_5 \in \mathcal{A}(G)$, $U \mid V_1 \dots V_5$, but U does not divide any proper subproduct. We set $W = U^{-1} V_1 \dots V_5$, and assert that $\min \mathfrak{L}(W) = 6$, which implies that

$$\mathfrak{t}(G) \geq \mathfrak{t}(G, U) \geq 1 + \min \mathfrak{L}(W) = 7.$$

Note $W = (-e_1 - e_2)^8(e_1 - e_2)^5(e_2 - e_1)^4(-e_1)^2$, in particular it has length 19. We determine the atoms $S \in \mathcal{A}(G)$ with $S \mid W$ and $|S| \geq 4$. Such an atom must not contain both $(e_1 - e_2)$ and $(e_2 - e_1)$, yet contains at least three distinct elements; consequently it contains $-e_1$. First, suppose the two elements besides $-e_1$ are $(-e_1 - e_2)$ and $(e_2 - e_1)$. We note that $(-e_1)(-e_1 - e_2)(e_2 - e_1)$ is a (minimal) zero-sum sequence, and thus the only minimal zero-sum sequence with this support. Thus, since $|S| \geq 4$, we have $\text{supp}(S) = \{-e_1, -e_1 - e_2, e_1 - e_2\}$. If the multiplicity of $-e_1$ is 1, we get the atom $(-e_1)(-e_1 - e_2)(e_1 - e_2)^2$, and if the this multiplicity is 2, we get the atom $(-e_1)^2(-e_1 - e_2)^2(e_1 - e_2)$.

Therefore, noting that the multiplicity of $-e_1$ in W is 2, we can infer that every factorization of W contains (counted with multiplicity) either one atom of length 5 and none of length 4 or none of length 5 and at most 2 of length 4. Thus, $\min \mathbf{L}(W)$ is at least the smaller of $1 + \lceil (19 - 5)/3 \rceil = 6$ and $2 + \lceil (19 - 2 \cdot 4)/3 \rceil = 6$; that is it is at least 6. \square

Lemma 4.11. *Let $G = C_4^r$ with $r \in \mathbb{N}$. Then $\mathbf{t}(H) = \mathbf{D}(G)$ if and only if $r = 1$.*

Proof. Let $r = 1$. Let $u \in \mathcal{A}(H)$ but not prime. We have to show that $\mathbf{t}(H, u) \leq \mathbf{D}(G) = 4$. Let $v_1, \dots, v_m \in \mathcal{A}(H)$ such that $u \mid v_1 \cdots v_m$, but u divides no proper subproduct. We set $w = u^{-1}v_1 \cdots v_m \in H$, and note that $m \leq |u|$ and $m \in [2, 4]$. If $m = 2$, then $|u^{-1}v_1v_2| \leq 6$, and hence $\min \mathbf{L}(w) \leq 3$. We set $U = \beta(u)$, $W = \beta(w)$, $V_i = \beta(v_i)$ for all $i \in [1, m]$, and distinguish the cases $m = 3$ and $m = 4$.

CASE 1: $m = 3$.

First suppose that $|U| = 3$. Then $U = g^2(2g)$ for some $g \in G$ with $\text{ord}(g) = 4$, and we may suppose that $g \mid V_1$, $g \mid V_2$, and $(2g) \mid V_3$. Then $|V_3| \leq 3$. Assume to the contrary that $\min \mathbf{L}(W) \geq 4$. Then $|W| \geq 8$, which implies that $|V_1| = |V_2| = 4$ and $|V_3| = 3$. Then $V_1 = V_2 = g^4$, and $V_3 \in \{(2g)g^2, (2g)(-g)^2\}$. In both cases we get that $\min \mathbf{L}(W) < 4$, a contradiction.

Now suppose that $|U| = 4$. Then $U = g^4$ for some $g \in G$ with $\text{ord}(g) = 4$, and we assume again that $\min \mathbf{L}(W) \geq 4$. This implies that $|W| = 8$, and hence, after renumbering if necessary, $V_1 \in \{g^4, g^2(2g)\}$, and $V_2 = V_3 = g^4$. Thus we get $\mathbf{L}(W) = \{2\}$, a contradiction.

CASE 2: $m = 4$.

Then $U = g^4$ for some $g \in G$ with $\text{ord}(g) = 4$. Thus $V_1, V_2, V_3, V_4 \in \{g(-g), g^2(2g), g^4\}$. Assume to the contrary that $\min \mathbf{L}(W) \geq 4$. Then $|W| \geq 8$, and at most two of the V_i are equal to $(-g)g$. Discussing all possibilities we quickly see that $\min \mathbf{L}(W) \leq 3$, a contradiction.

Now suppose that $r \geq 2$, and note that $\mathbf{D}(C_4^r) = \mathbf{D}^*(C_4^r) = 3r + 1$. Since $\mathbf{t}(H) \geq \mathbf{t}(G)$, it suffices to prove that $\mathbf{t}(G) > \mathbf{D}(G)$. Thus by Proposition 4.5 it is sufficient to show that $\mathbf{t}(C_4 \oplus C_4) > 7 = \mathbf{D}(C_4 \oplus C_4)$.

Let $G = C_4 \oplus C_4$ and let (e_1, e_2) be a basis of G . We define

$$\begin{aligned} V_1 = V_2 = V_3 &= e_1(-e_1 - e_2)^3(2e_1 - e_2), \\ V_4 = V_5 = V_6 &= e_2(-e_1 - e_2)^3(-e_1 + 2e_2), \\ V_7 &= (e_1 + e_2)^4, \quad \text{and} \quad U = e_1^3e_2^3(e_1 + e_2). \end{aligned}$$

Then $U, V_1, \dots, V_7 \in \mathcal{A}(G)$, $U \mid V_1 \cdots V_7$, but U does not divide any proper subproduct. We set $W = U^{-1}V_1 \cdots V_7$, and assert that $\min \mathbf{L}(W) = 7$, which implies that

$$\mathbf{t}(G) \geq \mathbf{t}(G, U) \geq 1 + \min \mathbf{L}(W) = 8.$$

First we determine the atoms $S \in \mathcal{A}(G)$ with $S \mid W$. Since

$$(2e_1 - e_2, -e_1 + 2e_2) = (e_1, e_2) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and the determinant of the transformation matrix equals -1 modulo 4, it follows that $(2e_1 - e_2, -e_1 + 2e_2)$ is independent, and hence the sequence $(2e_1 - e_2)^3(-e_1 + 2e_2)^3$ is zero-sum free. Now it is easy to check that

$$S_1 = (e_1 + e_2)(2e_1 - e_2)^3(-e_1 + 2e_2)^3, S_2 = (e_1 + e_2)^2(2e_1 - e_2)^2(-e_1 + 2e_2)^2, S_3 = (e_1 + e_2)^3(2e_1 - e_2)(-e_1 + 2e_2)$$

are the atoms S with $(-e_1 - e_2) \nmid S$ and $S \mid W$, and

$$S_4 = (-e_1 - e_2)(2e_1 - e_2)(-e_1 + 2e_2), S_5 = (-e_1 - e_2)(e_1 + e_2), S_6 = (-e_1 - e_2)^4$$

are the atoms S with $(-e_1 - e_2) \mid S \mid W$. We claim that

$$Z(W) = \{S_1 S_5^2 S_6^4, S_3 S_4^2 S_6^4, S_2 S_4 S_5 S_6^4, S_4^3 S_5^3 S_6^3\}$$

which implies that $L(W) = \{7, 9\}$. Clearly, it remains to show that the given factorizations are the only ones. Let $z \in Z(W)$. If $S_1 \mid z$, then obviously $z = S_1 S_5^2 S_6^4$. Suppose that $S_1 \nmid z$. If $S_3 \mid z$, then $z = S_3 S_4^2 S_6^4$. Suppose that $S_3 \nmid z$. If $S_2 \mid z$, then $z = S_2 S_4 S_5 S_6^4$. If also $S_2 \nmid z$, then $z = S_4^3 S_5^3 S_6^3$. \square

Theorem 4.12. *Let H be a Krull monoid with finite class group G such that every class contains a prime divisor.*

1. $\mathfrak{t}(G) = \mathfrak{D}^*(G)$ if and only if $G \in \{C_3, C_4, C_2^2, C_2^3\}$.
2. If one nontrivial class contains at least two distinct prime divisors, then $\mathfrak{t}(H) = \mathfrak{D}^*(G)$ if and only if $G \in \{C_2, C_3, C_4, C_2^2\}$.
3. Suppose that G has rank $\mathfrak{r}(G) = r$, and consider both, $\mathfrak{D}(G)$ and $\mathfrak{t}(H)$, as functions in r . Then there are constants $M_1, M_2, M_3, M_4 \in \mathbb{R}_{>0}$ (depending only on $\exp(G)$ but not on r) such that

$$M_1 r \leq \mathfrak{D}(G) \leq M_2 r \quad \text{and} \quad M_3 r^2 \leq \mathfrak{t}(H) \leq M_4 r^2.$$

In particular, $\mathfrak{t}(H)$ grows as the upper bound $1 + \mathfrak{D}(G)(\mathfrak{D}(G) - 1)/2$, given in Proposition 3.3.3.

Proof. If $|G| = 1$, then both, H and $\mathcal{B}(G)$, are factorial, whence $\mathfrak{t}(H) = \mathfrak{t}(G) = 0$, but we have $\mathfrak{D}^*(G) = 1$. From now on we suppose that $|G| > 1$, say $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$, where $r = \mathfrak{r}(G) \in \mathbb{N}$ is the rank of G , $n_1, \dots, n_r \in \mathbb{N}$, $1 < n_1 \mid \dots \mid n_r$, and $n = n_r = \exp(G)$ is the exponent of G .

1. If $|G| = 2$, then $\mathfrak{t}(G) = 0$ and $\mathfrak{D}^*(G) = 2$. Suppose that $|G| > 2$, and that $\mathfrak{t}(G) = \mathfrak{D}^*(G)$. By Lemma 4.1 we obtain that $\mathfrak{t}(G) \geq 1 + \exp(G)k^*(G)$. Therefore Lemma 4.4 implies that equality holds and that G is a p -group with $n_1 = \dots = n_r$, whence $G = C_n^r$. If $n \geq 5$, then $\mathfrak{t}(C_n) > n = \mathfrak{D}(C_n)$ by Lemma 4.1.2. Thus Proposition 4.5.2 implies that $\mathfrak{t}(C_n^r) > \mathfrak{D}(C_n^r)$. Therefore it remains to consider the cases where $n \in [2, 4]$. Lemmas 4.9, 4.10, and 4.11 show that the mentioned groups satisfy $\mathfrak{t}(G) = \mathfrak{D}^*(G)$, and that there are no other groups G with $\exp(G) \leq 4$ with this property.

2. If $|G| = 2$, then H is not factorial whence $\mathfrak{t}(H) \geq 2$, and Proposition 3.3.3 implies that $\mathfrak{t}(H) \leq 2$. If $G = C_2^3$, then Lemma 4.9 implies that $\mathfrak{t}(H) = \mathfrak{D}(G) + 1$. Since $\mathfrak{t}(H) \geq \mathfrak{t}(G) \geq \mathfrak{D}(G) \geq \mathfrak{D}^*(G)$, the remaining assertions follow from 1.

3. We have $1 + r(n_1 - 1) \leq \mathfrak{D}^*(G) \leq \mathfrak{D}(G) \leq \mathfrak{D}(C_n^r)$, and by [12, Theorem 5.5.5], we obtain that

$$\mathfrak{D}(C_n^r) \leq n + n \log n^{r-1} \leq (n \log n)r.$$

Thus there exist constants M_1, M_2 as required. Let p be a prime with $p \mid n_1$. Then G has a subgroup isomorphic to C_p^r and hence $\mathfrak{t}(C_p^r) \leq \mathfrak{t}(G)$. We intend to find a $M_3 \in \mathbb{R}_{>0}$ with $M_3 r^2 \leq \mathfrak{t}(C_p^r)$. If $p = 2$, this holds by Lemma 4.3.2. Let p be odd. Then there is an $s \in [r - 2, r]$ such that s is even and $\gcd(p, s - 1) = 1$, and hence Lemma 4.3.2 implies that

$$1 + 2s \left\lfloor \frac{s}{2p} \right\rfloor \leq \mathfrak{t}(C_p^s) \leq \mathfrak{t}(C_p^r).$$

Finally Proposition 3.3.3 implies that

$$\mathfrak{t}(H) \leq 1 + \frac{\mathfrak{D}(G)(\mathfrak{D}(G) - 1)}{2} \leq \mathfrak{D}(G)^2 \leq M_2^2 r^2. \quad \square$$

5. KRULL MONOIDS WHOSE CLASS GROUP IS EITHER CYCLIC OR AN ELEMENTARY 2-GROUP

In this section we study Krull monoids H whose class group G is either cyclic or an elementary 2-group. We get quite precise results, which confirm the general tendency of the tame degree indicated by Theorem 4.12. Suppose $|G| \geq 3$, that every class contains a prime divisor, and consider again the inequality

$$D(G) \leq \mathfrak{t}(H) \leq 1 + \frac{D(G)(D(G) - 1)}{2}.$$

In case of elementary 2-groups, $\mathfrak{t}(H)$ almost equals the upper bound and, apart from one exceptional case, we always have $\mathfrak{t}(H) = \mathfrak{t}(G)$. Suppose that G is cyclic of order $|G| = n \geq 5$. Then $D(G) = n \leq \mathfrak{t}(H) \leq n^2$ (for better lower bounds see Lemma 4.1). As expected, it turns out that the tame degree is close to the lower bound.

Theorem 5.1. *Let H be a Krull monoid whose class group G is an elementary 2-group, say $G \cong C_2^r$ with $r \in \mathbb{N}$, and suppose that every class contains a prime divisor. Then we have*

1. *If $r = 1$, then $\mathfrak{t}(H) = 2$ and $\mathfrak{t}(G) = 0$.*
2. *If $r = 3$, then $\mathfrak{t}(G) = 4$, and if one nontrivial class contains at least two distinct prime divisors, then $\mathfrak{t}(H) = 5$.*
- 3.

$$\mathfrak{t}(H) = \mathfrak{t}(G) \quad \begin{cases} = 1 + \frac{r^2}{2} & \text{if } r \geq 2 \text{ is even,} \\ \geq 2 + \frac{r(r-1)}{2} & \text{if } r \geq 5 \text{ is odd.} \end{cases}$$

Proof. We may suppose that H is reduced, and that $H \hookrightarrow \mathcal{F}(P)$ a divisor theory with class group G . All statements of 1. and 2. follow from Lemma 4.9 and from Theorem 4.12, and hence it remains to prove 3. The assertion for even $r \geq 2$ follows from [12, Corollary 6.5.6]. Suppose that $r \geq 5$ is odd. The lower bound for $\mathfrak{t}(G)$ follows from [12, Theorem 6.5.3]. So it remains to show that $\mathfrak{t}(H) = \mathfrak{t}(G)$. By Proposition 3.3, it suffices to show that $\mathfrak{t}(H) \leq \mathfrak{t}(G)$.

Let $u \in \mathcal{A}(H)$. We have to show that $\mathfrak{t}(H, u) \leq \mathfrak{t}(G)$. If $u \in P$, then $\mathfrak{t}(H, u) = 0$. Suppose that $u \notin P$, and let $a \in uH$, $v_1, \dots, v_n \in \mathcal{A}(H)$ with $z = v_1 \cdot \dots \cdot v_n \in Z(a)$ such that u divides no proper subproduct of $v_1 \cdot \dots \cdot v_n$, and such that $\mathfrak{t}(H, u) = \max\{n, 1 + \min L(w)\}$, where $w = u^{-1}v_1 \cdot \dots \cdot v_n$. If $n \leq 2$, then the assertion follows. Suppose that $n \geq 3$, and note that $n \leq |u|$. Then Lemma 4.7 implies that $\min L(w) \leq \frac{n(r-1)+1}{2}$. Thus we are done for $n \leq r$.

Suppose that $n = r + 1$. Then $u = p_1 \cdot \dots \cdot p_{r+1}$ and, for all $i \in [1, r + 1]$, $v_i = p_i a_i$ where $p_i \in P$ and $a_i \in \mathcal{F}(P) \setminus \{1\}$. For $i \in [1, r + 1]$, we set $V_i = \beta(v_i)$, $g_i = [p_i]$ and we set $U = \beta(u)$. Then $U \mid V_1 \cdot \dots \cdot V_{r+1}$. After renumbering if necessary we may assume that there is some $m \in [1, r + 1]$ such that $U \mid V_1 \cdot \dots \cdot V_m$ but U does not divide any proper subproduct.

If $m = 1$, then $V_1 = U$, $\overline{\beta}(z) = V_1 \cdot \dots \cdot V_{r+1} \in UZ(G)$, and by definition of the tame degree in the fibres, there is a $z' \in Z(a) \cap uZ(H)$ with $\overline{\beta}(z') = \overline{\beta}(z)$ and $d(z, z') \leq \mathfrak{t}(H, u, \beta)$. By Lemma 3.2, we get $\mathfrak{t}(H, u, \beta) \leq 1 + D(G) = r + 2 \leq \mathfrak{t}(G)$.

Suppose that $m \geq 2$. There exist $U_2, \dots, U_k \in \mathcal{A}(G)$ such that $V_1 \cdot \dots \cdot V_m = UU_2 \cdot \dots \cdot U_k$ and

$$\max\{k, m\} = d(V_1 \cdot \dots \cdot V_m, UU_2 \cdot \dots \cdot U_k) \leq \mathfrak{t}(G, U) \leq \mathfrak{t}(G).$$

Since $\beta(u^{-1}a) = U_2 \cdot \dots \cdot U_k V_{m+1} \cdot \dots \cdot V_{r+1}$ and $\beta: H \rightarrow \mathcal{B}(G)$ is a transfer homomorphism, there exist $u_2, \dots, u_k, w_{m+1}, \dots, w_{r+1} \in \mathcal{A}(H)$ such that $\beta(u_i) = U_i$ for all $i \in [2, k]$, $\beta(w_j) = V_j$ for all $j \in [m + 1, r + 1]$ and $u^{-1}a = u_2 \cdot \dots \cdot u_k w_{m+1} \cdot \dots \cdot w_{r+1}$. Then

$$z' = uu_2 \cdot \dots \cdot u_k w_{m+1} \cdot \dots \cdot w_{r+1} \in Z(a) \cap uZ(H) \quad \text{and} \quad d(z, z') \leq \max\{r + 1, k + r + 1 - m\}.$$

If $m = r + 1$, this implies $d(z, z') \leq \mathfrak{t}(G)$. If $m = 2$, then

$$k - 1 \leq \left\lfloor \frac{|V_1 V_2| - |U|}{2} \right\rfloor \leq \left\lfloor \frac{2D(G) - 3}{2} \right\rfloor = r - 1,$$

and

$$k + 1 + r - m \leq 2r - 1 \leq 2 + \frac{r(r-1)}{2} \leq \mathfrak{t}(G).$$

Suppose that $m \in [3, r]$. Then Lemma 4.7 (applied with $H = \mathcal{B}(G)$ and $n = m \geq 3$) implies that $k - 1 \leq \lfloor \frac{m(r-1)+1}{2} \rfloor$. If $m \leq r - 1$, then

$$k + r + 1 - m \leq 2 + \frac{m(r-1)+1}{2} + r - m \leq 2 + \frac{r(r-1)}{2} \leq \mathfrak{t}(G),$$

If $m = r$, then $k - 1 \leq \frac{r(r-1)}{2}$ and

$$k + r + 1 - m = k + 1 \leq 2 + \frac{r(r-1)}{2} \leq \mathfrak{t}(G).$$

Thus in both cases we get $\mathfrak{d}(z, z') \leq \mathfrak{t}(G)$. \square

For $m, n \in \mathbb{N}$, let $\omega(n)$ denote the number of distinct prime divisors of n , and let $\phi_m(n)$ denote the number of integers $a \in [1, m]$ with $\gcd(a, n) = 1$; this function is sometimes called Legendre's totient function. Thus $\phi_n(n) = \phi(n)$ is Euler's totient function.

Theorem 5.2. *Let H be a Krull monoid having a cyclic class group G of order $|G| = n \geq 5$.*

1. *If $n = p \in \mathbb{P}$, then $\mathfrak{t}(H) \leq 1 + \frac{2(p-1)p}{p+5} + 2(p-1)(\frac{1}{2} + \log(\frac{p+3}{2}))$.*
2. *If $n = p^\alpha$, where $p \in \mathbb{P}$ and $\alpha \geq 2$, then*

$$\mathfrak{t}(H) \leq 1 - 2\alpha + \frac{2p^{\alpha+1}}{p-1} + 2\alpha n + 3 \sum_{i=1}^{\alpha} (p^i - 1) \log \frac{p^i}{2}.$$

3. *If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $r \geq 2$, $\alpha_1, \dots, \alpha_r \in \mathbb{N}$, and $p_1, \dots, p_r \in \mathbb{P}$ are distinct, then $\mathfrak{t}(H) \leq$*

$$1 + \frac{4.3}{2} \sum_{1 < d|n} (d-1)+n \sum_{1 < d|n, d \leq 4375} \frac{d}{2} + n \sum_{4376 \leq d|n} 2^{\omega(d)+1} \sqrt{2\omega(d)+3.3} \sum_{1 < d|n} (d-1) \log(\lfloor \frac{d+1}{2^{\omega(d)+1} \sqrt{2\omega(d)-1} + 1} - 1 \rfloor).$$

We need a new combinatorial invariant and a series of lemmas. The proof of Theorem 5.2 will be given at the end of this section.

Definition 5.3. Let G be a finite abelian group. For every $t \in [2, \mathfrak{D}(G)]$, let $\mathfrak{m}(G, t)$ denote the smallest integer $\ell \in \mathbb{N}$ such that every sequence S over $G \setminus \{0\}$ of length $|S| \geq \ell$ and in addition satisfying $\nu_g(S) \leq \text{ord}(g)$ for each $g \in G \setminus \{0\}$, has a minimal zero-sum subsequence T of length $|T| \geq t$.

The idea behind defining this constant is to somehow quantify how easy or hard it is for a given group G to avoid the existence of long minimal zero-sum subsequences. While it is clear that some additional condition, beyond the usual one on the length, is needed to make this definition a meaningful one, regarding the precise condition there is some flexibility. The one we choose is, except for excluding 0, the most permissive one that seems reasonable. If one cares about minimal zero-sum sequences one never has a need for an element more than its order times. To exclude 0 makes sense for the present application and more generally is convenient; the variant of the constant where 0 would be admitted (with multiplicity 1) would merely differ by exactly 1 from the current version.

The following lemma establishes some basic properties of this new invariant.

Lemma 5.4. *Let G be a finite abelian group with $|G| > 1$.*

1. *For every $t \in [2, \mathfrak{D}(G)]$, we have $\mathfrak{m}(G, t) \geq \mathfrak{D}(G)$.*
2. *We have $\mathfrak{m}(G, 2) = \mathfrak{D}(G)$ and $\mathfrak{m}(G, 3) \in [2\mathfrak{D}^*(G) - 1, 2\mathfrak{D}(G) - 1]$.*

Proof. 1. By definition of $D(G)$, there is a zero-sum free sequence S over G of length $|S| = D(G) - 1$; note that, S being zero-sum free, $v_g(S) \leq \text{ord}(g)$ for each g . Since such a sequence does not satisfy the defining property of $m(G, t)$, it follows that $m(G, t) \geq D(G)$ for every $t \in [2, D(G)]$.

2. Every sequence S over $G \setminus \{0\}$ of length $D(G)$ has a zero-sum subsequence, and hence a minimal zero-sum subsequence T . Since $0 \nmid S$, we get $|T| \geq 2$. Thus $m(G, 2) \leq D(G)$, and equality follows by 1.

In order to show that $m(G, 3) \leq 2D(G) - 1$, let S be a sequence of length $|S| \geq 2D(G) - 1$. We write S in the form $S = S_1S_2T_1T_2$ where, for $i \in [1, 2]$, S_i is a sequence over G such that $\text{supp}(S_i) \cap \text{supp}(-S_i) = \emptyset$ and T_i is a squarefree sequence over G containing only elements of order 2. (Recall that the multiplicity of an element of order 2 in S is at most 2, and also note that S_1S_2 cannot contain elements of order 2 appearing in S .)

Without restriction we may suppose that $|S_1| \geq |S_2|$ and $|T_1| \geq |T_2|$. Then $|S_1T_1| \geq D(G)$, and S_1T_1 thus contains a minimal zero-sum subsequence T . By construction we have $|T| \geq 3$.

Finally, we verify that $m(G, 3) \geq 2D^*(G) - 1$. Suppose that $G = C_{n_1} \oplus \dots \oplus C_{n_r}$ with $1 < n_1 \mid \dots \mid n_r$, and let (e_1, \dots, e_r) be a basis of G with $\text{ord}(e_i) = n_i$ for all $i \in [1, r]$. Then

$$S = \prod_{i=1}^r e_i^{n_i-1}$$

is a zero-sum free sequence of length $|S| = D^*(G) - 1$, and the sequence $(-S)S$ fulfills the additional condition (for slightly different reasons in the cases $\text{ord}(e_i) = 2$ and $\text{ord}(e_i) > 2$), and has no minimal zero-sum subsequence T of length $|T| \geq 3$. Thus it follows that $m(G, 3) > |(-S)S| = 2D^*(G) - 2$. \square

From now on till the rest of this section, let G be a finite cyclic group of order $|G| = n \geq 5$, $G^\bullet = G \setminus \{0\}$, and $m \in [1, n]$.

Lemma 5.5. *Let S be a sequence over G such that $\text{ord}(g) = n$ for all $g \in \text{supp}(S)$. If $|S| \geq \frac{\phi(n)(n-1)+1}{\phi_m(n)}$, then S has a minimal zero-sum subsequence of length at least $\lceil \frac{n}{m} \rceil$.*

Proof. For $e \in G$ with $\text{ord}(e) = n$, we have $S = m_1^e e \cdot \dots \cdot m_k^e e$ where $m_1^e, \dots, m_k^e \in [1, n]$. For every $\ell \in [2, n-1]$, let $S(\ell, e)$ denote the subsequence consisting of all terms $m_k^e e$ with $m_k^e \in [1, \ell]$. Clearly, if

$$(5.1) \quad |S(\ell, e)| \geq n$$

then $S(\ell, e)$ has a minimal zero-sum subsequence of length at least $\lceil \frac{n}{\ell} \rceil$. So, it suffices to prove the existence of some e such that (5.1) holds with $\ell = m$. Since $\text{ord}(g) = n$ for each $g \mid S$ we have

$$\sum_{e \in G, \text{ord}(e)=n} S(m, e) = |S| \sum_{i \in [1, m], \text{gcd}(i, n)=1} 1 = \phi_m(n) |S|.$$

Therefore,

$$\max\{|S(m, e)| \mid e \in G, \text{ord}(e) = n\} \geq \frac{\phi_m(n) |S|}{\phi(n)}.$$

It follows from $|S| \geq \frac{\phi(n)(n-1)+1}{\phi_m(n)}$ that $\max\{|S(m, e)| \mid e \in G, \text{ord}(e) = n\} > n - 1$. This proves (5.1) holds for some e and completes the proof. \square

The following technical lemma establishes some bounds on sums that is needed several times later on; the somewhat unusual indexing is convenient then.

Lemma 5.6. *Let c_2, \dots, c_M be non-negative reals. Let C be such that $\sum_{i=2}^m ic_i \leq Cm$ for each $m \in [2, M]$. Then $\sum_{i=2}^m c_i \leq C(1 + \sum_{i=3}^m 1/i)$ for each $m \in [2, M]$.*

Proof. The argument is by induction on M . For $M = 2$ the claim is obvious. Consider $M \geq 3$. Set $K_m = \sum_{i=2}^m c_i$. It suffices to show the claimed bound for K_M (for the others the claim is clear by hypothesis). Note that $MK_M = \sum_{m=2}^{M-1} K_m + \sum_{i=2}^M ic_i$. Thus,

$$MK_M \leq C \sum_{m=2}^{M-1} (1 + \sum_{i=3}^m 1/i) + CM = CM(1 + \sum_{i=3}^M 1/i),$$

where the last equality can be seen, for example, by another inductive argument. \square

Lemma 5.7. *Let $n = p \in \mathbb{P}$.*

1. *For every $t \in [1, p-1]$, we have $m(G, t+1) \leq \left\lfloor \frac{(p-1)^2}{[p/t]} \right\rfloor + 1$.*
2. *If S is a zero-sum sequence over G^\bullet , then $\min \mathsf{L}(S) \leq \min\{\frac{|S|}{2}, \frac{2|S|}{p+5} + 2(p-1)(\frac{1}{2} + \log(\frac{p+3}{2}))\}$.*

Proof. 1. This follows from Lemma 5.5.

2. Clearly, we have $\min \mathsf{L}(S) \leq \max \mathsf{L}(S) \leq \frac{|S|}{2}$. Thus it suffices to prove that $\min \mathsf{L}(S)$ is bounded above by the second term in the above set. To do so, we show that there exists a factorization $S = U_1 \cdot \dots \cdot U_t$, where $U_1, \dots, U_t \in \mathcal{A}(G)$ and t is bounded above by the given term. We construct U_1, \dots, U_t recursively. Indeed, for $i \in [1, t]$, let U_i be a minimal zero-sum subsequence of $S(\prod_{j=1}^{i-1} U_j)^{-1}$, whose length is maximal possible. Now we use 1. to obtain an upper bound on t . For every $k \in [2, p]$, let n_k be the number of U_i such that $|U_i| = k$. For every $m \in [2, p-1]$, the construction of U_i and 1. imply that

$$(5.2) \quad \sum_{i=2}^m in_i \leq \left\lfloor \frac{(p-1)^2}{[p/m]} \right\rfloor.$$

If $m \leq \frac{p+3}{2}$, then $[p/m] \geq \frac{p-m+1}{m} \geq \frac{p-1}{2m}$. Therefore, from (5.2) we infer that

$$(5.3) \quad \sum_{i=2}^m in_i \leq 2(p-1)m$$

holds for every $m \in [2, \frac{p+3}{2}]$.

By equation (5.3) and Lemma 5.6 we obtain that

$$\sum_{i=2}^m n_i \leq 2(p-1)(1 + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{m})$$

holds for every $m \in [2, \frac{p+3}{2}]$. Especially,

$$\sum_{i=2}^{\frac{p+3}{2}} n_i \leq 2(p-1)(\frac{1}{2} + \sum_{i=2}^{\frac{p+3}{2}} \frac{1}{i}) \leq 2(p-1)(\frac{1}{2} + \log(\frac{p+3}{2})).$$

Since $\sum_{i \geq \frac{p+5}{2}} in_i \leq |S|$, we have $\sum_{i \geq \frac{p+5}{2}} n_i \leq \frac{2|S|}{p+5}$. Hence,

$$t = \sum_{i=2}^p n_i \leq \frac{2|S|}{p+5} + 2(p-1)(\frac{1}{2} + \log(\frac{p+3}{2})). \quad \square$$

Lemma 5.8. *Let $n = p^\alpha$, where $p \in \mathbb{P}$ and $\alpha \geq 2$, and let S be a zero-sum sequence over G such that $\text{ord}(g) = n$ for all $g \in \text{supp}(S)$. Then $\min \mathsf{L}(S) \leq \min\{\frac{|S|}{2}, \frac{2|S|}{n+1} + 3(n-1)(\frac{1}{2} + \log(\frac{n}{2}))\}$.*

Proof. As in Lemma 5.7.2., it suffices to show that there exists a factorization $S = U_1 \cdot \dots \cdot U_t$, where $U_1, \dots, U_t \in \mathcal{A}(G)$ and t is bounded above by the second term in the above set. We construct U_1, \dots, U_t recursively. Indeed, for $i \in [1, t]$, let U_i be a minimal zero-sum subsequence of $S(\prod_{j=1}^{i-1} U_j)^{-1}$, whose length is maximal possible. We are going to use Lemma 5.5 to get an upper bound on t . For every $k \in [2, n]$, let n_k be the number of U_i such that $|U_i| = k$. For every $m \in [3, n-1]$, the construction of U_i and Lemma 5.5 imply that

$$(5.4) \quad \sum_{i=2}^{m-1} in_i \leq \frac{\phi(n)(n-1)}{\phi_{\lfloor \frac{n}{m} \rfloor}(n)}$$

If $m \leq \frac{n+2}{2}$ then $\phi_{\lfloor \frac{n}{m} \rfloor}(n) = \lfloor \frac{n}{m} \rfloor - \lfloor \frac{\lfloor \frac{n}{m} \rfloor}{p} \rfloor \geq \lfloor \frac{n}{m} \rfloor (1 - \frac{1}{p}) \geq \frac{n-m+1}{m} (1 - \frac{1}{p}) \geq \frac{n}{2m} (1 - \frac{1}{p}) = \frac{\phi(n)}{2m}$. It follows from (5.4) that

$$\sum_{i=2}^{m-1} in_i \leq 2m(n-1) \leq 3(m-1)(n-1).$$

Therefore, for every $m \in [2, \frac{n}{2}]$, we have

$$(5.5) \quad \sum_{i=2}^m in_i \leq 3(n-1)m$$

It follows from (5.5), applying Lemma 5.6, that

$$\sum_{i=2}^m n_i \leq 3(n-1) \left(1 + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{m}\right)$$

holds for every $m \in [2, \frac{n}{2}]$. Especially,

$$\sum_{i=2}^{\frac{n}{2}} n_i \leq 3(n-1) \left(\frac{1}{2} + \sum_{i=2}^{\frac{n}{2}} \frac{1}{i}\right) \leq 3(n-1) \left(\frac{1}{2} + \log\left(\frac{n}{2}\right)\right).$$

Since $\sum_{i > \frac{n}{2}} in_i = \sum_{i \geq \frac{n+1}{2}} in_i \leq |S|$, $\sum_{i > \frac{n}{2}} n_i \leq \frac{|S|}{\frac{n+1}{2}}$. Hence,

$$t = \sum_{i=2}^n n_i \leq \frac{2|S|}{n+1} + 3(n-1) \left(\frac{1}{2} + \log\left(\frac{n}{2}\right)\right). \quad \square$$

Lemma 5.9. *Let $n = p^\alpha$, where $p \in \mathbb{P}$ and $\alpha \geq 2$, and let S be a zero-sum sequence over G^\bullet . For every positive divisor $d > 1$ of n , let N_d denote the number of the terms of S which have order d . Then $\min L(S) \leq \min\left\{\frac{|S|}{2}, -2\alpha + \frac{2p^{\alpha+1}}{p-1} + \sum_{i=1}^{\alpha} \frac{2N_{p^i}}{p^i+1} + 3 \sum_{i=1}^{\alpha} (p^i - 1) \log \frac{p^i}{2}\right\}$.*

Proof. Clearly, it suffices to prove that $\min L(S)$ is bounded above by the second term in the above set. For every $i \in [1, \alpha]$, let S_i denote the subsequence of S consisting of all terms with order p^i , let T_i be a zero-sum subsequence of S_i with maximal possible length, and set $T'_i = S_i T_i^{-1}$. Therefore

$$S = S_1 \cdot \dots \cdot S_\alpha = \prod_{i=1}^{\alpha} T'_i \prod_{i=1}^{\alpha} T_i \quad \text{and} \quad \prod_{i=1}^{\alpha} T'_i \quad \text{has sum zero.}$$

By the maximality of T_i we infer that $|T'_i| \leq p^i - 1$ for every $i \in [1, \alpha]$. Hence,

$$(5.6) \quad \left| \prod_{i=1}^{\alpha} T'_i \right| \leq \sum_{i=1}^{\alpha} (p^i - 1) \leq p^\alpha \frac{p}{p-1} - \alpha.$$

Therefore,

$$\min \mathsf{L}(S) \leq \min \mathsf{L}\left(\prod_{i=1}^{\alpha} T'_i\right) + \sum_{i=1}^{\alpha} \min \mathsf{L}(T_i) \leq \frac{|\prod_{i=1}^{\alpha} T'_i|}{2} + \sum_{i=1}^{\alpha} \min \mathsf{L}(T_i).$$

It follows from (5.6) that

$$(5.7) \quad \min \mathsf{L}(S) \leq \frac{p^{\alpha} \frac{p}{p-1} - \alpha}{2} + \sum_{i=1}^{\alpha} \min \mathsf{L}(T_i).$$

By Lemma 5.7.2 and Lemma 5.8 we obtain that

$$\min \mathsf{L}(T_i) \leq \frac{2|T_i|}{p^i + 1} + 3(p^i - 1)\left(\frac{1}{2} + \log \frac{p^i}{2}\right)$$

holds for every $i \in [1, \alpha]$. It follows from (5.6) that

$$\begin{aligned} \min \mathsf{L}(S) &\leq \frac{p^{\alpha} \frac{p}{p-1} - \alpha}{2} + \sum_{i=1}^{\alpha} \left(\frac{2|T_i|}{p^i + 1} + 3(p^i - 1)\left(\frac{1}{2} + \log \frac{p^i}{2}\right)\right) \\ &= \frac{p^{\alpha+1} - \alpha(p-1)}{2(p-1)} + \sum_{i=1}^{\alpha} \frac{2|T_i|}{p^i + 1} + \frac{3}{2} \sum_{i=1}^{\alpha} (p^i - 1) + 3 \sum_{i=1}^{\alpha} (p^i - 1) \log \frac{p^i}{2} \\ &= 2 \frac{p^{\alpha+1} - \alpha(p-1)}{p-1} + \sum_{i=1}^{\alpha} \frac{2|T_i|}{p^i + 1} + 3 \sum_{i=1}^{\alpha} (p^i - 1) \log \frac{p^i}{2} \\ &\leq 2 \frac{p^{\alpha+1} - \alpha(p-1)}{p-1} + \sum_{i=1}^{\alpha} \frac{2N p^i}{p^i + 1} + 3 \sum_{i=1}^{\alpha} (p^i - 1) \log \frac{p^i}{2} \\ &= -2\alpha + \frac{2p^{\alpha+1}}{p-1} + \sum_{i=1}^{\alpha} \frac{2N p^i}{p^i + 1} + 3 \sum_{i=1}^{\alpha} (p^i - 1) \log \frac{p^i}{2}. \quad \square \end{aligned}$$

We need certain bounds for Legendre's totient function. We establish what we need in the two subsequent lemmas in a self-contained way.

Lemma 5.10. *Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, where $s \geq 2$, $\alpha_1, \dots, \alpha_s \in \mathbb{N}$, and $p_1, \dots, p_s \in \mathbb{P}$ are distinct. If $m \geq 2^{s+1} \sqrt{2s-1}$ then $\phi_m(n) \geq \frac{m}{2} \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) = \frac{m\phi(n)}{2n}$.*

Proof. By the inclusion-exclusion principle we know that

$$\begin{aligned} \phi_m(n) &= m - \sum_{i=1}^s \lfloor \frac{m}{p_i} \rfloor + \sum_{1 \leq i < j \leq s} \lfloor \frac{m}{p_i p_j} \rfloor - \dots + (-1)^s \lfloor \frac{m}{p_1 \cdots p_s} \rfloor \\ &\geq m - \sum_{i=1}^s \frac{m}{p_i} + \sum_{1 \leq i < j \leq s} \left(\frac{m}{p_i p_j} - 1\right) - \sum_{1 \leq i < j < k} \left(\frac{m}{p_i p_j p_k} - 1\right) + \sum_{1 \leq i < j < k < l} \left(\frac{m}{p_i p_j p_k p_l} - 1\right) - \dots \\ &= m \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) - \binom{s}{2} + \binom{s}{4} - \dots = m \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) - 2^{s-1}. \end{aligned}$$

Therefore,

$$(5.8) \quad \phi_m(n) \geq m \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) - 2^{s-1}.$$

It is easy to see that $p_i \geq 2i - 1$ for all $i \in [2, s]$. Therefore,

$$(5.9) \quad \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) \geq \frac{1}{2} \prod_{i=2}^s \frac{2i-2}{2i-1}.$$

Since $\frac{2i-2}{2i-1} \geq \frac{2i-3}{2i-2}$ holds for every $i \in [2, s]$, we obtain that

$$\left(\prod_{i=2}^s \frac{2i-2}{2i-1}\right)^2 \geq \prod_{i=2}^s \frac{2i-2}{2i-1} \prod_{i=2}^s \frac{2i-3}{2i-2} = \prod_{i=1}^{2s-2} \frac{i}{i+1} = \frac{1}{2s-1}.$$

It follows that

$$(5.10) \quad \prod_{i=2}^s \frac{2i-2}{2i-1} \geq \frac{1}{\sqrt{2s-1}}, \quad \text{and hence by (5.9)} \quad \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) \geq \frac{1}{2\sqrt{2s-1}}.$$

Since $m \geq 2^{s+1} \sqrt{2s-1}$, from (5.10) we deduce that $\frac{m}{2} \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) \geq 2^{s-1}$. It follows from (5.8) that $\phi_m(n) \geq m \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) - 2^{s-1} \geq \frac{m}{2} \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right)$. \square

Lemma 5.11. *Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, where $s \geq 2$, $\alpha_1, \dots, \alpha_s \in \mathbb{N}$, and $p_1, \dots, p_s \in \mathbb{P}$ are distinct. If $t \leq \frac{n+1}{2^{s+1}\sqrt{2s-1}+1}$ then $\phi_{\lfloor \frac{n}{t} \rfloor}(n) \geq \frac{n}{2.2t} \prod_{i=1}^s (1 - \frac{1}{p_i}) = \frac{\phi(n)}{2.2t}$.*

Proof. From $t \leq \frac{n+1}{2^{s+1}\sqrt{2s-1}+1}$ we obtain that $\lfloor \frac{n}{t} \rfloor \geq \frac{n-t+1}{t} \geq 2^{s+1}\sqrt{2s-1}$. Lemma 5.10 implies that

$$(5.11) \quad \phi_{\lfloor \frac{n}{t} \rfloor}(n) \geq \frac{1}{2} \lfloor \frac{n}{t} \rfloor \frac{\phi(n)}{n} \geq \frac{n-t+1}{2t} \frac{\phi(n)}{n}.$$

Since $s \geq 2$, we infer that $t \leq \frac{n+1}{2^{s+1}\sqrt{2s-1}+1} \leq \frac{n+1}{8\sqrt{3}+1} < \frac{n+11}{11}$. Therefore, $\frac{n-t+1}{2t} \geq \frac{n}{2.2t}$ and the result follows from (5.11). \square

Lemma 5.12. *Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, where $s \geq 2$, $\alpha_1, \dots, \alpha_s \in \mathbb{N}$, and $p_1, \dots, p_s \in \mathbb{P}$ are distinct, and let S be a zero-sum sequence over G such that $\text{ord}(g) = n$ for all $g \in \text{supp}(S)$. Then $\min \mathbf{L}(S) \leq \min\{\frac{|S|}{2}, \frac{|S|}{\lfloor \frac{n+1}{2^{s+1}\sqrt{2s-1}+1} \rfloor} + 3.3(n-1)(\frac{1}{2} + \log(\lfloor \frac{n+1}{2^{s+1}\sqrt{2s-1}+1} - 1 \rfloor))\}$.*

Proof. For ease of notation, set $u = \frac{n+1}{2^{s+1}\sqrt{2s-1}+1}$. It suffices to show that there is a factorization $S = U_1 \cdots U_t$, where $U_1, \dots, U_t \in \mathcal{A}(G)$ and t is bounded above by the second term in the above set, and again we construct U_1, \dots, U_t recursively. For $i \in [1, t]$, let U_i be a minimal zero-sum subsequence of $S(\prod_{j=1}^{i-1} U_j)^{-1}$ with maximal possible length. We use Lemma 5.5 and Lemma 5.11 to study t . For every $k \in [2, n]$, let n_k be the number of U_i such that $|U_i| = k$. For every $m \in [3, n-1]$, the construction of U_i and Lemma 5.5 imply that

$$(5.12) \quad \sum_{i=2}^{m-1} in_i \leq \frac{\phi(n)(n-1)}{\phi_{\lfloor \frac{n}{m} \rfloor}(n)}$$

If $m \leq u$, then by Lemma 5.11 we have $\phi_{\lfloor \frac{n}{m} \rfloor}(n) \geq \frac{\phi(n)}{2.2m}$. It follows from (5.12) that

$$\sum_{i=2}^{m-1} in_i \leq 2.2m(n-1) \leq 3.3(m-1)(n-1).$$

Therefore, for every $m \in [2, u-1]$, we have

$$(5.13) \quad \sum_{i=2}^m in_i \leq 3.3(n-1)m.$$

Applying Lemma 5.6 we deduce that

$$\sum_{i=2}^m n_i \leq 3.3(1 + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{m})(n-1)$$

holds for every $m \in [2, u-1]$. Especially,

$$\sum_{i=2}^{\lfloor u-1 \rfloor} n_i \leq 3.3(n-1)(\frac{1}{2} + \sum_{i=2}^{\lfloor u-1 \rfloor} \frac{1}{i}) \leq 3.3(n-1)(\frac{1}{2} + \log(\lfloor u-1 \rfloor)).$$

Since $\sum_{i \geq \lfloor u \rfloor} in_i \leq |S|$, $\sum_{i \geq \lfloor u \rfloor} n_i \leq \frac{|S|}{\lfloor u \rfloor}$. Hence,

$$t = \sum_{i=2}^n n_i \leq \frac{|S|}{\lfloor u \rfloor} + 3.3(n-1)(\frac{1}{2} + \log(\lfloor u-1 \rfloor)). \quad \square$$

Lemma 5.13. *Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, where $s \geq 2$, $\alpha_1, \dots, \alpha_s \in \mathbb{N}$, and $p_1, \dots, p_s \in \mathbb{P}$ are distinct. If $n \geq 4376$, then $\lfloor \frac{n+1}{2^{s+1}\sqrt{2s-1}+1} \rfloor \geq \frac{n}{2^{s+1}\sqrt{2s}}$.*

Proof. Clearly, it suffices to prove that

$$\frac{n+1}{2^{s+1}\sqrt{2s-1}+1} \geq \frac{n}{2^{s+1}\sqrt{2s}} + 1$$

which will follow from

$$(5.14) \quad \left(\frac{2^{s+1}}{\sqrt{2s} + \sqrt{2s-1}} - 1\right)n \geq 2^{2s+2}\sqrt{2s(2s-1)}.$$

By a straightforward computation we get that (5.14) holds for $(s = 2 \text{ and } n \geq 250)$, for $(s = 3 \text{ and } n \geq 656)$, for $(s = 4 \text{ and } n \geq 1707)$, and for $(s = 5 \text{ and } n \geq 4376)$. If $s = 6$, then $n \geq 2 \times 3 \times 5 \times 7 \times 11 \times 13 = 30030$. Again by a straightforward computation we get that (5.14) holds. Now we proceed by induction. Assume that (5.14) holds for some $s \geq 6$. Then it holds for $s+1$ because

$$\begin{aligned} \left(\frac{2^{s+2}}{\sqrt{2(s+1)} + \sqrt{2s+1}} - 1\right)n &\geq \left(\frac{2^{s+1}}{\sqrt{2s} + \sqrt{2s-1}} - 1\right)\frac{n}{p_{s+1}^{\alpha_{s+1}}} p_{s+1}^{\alpha_{s+1}} \\ &\geq (2^{2s+2}\sqrt{2s(2s-1)})p_{s+1}^{\alpha_{s+1}} \\ &\geq (2^{2s+2}\sqrt{2s(2s-1)}) \times 17 \\ &\geq 2^{2s+4}\sqrt{2(s+1)(2s+1)}. \end{aligned} \quad \square$$

Lemma 5.14. *Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $r \geq 2$, $\alpha_1, \dots, \alpha_r \in \mathbb{N}$, and $p_1, \dots, p_r \in \mathbb{P}$ are distinct, and let S be a zero-sum sequence over G^\bullet . For every divisor $d > 1$ of n , let N_d denote the number of the terms of S which have order d . Then $\min \mathbf{L}(S) \leq \min\{\frac{|S|}{2}, \frac{4.3}{2} \sum_{1 < d|n} (d-1) + \sum_{1 < d|n, d \leq 4375} \frac{N_d}{2} + \sum_{4376 \leq d|n} \frac{2^{\omega(d)+1}\sqrt{2\omega(d)}N_d}{d} + 3.3 \sum_{4376 \leq d|n} (d-1) \log(\lfloor \frac{d+1}{2^{\omega(d)+1}\sqrt{2\omega(d)-1}+1} - 1 \rfloor)\}$.*

Proof. It suffices to show that $\min \mathbf{L}(S)$ is bounded above by the second term in the above set. For every $1 < d|n$, let S_d denote the subsequence of S consisting of all terms with order d , let T_d be a zero-sum subsequence of S_d with maximal possible length, and set $T'_d = S_d T_d^{-1}$. Therefore

$$S = \prod_{1 < d|n} T'_d \prod_{1 < d|n} T_d \quad \text{and} \quad \prod_{1 < d|n} T'_d \quad \text{has sum zero.}$$

By the maximality of T_d we infer that $|T'_d| \leq d-1$ for every $1 < d|n$, and hence,

$$(5.15) \quad \left| \prod_{1 < d|n} T'_d \right| \leq \sum_{1 < d|n} (d-1).$$

Therefore,

$$\min \mathbf{L}(S) \leq \min \mathbf{L}\left(\prod_{1 < d|n} T'_d\right) + \sum_{1 < d|n} \min \mathbf{L}(T_d) \leq \frac{|\prod_{1 < d|n} T'_d|}{2} + \sum_{1 < d|n} \min \mathbf{L}(T_d).$$

It follows from (5.15) that

$$(5.16) \quad \min \mathbf{L}(S) \leq \frac{\sum_{1 < d|n} (d-1)}{2} + \sum_{1 < d|n} \min \mathbf{L}(T_d).$$

If $\omega(d) \geq 2$ then by Lemma 5.12 and Lemma 5.13 we obtain that

$$(5.17) \quad \min \mathbf{L}(T_d) \leq \frac{2^{\omega(d)+1}\sqrt{2\omega(d)}N_d}{d} + 3.3(d-1)\left(\frac{1}{2} + \log\left(\lfloor \frac{d+1}{2^{\omega(d)+1}\sqrt{2\omega(d)-1}+1} - 1 \rfloor\right)\right)$$

holds for every $4376 \leq d|n$. By Lemma 5.7.2 and Lemma 5.8 we obtain that (5.17) is true for all $1 < d|n$ with $\omega(d) = 1$. It follows from (5.16) that

$$\begin{aligned} \min L(S) &\leq \frac{\sum_{1 < d|n} d^{-1}}{2} + \sum_{1 < d|n} \min L(T_d) \\ &\leq \frac{\sum_{1 < d|n} d^{-1}}{2} + \sum_{1 < d|n, d \leq 4375} \frac{N_d}{2} + \sum_{4376 \leq d|n} \left(\frac{2^{\omega(d)+1} \sqrt{2\omega(d)} N_d}{d} + 3.3(d-1) \left(\frac{1}{2} + \log \left(\lfloor \frac{d+1}{2^{\omega(d)+1} \sqrt{2\omega(d)-1+1}} - 1 \rfloor \right) \right) \right) \\ &\leq \frac{4.3}{2} \sum_{1 < d|n} (d-1) + \sum_{1 < d|n, d \leq 4375} \frac{N_d}{2} + \sum_{4376 \leq d|n} \frac{2^{\omega(d)+1} \sqrt{2\omega(d)} N_d}{d} \\ &\quad + 3.3 \sum_{4376 \leq d|n} (d-1) \log \left(\lfloor \frac{d+1}{2^{\omega(d)+1} \sqrt{2\omega(d)-1+1}} - 1 \rfloor \right). \quad \square \end{aligned}$$

Proof of Theorem 5.2. We may suppose that H is reduced and that $H \hookrightarrow \mathcal{F}(P)$ is a divisor theory with class group G . Let $u, v_1, \dots, v_m, u_2, \dots, u_\ell \in \mathcal{A}(H)$ be such that $u | v_1 \cdots v_m$, but u divides no proper subproduct, that $v_1 \cdots v_m = uu_2 \cdots u_\ell$, and that $\max\{\ell, m\} = \mathfrak{t}(H, u) = \mathfrak{t}(H)$. By Lemma 4.1.2, we have $\mathfrak{t}(H) \geq \mathfrak{t}(G) > n \geq |u| \geq m$, and hence we get $1 + \min L(w) = \ell = \mathfrak{t}(H) > n$, where $w = u^{-1}v_1 \cdots v_m$. For $i \in [1, m]$, we set $v_i = s_i a_i$ with $a_i, s_i \in \mathcal{F}(P) \setminus \{1\}$ and $A_i = \beta(a_i)$. We set $S = A_1 \cdots A_m$, and observe that $S = \beta(w)$ and that $|S| \leq m(n-1) \leq n(n-1)$. We have to show that $1 + \min L(S)$ is bounded above by the terms given in the statement of the theorem.

CASE 1: $n = p \in \mathbb{P}$.

This follows from Lemma 5.7.2.

CASE 2: $n = p^\alpha$, where $p \in \mathbb{P}$ and $\alpha \geq 2$.

For every divisor $d > 1$ of n , let N_d denote the number of the terms of S which has order d . Since A_i is zero-sum free, we infer that A_i has at most $d-1$ terms which have order d , and hence $N_d \leq m(d-1) \leq n(d-1)$. Thus it follows from Lemma 5.9 that

$$\begin{aligned} \min L(S) &\leq -2\alpha + \frac{2p^{\alpha+1}}{p-1} + \sum_{i=1}^{\alpha} \frac{2N_{p^i}}{p^i+1} + 3 \sum_{i=1}^{\alpha} (p^i - 1) \log \frac{p^i}{2} \\ &\leq -2\alpha + \frac{2p^{\alpha+1}}{p-1} + \sum_{i=1}^{\alpha} \frac{2n(p^i-1)}{p^i+1} + 3 \sum_{i=1}^{\alpha} (p^i - 1) \log \frac{p^i}{2} \\ &\leq -2\alpha + \frac{2p^{\alpha+1}}{p-1} + 2\alpha n + 3 \sum_{i=1}^{\alpha} (p^i - 1) \log \frac{p^i}{2}. \end{aligned}$$

CASE 3: $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $r \geq 2$, $\alpha_1, \dots, \alpha_r \in \mathbb{N}$, and $p_1, \dots, p_r \in \mathbb{P}$ are distinct.

For every divisor $d > 1$ of n , let N_d denote the number of the terms of S which have order d . Since A_i is zero-sum free, we infer that A_i has at most $d-1$ terms whose order divide d . Therefore, $N_d \leq m(d-1) \leq n(d-1)$ for each $1 < d|n$. Now the result follows from Lemma 5.14. \square

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CENTER FOR COMBINATORICS, NANKAI UNIVERSITY, TIANJIN 300071, P.R. CHINA
E-mail address: wdgao_1963@aliyun.com

INSTITUT FÜR MATHEMATIK UND WISSENSCHAFTLICHES RECHNEN, KARL–FRANZENS–UNIVERSITÄT GRAZ, HEINRICH-STRASSE 36, 8010 GRAZ, AUSTRIA
E-mail address: alfred.geroldinger@uni-graz.at

UNIVERSITÉ PARIS 13, SORBONNE PARIS CITÉ, LAGA, CNRS, UMR 7539, UNIVERSITÉ PARIS 8, F-93430, VILLETANEUSE, FRANCE
E-mail address: schmid@math.univ-paris13.fr