

# NATURAL VIBRATIONS OF SUSPENDED CABLES WITH FLEXIBLE SUPPORTS

GIUSEPPE REGA†

Istituto di Scienza delle Costruzioni, Università dell'Aquila, L'Aquila 67100, Italy

and

ANGELO LUONGO‡

Istituto di Scienza delle Costruzioni, Università di Roma, Roma 00184, Italy

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**Abstract**—A finite difference algorithm is applied to the study of *in-plane* natural vibrations of an *inextensible* cable with symmetric movable supports. Attention is given mainly to the analysis of the influence of support flexibility on the dynamic phenomenon. Several parametric studies are carried out and the analogy between the two cable models with lumped or distributed flexibility is shown. The main results regarding the influence of the elastic cable parameters, reported by earlier investigators, are obtained by lumping the cable flexibility at the supports. The algorithm can thus be applied over the whole sag/span ratio range and appears as a possible *alternative to finite element models*. Finally, the relative importance of cable and real support flexibilities in the dynamic behaviour of the system is examined. Besides numerical results, reference is made for the description of the transition phenomenon to a synthetic mechanical description based on Irvine's theory.

## 1. INTRODUCTION

The study of the natural vibrations of suspended cables through continuous models involves restrictive hypotheses of various kinds, even though the linearized theory is used [1-4]. On the other hand, discretization into finite elements allows quite accurate handling of the mechanical problem [7,9] and is doubtless the most efficient tool for structural analysis in this as in other fields. Nevertheless, the comprehension and essential description of some important aspects of the dynamic behaviour of suspended cables must be ascribed above all to the continuous treatment proposed by Irvine [4], though his theory applies only to cables having sag/span ratio less than about 1/8 (parabolic static profile) and involves some approximations whose reliability is not confirmed completely by the numerical results [11].

It is also interesting to note that, although only small vibrations are concerned, very little investigation into the influence of support flexibility on the system's dynamic behaviour has been made till now [8]. The reason is twofold. First, because the purpose of most studies in the literature is accurately to describe the behaviour of the cable itself, thus eliminating other dynamic effects; second, because it is not easy to account for the flexibility of the actual supporting structures in a sufficiently general way. In fact, in a first stage, general information is needed about the modification of the dynamic behaviour of the system associated with support flexibility, rather than analysis of specific structural problems.

In [11] the writers developed an algorithm to obtain linearized in-plane frequencies, mode shapes, and cable

tension changes of an inextensible cable with symmetric movable supports. No hypothesis was made as regards the cable sag/span ratio. The inextensible model is not satisfactory in the low range of sag/span ratios [4, 5], but it is useful in order to predict the influence of support flexibility, since the system is free of the dynamic effects associated with cable elasticity.

In the present paper the algorithm is used to study this influence. After reviewing its relevant features, the main characteristics of the dynamic behaviour of the cable are discussed briefly, referring to a synthetic mechanical description based on Irvine's theory. Some parametric studies are then made to analyse the possibility of assuming structural models with either lumped or distributed flexibility in the treatment of the problem. As a consequence of the investigation performed, the range of validity of the algorithm is widened to include low values of sag/span ratio, i.e. values for which crossover of symmetric and antisymmetric mode shapes occurs. The results obtained are compared with those of earlier investigators. Finally, the relative importance of cable and real support flexibilities is examined both from a qualitative and a numerical point of view.

## 2. MODELLING OF THE INEXTENSIBLE CABLE

The linearized equations of the in-plane motion for a cable element  $ds$  are (Fig. 1):

$$\frac{\partial}{\partial s} \left( T \frac{\partial u}{\partial s} + \tau \frac{dx}{ds} \right) = m \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial}{\partial s} \left( T \frac{\partial v}{\partial s} + \tau \frac{dy}{ds} \right) = m \frac{\partial^2 v}{\partial t^2}$$

where  $m$  is the mass per unit length,  $u(s, t)$  and  $v(s, t)$  are, respectively, the longitudinal and vertical com-

†Associate Professor.

‡Research Assistant.

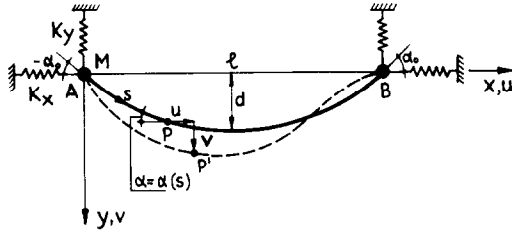


Fig. 1. Cable configuration.

ponents of the displacement,  $T(s)$  is the cable tension in the static equilibrium configuration,  $\tau(s, t)$  is the additional tension caused by the motion. The inextensibility condition is expressed by the equation of constraint:

$$\frac{dx}{ds} \frac{\partial u}{\partial s} + \frac{dy}{ds} \frac{\partial v}{\partial s} = 0.$$

Let  $u, v, \tau$  be proportional to  $\exp(-i\omega t)$ , where  $\omega$  is the natural frequency of vibration. Furthermore, let the inextensible catenary describe the static equilibrium configuration. By changing the independent variable to  $\alpha$  and referring to locally tangential  $\xi(\alpha)$  and normal  $\eta(\alpha)$  displacement components, Saxon and Cahn[2] obtained the fourth order differential equation:

$$\begin{aligned} (\cos \alpha) \xi^{IV} - (2 \sin \alpha) \xi^{III} + \left( \cos \alpha + \frac{\Lambda^2}{\cos^2 \alpha} \right) \xi^{II} \\ + \left( \frac{\Lambda^2}{\cos^3 \alpha} - 1 \right) (2 \sin \alpha) \xi^I - \left( \frac{\Lambda^2}{\cos^2 \alpha} \right) \xi = 0 \end{aligned} \quad (1)$$

where  $\Lambda^2 = \omega^2 c/g$ ;  $c = l_c/2t\alpha_0$  is the parameter of catenary and  $l_c$  is the length of the cable. Once integration of eqn (1) is performed,  $\eta$  and  $\tau$  can be obtained as follows:

$$\begin{aligned} \eta &= \xi^I & (2) \\ \tau &= mg \left[ (-\cos \alpha) \xi^{III} + (\sin \alpha) \xi^{II} - \left( \cos \alpha + \frac{\Lambda^2}{\cos^2 \alpha} \right) \xi^I \right. \\ &\quad \left. + (\sin \alpha) \xi \right] & (3) \end{aligned}$$

The differential system is then uncoupled by the introduction of intrinsic displacement components  $\xi, \eta$  which are simply related by eqn (2). The boundary conditions are as follows:

—with fixed supports

$$\xi(\pm \alpha_0) = \xi^I(\pm \alpha_0) = 0 \quad (4a)$$

—with movable supports

$$\begin{aligned} (-mg \cos^2 \alpha_0) \xi^{III} - (K_x \sin \alpha_0 + mg \cos^2 \alpha_0) \xi^I \\ + (K_x \cos \alpha_0) \xi \\ + \Lambda^2 \left[ \left( \frac{Mg}{c} \sin \alpha_0 - \frac{mg}{\cos \alpha_0} \right) \xi^I - \left( \frac{Mg}{c} \cos \alpha_0 \right) \xi \right] = 0 \\ (-mg \sin \alpha_0 \cos \alpha_0) \xi^{III} + mg \xi^{II} \\ + (K_y \cos \alpha_0 - mg \sin \alpha_0 \cos \alpha_0) \xi^I \\ + (K_y \sin \alpha_0 + mg) \xi \\ + \Lambda^2 \left[ \left( -\frac{Mg}{c} \cos \alpha_0 - mg \frac{\sin \alpha_0}{\cos^2 \alpha_0} \right) \xi^I \right. \end{aligned} \quad (4b)$$

$$\left. - \left( \frac{Mg}{c} \sin \alpha_0 \right) \xi \right] = 0.$$

In a first stage of approximation,  $M$  is the equivalent mass of the actual supporting structure,  $K_x$  and  $K_y$  are the elasticity constants in the  $x, y$  directions.

The assumption of supports at the same level introduces some interesting symmetry features into the problem. In fact, separate solutions can be obtained for antisymmetric and symmetric modes by numerically integrating eqn (1) on half cable length, under proper boundary conditions, with a suitable halving of the d.o.f. number of the discretized problem. By using a finite difference approach, a set of algebraic equations is obtained, which reads, in matrix form:

$$([A] + \Lambda^2[B])\{\xi\} = 0 \quad (5)$$

Matrices  $[A]$  and  $[B]$  have five and three non-zero diagonals respectively. Due to discretization of the continuum problem they are not symmetric except for a very large number  $n$  of intervals. For this reason, and in order to take advantage of the narrow banded nature of the matrices, the eigenvalue problem (5) is solved through iteration on its characteristic polynomial. More efficient methods, such as the  $QR$  one, do not exploit banding and therefore require a much larger number of operations. The zeros of the polynomial are determined by means of the generalized secant method, which converges under not excessively severe conditions.

The algorithm does not give double eigenvalues, which exist, however, only for certain cable configurations and provided cable elasticity is taken into account. As will be shown later, if the inextensible model is referred to, they occur with movable supports only. In this case a definite advantage is gained by separating the analysis of symmetric and antisymmetric modes in order to utilize the proposed algorithm.

Once the  $\xi$  modal components are known in each node of the discretized cable, the  $\eta$  components and the relative cable tension change  $\tau/T$  associated with a given maximum vertical displacement can be determined by the difference form of eqns (2) and (3).

The cable with fixed supports chosen for numerically testing the algorithm is one which has been investigated by other researchers. Its properties are shown in Table 1. Results obtained for the natural frequencies, mode shapes and relative cable tension change are extensively reported in [11]; only a few of them are mentioned here.

In Table 2, the frequencies determined using the finite difference approach ( $n=64$ ) are compared with those obtained by other authors using both continuous and discrete methods. Good agreement with the more reliable values can be seen, i.e. with those determined by West *et al.* [5] from extrapolating their curves to an infinite number of elastic cable segments (initial value problem) and those obtained by Gambhir and Batchelor [10] using a curved finite element model.

Curves of frequency convergence vs the cable d.o.f. number show that a discretization with  $n=16$  is sufficiently accurate; this is the value used in the numerical analysis reported below.

### 3. DYNAMIC BEHAVIOR OF THE CABLE

Various authors [4, 5, 8] have shown that when the inextensible cable is used the important phenomenon of

Table 1. Cable properties

$l = 853.44 \text{ m}$	$\alpha_0 = 18^\circ 30'$
$d = 70.71 \text{ m}$	$A = 1235.5 \text{ cm}^2$
$d/l = 1/12$	$E = 1.82 \cdot 10^6 \text{ Kg/cm}^2$
$l_c = 868.28 \text{ m}$	$m = 98.6 \text{ Kg} \cdot \text{s}^2 / \text{m}^2$

transition of modes that occurs at either low sag or high span values cannot be observed. Only if cable elasticity is taken into account does crossover of the first two modes occur, and the discrepancy between the sagged cable and the taut string theories, which exists with the inextensible model as the sag/span ratio reduces to zero, is overcome. The accuracy of the inextensible model in determining the symmetric frequencies is quite good for sag/span ratios greater than a certain value ( $\approx 1/15$ ), at least for the lower modes; it is good as  $d/l \rightarrow 0$  as well, apart from the lack of the taut string symmetric mode; however, it is very poor for values of  $d/l < 1/15$  and not very low. On the contrary, no problems arise for the antisymmetric modes.

The dynamic behaviour of the symmetric modes is synthetically expressed by Irvine's elastic and geometric parameter  $\lambda^2$  which has the value  $\lambda_{in}^2 = (2n\pi)^2$  at the  $n$ th transition point. A significant mechanical description of the transition phenomenon, which accounts for the influence of the cable parameters as well, can be made as follows. Let

$$D_c = \frac{l}{2EA} \tag{6}$$

be the cable flexibility parameter and

$$D_{in} = \left(\frac{8d}{l}\right)^3 \frac{1}{2\lambda_{in}^2 mg} \quad (n = 1, 2, \dots) \tag{7}$$

the minimum, or crossover flexibility for each mode;  $E$  and  $A$  are the cable elasticity modulus and area respectively. Irvine's observations can be restated in the following equivalent form. If  $D_c > D_{in} (\lambda^2 < \lambda_{in}^2)$ , the frequency of the  $n$ th antisymmetric mode is greater than the  $n$ th symmetric one; if  $D_c < D_{in} (\lambda^2 > \lambda_{in}^2)$ , the frequency of the symmetric mode is greater ("natural" mode-order); finally, the value  $D_c = D_{in}$  is the lowest flexibility the cable must have in order for inversion of the  $n$ th pair of modes to occur. If all but one of the cable parameters are held constant and the curves  $D_c$  and  $D_{in}$  ( $n = 1, 2, \dots$ ) are plotted against the free parameter in a reference system  $D, E$  (or  $d, l, A, m$ ), their intersection point gives the value of the free parameter for which the  $n$ th frequency undergoes crossover. The  $D_{in}$  curve separates the two domains where cable configurations with and without crossover of the  $n$ th mode occur.

The first four frequencies according to Irvine's theory are plotted vs  $E$  in Fig. 2(a) for two different cable span lengths. The antisymmetric frequencies are insensitive to variation in cable elasticity, while the symmetric ones undergo transition for low  $E$  values. The smaller the span is, the lower the  $E$  value is. The influence of span variation is negligible for both low values of  $E$  (inversion of all modes always occurs) and high ones (it never occurs, as with the inextensible model). In the  $D, E$  reference system (Fig. 2b) these observations read as follows. If  $E < E_{in}$ , the cable flexibility is sufficient for the  $n$ th transition to occur; as  $E$  increases it approaches the  $D_{in}$  value; if  $E > E_{in}$ , the "natural" succession of two modes takes place. Crossover is facilitated by an increase of the span length, i.e. it can occur for higher values of  $E$  as well. In fact, greater span length increases cable flexibility and reduces  $D_{in}$  flexibilities. The inextensible model corresponds to  $D_c = 0$ . Its accuracy is good for high values of  $D_{in}$  only, corresponding to large sag, small span, or low modes, i.e. geometric configurations such that cable elasticity does not make  $D_c$  comparable with  $D_{in}$ .

Table 2. Comparison of frequencies obtained by various authors

	ANTISYMM. MODES			SYMM. MODES		
	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_1$	$\omega_2$	$\omega_3$
Finite Diff. (64 intervals)	0.8113	1.6385	2.4979	1.1739	2.0223	2.8558
Saxon & Cahn [2]	0.803	1.671	2.452	1.185	2.042	—
Pugsley [1]	0.811	1.647	—	1.148	—	—
Irvine ( $E = \infty$ )	0.827	1.654	2.481	1.183	2.035	2.871
Irvine ( $E = 1.82 \cdot 10^6$ )	0.827	1.654	2.481	1.176	2.019	2.832
West [5]	0.800	1.630	2.450	1.160	1.990	—
St. Line Elem. (16 elements) [10]	0.809	1.680	2.608	1.185	2.090	—
Curved Elem. (6 elements) [10]	0.795	1.627	2.460	1.155	1.998	—

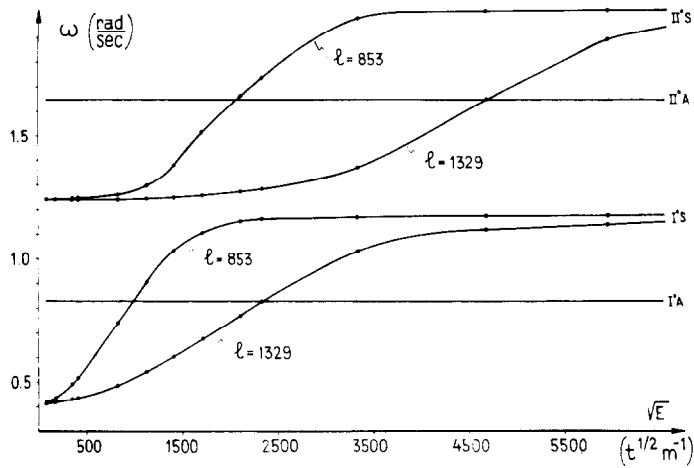


Fig. 2(a). Frequencies vs modulus of elasticity.

#### 4. SUSPENDED CABLES WITH LUMPED OR DISTRIBUTED FLEXIBILITY

A numerical investigation is made herein, by means of the proposed algorithm, to account for the influence of the cable and support parameters on the natural frequencies of the system.

Figure 3 shows a plot of the system frequencies  $\omega$  vs the support frequencies  $\omega_s = \sqrt{K/M}$ ; a number of mode shapes are also plotted at certain points.  $K$  and  $M$  refer to the horizontal spring equivalent to the actual supporting structure, which is assumed, for the sake of simplicity, of steel rectangular thin-walled section.

Although the inextensible cable is referred to, a dynamic behaviour analogous to that of the elastic cable with fixed supports can be noticed. For high values of  $\omega_s$ , the mode shapes are the same as those of the suspended cable, whether elastic or inextensible, with fixed supports (natural mode-order). As  $\omega_s$  is decreased, crossover of symmetric frequencies with antisymmetric ones can be observed. A transition in modal (vertical) configuration of the second symmetric mode from four nodal points to two nodal points occurs at  $\omega_s = \omega_{s2}$ ; then a transition of the first one from two nodal points to no nodal points (taut string mode) occurs at  $\omega_s = \omega_{s1}$ .

As  $\omega_s$  further decreases with respect to each transition point, a new phenomenon occurs. Both antisymmetric modes show a steep decrease of frequencies. A variation occurs in the modal configuration of the second mode, as well as the higher modes not plotted in the figure; each loses two nodes, while preserving antisymmetry. This phenomenon is associated with a new spurious mode, henceforth termed "support" type, characterized by increasing values of the horizontal displacement of the supports, due either to their low stiffness or large mass. The modal configuration of the new mode is the same as that of the first antisymmetric mode. The cable continues to vibrate with the classical vertical eigenfunction with one nodal point, but at the same time it shifts horizontally. It is interesting to note that, if the support type mode is omitted, the same frequencies and mode order are found for low values of  $\omega_s$ , as occurs in the case of elastic cable with fixed supports when small  $d$  or  $E$  values or large  $l$  values are considered.

The numerical analysis just discussed has been repeated by varying the support stiffness with different

constant values of mass. The same dynamic behaviour has been found. Besides, it has been observed that the computed frequencies are unaltered by a variation of  $M$  for values of  $K$  which are not very low; for low  $K$  values instead the higher values of  $M$  facilitate the support type mode, i.e. it can occur with stiffer supports as well. Therefore, the phenomenon of crossover of symmetric frequencies seems to be related to a flexibility parameter only, while crossover of antisymmetric ones depends on both flexibility and mass parameters of the supports. In any case, the behaviour of the system does not change even when massless supports are considered.

The analogy of the two models of elastic cable with fixed supports and inextensible cable with elastic supports clearly stands out in the comparison of Figs. 2(a) and 4. In the latter, the influence of support stiffness  $K$  on the natural frequencies is studied for the same cable as in Fig. 2(a), now considered inextensible. To this end, reference is made to the fictitious modulus  $E^* = Kl_c/2A$  of an elastic cable with fixed supports, kinematically equivalent to the inextensible one. The behaviour of the two models as system stiffness decreases is the same: putting into the system a certain flexibility, whether distributed along the cable or lumped at the supports, is sufficient for the taut string mode to occur. With a given cable length, lower flexibility values suffice to make crossover of higher modes possible. For greater values of  $l_c$ , all the transition points shift towards higher system stiffness. Similar results are shown by the curves in Fig. 5, where the system frequencies are plotted vs the length of the inextensible cable for two constant values of support stiffness. Each curve is quite similar to that of the elastic cable with fixed supports presented by various authors [5, 10]. Further, as already discussed, a decrease in system stiffness allows transition of all modes to occur with lower cable length as well.

A conclusion can thus be drawn: the important phenomenon of crossover is essentially a matter of kinematical compatibility. In fact, the taut string mode cannot be obtained with the inextensible model unless the support flexibility permits such displacements as to allow it. In the same way, the taut mode is obtainable with the elastic cable, provided extensibility is sufficient for it to take place. The analogy no longer holds in the stiffness range, variable with the mode number and the

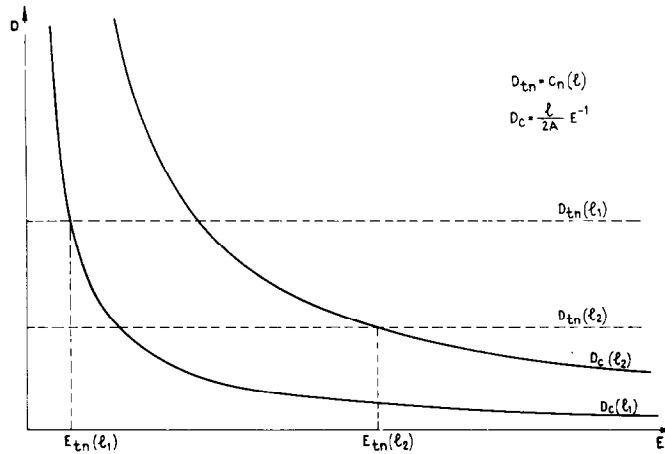


Fig. 2(b). Description of the dynamic behaviour of the cable in the  $D, E$  reference system.

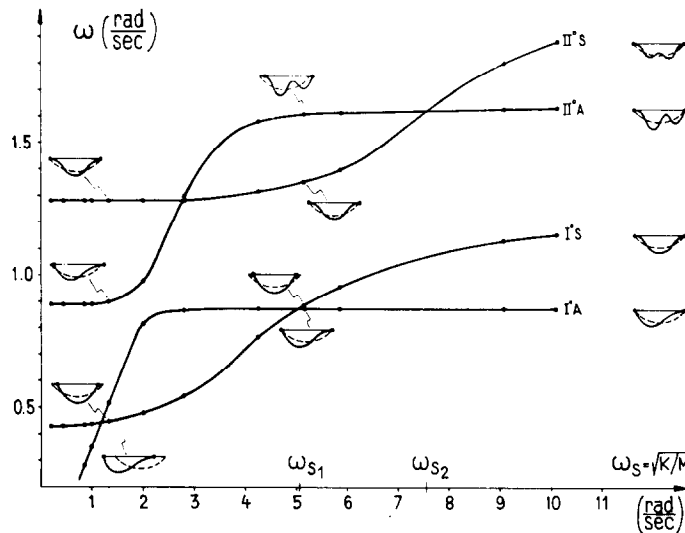


Fig. 3. System frequencies vs support frequencies.

support mass value, where the sudden decrease of the antisymmetric frequencies occurs (Fig. 4). This phenomenon is peculiar to the model with lumped flexibility. The analogy still holds for the antisymmetric modes on both sides of these narrow ranges, provided the support type mode is omitted; as far as the symmetric modes are concerned, it is valid over the whole stiffness field.

In this respect, it must be observed that this analogy depends on the fact that the dynamic effect of the system flexibility occurs, for each pair of modes, before the support inertia effect. That is, fairly small displacements of the supports allow transition of the symmetric modes when the inertia forces are still negligible; only afterwards, as displacements increase, do the flexibility and inertia effects join together to make transition of the antisymmetric modes possible.

5. ACCOUNTING FOR CABLE ELASTICITY BY MEANS OF THE INEXTENSIBLE MODEL

In order to use the analogy just established in technical

applications, it must be verified in quantitative terms as well.

To this end a parametric analysis is made of a cable with elastic, massless supports, with equivalent stiffness:

$$K_{eq.} = \frac{2EA}{l_c} \tag{8}$$

Referring once again to the basic configuration of the cable, the influence of the independent variation of the parameters sag, span and mass is analysed. The results obtained are compared with those obtained by Irvine's theory [4, 6].

The study of sag variation is shown in Fig. 6. The range of values of the sag/span ratio investigated is very wide (1/90-1/5), and extends well below the range of validity of the inextensible cable theory. The results obtained are very close to those of Irvine. The small differences observed are of the same order as those found for the standard configuration with fixed supports, where elasticity is of small effect due to the  $d/l$  ratio

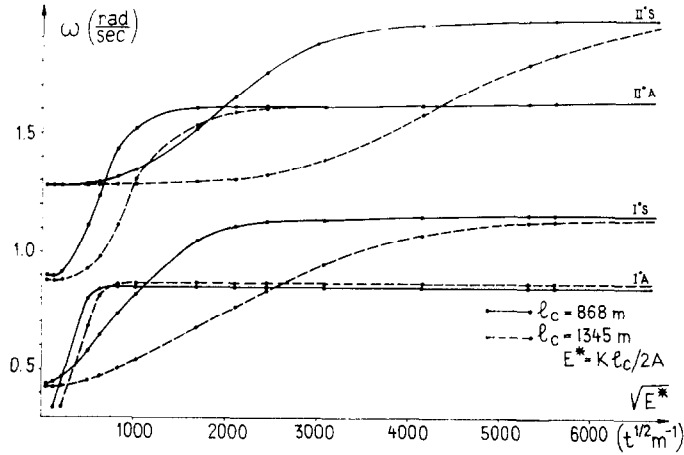


Fig. 4. Influence of support stiffness on the system frequencies.

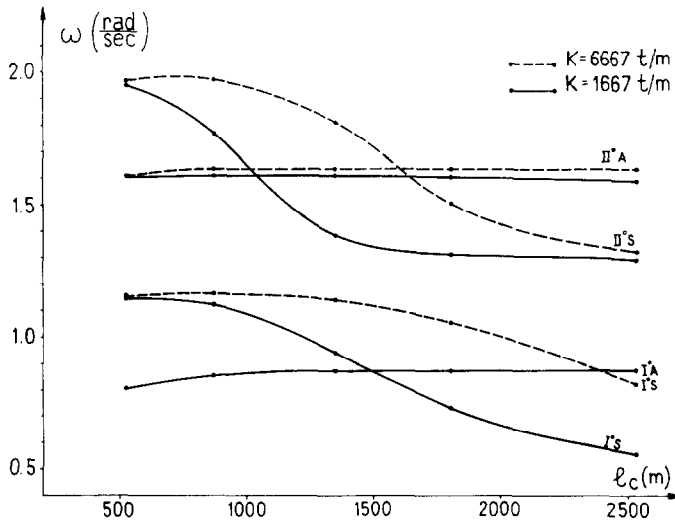


Fig. 5. System frequencies vs cable length.

value. In both cases, they can be attributed for the most part to the rather rough discretization adopted. By considering that the total elongation of the cable is concentrated at the supports, all the symmetric mode shapes of the elastic cable are obtained.

Figure 7 clearly shows the excellent approximation of the model. The significance of the scale factor is brought out; it ceases at around  $d/l = 0.09$  for the fourth mode, thus confirming the results of Gambhir and Batchelor[10]. For high sag/span ratios, the curves all have a slightly negative slope, as is the case with the inextensible model with fixed supports (Pugsley[1], Saxon and Cahn[2]). Therefore elastic supports do not alter the results obtained with the fixed support model, which are already good in this field. Irvine's curves, on the contrary, must be limited to  $d/l$  values which are not too high, otherwise they do not show the negative slope found by other authors[5, 10] as well.

As for mass variation, it is easily demonstrated that the frequencies do not depend on it. In fact, the boundary conditions (4) are the only equations in which mass

$m$  appears. In the case being examined ( $M = 0$ ), they depend only on the  $K/m$  ratio, or  $A/m$ , due to eqn (8). If the specific weight is held constant, the whole system of equations is independent of  $m$ , as are the eigenvalues  $\omega$ , and the findings of the other authors are obtained[5, 7, 10].

The proposed model compares positively with the elastic model, and appears as a possible alternative to finite element models due to the good results it provides in the whole range of  $d/l$  ratio values. It is important to note that the parametric analysis does not show the anomaly of transition of antisymmetric modes, seen in the preceding section. In technical applications, in fact,  $K_{eq}$  is usually much larger than the stiffness which causes this phenomenon to occur.

*There are two main advantages to the proposed model:*

(a) *Cable vibrations can be studied treating the cable as inextensible. From the quantitative point of view, this implies reducing the d.o.f. number of the continuum by one infinity, i.e. halving it in the discretized system. Interesting results might be obtained, in this direction, in*

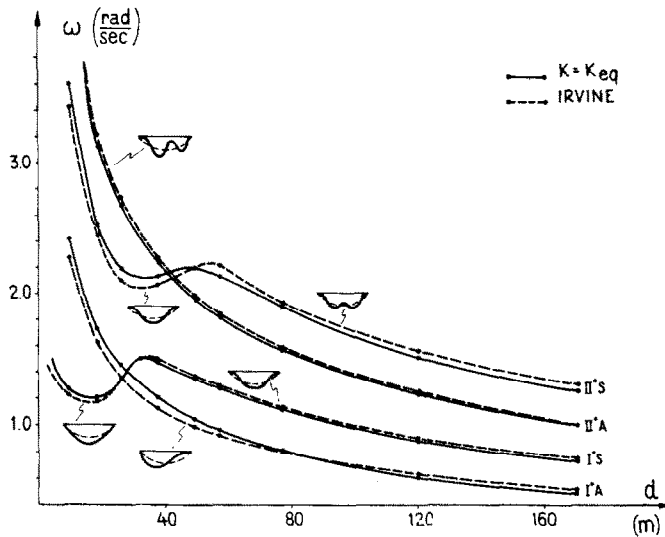


Fig. 6. Cable frequencies vs sag.

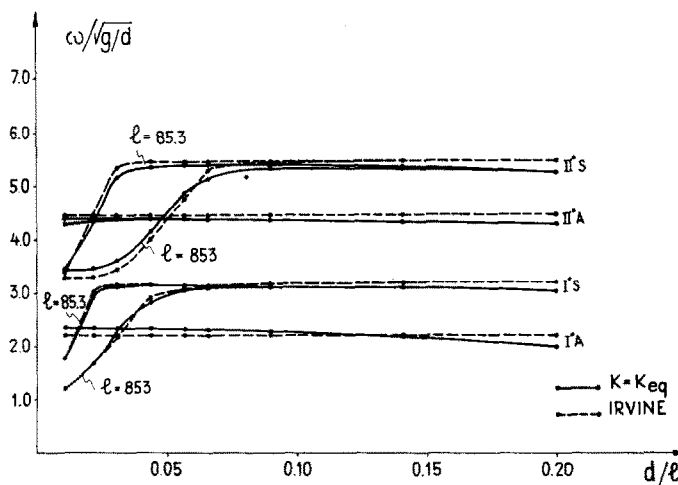


Fig. 7. Normalized cable frequencies vs sag/span ratio.

the dynamic analysis of more complex structures, such as cable systems.

(b) It is evident, qualitatively speaking, that the elasticity of the supports is not a new problem from the dynamic point of view, or at any rate a problem different from that of cable elasticity. *It is thus possible to refer to the support (or cable) flexibility only, considering it as inclusive of its own flexibility and that of the cable (or support) as well.* The two quantities can be directly compared and their relative importance evaluated. This means examining the possibility of disregarding one or both of them in technical applications, and of adopting simplified models of the mechanical system.

6. RELATIVE IMPORTANCE OF CABLE AND SUPPORT FLEXIBILITY

The analogy just drawn is very useful in determining the natural in-plane frequencies of an elastic cable with movable supports. Apart from finite element models,

two practical possibilities exist, in each of which a mechanical model simpler than the real one is used. Reference can be made either to the inextensible model by lumping the whole system flexibility at the supports, or to the elastic cable by defining a modulus  $E^*$  to account for the support flexibility as well.

Primary attention is given to the first approach as it is more consistent with the treatment outlined herein and allows use of the proposed algorithm. The second approach is mentioned only briefly. Let

$$D_{TOT.} = D_c + D_s = \frac{l}{2EA} + D_s$$

be the whole system flexibility, inclusive of the supports and the cable. This latter is taken into account as an additional support flexibility in the equivalent model (Fig. 8) by putting  $l_c \approx l$ . The role played by this approximation is very modest indeed, as is shown by the numerical investigation performed. Besides, it is

thoroughly consistent with the limits of applicability of Irvine's theory, referred to below, once again.

The results obtained for the inextensible model with elastic supports are still quantitatively valid, provided  $D_{TOT}$  is substituted for  $D_s$ . The system has a greater flexibility than it has with the inextensible cable (or fixed supports); crossover is thus expected to be facilitated by the cable (or support) elasticity. The dependence of the support flexibility on the cable geometric parameters must necessarily be specified, e.g.  $D_s = D_s(l)$ . It is not easy to account in a general way both for the several possible support typologies and for the various external actions (wind, earthquakes) that affect their design together with the static cable tension. Nevertheless,  $D_s$  certainly decreases with respect to  $l$ , at least for values of  $l$  which are not so low that the importance of the other actions becomes greater than that of static tension. In this latter range, however, the influence of system flexibility is very low, since all  $D_n$  flexibilities reach high values, i.e. the system's dynamic behavior is nearly the same as that of the inextensible cable with fixed supports.

A numerical investigation is performed on the reference cable by varying the span length in the range 500–2500 m. By accounting for the cable static tension and the wind effect the following relationship is obtained for the supporting structure:  $D_s = c_0 l^{-3}$  where  $c_0 = 3.697 \cdot 10^5 \text{ m}^4/\text{t}$ . Since it is associated with a particular type of support, no general meaning can be attributed to it; however, it is useful for a numerical analysis of the problem.

Table 3 shows the cable, support and total flexibilities for each span value.  $D_s$ , initially higher than  $D_c$ , is comparable with the latter for span values up to about 1500 m and can thus be said to play an important role in this  $l$ -range.

Plots of system frequencies vs  $l$  are shown in Fig. 9(a, b) for the four models that can be considered. The analysis of the results is made referring to Fig. 10 as well, which illustrates the phenomenon in the  $D, l$  plane on the basis of the mechanical description of Irvine's theory discussed in Section 3; in fact, it appears very useful in order synthetically to analyse the problem. The two inextensible models with and without support flexibility are compared in Fig. 9(a). The order of the first two symmetric and antisymmetric frequencies is the natural one, without crossover, for both fixed and movable supports, while these latter substantially modify the dynamic behaviour of the higher modes, whose frequency order is inverted for each  $l$  value. The results obtained are easily understood by considering the assumed  $D_s(l)$  relationship is of the same type as the  $D_{t_n}(l)$  ones and that

$D_3 < D_s < D_{t_2} < D_{t_1}$ . Of course they are not general. For stiffer supports, in fact, crossover of the third mode can occur, while for more flexible ones crossover of the first two modes can take place.

The two elastic models with and without support flexibility are compared in Fig. 9(b). From a general point of view, cable extensibility can be said to play a more important role than that of supports in the system's dynamic behaviour. In fact, the former is sufficient by itself to cause crossover of all modes as the span increases, while the latter causes significant transitions only under certain conditions. As far as the assumed  $D_s$  curve is concerned, three  $l$ -ranges can be distinguished.

(a)  $l < l_2$ :  $D_s$  furnishes the greater contribution to system flexibility. All but the first and second modes have already undergone crossover. The actual values of the higher frequencies, as well as their order, differ from those obtained using the elastic cable with fixed supports. This case, however, seldom occurs in practice.

(b)  $l > l_1$ :  $D_c$  furnishes the greater contribution to system flexibility. Crossover of all modes has occurred. The taut string mode occurs at a span value ( $l_1$ ) lower than in the case of elastic cable with fixed supports.

(c)  $l_1 < l < l_2$ : the two flexibilities are comparable. Taking support flexibility into account causes shifting of the transition points of the lower modes towards lower span length.

As far as the numerical analysis performed is concerned, the differences are quite high at the second and third symmetric mode.

The results obtained with the lumped flexibility model can be obtained by means of Irvine's theory as well. By equating the flexibilities of the actual system and of the distributed flexibility model, the fictitious cable modulus:

$$E^* = E \frac{1}{1 + \frac{D_s}{D_c}} = \frac{El}{l + 2EAD_s} \tag{9}$$

can be defined, which accounts for support flexibility as well. After calculating Irvine's  $\lambda^2$ , the system's natural frequencies are easily obtained [4, 6]. The influence of support flexibility on the system's dynamic behaviour stands out clearly in eqn (9). For  $D_s \approx D_c$ ,  $E^* = E/2$  is obtained; the associated frequency modifications can be observed in Fig. 2(a).

Finally, two considerations must be made which limit, to a certain extent, the equivalence between the two cable models.

First, the support mass is taken into account in the lumped model but not in the distributed one; a more extensive formulation of the equivalence, however, would allow this limitation to be overcome. Second, and more important from a technical point of view, the proposed algorithm (lumped model) can be used over the

Table 3. Cable, support and total flexibilities for various span lengths

l	$D_s$	$D_c$	$D_{tot} = D_s + D_c$
501.5	$2.931 \cdot 10^{-3}$	$0.116 \cdot 10^{-3}$	$3.047 \cdot 10^{-3}$
853.4	$0.595 \cdot 10^{-3}$	$0.190 \cdot 10^{-3}$	$0.785 \cdot 10^{-3}$
1329.7	$0.157 \cdot 10^{-3}$	$0.296 \cdot 10^{-3}$	$0.453 \cdot 10^{-3}$
1779.6	$0.656 \cdot 10^{-4}$	$3.959 \cdot 10^{-4}$	$4.615 \cdot 10^{-4}$
2513.3	$0.233 \cdot 10^{-4}$	$5.591 \cdot 10^{-4}$	$5.824 \cdot 10^{-4}$

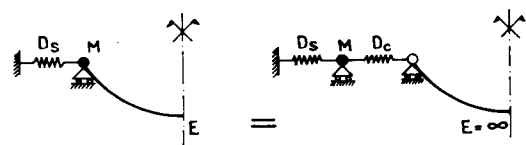
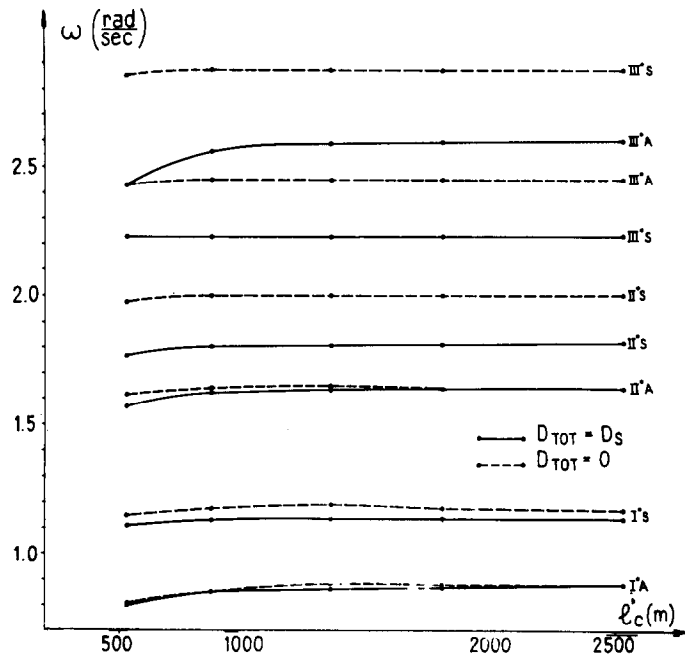
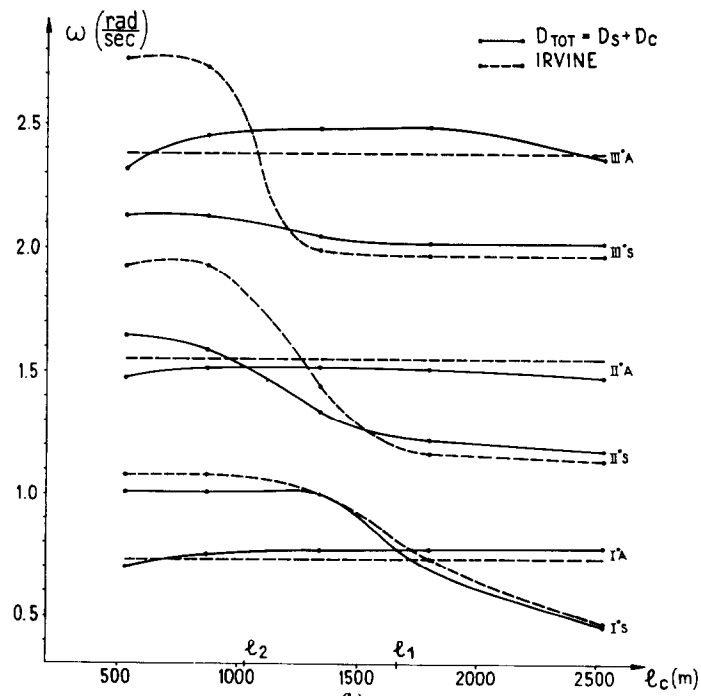


Fig. 8. Model with lumped flexibility equivalent to the extensible cable with movable supports.





(a)



(b)

Fig. 9. System frequencies vs span length: inextensible model with ( $D = D_s$ ) and without ( $D = 0$ ) support flexibility (Fig. 9(a)), elastic model with ( $D = D_s + D_c$ ) and without ( $D = D_c$ ) support flexibility (Fig. 9(b)).

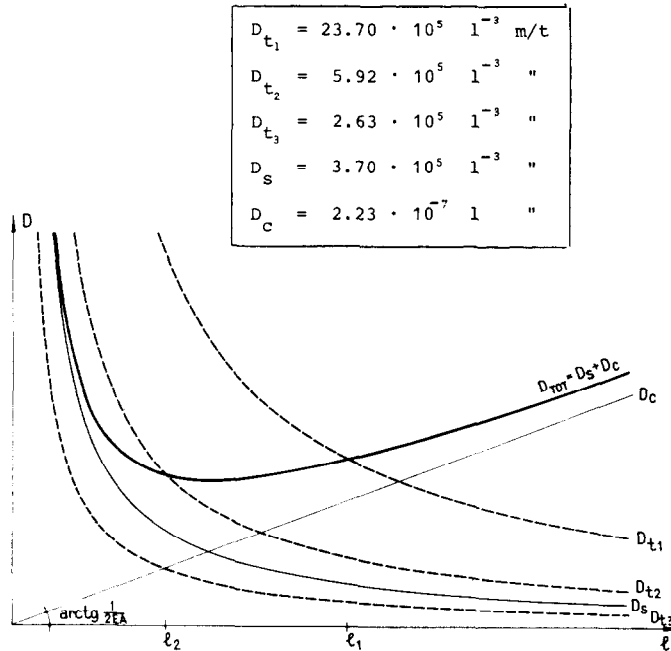


Fig. 10. Description of the dynamic behaviour of the system in the  $D, l$  reference system.

whole  $d/l$  range, while use of Irvine's theory for the distributed model is restricted to  $d/l$  values less than about 1/8. Of course, finite element models make it possible to overcome this problem.

#### 7. CONCLUSIONS

The present paper applies a finite difference algorithm to the study of linearized in-plane natural vibrations of an inextensible cable with symmetric movable supports.

Attention has been given mainly to the analysis of the influence of support flexibility on the dynamic phenomenon. Several parametric studies have been carried out and the analogy between the dynamic behaviour of cable models with lumped or distributed flexibility has been shown. For certain values of geometric and elastic parameters, both models show the important phenomenon of transition between symmetric and antisymmetric mode shapes, which is lacking, on the contrary, when the inextensible cable with fixed supports is considered. The conclusion has been drawn that crossover of mode shapes is substantially a matter of kinematical compatibility.

The main results regarding the influence of the elastic cable parameters, reported by previous investigators, have been obtained by lumping the cable flexibility at the supports. By means of this technique, the algorithm can be applied over the whole  $d/l$  range, and appears as a possible alternative to finite element models as well, due to halving of the d.o.f. number of the discretized problem associated with the assumption of inextensibility.

Finally, by applying the above mentioned analogy, natural vibrations of elastic suspended cables with flexible supports have been studied. The relative importance of the two flexibilities in the dynamic behaviour of the system and the possibility of assuming simplified models in technical applications have been examined. It has been shown that cable extensibility plays a more important role than support flexibility. However, due to the latter, remarkable differences have been observed with respect

to the case of elastic cable with fixed supports. Transition points of the lower modes shift towards lower span length, and frequency values and order of higher symmetric modes can differ from the natural ones for low span values.

The results obtained throughout the study have been compared with the findings of earlier investigators who treated the problem with both continuous and discrete approaches. Furthermore, besides numerical results, reference has been made to a synthetic description of the dynamic phenomenon based on Irvine's theory.

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