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# CONSTRUCTIBLE CHARACTERS AND $b$ -INVARIANT

by

CÉDRIC BONNAFÉ

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Let  $(W, S)$  be a finite Coxeter system and let  $\varphi : S \rightarrow \mathbb{R}_{>0}$  be a *weight function* that is, a map such that  $\varphi(s) = \varphi(t)$  whenever  $s$  and  $t$  are conjugate in  $W$ . Associated with this datum, G. Lusztig has defined [Lu3, §22] a notion of *constructible characters* of  $W$ : it is conjectured that a character is constructible if and only if it is the character afforded by a Kazhdan-Lusztig left cell (defined using the weight function  $\varphi$ ). These constructible characters depend heavily on  $\varphi$  so we will call them the  *$\varphi$ -constructible characters of  $W$* : the set of  $\varphi$ -constructible characters will be denoted by  $\text{Cons}_\varphi^{\text{Lus}}(W)$ . We shall also define a graph  $\mathcal{G}_{W,\varphi}^{\text{Lus}}$  as follows: the vertices of  $\mathcal{G}_{W,\varphi}^{\text{Lus}}$  are the irreducible characters and two irreducible characters  $\chi$  and  $\chi'$  are joined in this graph if there exists a  $\varphi$ -constructible character  $\gamma$  of  $W$  such that  $\chi$  and  $\chi'$  both occur as constituents of  $\gamma$ . The connected components of  $\mathcal{G}_{W,\varphi}^{\text{Lus}}$  (viewed as subsets of  $\text{Irr}(W)$ ) will be called the *Lusztig  $\varphi$ -families*: the set of Lusztig  $\varphi$ -families will be denoted by  $\text{Fam}_\varphi^{\text{Lus}}(W)$ . If  $\mathcal{F} \in \text{Fam}_\varphi^{\text{Lus}}(W)$ , we denote by  $\text{Cons}_\varphi^{\text{Lus}}(\mathcal{F})$  the set of  $\varphi$ -constructible characters of  $W$  all of whose irreducible components belong to  $\mathcal{F}$ .

On the other hand, using the theory of rational Cherednik algebras at  $t = 0$  and the geometry of the Calogero-Moser space associated with  $(W, \varphi)$ , R. Rouquier and the author (see [BoRo1] and [BoRo2]) have defined a notion of *Calogero-Moser  $\varphi$ -cells* of  $W$ , a notion of *Calogero-Moser  $\varphi$ -cellular characters* of  $W$  (whose set is denoted by  $\text{Cell}_\varphi^{\text{CM}}(W)$ ) and a notion of *Calogero-Moser  $\varphi$ -families* (whose set is denoted by  $\text{Fam}_\varphi^{\text{CM}}(W)$ ).

**Conjecture (see [BoRo1], [BoRo2] and [GoMa]).** *With the above notation,*

$$\text{Cons}_\varphi^{\text{Lus}}(W) = \text{Cell}_\varphi^{\text{CM}}(W) \quad \text{and} \quad \text{Fam}_\varphi^{\text{Lus}}(W) = \text{Fam}_\varphi^{\text{CM}}(W)$$

*for every weight function  $\varphi : S \rightarrow \mathbb{R}_{>0}$ .*

The statement about families in this conjecture holds for classical Weyl groups thanks to a case-by-case analysis relying on [Lu3, §22] (for the computation of Lusztig  $\varphi$ -families), [GoMa] (for the computation of Calogero-Moser  $\varphi$ -families in type  $A$  and  $B$ ) and [Be2] (for the computation of the Calogero-Moser  $\varphi$ -families in type  $D$ ). It also holds whenever  $|S| = 2$  (see [Lu3, §17 and Lemma 22.2] and [Be1, §6.10]).

The statement about constructible characters is much more difficult to establish, as the computation of Calogero-Moser  $\varphi$ -cellular characters is at that time out of reach: it has been proved whenever the Calogero-Moser space associated with  $(W, S, \varphi)$  is smooth [BoRo2, Theorem 14.4.1] (it has also been checked if  $W$  is of type  $B_2$ ...).

Our aim in this paper is to show that this conjecture is compatible with properties of the  $b$ -invariant (as defined below). With each irreducible character  $\chi$  of  $W$  is associated its *fake degree*  $f_\chi(\mathbf{t})$ , using the invariant theory of  $W$  (see for instance [BoRo2, Definition 1.5.7]). Let us denote by  $b_\chi$  the valuation of  $f_\chi(\mathbf{t})$ :  $b_\chi$  is called the  *$b$ -invariant* of  $\chi$ . For instance,  $b_1 = 0$  and  $b_\varepsilon$  is the number of reflections of  $W$  (here,  $\varepsilon : W \rightarrow \{1, -1\}$  denotes the sign character). Also,  $b_\chi = 1$  if and only if  $\chi$  is an irreducible constituent of the canonical reflection representation of  $W$ . The following result is proved in [BoRo2, Theorems 9.6.1 and 12.3.14]:

**Theorem CM.** *Let  $\varphi : S \rightarrow \mathbb{R}_{>0}$  be a weight function. Then:*

- (a) *If  $\mathcal{F} \in \text{Fam}_\varphi^{\text{CM}}(W)$ , then there exists a unique  $\chi \in \mathcal{F}$  with minimal  $b$ -invariant.*
- (b) *If  $\gamma \in \text{Cons}_\varphi^{\text{CM}}(W)$ , then there exists a unique irreducible constituent  $\chi$  of  $\gamma$  with minimal  $b$ -invariant.*

The next theorem is proved in [Lu2, Theorem 5.25 and its proof] (see also [Lu1] for the first occurrence of the *special* representations):

**Theorem (Lusztig).** *Assume that  $\varphi$  is constant. Then:*

- (a) *If  $\mathcal{F} \in \text{Fam}_\varphi^{\text{Lus}}(W)$ , then there exists a unique  $\chi_{\mathcal{F}} \in \mathcal{F}$  with minimal  $b$ -invariant ( $\chi_{\mathcal{F}}$  is called the **special** character of  $\mathcal{F}$ ).*
- (b) *If  $\gamma \in \text{Cons}_\varphi^{\text{Lus}}(\mathcal{F})$ , then  $\chi_{\mathcal{F}}$  is an irreducible constituent of  $\gamma$  (and, of course, among the irreducible constituents of  $\gamma$ ,  $\chi_{\mathcal{F}}$  is the unique one with minimal  $b$ -invariant).*

It turns out that, for general  $\varphi$ , there might exist Lusztig  $\varphi$ -families  $\mathcal{F}$  such that no element of  $\mathcal{F}$  occurs as an irreducible constituent of *all* the  $\varphi$ -constructible characters in  $\text{Cons}_\varphi^{\text{Lus}}(\mathcal{F})$  (this already occurs in type  $B_3$ , and the reader can also check this fact in type  $F_4$ , using the tables given by Geck [Ge, Table 2]). Nevertheless, we will prove in this paper the following result, which is compatible with the above conjecture and the above theorems:

**Theorem L.** *Let  $\varphi : S \rightarrow \mathbb{R}_{>0}$  be a weight function. Then:*

- (a) *If  $\mathcal{F} \in \text{Fam}_{\varphi}^{\text{Lus}}(W)$ , then there exists a unique  $\chi \in \mathcal{F}$  with minimal  $b$ -invariant.*
- (b) *If  $\gamma \in \text{Cons}_{\varphi}^{\text{Lus}}(W)$ , then there exists a unique irreducible constituent  $\chi$  of  $\gamma$  with minimal  $b$ -invariant.*

The proof of Theorem CM is general and conceptual, while our proof of Theorem L goes through a case-by-case analysis, based on Lusztig’s description of  $\varphi$ -constructible characters and Lusztig  $\varphi$ -families [Lu3, §22].

REMARK 0 - Let  $\gamma_{\chi}$  denote the coefficient of  $\mathbf{t}^{b_{\chi}}$  in  $F_{\chi}(\mathbf{t})$ . Then it has been noticed by Lusztig [Lu1, §2, Page 325] that  $\gamma_{\chi} = 1$  whenever  $\chi$  is special.

As the only irreducible Coxeter systems affording possibly unequal parameters are of type  $I_2(2m)$ ,  $F_4$  or  $B_n$ , and as  $\gamma_{\chi} = 1$  for any character  $\chi$  in these groups, we can conclude that, in general (equal or unequal parameters),  $\gamma_{\chi} = 1$  for all the characters  $\chi$  with minimal  $b$ -invariant constructed in Theorem L (for both (a) and (b)).

The same property holds for the characters  $\chi$  with minimal  $b$ -invariant constructed in Theorem CM (in this case, the proof is again general and conceptual [BoRo2]). ■

### 1. Proof of Theorem L

**1.A. Reduction.** — It is easily seen that the proof of Theorem L may be reduced to the case where  $(W, S)$  is irreducible. If  $W$  is of type  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$ ,  $E_8$ ,  $H_3$  or  $H_4$ , then  $\varphi$  is necessarily constant and Theorem L follows immediately from Lusztig’s Theorem. If  $W$  is dihedral, then Theorem L is easily checked using [Lu3, §17 and Lemma 22.2]. If  $W$  is of type  $F_4$ , then Theorem L follows from inspection of [Ge, Table 2]. Therefore, this shows that we may, and we will, assume that  $W$  is of type  $B_n$ , with  $n \geq 2$ . Write  $S = \{t, s_1, s_2, \dots, s_{n-1}\}$  in such a way that the Dynkin diagram of  $(W, S)$  is



Write  $b = \varphi(t)$  and  $a = \varphi(s_1) = \varphi(s_2) = \dots = \varphi(s_{n-1})$ . If  $b \notin a\mathbb{N}^*$ , then  $\text{Cons}_{\varphi}^{\text{Lus}}(W) = \text{Irr}(W)$  (see [Lu3, Proposition 22.25]) and Theorem L becomes obvious. So we may assume that  $b = ra$  with  $r \in \mathbb{N}^*$ . To summarize:

**Hypothesis.** *From now on, and until the end of this section, we will assume that  $(W, S)$  is of type  $B_n$ , with  $n \geq 2$ , that  $S = \{t, s_1, s_2, \dots, s_{n-1}\}$  is such that the Dynkin diagram of  $(W, S)$  is given by (#), that  $\varphi(t) = r\varphi(s_1) = r\varphi(s_2) = \dots = r\varphi(s_{n-1}) = 1$  with  $r \in \mathbb{N}^*$ .*

**1.B. Admissible involutions.** — Let  $l \geq 0$  and let  $Z$  be a totally ordered set of size  $2l + r$ . We shall define by induction on  $l$  what is an  $r$ -admissible involution of  $Z$ . Let  $\iota : Z \rightarrow Z$  be an involution. Then  $\iota$  is said  $r$ -admissible if it has  $r$  fixed points and, if  $l \geq 1$ , there exist two consecutive elements  $b$  and  $c$  of  $Z$  such that  $\iota(b) = c$  and the restriction of  $\iota$  to  $Z \setminus \{b, c\}$  is  $r$ -admissible.

Note that, if  $\iota$  is an  $r$ -admissible involution and if  $\iota(b) = c > b$  and  $\iota(z) = z$ , then  $z < b$  or  $z > c$  (this is easily proved by induction on  $|Z|$ ).

**1.C. Symbols.** — We shall denote by  $\mathbf{Sym}_k(r)$  the set of *symbols*  $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$  where  $\beta = (\beta_1 < \beta_2 < \dots < \beta_{k+r})$  and  $\gamma = (\gamma_1 < \gamma_2 < \dots < \gamma_k)$  are increasing sequences of *non-zero* natural numbers. We set

$$|\Lambda| = \sum_{i=1}^{k+r} (\beta_i - i) + \sum_{j=1}^k (\gamma_j - j)$$

and 
$$\mathbf{b}(\Lambda) = \sum_{i=1}^{k+r} (2k + 2r - 2i)(\beta_i - i) + \sum_{j=1}^k (2k + 1 - 2j)(\gamma_j - j).$$

The number  $\mathbf{b}(\Lambda)$  will be called the ***b*-invariant** of  $\Lambda$ . For simplifying our arguments, we shall define

$$\nabla_{k,r} = \sum_{i=1}^{k+r} (2k + 2r - 2i)i + \sum_{j=1}^k (2k + 1 - 2j)j$$

so that

$$\mathbf{b}(\Lambda) = \sum_{i=1}^{k+r} (2k + 2r - 2i)\beta_i + \sum_{j=1}^k (2k + 1 - 2j)\gamma_j - \nabla_{k,r}.$$

By abuse of notation, we denote by  $\beta \cap \gamma$  the set  $\{\beta_1, \beta_2, \dots, \beta_{k+r}\} \cap \{\gamma_1, \gamma_2, \dots, \gamma_k\}$  and by  $\beta \cup \gamma$  the set  $\{\beta_1, \beta_2, \dots, \beta_{k+r}\} \cup \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ . We also set  $\beta \dot{+} \gamma = (\beta \cup \gamma) \setminus (\beta \cap \gamma)$ .

Now, let  $\mathbf{z}'(\Lambda) = (\beta_1, \beta_2, \dots, \beta_r, \gamma_1, \beta_{r+1}, \gamma_2, \beta_{r+2}, \dots, \gamma_k, \beta_{r+k})$ . We shall write

$$\mathbf{z}'(\Lambda) = (z'_1(\Lambda), z'_2(\Lambda), \dots, z'_{2k+r}(\Lambda)),$$

so that

$$\begin{aligned} \mathbf{b}(\Lambda) &= \sum_{i=1}^r (2k + 2r - 2i)z'_i(\Lambda) + \sum_{i=r+1}^{2k+r} (2k + r - i)z'_i(\Lambda) - \nabla_{k,r} \\ (\clubsuit) \quad &= \sum_{i=1}^r (r - i)z'_i(\Lambda) + \sum_{i=1}^{2k+r} (2k + r - i)z'_i(\Lambda) - \nabla_{k,r} \\ &= \sum_{i=1}^r \left( \sum_{j=1}^i z'_j(\Lambda) \right) + \sum_{i=1}^{2k+r-1} \left( \sum_{j=1}^i z'_j(\Lambda) \right) - \nabla_{k,r}. \end{aligned}$$

**1.D. Families of symbols.** — We denote by  $\mathbf{z}(\Lambda)$  the sequence  $z_1 \leq z_2 \leq \dots \leq z_{2k+r}$  obtained after rewriting the sequence  $(\beta_1, \beta_2, \dots, \beta_{k+r}, \gamma_1, \gamma_2, \dots, \gamma_k)$  in non-decreasing order.

REMARK 1 - Note that the sequence  $\mathbf{z}'(\Lambda)$  determines the symbol  $\Lambda$ , contrarily to the sequence  $\mathbf{z}(\Lambda)$ . However,  $\mathbf{z}(\Lambda)$  determines completely  $|\Lambda|$  thanks to the formula  $|\Lambda| = \sum_{z \in \mathbf{z}(\Lambda)} z - r(r+1)/2 - (k+r)(k+r+1)/2$ .  $\square$

We say that two symbols  $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$  and  $\Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}$  in  $\mathbf{Sym}_k(r)$  are in the same family if  $\mathbf{z}(\Lambda) = \mathbf{z}(\Lambda')$ . Note that this is equivalent to say that  $\beta \cap \gamma = \beta' \cap \gamma'$  and  $\beta \cup \gamma = \beta' \cup \gamma'$ . If  $\mathcal{F}$  is the family of  $\Lambda$ , we set  $X_{\mathcal{F}} = \beta \cap \gamma$  and  $Z_{\mathcal{F}} = \beta \dot{+} \gamma$ : note that  $X_{\mathcal{F}}$  and  $Z_{\mathcal{F}}$  depend only on  $\mathcal{F}$  (and not on the particular choice of  $\Lambda \in \mathcal{F}$ ).

If  $\iota$  is an  $r$ -admissible involution of  $Z_{\mathcal{F}}$ , we denote by  $\mathcal{F}_{\iota}$  the set of symbols  $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$  in  $\mathcal{F}$  such that  $|\beta \cap \omega| = 1$  for all  $\iota$ -orbits  $\omega$ .

**1.E. Lusztig families, constructible characters.** — Let  $\Lambda \in \mathbf{Sym}_k(r)$  be such that  $|\Lambda| = n$ . Let  $\text{Bip}(n)$  be the set of bipartitions of  $n$ . We set

$$\lambda_1(\Lambda) = (\beta_{k+r} - (k+r) \geq \dots \geq \beta_2 - 2 \geq \beta_1 - 1),$$

$$\lambda_2(\Lambda) = (\gamma_k - k \geq \dots \geq \gamma_2 - 2 \geq \gamma_1 - 1)$$

and

$$\lambda(\Lambda) = (\lambda_1(\Lambda), \lambda_2(\Lambda)).$$

Then  $\lambda(\Lambda)$  is a bipartition of  $n$ . We denote by  $\chi_{\Lambda}$  the irreducible character of  $W$  denoted by  $\chi_{\lambda(\Lambda)}$  in [Lu3, §22] or in [GePf, §5.5.3]. Then [GePf, §5.5.3]

$$(\diamond) \quad b_{\chi_{\Lambda}} = \mathbf{b}(\Lambda).$$

With these notation, Lusztig described the  $\varphi$ -constructible characters in [Lu3, Proposition 22.24], from which the description of Lusztig  $\varphi$ -families follow by using [Lu3, Lemma 22.22]:

**Theorem 2 (Lusztig).** *Let  $\mathcal{F}_{\text{Lus}}$  be a Lusztig  $\varphi$ -family and let  $\gamma \in \text{Cons}_{\varphi}^{\text{Lus}}(\mathcal{F}_{\text{Lus}})$ . If we choose  $k$  sufficiently large, then:*

(a) *There exists a family  $\mathcal{F}$  of symbols in  $\mathbf{Sym}_k(r)$  such that*

$$\mathcal{F}_{\text{Lus}} = \{\chi_{\Lambda} \mid \Lambda \in \mathcal{F}\}.$$

(b) *There exists an  $r$ -admissible involution  $\iota$  of  $Z_{\mathcal{F}}$  such that*

$$\gamma = \sum_{\Lambda \in \mathcal{F}_{\iota}} \chi_{\Lambda}.$$

**Definition 3.** The symbol  $\Lambda$  is said *special* if  $\mathbf{z}(\Lambda) = \mathbf{z}'(\Lambda)$ .

REMARK 4 - According to Remark 1, there is a unique special symbol in each family. It will be denoted by  $\Lambda_{\mathcal{F}}$ . Finally, note that, if  $\Lambda, \Lambda'$  belong to the same family, then  $|\Lambda| = |\Lambda'|$ .  $\square$

Now, Theorem L follows from Theorem 2, Formula ( $\diamond$ ) and the following next Theorem:

**Theorem 5.** Let  $\mathcal{F}$  be a family of symbols in  $\mathbf{Sym}_k(r)$ , let  $\iota$  be an  $r$ -admissible involution of  $Z_{\mathcal{F}}$  and let  $\Lambda \in \mathcal{F}$ . Then:

- (a)  $\mathbf{b}(\Lambda) \geq \mathbf{b}(\Lambda_{\mathcal{F}})$  with equality if and only if  $\Lambda = \Lambda_{\mathcal{F}}$ .
- (b) There is a unique symbol  $\Lambda_{\mathcal{F},\iota}$  in  $\mathcal{F}_{\iota}$  such that, if  $\Lambda \in \mathcal{F}_{\iota}$ , then  $\mathbf{b}(\Lambda) \geq \mathbf{b}(\Lambda_{\mathcal{F},\iota})$ , with equality if and only if  $\Lambda = \Lambda_{\mathcal{F},\iota}$ .

**1.F. Proof of Theorem 5(a).** — First, note that  $\mathbf{z}(\Lambda) = \mathbf{z}(\Lambda_{\mathcal{F}}) = \mathbf{z}'(\Lambda_{\mathcal{F}})$ . As  $\mathbf{z}'(\Lambda)$  is a permutation of the non-decreasing sequence  $\mathbf{z}'(\Lambda_{\mathcal{F}})$ , we have

$$\sum_{j=1}^i z'_j(\Lambda) \geq \sum_{j=1}^i z'_j(\Lambda_{\mathcal{F}})$$

for all  $i \in \{1, 2, \dots, 2k+r\}$ . So, it follows from ( $\clubsuit$ ) that

$$\mathbf{b}(\Lambda) - \mathbf{b}(\Lambda_{\mathcal{F}}) = \sum_{i=1}^{r-1} \left( \sum_{j=1}^i (z'_j(\Lambda) - z'_j(\Lambda_{\mathcal{F}})) \right) + \sum_{i=1}^{2k+r-1} \left( \sum_{j=1}^i (z'_j(\Lambda) - z'_j(\Lambda_{\mathcal{F}})) \right).$$

So  $\mathbf{b}(\Lambda) \geq \mathbf{b}(\Lambda_{\mathcal{F}})$  with equality only whenever  $\sum_{j=1}^i z'_j(\Lambda) = \sum_{j=1}^i z'_j(\Lambda_{\mathcal{F}})$  for all  $i \in \{1, 2, \dots, 2k+r\}$ . The proof of Theorem 5(a) is complete.

**1.G. Reduction for the proof of Theorem 5(b).** — First, assume that  $X_{\mathcal{F}} \neq \emptyset$ . Let  $b \in X_{\mathcal{F}}$  and let  $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathcal{F}$ . Then  $b \in \beta \cap \gamma = X_{\mathcal{F}}$  and we denote by  $\beta[b]$  the sequence obtained by removing  $b$  to  $\beta$ . Similarly, let  $\Lambda[b] = \begin{pmatrix} \beta[b] \\ \gamma[b] \end{pmatrix}$ .

Then  $\Lambda[b] \in \mathbf{Sym}_{k-1}(r)$  and

$$(\heartsuit) \quad \mathbf{b}(\Lambda) = \mathbf{b}(\Lambda[b]) + \nabla_{k,r} - \nabla_{k-1,r} + b \left( 4k + 2r + 1 - \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z \leq b}} 2 \right) + 2 \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z < b}} z.$$

*Proof of ( $\heartsuit$ ).* Let  $i_0$  and  $j_0$  be such that  $\beta_{i_0} = b$  and  $\gamma_{j_0} = b$ . Then

$$\mathbf{b}(\Lambda) - \mathbf{b}(\Lambda[b]) = \nabla_{k,r} - \nabla_{k-1,r} + (2k + 2r - 2i_0)b + \sum_{i=1}^{i_0-1} 2\beta_i + (2k + 1 - 2j_0)b + \sum_{j=1}^{j_0-1} 2\gamma_j.$$

But the numbers  $\beta_1, \beta_2, \dots, \beta_{i_0}, \gamma_1, \gamma_2, \dots, \gamma_{j_0}$  are exactly the elements of the sequence  $\mathbf{z}(\Lambda)$  which are  $\leq b$ . So

$$i_0 + j_0 = \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z \leq b}} 1$$

and

$$\sum_{i=1}^{i_0-1} \beta_i + \sum_{j=1}^{j_0-1} \gamma_j = \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z < b}} z.$$

This shows  $(\heartsuit)$ . ■

Now, the family of  $\Lambda[b]$  depends only on the family of  $\Lambda$  (and not on  $\Lambda$  itself): indeed,  $\mathbf{z}(\Lambda[b])$  is obtained from  $\mathbf{z}(\Lambda)$  by removing the two entries equal to  $b$ . We will denote by  $\mathcal{F}[b]$  the family of  $\Lambda[b]$ . Moreover,  $Z_{\mathcal{F}[b]} = Z_{\mathcal{F}}$  and the map  $\Lambda \mapsto \Lambda[b]$  induces a bijection between  $\mathcal{F}$  and  $\mathcal{F}[b]$ , and also induces a bijection between  $\mathcal{F}_i$  and  $\mathcal{F}[b]_i$ .

Moreover, the formula  $(\heartsuit)$  shows that the difference between  $\mathbf{b}(\Lambda)$  and  $\mathbf{b}(\Lambda[b])$  depends only on  $b$  and  $\mathcal{F}$ , so proving Lemma 6 for the pair  $(\mathcal{F}, \iota)$  is equivalent to proving Lemma 6 for the pair  $(\mathcal{F}[b], \iota)$ . By applying several times this principle if necessary, this means that we may, and we will, assume that

$$X_{\mathcal{F}} = \emptyset.$$

**1.H. Proof of Theorem 5(b).** — We denote by  $f_r < \dots < f_1$  the elements of  $Z_{\mathcal{F}}$  which are fixed by  $\iota$ . We also set  $f_{r+1} = 0$  and  $f_0 = \infty$ . As  $\iota$  is  $r$ -admissible, the set  $Z_{\mathcal{F}}^{(d)} = \{z \in Z_{\mathcal{F}} \mid f_{d+1} < z < f_d\}$  is  $\iota$ -stable and contains no  $\iota$ -fixed point (for  $d \in \{0, 1, \dots, r\}$ ). Let  $k_d = |Z_{\mathcal{F}}^{(d)}|/2$  and let  $\iota_d$  be the restriction of  $\iota$  to  $Z_{\mathcal{F}}^{(d)}$ . Then  $\iota_d$  is a 0-admissible involution of  $Z_{\mathcal{F}}^{(d)}$ .

If  $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathcal{F}_i$ , we set  $\beta^{(d)} = \beta \cap Z_{\mathcal{F}}^{(d)}$ ,  $\gamma^{(d)} = \gamma \cap Z_{\mathcal{F}}^{(d)}$  and  $\Lambda^{(d)} = \begin{pmatrix} \beta^{(d)} \\ \gamma^{(d)} \end{pmatrix}$ . Then  $\Lambda^{(d)} \in \mathbf{Sym}_{k_d}(0)$  and, if  $\mathcal{F}^{(d)}$  denotes the family of  $\Lambda^{(d)}$ , then  $\Lambda^{(d)} \in \mathcal{F}_{i_d}^{(d)}$ .

Now, if  $\Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix} \in \mathbf{Sym}_{k'}(0)$ , we set

$$\mathbf{b}_d(\Lambda') = \sum_{i=1}^{k'} (2k' + 2d - 2i) \beta'_i + \sum_{j=1}^{k'} (2k' + 1 - 2j) \gamma'_j.$$

The number  $\mathbf{b}_d(\Lambda')$  is called the  $\mathbf{b}_d$ -invariant of  $\Lambda'$ . It then follows from the definition of  $\mathbf{b}$  and  $\nabla_{k,r}$  that

$$(\spadesuit) \quad \mathbf{b}(\Lambda) = \sum_{d=0}^r \mathbf{b}_d(\Lambda^{(d)}) - \nabla_{k,r} + \sum_{d=1}^r 2(k_0 + k_1 + \dots + k_{d-1}) \left( f_d + \sum_{z \in Z^{(d)}} z \right).$$



Since the map

$$\begin{aligned} \mathcal{F}_\iota &\longrightarrow \prod_{d=0}^r \mathcal{F}_{\iota_d}^{(d)} \\ \Lambda &\longmapsto (\Lambda^{(0)}, \Lambda^{(1)}, \dots, \Lambda^{(d)}) \end{aligned}$$

is bijective and since  $\mathbf{b}(\Lambda) - \sum_{d=0}^r \mathbf{b}_d(\Lambda^{(d)})$  depends only on  $(\mathcal{F}, \iota)$  and not on  $\Lambda$  (as shown by the formula ( $\spadesuit$ )), Theorem 5(b) will follow from the following lemma :

**Lemma 6.** *There exists a unique symbol in  $\mathcal{F}_{\iota_d}^{(d)}$  with minimal  $\mathbf{b}_d$ -invariant.*

The proof of Lemma 6 will be given in the next section.

## 2. Minimal $\mathbf{b}_d$ -invariant

For simplifying notation, we set  $Z = Z_{\mathcal{F}}^{(d)}$ ,  $l = k_d$ ,  $\mathcal{G} = \mathcal{F}^{(d)}$  and  $J = \iota_d$ . Let us write  $Z = \{z_1, z_2, \dots, z_{2l}\}$  with  $z_1 < z_2 < \dots < z_{2l}$ . Recall from the previous section that  $J$  is a 0-admissible involution of  $Z$ .

**2.A. Construction.** — We shall define by induction on  $l \geq 0$  a symbol  $\Lambda_J^{(d)}(Z) \in \mathcal{G}_J$ . If  $l = 0$ , then  $\Lambda_J^{(d)}(Z)$  is obviously empty. So assume now that, for any set of non-zero integers  $Z'$  of order  $2(l-1)$ , for any 0-admissible involution  $J'$  of  $Z'$  and any  $d' \geq 0$ , we have defined a symbol  $\Lambda_{J'}^{(d')}(Z')$ . Then  $\Lambda_J^{(d)}(Z) = \begin{pmatrix} \beta_J^{(d)}(Z) \\ \gamma_J^{(d)}(Z) \end{pmatrix}$  is defined as follows: let  $Z' = Z \setminus \{z_1, \iota(z_1)\}$ ,  $J'$  the restriction of  $J$  to  $Z'$  and let

$$d' = \begin{cases} d-1 & \text{if } d \geq 1, \\ 1 & \text{if } d = 0. \end{cases}$$

Then  $|Z'| = 2(l-1)$  and  $J'$  is 0-admissible. So  $\Lambda_{J'}^{(d')}(Z') = \begin{pmatrix} \beta_{J'}^{(d')}(Z') \\ \gamma_{J'}^{(d')}(Z') \end{pmatrix}$  is well-defined by the induction hypothesis. We then set

$$\beta_J^{(d)}(Z) = \begin{cases} \beta_{J'}^{(d')}(Z') \cup \{z_1\} & \text{if } d \geq 1, \\ \beta_{J'}^{(d')}(Z') \cup \{J(z_1)\} & \text{if } d = 0, \end{cases}$$

and

$$\gamma_J^{(d)}(Z) = \begin{cases} \gamma_{J'}^{(d')}(Z') \cup \{J(z_1)\} & \text{if } d \geq 1, \\ \gamma_{J'}^{(d')}(Z') \cup \{z_1\} & \text{if } d = 0. \end{cases}$$

Then Lemma 6 is implied by the next lemma :

**Lemma 6<sup>+</sup>.** *Let  $\Lambda \in \mathcal{G}_J$ . Then  $\mathbf{b}_d(\Lambda) \geq \mathbf{b}_d(\Lambda_J^{(d)}(Z))$  with equality if and only if  $\Lambda = \Lambda_J^{(d)}(Z)$ .*

The rest of this section is devoted to the proof of Lemma 6<sup>+</sup>. We will first prove Lemma 6<sup>+</sup> whenever  $d \in \{0, 1\}$  using Lusztig's Theorem. We will then turn to the general case, which will be handled by induction on  $l = |Z|/2$ . We fix  $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathcal{G}_l$ .

**2.B. Proof of Lemma 6<sup>+</sup> whenever  $d = 1$ .** — Let  $z$  be a natural number strictly bigger than all the elements of  $Z$ . Let  $\tilde{\Lambda} = \begin{pmatrix} \beta \cup \{z\} \\ \gamma \end{pmatrix} \in \mathbf{Sym}_k(1)$ . Then  $\mathbf{b}_1(\Lambda) = \mathbf{b}(\tilde{\Lambda}) + C$ , where  $C$  depends only on  $Z$ . Let  $\tilde{\Lambda}_0 = \begin{pmatrix} z_1, z_3, \dots, z_{2l-1}, z \\ z_2, \dots, z_{2l} \end{pmatrix}$ . Since  $J$  is 0-admissible, it is easily seen that, if  $J(z_i) = z_j$ , then  $j - i$  is odd. So  $\tilde{\Lambda}_0 \in \mathcal{G}_j$ . But, by [Lu1, §5],  $\mathbf{b}(\tilde{\Lambda}) \geq \mathbf{b}(\tilde{\Lambda}_0)$  with equality if and only if  $\tilde{\Lambda} = \tilde{\Lambda}_0$ . So it is sufficient to notice that  $\overline{\Lambda_j^{(1)}(Z)} = \tilde{\Lambda}_0$ , which is easily checked.

**2.C. Proof of Lemma 6<sup>+</sup> whenever  $d = 0$ .** — Assume in this subsection, and only in this subsection, that  $d = 0$  or 1. We denote by  $\Lambda^{\text{op}} = \begin{pmatrix} \gamma \\ \beta \end{pmatrix} \in \mathcal{G}_j$ . It is readily seen from the construction that  $\Lambda_j^{(0)}(Z)^{\text{op}} = \Lambda_j^{(1)}(Z)$  and that

$$\mathbf{b}_1(\Lambda) = \mathbf{b}_0(\Lambda^{\text{op}}) + \sum_{z \in Z} z.$$

So Lemma 6<sup>+</sup> for  $d = 0$  follows from Lemma 6<sup>+</sup> for  $d = 1$ .

**2.D. Proof of Lemma 6<sup>+</sup> whenever  $d \geq 2$ .** — Assume now, and until the end of this section, that  $d \geq 2$ . We shall prove Lemma 6<sup>+</sup> by induction on  $l = |Z|/2$ . The result is obvious if  $l = 0$ , as well as if  $l = 1$ . So we assume that  $l \geq 2$  and that Lemma 6<sup>+</sup> holds for  $l' \leq l - 1$ . Write  $J(z_1) = z_{2m}$ , where  $m \leq l$  (note that  $J(z_1) \notin \{z_1, z_3, z_5, \dots, z_{2l-1}\}$  since  $J$  is 0-admissible).

Assume first that  $m < l$ . Then  $Z$  can be written as the union  $Z = Z^+ \dot{\cup} Z^-$ , where  $Z^+ = \{z_1, z_2, \dots, z_{2m}\}$  and  $Z^- = \{z_{2m+1}, z_{2m+2}, \dots, z_{2l}\}$  are  $J$ -stable (since  $J$  is 0-admissible). If  $\varepsilon \in \{+, -\}$ , let  $J^\varepsilon$  denote the restriction of  $J$  to  $Z^\varepsilon$ , let  $\beta^\varepsilon = \beta \cap Z^\varepsilon$ ,  $\gamma^\varepsilon = \gamma \cap Z^\varepsilon$  and  $\Lambda^\varepsilon = \begin{pmatrix} \beta^\varepsilon \\ \gamma^\varepsilon \end{pmatrix}$ , and let  $\mathcal{G}^\varepsilon$  denote the family of  $\Lambda^\varepsilon$ . Then it is easily seen that  $\Lambda^\varepsilon \in \mathcal{G}_{j^\varepsilon}^\varepsilon$ , that  $\mathbf{b}_d(\Lambda) - (\mathbf{b}_d(\Lambda^+) + \mathbf{b}_d(\Lambda^-))$  depends only on  $(\mathcal{G}, J)$  and that  $\Lambda_j^{(d)}(Z)^\varepsilon = \Lambda_{j^\varepsilon}^{(d)}(Z^\varepsilon)$ . By the induction hypothesis,  $\mathbf{b}_d(\Lambda^\varepsilon) \geq \mathbf{b}_d(\Lambda_{j^\varepsilon}^{(d)}(Z^\varepsilon))$  with equality if and only if  $\Lambda^\varepsilon = \Lambda_{j^\varepsilon}^{(d)}(Z^\varepsilon)$ . So the result follows in this case. This means that we may, and we will, work under the following hypothesis:

**Hypothesis.** From now on, and until the end of this section, we assume that  $J(z_1) = z_{2l}$ .

As in the construction of  $\Lambda_j^{(d)}(Z)$ , let  $Z' = Z \setminus \{z_1, z_{2l}\} = \{z_2, z_3, \dots, z_{2l-1}\}$ , let  $j'$  denote the restriction of  $j$  to  $Z'$  and let

$$d' = \begin{cases} d-1 & \text{if } d \geq 1, \\ 1 & \text{if } d = 0. \end{cases}$$

Then  $|Z'| = 2(l-1)$  and  $j'$  is 0-admissible. Let  $\Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}$  where  $\beta' = \beta \setminus \{z_1, z_{2l}\}$  and  $\gamma' = \gamma \setminus \{z_1, z_{2l}\}$ . Since  $d \geq 2$ , we have  $z_1 \in \beta_j^{(d)}(Z)$  and  $z_{2l} \in \gamma_j^{(d)}(Z)$ . This implies that

$$(\star) \quad \mathbf{b}_d(\Lambda_j^{(d)}(Z)) = \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) + z_{2l} + 2(l+d)z_1 + 2 \sum_{z \in Z'} z.$$

If  $z_1 \in \beta$ , then  $\Lambda = \Lambda_j^{(d)}(Z)$  if and only if  $\Lambda' = \Lambda_{j'}^{(d')}(Z')$  and again

$$\mathbf{b}_d(\Lambda) = \mathbf{b}_{d-1}(\Lambda') + z_{2l} + 2(l+d)z_1 + 2 \sum_{z \in Z'} z.$$

So the result follows from  $(\star)$  and from the induction hypothesis.

This means that we may, and we will, assume that  $z_1 \in \gamma$ . In this case,

$$\mathbf{b}_d(\Lambda) = \mathbf{b}_{d+1}(\Lambda') + 2dz_{2l} + (2l+1)z_1.$$

Then it follows from  $(\star)$  that

$$\mathbf{b}_d(\Lambda) - \mathbf{b}_d(\Lambda_j^{(d)}(Z)) = \mathbf{b}_{d+1}(\Lambda') - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) + (2d-1)(z_{2l} - z_1) - 2 \sum_{z \in Z'} z.$$

So, by the induction hypothesis,

$$\mathbf{b}_d(\Lambda) - \mathbf{b}_d(\Lambda_j^{(d)}(Z)) \geq \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) + (2d-1)(z_{2l} - z_1) - 2 \sum_{z \in Z'} z.$$

Since  $z_{2l} - z_1 > z_{2l-1} - z_2$ , it is sufficient to show that

$$(?) \quad \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) \geq -(2d-1)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z.$$

This will be proved by induction on the size of  $Z'$ . First, if  $j(z_2) < z_{2l}$ , then we can separate  $Z'$  into two  $j'$ -stable subsets and a similar argument as before allows to conclude thanks to the induction hypothesis.

So we assume that  $j'(z_2) = z_{2l-1}$ . Let  $Z'' = Z' \setminus \{z_2, z_{2l-1}\}$  and let  $j''$  denote the restriction of  $j'$  to  $Z''$ . Since  $z_2 \in \beta_{j'}^{(d+1)}(Z')$ , we can apply  $(\star)$  one step further to get

$$\begin{aligned} \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) &= \mathbf{b}_d(\Lambda_{j''}^{(d)}(Z'')) + z_{2l-1} + 2(l+d)z_2 + 2 \sum_{z \in Z''} z \\ &\quad - (\mathbf{b}_{d-2}(\Lambda_{j''}^{(d-2)}(Z'')) + z_{2l-1} + 2(l+d-2)z_2 + 2 \sum_{z \in Z''} z) \\ &= \mathbf{b}_d(\Lambda_{j''}^{(d)}(Z'')) - \mathbf{b}_{d-2}(\Lambda_{j''}^{(d-2)}(Z'')) + 4z_2. \end{aligned}$$

So, by the induction hypothesis,

$$\begin{aligned}
 \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) &\geq -(2d-3)(z_{2l-2} - z_3) + 2 \sum_{z \in Z''} z + 4z_2 \\
 &\geq -(2d-3)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z + 2z_2 - 2z_{2l-1} \\
 &= -(2d-1)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z,
 \end{aligned}$$

as desired. This shows (?) and completes the proof of Lemma 6<sup>+</sup>.

### 3. Complex reflection groups

If  $\mathscr{W}$  is a complex reflection group, then R. Rouquier and the author have also defined Calogero-Moser cellular characters and Calogero-Moser families (see [BoRo1] or [BoRo2]). If  $\mathscr{W}$  is of type  $G(l, 1, n)$  (in Shephard-Todd classification), then Leclerc and Miyachi [LeMi, §6.3] proposed, in link with canonical bases of  $U_v(\mathfrak{sl}_\infty)$ -modules, a family of characters that could be good analogue of constructible characters: let us call them the *Leclerc-Miyachi constructible characters* of  $G(l, 1, n)$ . If  $l = 2$ , then they coincide with constructible characters [LeMi, Theorem 10].

Of course, it would be interesting to know if Calogero-Moser cellular characters coincide with the Leclerc-Miyachi ones: this seems rather complicated but it should be at least possible to check if the Leclerc-Miyachi constructible characters satisfy the analogous properties with respect to the  $b$ -invariant.

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