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Faustmann Rotation and population dynamics in the presence of a risk of destructive events

Patrice Loisel ^{*}†

Abstract

The impact of the presence of risk of destructive event on the silvicultural practice of a forest stand is investigated. For that, we consider a model of population dynamics. This model has allowed us to make the comparison without and with risk, and highlight the influence of the presence of risk of destructive event on optimal thinning and optimal rotation period.

Keywords : Faustmann rotation ; optimal cutting age ; model ; thinning ; natural risk

JEL classification codes : C61, D81, Q23

Introduction

In terms of forest management, the first question that arises is : what is the optimal rotation period ? In the case where a calculation method to predict earnings for various rotation period is available, Faustmann (1849) proposed a formalism based on the expected discounted income. Many authors have successively improved or reformulated it ([Ohlin ,1921], [Pearse, 1967], [Clark, 1976]). The risk of destruction has been introduced to forest stands by Martell (1980) and Routledge (1980) in discrete time. Thereafter, Reed (1984) has studied the optimal forest rotation in continuous time with the risk of fire. Thorsen and Helles (1998) analysed endogeneous risk. Buongiorno (2001) proposed a generalization of Faustmann approach using Markov Decision Process Models. Peyron and Heshmatol-Vezin (2003) were interested in natural risks incurred by forests in discrete time. More recently, Goodnow et al. (2008) took into account, in the stand management, that the thinning regime affect the proportion of standing trees damaged for ice-damaged, Amacher et al. (2008) highlighted the influence of silvicultural practices of a landowner on damage if fire occurs.

Moreover, many authors have been studying ([Näslund, 1969], [Schreuder, 1971], [Clark, 1976], [Kao and Brodie, 1980], [Roise, 1986], [Haight et al., 1992]), the determination of optimal thinning and cutting age, under certainty, using whole-stand models. More recently, Hyytiainen and Tahvonen (2003) studied, among other elements, the influence of the rate of interest on the rotation period, Cao et al. (2006) analysed

^{*}INRA, UMR 729 MISTEA, F-34060 Montpellier, France

[†]SupAgro, UMR 729 MISTEA, F-34060 Montpellier, France

the effect of initial stand states on optimal thinning regime and rotation. Touza et al. (2008) investigated the ecology impact and interaction with the management.

For the absence of risk of destructive events, all the production cycles are carried out to the same cutting age. When the risk of destructive event exists and is taken into account, we assume, as most authors cited above that the operator systematically decides to interrupt the current cycle and begins a new cycle. The first question that follows is about the impact of presence of risk of destructive event on silviculture. A second question, which is linked to the first one, is : what is the consequence of silviculture over the rotation period ? To take into account thinning in the calculation of the land value we used a model of population dynamics. Contrary to Xabadia and Goetz (2010) works based on age-structured models, we consider a simplified model of average individual type i.e. a model based on characteristics of medium trees, in order to facilitate interpretation and analyse more specifically the impact of a risk of destruction on silviculture. The study generalizes easily to more complex and realistic models.

In a first part, we determine the land value without or with the presence of risk. In a second part we discuss the silviculture first in the reference limit case where individual tree growth is independent of tree-density (no density dependent growth) and then where individual tree-growth depends on the density (density dependent growth). Under the assumption of no density dependent growth and for a fixed period of rotation, we compare the results in the absence and presence of risk. Then we deduce the impact of the presence of a risk to forestry. In a third part, under the assumption of density dependent growth and using the optimal rotation period, we simulate a *Eucalyptus* stand. This study, by taking into account explicitly implemented thinning, allows us to deepen the results obtained by Reed (1984) and clarify the underlying assumptions (not taking into account thinning and cleanup costs proportional to the damage). The interest to introduce a model of population dynamics and to use the proposed method is justified by the possibility of taking into account thinning and clearing costs depending on the severity of damage therefore.

The land value

In the first part we study the land value without the risk of destructive event through a model of population dynamics. Then we study the same land value with the risk of destructive event.

Without the risk of destructive event

We first consider a stand without the presence of risk of destructive event. The study of this case will allow us to define a benchmark management of the stand.

For a cutting age, or rotation period T and a rate of thinning by unit of time $h(\cdot)$, the land value W_0 (up to a constant c_1) is the discounted value of cutting incomes net cost of replanting :

$$W_0 = \sum_{i=1}^{+\infty} (\mathcal{V}(h(\cdot), T) - c_1) e^{-i\delta T} = \frac{(\mathcal{V}(h(\cdot), T) - c_1) e^{-\delta T}}{1 - e^{-\delta T}} = \frac{\mathcal{V}(h(\cdot), T) - c_1}{e^{\delta T} - 1}$$

where $\mathcal{V}(h(\cdot), T)$ is the income generated by the cutting at time T and c_1 is the cost of replanting.

$\mathcal{V}(h(.), T)$ is by definition the sum of the thinning income on period $[0, T]$ and the income at final cutting age T .

The land value can also be interpreted as, W_0 is the instantaneous value of income from the forest in time T discounted at the initial time and is solution of :

$$W_0 = (W_0 + \mathcal{V}(h(.), T) - c_1)e^{-\delta T}$$

A model of population dynamic

To take into account the thinning in the calculation of the land value, i.e. to express $\mathcal{V}(h(.), T)$, we introduce a model of population dynamics. The considered model is an average tree model : the state variables are the number n of trees per hectare and the averaged tree-basal area s measured at breast height (1.30 meter from the ground). The evolution of these two variables is governed by the system of ordinary differential equations :

$$\begin{aligned}\frac{dn(t)}{dt} &= -(m(t) + h(t))n(t) \\ \frac{ds(t)}{dt} &= G(n(t), s(t), t)\end{aligned}$$

where $m(.)$ is the natural mortality and $G(., ., .)$ is the possibly density dependent growth function : individual tree-growth depends on the tree density $n(t)$.

The considered population dynamic model permits to explicitly manage the trees. The use of more complex and realistic models with different classes of tree basal areas, instead of the model presented, would not cause any additional methodological problem. In order to facilitate interpretation and analysis of properties and results, we chose to assume identical the characteristics of the trees.

Link with stand models

Such models can be linked with stand models in the following way, let the basal area of the stand $S = n.s$ and assume that $nG(n, s, t) = g(n, s)\Gamma(t)$ and that $g(n, s) = g_0(S)$ depends only on $S = n.s$. Then the dynamics of the basal area is given by :

$$\frac{dS(t)}{dt} = g_0(S(t))\Gamma(t) - (m(t) + h(t))S(t)$$

In the particular case of $m = 0$ and of the price p proportional to s , by substituting first the basal area to the volume and secondly hS to h we obtain the Clark's model (Clark, 1976 pp 263-269).

The forest income

Once chosen the model of population dynamics we can express the total income. Total income $\mathcal{V}(h(.), T)$ for fixed thinning $h(.)$ is the sum of the thinning income $H(n(.), s(.), h(.), T)$ on the period $[0, T]$ and the final income $V_0(n(T), s(T))$:

$$\mathcal{V}(h(.), T) = H(n(.), s(.), h(.), T) + V_0(n(T), s(T))$$

The thinning income $H(n(.), s(.), h(.), t)$ on $[0, t]$ actualized to time t is :

$$H(n(.), s(.), h(.), t) = \int_0^t p(s(u))h(u)n(u)e^{\delta(t-u)}du$$

where $n(.)$ and $s(.)$ are solutions of the dynamic model.

Calculation of the land value

The land value is given by :

$$W_0 = \frac{\mathcal{V}(h(.), T) - c_1}{e^{\delta T} - 1}$$

$$\text{with : } \mathcal{V}(h(.), T) = \int_0^T p(s(u))h(u)n(u)e^{\delta(T-u)}du + V_0(n(T), s(T))$$

The maximal value of the land value defined by Faustmann is obtained by solving the problem :

$$(\mathcal{P}_{F_0}) : \quad \max_{h(.), T} \frac{\mathcal{V}(h(.), T) - c_1}{e^{\delta T} - 1}$$

The maximization of the Faustmann value W_0 taking into account thinning $h(.)$ for the cutting age T can be decomposed in two steps : first we maximize $\mathcal{V}(h(.), T)$ with respect to $h(.)$:

$$(\mathcal{S}_0) \quad \mathcal{V}_0(T) = \max_{h(.)} \mathcal{V}(h(.), T)$$

then we maximize $\frac{\mathcal{V}_0(T) - c_1}{e^{\delta T} - 1}$ with respect to T with $\mathcal{V}_0(T)$ resulting from the first step.

In the presence of risk of destructive event

Following Reed (1976), we suppose that destructive events occur in a Poisson process i.e. that destructive events occur independently of one another, and randomly in time.

The distribution of the destructive event time is an exponential with mean $\frac{1}{\lambda}$: $F(x) = 1 - e^{-\lambda x}$ where λ is the expected number of destructive events per unit time. No assumption is made on the type of destructive events. Destructive event can also be a storm, a fire, a disease or an insect infestation. We assume that the event is always of the same type.

We assume that θ_t is the proportion of damaged trees following a destructive event and θ' is the rate depreciation of timber due to an influx of wood on the market. θ and θ' are random variables positively correlated : if $\theta = 0$ (no damage) then $\theta' = 0$ and more θ is high, more θ' is also. Depending on the type of risk, θ_t and thus θ'_t may be conditionally distributed to $n(\cdot)$, $s(\cdot)$, to thinning $h(\cdot)$ or to another control parameter. We define the expectations α of $(1 - \theta_t)$ and α_p of $(1 - \theta_t)(1 - \theta'_t)$, due to the assumptions on the random variables θ_t and θ'_t we deduce $\alpha_p \leq \alpha$. In order to lighten the presentation we omit the $n(\cdot)$, $s(\cdot)$, $h(\cdot)$ dependancy of α and α_p . The modelling is suitable for different types of destructive risk : the specificity is reflected in the choice of random variables θ , θ' distribution laws and therefore of functions α and α_p .

For a cutting age T and a fixed thinning $h(\cdot)$, the land value W_0 is the actualized value at initial time of the sum of two terms. The first one is the expectation (with respect to the time of event) of the sum of the land value and the expectation (with respect to θ_t and θ'_t) of the total income $\mathcal{V}_1(h(\cdot), t)$ for the period $[0, t]$ minus clearing costs $\mathcal{C}(n(t), t)$ in case of a destructive event at time t . The second one is the sum of the land value and the total income $\mathcal{V}(h(\cdot), T)$ for the period $[0, T]$ in case of no destructive event. Then W_0 is the solution of the following equation :

$$W_0 = \int_0^T [W_0 + \mathcal{V}_1(h(\cdot), t) - c_1 - \mathcal{C}(n(t), t)]e^{-\delta t} dF(t) + (W_0 + \mathcal{V}(h(\cdot), T) - c_1)e^{-\delta T}(1 - F(T)) \quad (1)$$

where $\mathcal{V}_1(h(\cdot), t) = E(\mathcal{V}(h(\cdot), \theta, \theta', t))$ with $\mathcal{V}(h(\cdot), \theta, \theta', t)$ the total income for θ and θ' , $\mathcal{C}(n(t), t) = E(\mathcal{C}_n(\theta_t, n(t)))$ with $\mathcal{C}_n(\theta_t, n(t))$ the clearing costs for θ_t .

The forest income

$\mathcal{V}(h(\cdot), \theta, \theta', t)$ is the sum of the thinning income $H(n(\cdot), s(\cdot), h(\cdot), t)$ during $[0, t]$ actualized at time t and the total income $\mathcal{V}_F(\theta, \theta', t)$ at time t :

$$\mathcal{V}(h(\cdot), \theta, \theta', t) = H(n(\cdot), s(\cdot), h(\cdot), t) + \mathcal{V}_F(\theta, \theta', t)$$

The final income $\mathcal{V}_F(\theta, \theta', t)$ is assumed proportional to the final income without risk of destructive event $V_0(n(T), s(T))$ and is given by : $\mathcal{V}_F(\theta, \theta', t) = (1 - \theta)(1 - \theta')V_0(n(t), s(t))$. From definition of α and α_p we deduce the total income expectation :

$$\mathcal{V}_1(h(\cdot), t) = E(\mathcal{V}(h(\cdot), \theta, \theta', t)) = H(n(\cdot), s(\cdot), h(\cdot), t) + \alpha_p(t)V_0(n(t), s(t))$$

In Reed (1976), the total income generated in case of risk of destructive event is assumed proportional to the income generated in the case without risk, which is not longer true if one wants to take the thinning into account.

The clearing costs

The clearing costs $\mathcal{C}_n$ are of two types : the first one concerns the damaged trees $\theta_t n(t)$ and the second one concerns the survival trees $(1 - \theta_t)n(t)$. The respective contribution of each type of cost depends on the type of destructive event. Assuming a linear dependency (Amacher et al., 2008), we then deduce the expression of the clearing costs :

$$\mathcal{C}_n(\theta_t, n(t)) = c_2 + c_d \theta_t n(t) + c_s (1 - \theta_t) n(t)$$

From the definition of α , we deduce the clearing costs expectation :

$$\mathcal{C}(n(t), t) = E(\mathcal{C}_n(\theta_t, n(t))) = c_2 + c_n(t) n(t)$$

where c_n is defined by $c_n(t) = c_d(1 - \alpha(t)) + c_s \alpha(t)$. Thus, $\min(c_d, c_s) \leq c_n(t) \leq \max(c_d, c_s)$.

Except for the fire risk (Amacher et al., 1976), the clearing costs mainly concerns the damaged trees.

Remark : Reed (1976) assumed that clearing costs were independent of the number of damaged trees. Using a model of population dynamics allows us to take into account the clearing costs proportional to the damage.

The salvageable function α

The salvageable function α dependency is specific of the type of risk of destructive event. Hence :

- for a storm risk, following Hanewinkel (2008), α is a decreasing function of the ratio height-diameter $\frac{H}{d}$ (where d is the tree-diameter $d = \sqrt{\frac{4s}{\pi}}$) and the time t : then α can be written as $\alpha(\frac{H(t)}{\sqrt{s(t)}}, t)$. More generally α is a function of $n(t)$ and $s(t)$.

- for ice damage risk, following Goodnow et al. (2008), α depends on the planting density n_0 and on the rate of thinning $h(\cdot)$.

- for a fire risk, following Amacher et al. (2008), α depends on the planting density n_0 and on the level of preventive intermediate treatment effort $z(\cdot)$. In that case, $z(\cdot)$ is an additional control parameter.

Calculation of the land value

From (1) and expression of $\mathcal{V}_1(h(\cdot), t)$ we deduce the land value :

Proposition 1 : *In the presence of the risk of destructive event the land value is given by :*

$$W_0 = \frac{\delta + \lambda}{\delta} \frac{\tilde{\mathcal{V}}_1(h(\cdot), T) - c_1}{e^{(\delta+\lambda)T} - 1} - \frac{\lambda}{\delta}(c_1 + c_2)$$

$$\text{where : } \tilde{\mathcal{V}}_1(h(\cdot), T) = \int_0^T [p(s(t))h(t)n(t) + \lambda\alpha_p(t)V_0(n(t), s(t)) - \lambda c_n(t)n(t)]e^{(\delta+\lambda)(T-t)}dt + V_0(n(T), s(T)).$$

Proof : From (1) and expressions of $\mathcal{V}(h(\cdot), t)$ and $\mathcal{V}_1(h(\cdot), t)$:

$$\begin{aligned} W_0 = \int_0^T [W_0 + H(n(\cdot), s(\cdot), h(\cdot), t) + \alpha_p(t)V_0(n(t), s(t)) - c_1 - c_2 - c_n(t)n(t)]e^{-\delta t}dF(t) \\ + (W_0 + H(n(\cdot), s(\cdot), h(\cdot), T) + V_0(n(T), s(T)) - c_1)e^{-\delta T}(1 - F(T)) \end{aligned}$$

We then deduce the land value :

$$W_0 = \frac{\delta + \lambda}{\delta} \frac{J_0(n(\cdot), s(\cdot), h(\cdot), T) + V_0(n(T), s(T)) - c_1}{e^{(\delta+\lambda)T} - 1} - \frac{\lambda}{\delta}(c_1 + c_2)$$

where $J_0(n(\cdot), s(\cdot), h(\cdot), T)$ has the following value :

$$\int_0^T [H(n(\cdot), s(\cdot), h(\cdot), t) + \alpha_p(t)V_0(n(t), s(t)) - c_n(t)n(t)]e^{\delta(T-t)}de^{\lambda(T-t)} + H(n(\cdot), s(\cdot), h(\cdot), T).$$

After changing the integration order in $\int_0^T H(n(\cdot), s(\cdot), h(\cdot), t)e^{\delta(T-t)}de^{\lambda(T-t)}$, J_0 becomes :

$$J_0(n(\cdot), s(\cdot), h(\cdot), T) = \int_0^T [p(s(t))h(t)n(t) + \lambda\alpha_p(t)V_0(n(t), s(t)) - \lambda c_n(t)n(t)]e^{(\delta+\lambda)(T-t)}dt$$

then the result. □

As in the case without risk, we find that the land value can be deduced from the income $\tilde{\mathcal{V}}_1(h(\cdot), T)$. The following differences occur : first the rate of interest δ is replaced by $\delta + \lambda$, secondly $\mathcal{V}(h(\cdot), T)$ is replaced by a modified expression of the income $\tilde{\mathcal{V}}_1(h(\cdot), T)$. The second difference is reflected by the substitution in the case without risk of term $H(n(\cdot), s(\cdot), h(\cdot), T)$ by the term $J_0(n(\cdot), s(\cdot), h(\cdot), T)$, or even more precisely, the substitution of $p(s(t))h(t)n(t)$ by $p(s(t))h(t)n(t) + \lambda(\alpha_p(t)V_0(n(t), s(t)) - c_n(t)n(t))$.

The maximal value of the land value is obtained by solving :

$$(\mathcal{P}_{F_1}) : \max_{h(\cdot), T} W_0 = \frac{\delta + \lambda}{\delta} \frac{\tilde{\mathcal{V}}_1(h(\cdot), T) - c_1}{e^{(\delta+\lambda)T} - 1} - \frac{\lambda}{\delta}(c_1 + c_2)$$

As in the case without risk, the maximal value of the land value with respect to the thinning $h(\cdot)$ and the cutting age T can be decomposed in two steps : first we maximize $\tilde{\mathcal{V}}_1(h(\cdot), T)$ with respect to $h(\cdot)$:

$$(\mathcal{S}_1) \quad \mathcal{V}_1(T) = \max_{h(\cdot)} \tilde{\mathcal{V}}_1(h(\cdot), T)$$

then we maximize $\frac{\delta + \lambda}{\delta} \frac{\mathcal{V}_1(T) - c_1}{e^{(\delta+\lambda)T} - 1} - \frac{\lambda}{\delta}(c_1 + c_2)$ with respect to T with $\mathcal{V}_1(T)$ resulting from the first step.

Silviculture for a fixed cutting age T and a no density dependent growth

We consider first the limiting case of no density dependent growth which will be used as a reference in the case study of density dependent growth.

Without risk of destructive event

Let us consider the case where individual growth is not density dependent. In this case the evolution of the tree-basal area s does not depend on the tree-number n , then does not depend on the silviculture.

We study the maximization of the land value with respect to thinning $h(\cdot)$ for a fixed cutting age T :

$$(\mathcal{S}_0) \quad \max_{h(\cdot)} \mathcal{V}(h(\cdot), T) = \int_0^T p(s(t))h(t)n(t)e^{\delta(T-t)}dt + V_0(n(T), s(T))$$

with the constraint $0 \leq h(t) \leq \bar{h}$.

Assume that the final income is given by : $V_0(n, s) = p(s)n$. From the no density dependence of the individual growth, $p(s(t))$ is independent of the thinnings and only depends on t . Then we define $R(t) = p(s(t))$. We denote the functions π_0 and Π_0 : $\pi_0(t) = R'(t) - (\delta + m(t))R(t)$ and $\Pi_0(t) = \int_t^T e^{\int_u^T (\delta + m(u') + \bar{h})du'} \pi_0(u)du$. Applying the maximum Pontryagin Principle to the problem $\mathcal{P}_0$ (see Appendix A) we can deduce the proposition :

Proposition 2 : Assume π_0 is decreasing, the individual tree-growth is not density dependent. Consider a fixed cutting age T , then the optimal thinnings are given by :

- if $\pi_0(T) \geq 0$ then $h_* \equiv 0$

- if $\pi_0(T) < 0$ then it exists $0 \leq t_* < T$ such that $h_*(t) = 0$ for $t < t_*$ and $h(t) = \bar{h}$ pour $t > t_*$. Moreover if $\Pi_0(0) > 0$ then $t_* = 0$, else t_* is the unique solution of :

$$\Pi_0(t_*) = \int_{t_*}^T e^{\int_u^T (\delta + m(u') + \bar{h})du'} \pi_0(u)du = 0$$

Remark : The commutation time t_* is a function of the cutting age T .

In the presence of risk of destructive event

We study the maximization of the land value with respect to thinning $h(\cdot)$ for a fixed cutting age T :

$$(\mathcal{S}_1) \quad \max_{h(\cdot)} \tilde{\mathcal{V}}_1(h(\cdot), T) = J_0(n(\cdot), s(\cdot), h(\cdot), T) + V_0(n(T), s(T))$$

with the constraint $0 \leq h \leq \bar{h}$.

In this paragraph, we consider a function α depending only of time t . The chosen function α can be interpreted as the expectation with respect to the state variables of a more precise function depending on the state variables.

In case of a storm risk, α is a function of the ratio height-diameter and time t . For a no density dependent growth, the tree-basal area s and hence the diameter is an explicit function of time t , then α depends only on t . In this case, the approximate function is exact.

Due to the difference in the criterion without and with risk, we deduce that, for the fixed rotation period T , the silviculture differs and depends explicitly on λ , α and α_p . So we will pay attention to the consequence for the silvicultural practice.

We consider, as in the case without risk, a no density dependent growth for the trees to facilitate the comparaison.

Let denote the functions π_λ and Π_λ : $\pi_\lambda(t) = R'(t) - (\lambda(1 - \alpha_p(t)) + \delta + m(t))R(t) - \lambda c_n(t)$ and $\Pi_\lambda(t) = \int_t^T e^{\int_u^T (\lambda + \delta + m(u') + \bar{h}) du'} \pi_\lambda(u) du$. Applying the maximum Pontryagin Principle to the problem $\mathcal{P}_\lambda$ (see appendix A) we deduce the proposition :

Proposition 3 : Assume π_λ is decreasing, the individual tree-growth is not density dependent. Consider a fixed cutting age T , then the optimal thinnings are given by :

- if $\pi_\lambda(T) \geq 0$ then $h_* \equiv 0$

- if $\pi_\lambda(T) < 0$ then it exists $0 \leq t_* < T$ such that $h_*(t) = 0$ for $t < t_*$ and $h(t) = \bar{h}$ for $t > t_*$.

Moreover if $\Pi_\lambda(0) > 0$ then $t_* = 0$, else t_* is the unique solution of :

$$\Pi_\lambda(t_*) = \int_{t_*}^T e^{\int_u^T (\delta + \lambda + m(u') + \bar{h}) du'} \pi_\lambda(u) du = 0$$

Comparaison : without and with presence of risk

By comparing the without and with risk criteria, we remark that the rate of interest δ is replaced by $\delta + \lambda$

and the rate of thinning $h(t)$ is replaced by $h(t) + \lambda(\alpha_p(t) - \frac{c_n(t)}{p(s(t))})$ if V_0 is defined by $V_0(n, s) = p(s)n$.

Moreover, from the definition of π_λ , we deduce that π_λ is a decreasing function of λ . Thus, at least, in the vicinity of the cutting age T , the greater λ , the more frequently $h(T)$ will be equal to $\bar{h}$ in the following sense : for $\lambda_1 < \lambda_2$, with the associated thinnings h_1 and h_2 , if $h_1(T) = \bar{h}$ then $h_2(T) = \bar{h}$.

By comparing the two propositions, without considering the clearing costs, the natural mortality $m(t)$ in the case without risk is replaced by the mortality due to events $m(t) + \lambda(1 - \alpha_p(t))$ in the presence of risk. It is equivalent also, from a mathematical point of view, to replace the fixed discount rate δ by the variable discount rate $\delta + \lambda(1 - \alpha_p(t))$ in the previous problem.

By comparing the results of the two propositions we deduce that, for a fixed rotation period T , it is usually best to do thinning at least at the end of the period in the presence of risk. This comparison confirms the previous property deduced from the definition of π_λ .

Comparing the results of the two proposals is permitted if the rotation periods are identical. If we consider the maximization problem, with respect to the rotation period, the rotation periods have no reason to be the same. In that case the comparison is not permitted and only simulations can allow us to compare the respective thinning. We will therefore perform simulations.

Silviculture for a density dependent growth

We now consider the case where individual growth is density dependent. If the growth is weakly density dependent, by continuity with the case of no density dependent growth, the obtained results are still valid at fixed rotation period T . For a greater density dependence, if $(G(n, s, t)n)'_n$ is sufficiently small (see Appendix B), the optimal thinnings are the same as in the previous case in the vicinity of T . If this is not the case, we cannot obtain analytical results for the solutions, then simulations are required.

We are interested in a stand of *Eucalyptus*. In the absence of more precise information on the structure of the salvageable function α for a *Eucalyptus* stand, we restricted our analysis to α constant. The function of individual growth is given by :

$$G(n, s, t) = \frac{0.7445(1 - e^{-0.482ns})}{n} \frac{dH(t)}{dt}$$

where $H(t)$ is the high at time t : $H(t) = H_0(1 - e^{-\frac{t}{H_0}})$ and $H_0 = 30$ the limited high which depends mainly on soil fertility (Saint André et al., 2002). The structure of the growth function is generic and can be used for other species.

For the clearing costs, we neglected the clearing costs of the second type ($c_n(t) = c_d(1 - \alpha(t))$).

The weight of the trees of basal area s and high h is given by : $v(s, H, t) = 0.29 + (127.8 + 0.32t)sH$ in kg (Saint André et al., 2005). The price is assumed to depend on the weight : $p(s, t) = 0.1v(s, H(t), t) - 0.25$.

The determination of the cutting age T is important because of its impact on silviculture. To better describe the silviculture in the presence of random risk, it is wiser to look at the effective cutting age $\mathcal{T}$ and the effective final tree-basal area $\mathcal{S}$. Thus we calculate the respective expectations and variances :

$$E(\mathcal{T}) = \int_0^T t dF(t) + T(1 - F(T)) = \frac{F(T)}{\lambda}$$

$$Var(\mathcal{T}) = \int_0^T (t - \frac{F(T)}{\lambda})^2 dF(t) + (T - \frac{F(T)}{\lambda})^2 (1 - F(T)) = \frac{2}{\lambda^2} (1 - F(T)) (F(T) - \lambda T) + \frac{F^2(T)}{\lambda^2}$$

$E(\mathcal{S}) = \int_0^T s(t) dF(t) + s(T)(1 - F(T))$ and $Var(\mathcal{S}) = \int_0^T s^2(t) dF(t) + s^2(T)(1 - F(T))$ can be derived from the simulations.

Results and Discussion

The unit of time for the rotation period is the month. We suppose : $m = 0.0042 \text{ month}^{-1}$, $\lambda = 0.0075 \text{ month}^{-1}$, $\delta = 0.0034 \text{ month}^{-1}$, $\bar{h} = 0.075 \text{ month}^{-1}$.

Assume first, that in case of destructive event, the destruction of stand is total ($\alpha = 0, \alpha_p = 0$). With the constraint of no thinning, we found (Table 1) the classical well established result : risk implies a shortening of the optimum rotation period. By considering the optimal thinning, the optimal rotation period is larger than in the case without risk but the expected effective rotation period is smaller and remains of the same order of magnitude as in the case without risk. This can be explained by the fact that, in the presence of risk, the optimal rotation period T is achieved with a relatively low probability : $1 - F(T) = e^{-\lambda T}$. To complete the study, the standard deviation of the effective rotation period was calculated. Its value doesn't vary and is about 20 months. The expected effective final tree-basal area varies only slightly depending on the scenarios.

If the destruction is only partial i.e. a portion of the stumpage is salvageable ($\alpha = 0.6, \alpha_p = 0.4$), without risk and for the two considered tree-density ($n = 650$ or 1650 stems/ha), there is no thinning for the optimal solution (Table 2). In presence of risk, with the constraint of no thinning, the optimal rotation period is close to the previous one. In contrast, if we optimize allowing thinning, the optimal rotation period is greater and thinnings are to be done. Similarly to the previous case of total destruction, the expected effective rotation period remains of the same magnitude as in the case without risk.

The initial tree density in the studied range did not influence the qualitative behavior of optimal management. Comparing Table 1 and Table 2 for a density of 650 stems/ha, we remark that, the greater the potential damage, the greater the cutting age will be and the earlier the beginning of the preventive thinning. From Table 2, we deduced that the greater the initial tree-density, the later the beginning of the preventive thinning will be and the lower the difference between the land value in presence of risk without and with thinning. In Table 3, for the commonly used cutting age value $T = 84$ months, we found similar properties for the different optimizations.

The presence of risk of partial or total destruction involves earlier thinning. Because of early thinning the amount of standing trees is smaller at time T . Thus, because of thinning, the rotation period T , can be extended and greater than the rotation period without risk. The earlier thinning provides a kind of self-insurance against risk. We therefore make endogenous the risk through optimization.

Hyytiäinen and Tahvonen (2003) showed, in another context under certainty, that a greater rate of interest may lengthen the optimal rotation for non optimal initial density. The density of 650 stem/ha is not optimal. We have showed that the risk involves in particular the substitution of the rate of interest by the rate of interest plus the expected number of events λ , the consequence being an increase of the rate of interest. Thus our result is consistent with the results of Hyytiäinen and Tahvonen. By observing the curves of the land value W_0 (for a density of 650 stems/ha and $\alpha = 0.6$, $\alpha_p = 0.4$), depending on the rotation problem T with optimal thinning, we find that the land value least varies in the vicinity of the optimal rotation period with risk (Figure 2) than without risk (Figure 1). This is another consequence of the fact that, with risk the optimal rotation period T is achieved with a relatively low probability.

We consider the case where the rotation period is determined by other considerations. We take the commonly used value $T = 84$ months (Table 3). Without and with risk, it is optimal to practice thinning. However, in the presence of risk, optimal thinning starts earlier.

The results obtained in Reed (1976) are valid only under the following assumptions : the manager does not practice thinning and the clearing costs in case of destructive event are fixed. The possibility to consider thinning and clearing costs depends on damage severity and therefore justifies the interest to introduce a model of population dynamics.

Conclusion

We have studied the management of a stand in the presence of risk of destructive event. In order to determine the optimal thinning relative to the Faustmann criterion, we have considered a model of population dynamics, the chosen model is of average tree type. This model has allowed us to make the comparison without and with risk and highlighted the influence of the presence of risk of destructive event on optimal thinning.

Specifically, the obtained land values, without or with the risk of destructive event, highlighted differences in the criteria to be maximized. In the case of no density dependent individual growth, we have highlighted the impact of the presence of risk on the strategies, generically regardless of the considered species.

In the case of density dependent growth, the calculations for a stand of *Eucalyptus* have shown that the presence of risk of destruction event involves earlier thinning and a greater rotation period, for the optimal strategy.

The obtained results are conditioned by the choice of an individual tree growth model and by the specification of a weight model and a price model of trees. Other studies using models adapted for other species would make the obtained results more generic.

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Appendix A. Solving the optimisation problem ($\mathcal{P}_\lambda$) with a no density dependent growth

We consider, for a fixed rotation period T , the following problem ($\mathcal{P}_\lambda$) :

$$\max_{h(\cdot)} \int_0^T [R(t)h(t) + \lambda(\alpha_p(t)R(t) - c_n(t))]n(t)e^{(\delta+\lambda)(T-t)}dt + R(T)n(T)$$

with the constraint $0 \leq h(t) \leq \bar{h}$.

We apply the maximum Pontryagin Principle, the Hamiltonian is :

$$H = [R(t)h + \lambda(\alpha_p(t)R(t) - c_n(t))]ne^{(\delta+\lambda)(T-t)} - \mu(m(t) + h)n$$

The first order conditions are :

$$h^*(t) \text{ maximizes } [R(t)e^{(\delta+\lambda)(T-t)} - \mu(t)]hn \text{ with } h \in [0, \bar{h}]$$

$$\frac{d\mu(t)}{dt} = -\frac{\partial H}{\partial n} = -[R(t)h(t) + \lambda(\alpha_p(t)R(t) - c_n(t))]e^{(\delta+\lambda)(T-t)} + \mu(t)(m(t) + h(t))$$

with the transversality condition $\mu(T) = R(T)$

We consider the evolution of the function $l(t)$ defined by : $l(t) = R(t)e^{(\delta+\lambda)(T-t)} - \mu(t)$.

$$\frac{dl(t)}{dt} = (R'(t) - (\delta + \lambda(1 - \alpha_p(t)) + m(t))R(t) - \lambda c_n(t))e^{(\delta+\lambda)(T-t)} + l(t)(m(t) + h(t))$$

and define the function π and Π by : $\pi(t) = R'(t) - (\delta + \lambda(1 - \alpha_p(t)) + m(t))R(t) - \lambda c_n(t)$ and $\Pi(t) = \int_t^T e^{\int_u^T (\delta+\lambda+m(u')+\bar{h})du'} \pi(u)du$,

We then deduce the following Proposition :

Proposition A.1 : Assume π is decreasing, the individual tree-growth is not density dependent. Consider a fixed cutting age T , then the optimal thinnings are given by :

- if $\pi(T) \geq 0$ then $h_* \equiv 0$

- if $\pi(T) < 0$ then il existe $0 \leq t_* < T$ such that $h_*(t) = 0$ for $t < t_*$ and $h(t) = \bar{h}$ for $t > t_*$. Moreover if $\Pi(0) > 0$ then $t_* = 0$, else t_* is the unique solution of :

$$\Pi(t_*) = \int_{t_*}^T e^{\int_u^T (\delta+\lambda+m(u')+\bar{h})du'} \pi(u)du = 0$$

Proof : From $l(T) = 0$ and considering the equation in backward time $t' = T - t$, we deduce that in the vicinity of T , $\frac{dl(t')}{dt'}$ and $\pi(t)$ are of opposit sign, then $\frac{dl(t)}{dt}$ and $\pi(t)$ have the same sign : hence $l(t) < 0$ (resp. > 0) if $\pi(t) \geq 0$ (resp. < 0). If $\pi(T_*) \geq 0$, $\pi(t) > 0$ for all t then $l(t) < 0$ for all t which implies $h(t) = 0$. If $\pi(T_*) < 0$, from $l(t) = -e^{-\int_{t_*}^T (m(u)+\bar{h})du} \Pi(t)$ we deduce that either $\Pi(0) \leq 0$, $l(t) < 0$ and $l(t)$ cannot change of sign or $\Pi(0) > 0$ and $l(t)$ changes of sign for a unique time t_* .

Appendix B. Solving the optimisation problem $(\mathcal{P}_\lambda)$ with a density dependent growth

We consider, for a fixed rotation period T , the following problem $(\mathcal{P}_\lambda)$:

$$\max_{h(\cdot)} \int_0^T [p(s(t))h(t) + \lambda(\alpha_p(t)p(s(t)) - c_n(t))]n(t)e^{(\delta+\lambda)(T-t)}dt + p(s(T))n(T)$$

with the constraint $0 \leq h(t) \leq \bar{h}$.

We apply the maximum Pontryagin Principle, the Hamiltonian is :

$$H = [p(s)h + \lambda(\alpha_p(t)p(s) - c_n(t))]ne^{(\delta+\lambda)(T-t)} - \mu_n(m(t) + h)n + \mu_s G(n, s)$$

The first order conditions are :

$$\begin{aligned} h^*(t) \text{ maximizes } [p(s(t))e^{(\delta+\lambda)(T-t)} - \mu(t)]hn \text{ with } h \in [0, \bar{h}] \\ \frac{d\mu_n(t)}{dt} = -\frac{\partial H}{\partial n} = -[p(s(t))h(t) + \lambda(\alpha_p(t)p(s(t)) - c_n(t))]e^{(\delta+\lambda)(T-t)} + \mu_n(t)(m(t) + h(t)) \\ - \mu_s(t)G_n(n(t), s(t), t) \\ \frac{d\mu_s(t)}{dt} = -\frac{\partial H}{\partial s} = -(p'(s(t))h(t) + \lambda\alpha_p(t)p'(s(t)))n(t)e^{(\delta+\lambda)(T-t)} - \mu_s(t)G_s(n(t), s(t), t) \end{aligned}$$

with the transversality conditions $\mu_n(T) = p(s(T))$ and $\mu_s(T) = p'(s(T))n(T)$.

We consider the evolution of the function $l(t)$ defined by : $l(t) = p(s(t))e^{(\delta+\lambda)(T-t)} - \mu_n(t)$.

$$\begin{aligned} \frac{dl(t)}{dt} = (p'(s(t))G(n(t), s(t)) - (\delta + \lambda(1 - \alpha_p(t)) + m(t))p(s(t)) - \lambda c_n(t))e^{(\delta+\lambda)(T-t)} \\ + \mu_s(t)G_n(n(t), s(t), t) + l(t)(m(t) + h(t)) \end{aligned}$$

which allow us to give the following result :

Proposition A.2 : *If $p(s) = Cs^a$ and $a(G(n, s, t)n)'_n \leq (\delta + \lambda(1 - \alpha_p(T)) + m(T))s(0)$ then the thinning $h(t) = \bar{h}$ is optimal in the vicinity of the cutting age T .*

Proof : From $l(T) = 0$ we deduce : $\frac{dl}{dt}(T) = -(\delta + \lambda(1 - \alpha_p(T)) + m(T))p(s(T)) - \lambda c_n(T) + p'(s(T))(G(n, s, t)n)'_n(T)$. From given conditions, $\frac{dl}{dt}(T) < 0$ then $l(t) > 0$ in the vicinity of T .

Table 1 : Optimized land value with respect to T . $\alpha = 0$, $\alpha_p = 0$.

Scenario	Optimal thinnings	Cutting age	Land value	Expected effective cutting age
	(month ⁻¹)	(month)	(euro)	(month)
650 stems/ha				
Without risk				
$\max_{h(\cdot), T}$	$h \equiv 0$	58.5	2137.5	58.5
With risk				
no thinning		58.5	658.1	47.3
$\max_T$, no thinning		54	673.3	44.4
$\max_{h(\cdot), T}$	$h(t) = \bar{h}, t \geq 36.5$	69.5	810.4	54.2

Table 2 : Optimized land value with respect to T . $\alpha = 0.6$, $\alpha_p = 0.4$.

Scenario	Optimal thinnings	Cutting age	Land value	Expected effective cutting age
	(month ⁻¹)	(month)	(euro)	(month)
650 stems/ha				
Without risk				
$\max_{h(\cdot), T}$	$h \equiv 0$	58.5	2137.5	58.5
With risk				
no thinning		58.5	1108.9	47.3
$\max_T$, no thinning		57.5	1109.8	46.7
$\max_{h(\cdot), T}$	$h(t) = \bar{h}, t \geq 43.5$	65.5	1149.0	51.7
1650 stems/ha				
Without risk				
$\max_{h(\cdot), T}$	$h \equiv 0$	59.5	2497.2	59.5
With risk				
no thinning		59.5	1230.7	48.0
$\max_T$, no thinning		58.5	1232.1	47.3
$\max_{h(\cdot), T}$	$h(t) = \bar{h}, t \geq 46.5$	64.5	1251.1	51.1

Table 3 : Land value for $T = 84$ months.

Scenario	Optimal thinnings (month ⁻¹)	Cutting age (month)	Land value (euro)	Expected effective cutting age (month)
650 stems/ha				
Without risk				
$\max_{h(.)}$	$h(t) = \bar{h}, t \geq 60$	84	1914.1	84
With risk				
$h(t) = \bar{h}, t \geq 60$		84	1018.4	62.3
$\max_{h(.)}$	$h(t), t \geq 43.5$	84	1099.3	62.3
1650 stems/ha				
Without risk				
$\max_{h(.)}$	$h(t) = \bar{h}, t \geq 64.5$	84	2224.0	84
With risk				
$h(t) = \bar{h}, t \geq 64.5$		84	1099.3	62.3
$\max_{h(.)}$	$h(t) = \bar{h}, t \geq 48.5$	84	1194.9	62.3

Fig 1. Without risk case. The abscissa shows the rotation period T in months, the ordinate the land value

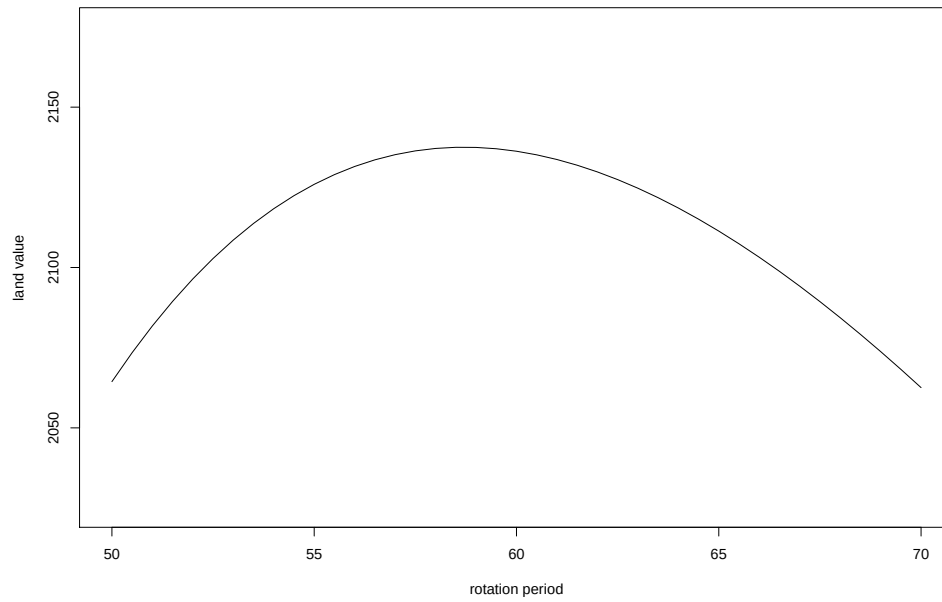


Fig 2. With risk case. The abscissa shows the rotation period T in months, the ordinate the land value

