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## Multiple zeta value cycles in low weight

Ismaël Soudères

ABSTRACT. In a recent work, the author has constructed two families of algebraic cycles in Bloch's cycle algebra over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  that are expected to correspond to multiple polylogarithms in one variable and have a good specialization at 1 related to multiple zeta values.

This is a short presentation, by the way of toy examples in low weight ( $\leq 5$ ), of this construction and could serve as an introduction to the general setting. Working in low weight also makes it possible to push ("by hand") the construction further. In particular, we will not only detail the construction of the cycles but we will also associate to these cycles explicit elements in the bar construction over the cycle algebra and make as explicit as possible the "bottom-left" coefficient of the Hodge realization period matrix. That is, in a few relevant cases we will associate to each cycle an integral showing how the specialization at 1 is related to multiple zeta values. We will be particularly interested in a new weight 3 example corresponding to  $-\zeta(2, 1)$ .

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## 1. Introduction

The multiple polylogarithm functions were defined in [Gon95] by the power series

$$Li_{k_1, \dots, k_m}(z_1, \dots, z_m) = \sum_{n_1 > \dots > n_m > 0} \frac{z_1^{n_1}}{n_1^{k_1}} \frac{z_2^{n_2}}{n_2^{k_2}} \dots \frac{z_m^{n_m}}{n_m^{k_m}} \quad (z_i \in \mathbb{C}, |z_i| < 1).$$

They admit an analytic continuation to a Zariski open subset of  $\mathbb{C}^m$ . The case  $m = 1$  is nothing but the classical *polylogarithm* functions. The case  $z_1 = z$  and  $z_2 = \dots = z_m = 1$  gives a one variable version of multiple polylogarithm functions

$$Li_{k_1, \dots, k_m}^{\mathbb{C}}(z) = Li_{k_1, \dots, k_m}(z, 1, \dots, 1) = \sum_{n_1 > \dots > n_m > 0} \frac{z^{n_1}}{n_1^{k_1} n_2^{k_2} \dots n_m^{k_m}}.$$

When  $k_1$  is greater or equal to 2, the series converge as  $z$  goes to 1 and one recovers the multiple zeta value

$$\zeta(k_1, \dots, k_m) = Li_{k_1, \dots, k_m}^{\mathbb{C}}(1) = Li_{k_1, \dots, k_m}(1, \dots, 1) = \sum_{n_1 > \dots > n_m > 0} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_m^{k_m}}.$$

To the tuple of integers  $(k_1, \dots, k_m)$  of weight  $n = \sum k_i$ , we can associate a tuple of 0's and 1's

$$(\varepsilon_n, \dots, \varepsilon_1) := \left( \underbrace{0, \dots, 0}_{k_1 - 1 \text{ times}}, 1, \dots, \underbrace{0, \dots, 0}_{k_m - 1 \text{ times}}, 1 \right)$$

which allows to write multiple polylogarithms as iterated integrals ( $z_i \neq 0$  for all  $i$ ):

$$Li_{k_1, \dots, k_m}^{\gamma}(z_1, \dots, z_m) = (-1)^m \int_{\Delta_{\gamma}} \frac{dt_1}{t_1 - \varepsilon_1 x_1} \wedge \dots \wedge \frac{dt_n}{t_n - \varepsilon_n x_n}$$

where  $\gamma$  is a path from 0 to 1 in  $\mathbb{C} \setminus \{x_1, \dots, x_n\}$ , the integration domain  $\Delta_{\gamma}$  is the associated real simplex consisting of all  $n$ -tuples of points  $(\gamma(t_1), \dots, \gamma(t_n))$  with  $t_i < t_j$  for  $i < j$  and where we have set  $x_n = z_1^{-1}$ ,  $x_1 = (z_1 \dots z_m)^{-1}$  and where, for all  $i$  such that  $k_1 + \dots + k_{l-1} + 1 \leq i < k_1 + \dots + k_l$ , we have set  $x_{n-i} = (z_1 \dots z_l)^{-1}$ . Classically,  $\gamma$  is the straight path from 0 to 1 :  $\gamma(t) = t$  and in this case the superscript will be omitted.

Bloch and Kriz in [BK94] have constructed an algebraic cycle avatar of the classical polylogarithm function. More recently in [GGL09], Gangl, Goncharov and Levin, using a combinatorial approach, have built algebraic cycles corresponding to the multiple polylogarithm values  $Li_{k_1, \dots, k_m}(z_1, \dots, z_m)$  with parameters  $z_i$  satisfying in particular that all the  $z_i$  but  $z_1$  have to be different from 1 and their methods do not give algebraic cycles corresponding to multiple zeta values.

The goal of the article [Sou12] was to develop a geometric construction for multiple polylogarithm cycles removing the previous obstruction which will allow to have multiple zeta cycles.

A general idea underlying this project consists of looking for cycles fibered over a larger base and not just point-wise cycles for some fixed parameter  $(z_1, \dots, z_m)$ . Levine in [Lev11] shows that there exists a short exact sequence relating the Bloch-Kriz Hopf algebra over  $\text{Spec}(\mathbb{Q})$ , its relative version over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  and the Hopf algebra associated to Goncharov and Deligne's motivic fundamental group over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  which contains motivic avatars of iterated integrals associated to the multiple polylogarithms in one variable.

As this one variable version of multiple polylogarithms gives multiple zeta values for  $z = 1$ , it is natural to investigate first the case of the Bloch-Kriz construction over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  in order to obtain algebraic cycles corresponding to multiple polylogarithms in one variable with a “good specialization” at 1.

This paper presents the main geometric tools in order to construct such algebraic cycles and applies the general construction described in [Sou12] to concrete examples up to weight 5. In these particular cases, one can easily go further in the description, lifting the obtained cycles to the bar constructions over the Bloch’s cycle algebra, describing the corresponding Bloch-Kriz motive and computing some associated integrals related to the Hodge realization. Those integrals give back multiple polylogarithms in one variable and their specialization at 1 give multiple zeta values.

The structure of the paper is organized as follows. In section 2 we review shortly the combinatorial context as it provides interesting relations for the bar elements associated to the cycles and an interesting relation with Goncharov’s motivic coproduct for motivic iterated integrals. Section 3 is devoted to the geometric situation and to the construction of the cycles after a presentation of the Bloch’s cycle algebra. Section 4, presents a combinatorial representation of the constructed cycles as parametrized cycles.

Section 5 recalls the definition of the bar construction over a commutative differential graded algebra and associates elements in the bar constructions (and a corresponding motive in the Bloch-Kriz construction) to the low weight examples of cycles. Finally in section 6, I follow Gangl, Goncharov and Levin’s algorithm associating an integral to some of the low weight algebraic cycles previously described.

## 2. Combinatorial situation

In this paper a *tree* is a planar finite tree whose internal vertices have valency  $\geq 3$  and where at each vertex a cyclic ordering of the incident edges is given. A *rooted tree* has a distinguished external vertex called the *root* and a *forest* is a disjoint union of trees.

Trees will be drawn with the convention that the cyclic ordering of the edges around an internal vertex is displayed in counterclockwise direction. The root vertex in the case of a rooted tree is displayed at the top.

**2.1. Trees, Lie algebras and Lyndon words.** Let  $\mathcal{T}^{tri}$  be the  $\mathbb{Q}$ -vector space generated by rooted trivalent trees with leaves decorated by 0 and 1 modulo the relation

$$\begin{array}{c} T_1 \\ \diagdown \quad \diagup \\ T_2 \quad T_3 \end{array} = - \begin{array}{c} T_1 \\ \diagup \quad \diagdown \\ T_3 \quad T_2 \end{array}$$

where the  $T_i$ ’s are subtrees (and  $T_1$  contains the root of the global tree). Note that in the above definition, the root is not decorated.

Define on  $\mathcal{T}^{tri}$  the internal law  $\frown$  by

$$\begin{array}{c} \odot \\ \diagdown \quad \diagup \\ T_1 \quad T_2 \end{array} \frown \begin{array}{c} \odot \\ \diagdown \quad \diagup \\ T_3 \quad T_4 \end{array} = \begin{array}{c} \odot \\ \diagdown \quad \diagup \\ \begin{array}{c} \diagdown \quad \diagup \\ T_1 \quad T_2 \quad T_3 \quad T_4 \end{array} \end{array} .$$

and extend it by bilinearity. One remarks that by definition  $\wedge$  is antisymmetric. Identifying  $\{0, 1\}$  with  $\{X_0, X_1\}$  by the obvious morphism and using the correspondence  $\wedge \leftrightarrow [, ]$ , this internal law allows us to identify the free Lie algebra  $\text{Lie}(X_0, X_1)$  with  $\mathcal{T}^{tri}$  modulo the Jacobi identity. Thus one can identify the (graded) dual of  $\text{Lie}(X_0, X_1)$  as a subspace of  $\mathcal{T}^{tri}$

$$\text{Lie}(X_0, X_1)^* \subset \mathcal{T}^{tri}.$$

A Lyndon word in 0 and 1 is a word in 0 and 1 strictly smaller than any of its nonempty proper right factors for the lexicographic order with  $0 < 1$  (for more details, see [Reu93]). The standard factorization  $[W]$  of a Lyndon word  $W$  is defined inductively by  $[0] = X_0$ ,  $[1] = X_1$  and otherwise by  $[W] = [[U], [V]]$  with  $W = UV$ ,  $U$  and  $V$  nontrivial and such that  $V$  is minimal. The sets of Lyndon brackets  $\{[W]\}$ , that is Lyndon words in standard factorization, form a basis of  $\text{Lie}(X_0, X_1)$  which can then be used to write the Lie bracket

$$[[U], [V]] = \sum_{\substack{W \text{ Lyndon} \\ \text{words}}} \alpha_{U,V}^W [W].$$

with  $U < V$  Lyndon words.

EXAMPLE 2.1. Lyndon words in letters  $0 < 1$  in lexicographic order are up to weight 5:

$$0 < 00001 < 0001 < 00011 < 001 < 00101 < 0011 < 00111 < \\ 01 < 01011 < 011 < 0111 < 01111 < 1$$

The above identification of  $\text{Lie}(X_0, X_1)$  as a quotient of  $\mathcal{T}^{tri}$  and the basis of Lyndon brackets allows us to define a family of trees dual to the Lyndon bracket basis beginning with  $T_{0^*} = \begin{array}{c} \circ \\ | \\ 0 \end{array}$  and  $T_{1^*} = \begin{array}{c} \circ \\ | \\ 1 \end{array}$  and then setting

$$(1) \quad T_{W^*} = \sum_{U < V} \alpha_{U,V}^W T_{U^*} \wedge T_{V^*}.$$

EXAMPLE 2.2. We give below the corresponding dual trees in weight 1, 2 and 3

$$T_{0^*} = \begin{array}{c} \circ \\ | \\ 0 \end{array}, \quad T_{1^*} = \begin{array}{c} \circ \\ | \\ 1 \end{array}, \quad T_{01^*} = \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 1 \end{array}, \quad T_{001^*} = \begin{array}{c} \circ \\ / \backslash \\ 0 \quad \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 1 \end{array} \end{array}, \quad T_{0111^*} = \begin{array}{c} \circ \\ / \backslash \\ \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 1 \end{array} \quad \begin{array}{c} \circ \\ / \backslash \\ 1 \quad 1 \end{array} \end{array}.$$

In weight 4 appears the first linear combination

$$T_{00001^*} = \begin{array}{c} \circ \\ / \backslash \\ \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 0 \end{array} \quad \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 1 \end{array} \end{array}, \quad T_{00111^*} = \begin{array}{c} \circ \\ / \backslash \\ \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 0 \end{array} \quad \begin{array}{c} \circ \\ / \backslash \\ 1 \quad 1 \end{array} \end{array} + \begin{array}{c} \circ \\ / \backslash \\ \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 0 \end{array} \quad \begin{array}{c} \circ \\ / \backslash \\ 1 \quad 1 \end{array} \end{array}, \quad T_{01111^*} = \begin{array}{c} \circ \\ / \backslash \\ \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 1 \end{array} \quad \begin{array}{c} \circ \\ / \backslash \\ 1 \quad 1 \end{array} \end{array},$$

due to the fact that both  $[0] \wedge [011]$  and  $[001] \wedge [1]$  are mapped onto  $[0011] = [X_0, [[X_0, X_1], X_1]]$  under the bracket map.

In weight 5, we will concentrate our attention to the two following examples

$$T_{00101^*} = \begin{array}{c} \circ \\ / \backslash \\ \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 0 \end{array} \quad \begin{array}{c} \circ \\ / \backslash \\ 1 \quad 0 \end{array} \end{array} - \begin{array}{c} \circ \\ / \backslash \\ \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 0 \end{array} \quad \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 1 \end{array} \end{array}$$

and

$$T_{01011^*} = \begin{array}{c} \circledast \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 1 \quad 0 \quad 1 \quad 1 \end{array} + \begin{array}{c} \circledast \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \quad 1 \end{array} + \begin{array}{c} \circledast \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \quad 1 \end{array} .$$

**2.2. Another differential on trees.** In [GGL09], Gangl, Goncharov and Levin introduced a differential  $d_{cy}$  on trees which reflects the differential in the Bloch’s cycle algebra  $\mathcal{N}_{\text{Spec}(\mathbb{Q})}$  (see Section 3). In their work they have shown that some particular linear combinations of trivalent trees attached to decompositions of polygons have decomposable differential. More precisely, the differential of these particular linear combinations of trees is a linear combination of products of the same type of linear combinations of trees. The elements  $T_{W^*}$  have a similar behavior under  $d_{cy}$ .

One begins by endowing trees with an extra structure.

**DEFINITION 2.3.** • An *orientation*  $\omega$  of a tree  $T$  (or a forest) is a numbering of the edges. That is if  $T$  has  $n$  edges and if  $E(T)$  denotes its set of edges,  $\omega$  is a map  $E(T) \rightarrow \{1, \dots, n\}$ .

- Let  $e(T)$  denote the cardinality of  $E(T)$ , that is the number of edges of  $T$ , and let  $we(T)$ , the weight of  $T$ , be the number of leaves of  $T$ . The degree of  $T$  is defined by  $deg(T) = 2we(T) - e(T)$ . We extend these definitions to forests by linearity.

**DEFINITION 2.4.**

- Let  $V^t$  be the  $\mathbb{Q}$ -vector space generated by a unit 1 and oriented forests of rooted trees  $T$  with root vertex decorated by:  $t$ , 0 or 1 and leaves decorated by 0 or 1.
- Let  $\cdot$  denote the product induced by the disjoint union of the trees and shift of the numbering for the orientation of the second factor. That is the product of  $(F_1, \omega_1)$  and  $(F_2, \omega_2)$  is the forest  $F = F_1 \sqcup F_2$  together with the numbering  $\omega$  satisfying  $\omega|_{E(F_1)} = \omega_1$  and  $\omega|_{E(F_2)} = \omega_2 + n_1$  where  $n_1 = e(F_1)$  is the number of edges in  $F_1$ . Note that here, by convention, the empty tree is 0 and the unit for  $\cdot$  is the extra generator 1.
- Define  $\mathcal{F}_{\mathbb{Q}}^\bullet$  to be the algebra  $V^t$  modulo the relations:

$$(T, \sigma(\omega)) = \varepsilon(\sigma)(T, \omega), \quad \begin{array}{c} \circledast \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ T_1 T_2 \end{array} = 0 \quad \text{and} \quad \begin{array}{c} \circledast \\ \bullet \\ \bullet \\ \bullet \\ 0 \end{array} = 0.$$

for any permutation  $\sigma$  and where  $\varepsilon(\sigma)$  denotes the usual signature of the permutation  $\sigma$ .

The algebra  $\mathcal{F}_{\mathbb{Q}}^\bullet$  endowed with the product  $\cdot$  is graded commutative because the orientation introduces signs into the usual disjoint union. Note that for any forest  $F$  one has  $(-1)^{\text{deg}(F)} = (-1)^{e(F)}$ .

- REMARK 2.5.**
- (1) There is an obvious direction on the edges of a rooted tree: away from the root.
  - (2) A rooted tree comes with a canonical numbering, starting from the root edge and induced by the cyclic ordering at each vertex.

EXAMPLE 2.6. With our convention, an example of this canonical ordering is shown at Figure 1; we recall that by convention we draw trees with the root at the top and the cyclic order at internal vertices counterclockwise.

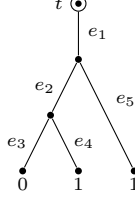


FIGURE 1. A tree with its canonical orientation, that is the canonical numbering of its edges.

Now, we define on  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$  a differential satisfying  $d^2 = 0$  and the Leibniz rule  $d((F_1, \omega_1) \cdot (F_2, \omega_2)) = d((F_1, \omega_1)) \cdot (F_2, \omega_2) + (-1)^{e(F_1)}(F_1, \omega_1) \cdot d((F_2, \omega_2))$ .

The set of rooted planar trees decorated as above endowed with their canonical orientation forms a set of representatives for the permutation relation and it generates  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$  as an algebra. Hence, we will define this differential first on these trees and then extend the definition by the Leibniz rule.

The differential of an oriented tree  $(T, \omega)$  is a linear combination of oriented forests where the trees appearing arise by contracting an edge of  $T$  and fall into two types depending on whether the edge is internal or not. We will need the notion of splitting.

DEFINITION 2.7. A *splitting* of a tree  $T$  at an internal vertex  $v$  is the disjoint union of the trees which arise as  $T_i \cup v$  where the  $T_i$  are the connected components of  $T \setminus v$ . Moreover

- the planar structure of  $T$  and its decorations of leaves induce on each  $T_i \cup v$  a planar structure and decorations of leaves ;
- an ordering of the edges of  $T$  induces an orientation of the forest  $\sqcup_i (T_i \cup v)$ ;
- if  $T$  has a root  $r$  then  $v$  becomes the root for all  $T_i \cup v$  which do not contain  $r$ , and if  $v$  has a decoration then it keeps its decoration in all the  $T_i \cup v$ .

DEFINITION 2.8. Let  $e$  be an edge of a tree  $T$ . The contraction of  $T$  along  $e$  denoted  $T/e$  is given as follows:

- (1) If the tree consists of a single edge, its contraction is the empty tree.
- (2) If  $e$  is an internal edge, then  $T/e$  is the tree obtained from  $T$  by contracting  $e$  and identifying the incident vertices to a single vertex.
- (3) If  $e$  is the edge containing the root vertex then  $T/e$  is the forest obtained by first contracting  $e$  to the internal incident vertex  $w$  (which inherits the decoration of the root) and then by splitting at  $w$ ;  $w$  becoming the new root of all trees in the forest  $T/e$ .
- (4) If  $e$  is an external edge not containing the root vertex then  $T/e$  is the forest obtained as follows: first one contracts  $e$  to the internal incident vertex  $w$  (which inherits the decoration of the leaf) and then one performs a splitting at  $w$ .

- (5) If  $T$  is endowed with its canonical orientation  $\omega$  there is a natural orientation  $i_e\omega$  on  $T/e$  given as follows :

$$\forall f \in E(T/e) \quad \begin{aligned} i_e\omega(f) &= \omega(f) && \text{if } \omega(f) < \omega(e) \\ i_e\omega(f) &= \omega(f) - 1 && \text{if } \omega(f) > \omega(e). \end{aligned}$$

EXAMPLE 2.9. Two examples are given below. In Figure 2, one contracts the root vertex and in Figure 3, a leaf is contracted.

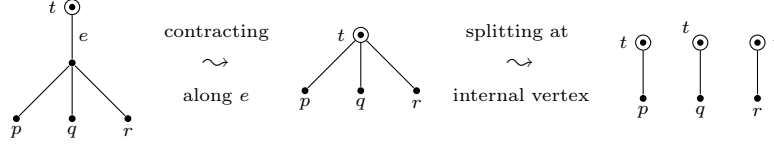


FIGURE 2. Contracting the root

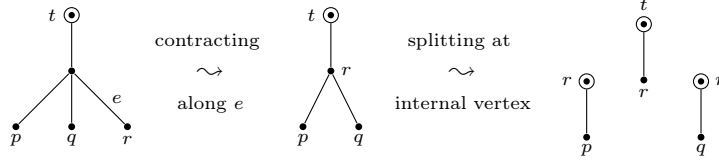


FIGURE 3. Contracting a leaf

DEFINITION 2.10. Let  $(T, \omega)$  be a tree endowed with its canonical orientation, one defines  $d_{cy}(T, \omega)$  as

$$d_{cy}(T, \omega) = \sum_{e \in E(T)} (-1)^{\omega(e)-1} (T/e, i_e\omega).$$

One extends  $d_{cy}$  to all oriented trees by the relation  $d_{cy}(T, \sigma \circ \omega) = \varepsilon(\sigma)d_{cy}(T, \omega)$  and to  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$  by linearity and the Leibniz rule.

In particular  $d_{cy}$  maps a tree with at most one edge to 0 (which corresponds by convention to the empty tree).

As proved in [GGL09],  $d_{cy}$ , extended with the Leibniz rule, induces a differential on  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$ .

PROPOSITION 2.11. *The map  $d_{cy} : \mathcal{F}_{\mathbb{Q}}^{\bullet} \rightarrow \mathcal{F}_{\mathbb{Q}}^{\bullet}$  makes  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$  into a commutative differential graded algebra. In particular  $d_{cy}^2 = 0$ .*

By an abuse of notation, for any Lyndon word  $U$  the image of  $T_{U^*}$  in  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$  with root vertex decorated by  $t$  and canonical orientation is also denoted by  $T_{U^*}$ . The image of  $T_{U^*}$  in  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$  with root vertex decorated by 1 and canonical orientation is denoted by  $T_{U^*}(1)$ .

The main result of the combinatorial aspects is the following.

THEOREM 2.12. *Let  $W$  be a Lyndon word. Then the following equality holds in  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$ :*

$$(ED-T) \quad d_{cy}(TW^*) = \sum_{U < V} \alpha_{U,V}^W T_{U^*} \cdot T_{V^*} + \sum_{U,V} \beta_{U,V}^W T_{U^*} \cdot T_{V^*}(1)$$



where the  $\alpha_{U,V}^W$  are the ones from Equation (1). In the above equation, coefficients  $\alpha$ 's and  $\beta$ 's are in  $\mathbb{Z}$ .

For a detailed proof, we refer to [Sou12]. The theorem mainly follows by induction from the combinatorics of the free Lie algebra  $\text{Lie}(X_0, X_1)$ :

- The terms in the first sum in the R.H.S of (ED-T) come from the contraction of the root edge which is nothing but the differential  $d_{Lie}$  dual to the bracket of  $\text{Lie}(X_0, X_1)$ .
- Using the inductive definition of  $T_{W^*}$  (cf. Equation (1)), one shows by induction that, as  $d_{Lie}^2 = 0$ , internal edges do not contribute.
- Terms in  $T_{V^*}(1)$  arise from leaves decorated by 1. The fact that terms arising from leaves decorated by 1 can be regrouped as a product  $T_{U^*} \cdot T_{V^*}(1)$  is due to a particular decomposition of some specific brackets in terms of the Lyndon basis.

EXAMPLE 2.13. As mentioned before, the trees are endowed with their canonical numbering. First one remarks that trees with only one edge are mapped to 0 so

$$d_{cy}(T_{0^*}) = d_{cy}\left(\begin{array}{c} \circ \\ | \\ 0 \end{array}\right) = 0 \quad \text{and} \quad d_{cy}(T_{1^*}) = d_{cy}\left(\begin{array}{c} \circ \\ | \\ 1 \end{array}\right) = 0.$$

We recall that a tree with root decorated by 0 is 0 in  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$ . As applying an odd permutation to the numbering changes the sign of the tree, the trivalency of the tree  $T_{W^*}$  shows that some trees coming from the computation of  $d_{cy}$  are 0 in  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$  because they contain a symmetric subtree; that is they contain a subtree of the form



where  $T$  is a trivalent tree.

Using the fact that the tree  $\begin{array}{c} \circ \\ | \\ 0 \end{array}$  is 0 in  $\mathcal{F}_{\mathbb{Q}}^{\bullet}$ , one computes in weight 2

$$d_{cy}(T_{01^*}) = d_{cy}\left(\begin{array}{c} \circ \\ | \\ \begin{array}{cc} \bullet & \bullet \\ / & \backslash \\ 0 & 1 \end{array} \end{array}\right) = \begin{array}{c} \circ \\ | \\ 0 \end{array} \cdot \begin{array}{c} \circ \\ | \\ 1 \end{array} = T_{0^*} T_{1^*}.$$

In weight 3, one has

$$d_{cy}(T_{001^*}) = d_{cy}\left(\begin{array}{c} \circ \\ | \\ \begin{array}{ccc} \bullet & \bullet & \bullet \\ / & | & \backslash \\ 0 & 0 & 1 \end{array} \end{array}\right) = \begin{array}{c} \circ \\ | \\ 0 \end{array} \cdot \begin{array}{c} \circ \\ | \\ \begin{array}{cc} \bullet & \bullet \\ / & \backslash \\ 0 & 1 \end{array} \end{array} = T_{0^*} T_{01^*}$$

and

$$\begin{aligned} d_{cy}(T_{011^*}) &= d_{cy}\left(\begin{array}{c} \circ \\ | \\ \begin{array}{ccc} \bullet & \bullet & \bullet \\ / & | & \backslash \\ 0 & 1 & 1 \end{array} \end{array}\right) = \begin{array}{c} \circ \\ | \\ \begin{array}{cc} \bullet & \bullet \\ / & \backslash \\ 0 & 1 \end{array} \end{array} \cdot \begin{array}{c} \circ \\ | \\ 1 \end{array} + \begin{array}{c} \circ \\ | \\ 1 \end{array} \cdot \begin{array}{c} \circ \\ | \\ \begin{array}{cc} \bullet & \bullet \\ / & \backslash \\ 0 & 1 \end{array} \end{array} \\ &= T_{01^*} T_{1^*} + T_{1^*} T_{01^*}(1). \end{aligned}$$

In weight 4, one can easily check that

$$d_{cy}(T_{0001^*}) = T_{0^*} T_{001^*} \quad \text{and} \quad d_{cy}(T_{0111^*}) = T_{011^*} T_{1^*} + T_{1^*} T_{011^*}(1).$$

The example of  $T_{0011^*}$  is more interesting.

$$(2) \quad d_{cy} \left( \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \end{array} + \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \end{array} \right) = \begin{array}{c} t \odot \\ | \\ \bullet \\ 0 \end{array} \cdot \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 1 \quad 1 \end{array} + \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \end{array} \cdot \begin{array}{c} t \odot \\ | \\ \bullet \\ 1 \end{array} \\ + \begin{array}{c} t \odot \\ | \\ \bullet \\ 1 \end{array} \cdot \begin{array}{c} 1 \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \end{array} + \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 1 \end{array} \cdot \begin{array}{c} 1 \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 1 \end{array}$$

That is:

$$(3) \quad d_{cy}(T_{0011^*}) = T_0^* T_{011^*} + T_{001^*} T_1^* + T_1^* T_{001^*}(1) + T_{01^*} T_{01^*}(1)$$

In the above equations, the term in  $T_{01^*} T_{01^*}(1)$  is coming from the last edge of the tree



appearing in  $T_{0011^*}$ .

Computing  $d_{cy}^2(T_{0011^*})$  (which is 0), the differential  $d_{cy}(T_{01^*} T_{01^*}(1))$  cancels with the term in  $T_0^* T_1^* T_{01^*}(1)$  arising from  $d_{cy}(T_0^* T_{011^*})$ . It can be thought of as the propagation of the weight 3 correction term  $T_1^* T_{01^*}(1)$  appearing in  $d_{cy}(T_{011^*})$ .

We give below an example in weight 5,  $d_{cy}(T_{01011^*})$  :

$$d_{cy} \left( \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 1 \quad 0 \quad 1 \quad 1 \end{array} + \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \quad 1 \end{array} + \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \quad 1 \end{array} \right) = \begin{array}{c} t \odot \\ | \\ \bullet \\ 0 \end{array} \cdot \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 1 \end{array} + \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 1 \quad 1 \end{array} \cdot \left( \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \end{array} + \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \end{array} \right) \cdot \begin{array}{c} t \odot \\ | \\ \bullet \\ 1 \end{array} \\ + \begin{array}{c} t \odot \\ | \\ \bullet \\ 1 \end{array} \cdot \left( \begin{array}{c} 1 \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \end{array} + \begin{array}{c} 1 \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 0 \quad 1 \quad 1 \end{array} \right) + 2 \cdot \begin{array}{c} t \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 1 \quad 1 \end{array} \cdot \begin{array}{c} 1 \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 0 \quad 1 \end{array}$$

where the last term arises from the part of the differential associated to edges  $e$  and  $f$ . The above equation can be written as

$$(4) \quad T_{01011^*} = T_{01^*} \cdot T_{011^*} + T_{0011^*} \cdot T_{1^*} + T_{1^*} \cdot T_{0011^*}(1) + 2T_{011^*} \cdot T_{01^*}(1)$$

### 3. Algebraic cycles

This section begins with the construction of the cycle complex (or cycle algebra) as presented in [Blo86, Blo97, BK94, Lev94]. Then, we give some properties of equidimensional cycles groups over  $\mathbb{A}^1$  and build some algebraic cycles corresponding to multiple polylogarithms in one variable.

Here the base field is  $\mathbb{Q}$  and the various structures have  $\mathbb{Q}$  coefficients.

**3.1. Construction of the cycle algebra.** Let  $\square^n$  be the algebraic  $n$ -cube

$$\square^n = (\mathbb{P}^1 \setminus \{1\})^n.$$

Insertion morphisms  $s_i^\varepsilon : \square^{n-1} \rightarrow \square^n$  are given by the identification

$$\square^{n-1} \simeq \square^{i-1} \times \{\varepsilon\} \times \square^{n-i}$$

for  $\varepsilon = 0, \infty$ . A *face*  $F$  of codimension  $p$  of  $\square^n$  is given by the equation  $x_{i_k} = \varepsilon_k$  for  $k$  in  $\{1, \dots, p\}$  and  $\varepsilon_k$  in  $\{0, \infty\}$  where  $x_1, \dots, x_n$  are the usual affine coordinates on  $\mathbb{P}^1$ . In particular, codimension 1 faces are given by the images of insertion morphisms.

Now, let  $X$  be a smooth irreducible quasi-projective variety over  $\mathbb{Q}$ .

**DEFINITION 3.1.** Let  $p$  and  $n$  be non-negative integers. Let  $\mathcal{Z}^p(X, n)$  be the free group generated by closed irreducible sub-varieties of  $X \times \square^n$  of codimension  $p$  which intersect all faces  $X \times F$  properly (where  $F$  is a face of  $\square^n$ ). That is:

$$\mathbb{Z} \left\langle Z \subset X \times \square^n \text{ such that } \begin{cases} Z \text{ is closed and irreducible} \\ \text{codim}_{X \times F}(Z \cap (X \times F)) = p \\ \text{or } Z \cap (X \times F) = \emptyset \end{cases} \right\rangle$$

A sub-variety  $Z$  of  $X \times \square^n$  as above is *admissible*. The insertion morphisms  $s_i^\varepsilon$  induce a well defined pull-back  $s_i^{\varepsilon*} : \mathcal{Z}^p(X, n) \rightarrow \mathcal{Z}^p(X, n-1)$  and a differential:

$$\partial = \sum_{i=1}^n (-1)^{i-1} (s_i^{0*} - s_i^{\infty*}) : \mathcal{Z}^p(X, n) \rightarrow \mathcal{Z}^p(X, n-1).$$

The permutation group  $\mathfrak{S}_n$  acts on  $\square^n$  by permutation of the factors. This action extends to an action of the semi-direct product  $G_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$  where each  $\mathbb{Z}/2\mathbb{Z}$  acts on  $\square^1$  by sending the usual affine coordinates  $x$  to  $1/x$ . The sign representation of  $\mathfrak{S}_n$  extends to a sign representation  $G_n \rightarrow \{\pm 1\}$ . Let  $\mathcal{A}lt_n \in \mathbb{Q}[G_n]$  be the corresponding projector; when the context is clear enough, we may drop the subscript  $n$ .

**DEFINITION 3.2.** Let  $p$  and  $k$  be integers as above. One defines

$$\mathcal{N}_X^k(p) = \mathcal{A}lt_{2p-k}(\mathcal{Z}^p(X, 2p-k) \otimes \mathbb{Q}).$$

We will refer to  $k$  as the *cohomological degree* and to  $p$  as the *weight*.

For our purpose, we will not only need admissible cycles but cycles in  $X \times \square^n$  whose fibers over  $X$  are also admissible.

**DEFINITION 3.3 (Equidimensionality).** Let  $X$  be an irreducible smooth variety.

- Let  $\mathcal{Z}_{eq}^p(X, n)$  denote the free abelian group generated by irreducible closed subvarieties  $Z \subset X \times \square^n$  such that for any face  $F$  of  $\square^n$ , the intersection  $Z \cap (X \times F)$  is empty or the restriction of  $p_1 : X \times \square^n \rightarrow X$  to

$$Z \cap (X \times F) \rightarrow X$$

is equidimensional of relative dimension  $\dim(F) - p$ .

- We say that elements of  $\mathcal{Z}_{eq}^p(X, n)$  are *equidimensional over  $X$  with respect to any face* or simply *equidimensional*.
- Following the definition of  $\mathcal{N}_X^k(p)$ , let  $\mathcal{N}_X^{eq, k}(p)$  denote

$$\mathcal{N}_X^{eq, k}(p) = \mathcal{A}lt_{2p-k}(\mathcal{Z}_{eq}^p(X, 2p-k) \otimes \mathbb{Q}).$$

- If  $Z$  is an irreducible closed subvariety of  $X \times \square^n$  satisfying the above condition,  $Z|_{t=x}$  will denote the fiber over the point  $x \in X$  of  $p_1$  restricted to  $Z$  that is  $Z \cap (\{x\} \times \square^n)$ .

Let  $C = \mathcal{A}lt(\sum q_i Z_i)$  be an element in  $\mathcal{N}_X^{eq, \bullet}$  with the  $Z_i$  as above and  $q_i$ 's in  $\mathbb{Q}$ . For a point  $x \in X$ , we will denote by  $C|_{t=x}$  the element of  $\mathcal{N}_X^\bullet$

$$C|_{t=x} = \mathcal{A}lt(\sum q_i Z_i|_{t=x})$$

which is well defined in both  $\mathcal{N}_X^\bullet$  and  $\mathcal{N}_{\{x\}}^\bullet$  by definition of the  $Z_i$ .

EXAMPLE 3.4. Consider the graph of the identity  $\mathbb{A}^1 \xrightarrow{t \rightarrow t} \mathbb{A}^1$  restricted to  $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{1\}$ . Let  $\Gamma_0$  be its embedding in  $\mathbb{A}^1 \times \square^1$ . Then  $\Gamma_0$  is of codimension 1 in  $\mathbb{A}^1 \times \square^1$  and is admissible as the intersection with the face  $x_1 = \infty$  is empty and the intersection with the face  $x_1 = 0$  is  $\{0\} \times \{0\}$  which is of codimension 1 in  $\mathbb{A}^1 \times \{0\}$ .

However,  $\Gamma_0$  is not equidimensional as

$$\Gamma_0 \cap (\mathbb{A}^1 \times \{0\}) = \{0\} \times \{0\}$$

is neither equidimensional over  $\mathbb{A}^1$  nor empty as the condition would require.

Applying the projector  $\mathcal{A}lt$  gives an element  $\overline{\mathcal{L}}_0$  in  $\mathcal{N}_{\mathbb{A}^1}^1(1)$ . Using the definition of  $\Gamma_0$  as a graph, one obtains a parametric representation (where the projector  $\mathcal{A}lt$  is omitted):

$$\overline{\mathcal{L}}_0 = [t; t] \subset \mathbb{A}^1 \times \square^1.$$

In the above notation the semicolon separates the base space coordinates from the cubical coordinates.

The morphisms  $s_i^{\varepsilon^*}$  induce morphisms  $\partial_i^\varepsilon : \mathcal{N}_X^k(p) \rightarrow \mathcal{N}_X^{k+1}(p)$  and the above differential  $\partial = \sum_i (-1)^{i-1} (\partial_i^0 - \partial_i^\infty)$  gives a complex

$$\mathcal{N}_X^\bullet(p) : \quad \cdots \rightarrow \mathcal{N}_X^k(p) \xrightarrow{\partial} \mathcal{N}_X^{k+1}(p) \rightarrow \cdots$$

DEFINITION 3.5. One defines the cycle complex as

$$\mathcal{N}_X^\bullet = \bigoplus_{p \geq 0} \mathcal{N}_X^p(p) = \mathbb{Q} \oplus \bigoplus_{p \geq 1} \mathcal{N}_X^p(p)$$

and as the differential restricts to equidimensional cycles, one also defines

$$\mathcal{N}_X^{eq, \bullet} = \bigoplus_{p \geq 0} \mathcal{N}_X^{eq, \bullet}(p).$$

The author refers sometimes to  $\mathcal{N}_X^\bullet$  as the cycle algebra because of another natural structure coming with this cubical cycle complex: the product structure.

Levine has shown in [Lev94][§5] or [Lev11][Example 4.3.2] the following proposition.

**PROPOSITION 3.6.** *Concatenation of the cube factors and pull-back by the diagonal*

$$X \times \square^n \times X \times \square^m \xrightarrow{\sim} X \times X \times \square^n \times \square^m \xrightarrow{\sim} X \times X \times \square^{n+m} \xleftarrow{\Delta_X} X \times \square^{n+m}$$

induce, after applying the Alt projector, a well-defined product:

$$\mathcal{N}_X^k(p) \otimes \mathcal{N}_X^l(q) \longrightarrow \mathcal{N}_X^{k+l}(p+q)$$

denoted by  $\cdot$ .

The complex  $\mathcal{N}_X^{eq, \bullet}$  is stable under this product law.

**REMARK 3.7.** The smoothness hypothesis on  $X$  allows us to consider the pull-back by the diagonal  $\Delta_X : X \rightarrow X \times X$  which is, in this case, of local complete intersection.

One has the following theorem (stated in [BK94, Blo97] for  $X = \text{Spec}(\mathbb{Q})$ ).

**THEOREM 3.8 ([Lev94]).** *The cycle complex  $\mathcal{N}_X^\bullet$  is an Adams graded, commutative differential graded algebra (Adams graded, c.d.g.a.). In weight  $p$ , its cohomology groups are the higher Chow groups of  $X$ :*

$$H^k(\mathcal{N}_X(p)) = \text{CH}^p(X, 2p - k)_{\mathbb{Q}},$$

where  $\text{CH}^p(X, 2p - k)_{\mathbb{Q}}$  stands for  $\text{CH}^p(X, 2p - k) \otimes \mathbb{Q}$ .

Moreover  $\mathcal{N}_X^{eq, \bullet}$  turns into a sub-Adams graded, c.d.g.a. Note that, in the graded algebra context, *commutative* always means *graded commutative*.

One has natural flat pull-backs and proper push-forwards on  $\mathcal{N}_X^\bullet$  (and on  $\mathcal{N}_X^{eq, \bullet}$ ). Comparison with higher Chow groups also gives on the cohomology groups both  $\mathbb{A}^1$ -homotopy invariance and the long exact sequence associated to an open and its closed complement. Writing  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  as  $\mathbb{A}^1 \setminus \{0, 1\}$ , one obtains the following description of  $H^*(\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet(p))$ :

$$\begin{aligned} H^k(\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet(p)) &\simeq H^k(\mathcal{N}_{\mathbb{Q}}^\bullet(p)) \oplus \\ &H^{k-1}(\mathcal{N}_{\mathbb{Q}}^\bullet(p-1)) \otimes \mathbb{Q}\mathcal{L}_0 \oplus H^{k-1}(\mathcal{N}_{\mathbb{Q}}^\bullet(p-1)) \otimes \mathbb{Q}\mathcal{L}_1, \end{aligned}$$

where  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are in cohomological degree 1 and weight 1 (that is of codimension 1). Their explicit description will be given later on.

Comparing the situation over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  and over  $\mathbb{A}^1$  comes as an important idea in our project as the desired cycles over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  need to admit a natural specialization at 1. In particular, we will need to work with equidimensional cycle and some of their properties are given in the next subsection.

**3.2. Equidimensional cycles.** The following result given in [Sou12] essentially follows from the definition and makes it easy to compare both situations.

**PROPOSITION 3.9.** *Let  $X_0$  be an open dense subset of  $X$  an irreducible smooth variety and let  $j : X_0 \rightarrow X$  be the inclusion. Then the restriction of cycles from*

$X$  to  $X_0$  induces a morphism of c.d.g.a. preserving the weight (that is the Adams grading)

$$j^* : \mathcal{N}_X^{eq, \bullet} \longrightarrow \mathcal{N}_{X_0}^{eq, \bullet}.$$

Moreover, let  $C$  be in  $\mathcal{N}_{X_0}^{\bullet}$  and write  $C$  in terms of the generators of the group  $\oplus \mathcal{Z}^*(X_0, \bullet)$  as

$$C = \sum_{i \in I} q_i Z_i, \quad q_i \in \mathbb{Q}$$

where  $I$  is a finite set. Assume that, for any  $i$ , the Zariski closure  $\overline{Z_i}$  of  $Z_i$  in  $X \times \square^{n_i}$  intersected with any face  $X \times F$  of  $X \times \square^{n_i}$  is equidimensional over  $X$  of relative dimension  $\dim(F) - p_i$ . Define  $C'$  as

$$C' = \sum_{i \in I} q_i \overline{Z_i},$$

then

$$C' \in \mathcal{N}_X^{eq, \bullet} \quad \text{and} \quad C = j^*(C') \in \mathcal{N}_{X_0}^{eq, \bullet}.$$

Below, we describe the main geometric fact that allows the construction of our cycles: pulling back by the multiplication induces a homotopy between identity and the zero section on the cycle algebra over  $\mathbb{A}^1$ .

Let  $m : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$  be the multiplication map sending  $(x, y)$  to  $xy$  and let  $\tau : \square^1 = \mathbb{P}^1 \setminus \{1\} \longrightarrow \mathbb{A}^1$  be the isomorphism sending the affine coordinate  $u$  to  $\frac{1}{1-u}$ . The map  $\tau$  sends  $\infty$  to 0, 0 to 1 and extends as a map from  $\mathbb{P}^1$  to  $\mathbb{P}^1$  sending 1 to  $\infty$ .

The maps  $m$  and  $\tau$  are in particular flat and equidimensional of relative dimension 1 and 0, respectively.

Consider the following commutative diagram for a positive integer  $n$

$$\begin{array}{ccc} \mathbb{A}^1 \times \square^1 \times \square^n & \xrightarrow{(m \circ (\text{id}_{\mathbb{A}^1} \times \tau)) \times \text{id}_{\square^n}} & \mathbb{A}^1 \times \square^n \\ p_{\mathbb{A}^1 \times \square^1} \downarrow & & \downarrow p_{\mathbb{A}^1} \\ \mathbb{A}^1 \times \square^1 & \xrightarrow{m \circ (\text{id}_{\mathbb{A}^1} \times \tau)} & \mathbb{A}^1 \\ p_{\mathbb{A}^1} \downarrow & & \\ \mathbb{A}^1 & & \end{array}$$

**PROPOSITION 3.10** (multiplication and equidimensionality). *In the following statement,  $p$ ,  $k$  and  $n$  will denote positive integers subject to the relation  $n = 2p - k$*

- the composition  $\tilde{m} = (m \circ (\text{id}_{\mathbb{A}^1} \times \tau)) \times \text{id}_{\square^n}$  induces a group morphism

$$\mathcal{Z}_{eq}^p(\mathbb{A}^1, n) \xrightarrow{\tilde{m}^*} \mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n)$$

which extends to a morphism of complexes for any  $p$

$$\mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}(p) \xrightarrow{\tilde{m}^*} \mathcal{N}_{\mathbb{A}^1 \times \square^1}^{eq, \bullet}(p).$$

- Moreover, one has a natural group morphism

$$h_{\mathbb{A}^1, n}^p : \mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n) \longrightarrow \mathcal{Z}_{eq}^p(\mathbb{A}^1, n+1)$$

given by regrouping the  $\square^1$  factors (as  $\square^n = (\square^1)^n$ ).

- The composition  $\mu^* = h_{\mathbb{A}^1, n}^p \circ \tilde{m}^*$  gives a linear map

$$\mu^* : \mathcal{N}_{\mathbb{A}^1}^{eq, k}(p) \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq, k-1}(p)$$

sending equidimensional cycles with empty fiber at 0 to equidimensional cycles with empty fiber at 0.

- Let  $\theta : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$  be the involution sending the natural affine coordinate  $t$  to  $1 - t$ . Twisting the map  $\tilde{m}$  by  $\theta$  gives a map  $\hat{m}$  via

$$\begin{array}{ccc} \mathbb{A}^1 \times \square^1 \times \square^n & \overset{\tilde{m}}{\dashrightarrow} & \mathbb{A}^1 \times \square^n \\ \theta \times \text{id}_{\square^{n+1}} \downarrow & & \uparrow \theta \times \text{id}_{\square^n} \\ \mathbb{A}^1 \times \square^1 \times \square^n & \xrightarrow{\hat{m}} & \mathbb{A}^1 \times \square^n \end{array}$$

and induces a linear map

$$\nu^* : \mathcal{N}_{\mathbb{A}^1}^{eq, k}(p) \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq, k-1}(p)$$

sending equidimensional cycles with empty fiber at 1 to equidimensional cycles with empty fiber at 1.

PROOF. It is enough to work with generators of  $\mathcal{Z}_{eq}^p(\mathbb{A}^1, n)$ . Let  $Z$  be an irreducible subvariety of  $\mathbb{A}^1 \times \square^n$  such that for any face  $F$  of  $\square^n$ , the first projection

$$p_{\mathbb{A}^1} : Z \cap (\mathbb{A}^1 \times F) \longrightarrow \mathbb{A}^1$$

is equidimensional of relative dimension  $\dim(F) - p$  or empty. Let  $F$  be a face of  $\square^n$ . We want first to show that under the projection  $\mathbb{A}^1 \times \square^1 \times \square^n \longrightarrow \mathbb{A}^1 \times \square^1$ ,

$$\tilde{m}^{-1}(Z) \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1 \times \square^1$$

is equidimensional of relative dimension  $\dim(F) - p$  or empty. This follows from the fact that  $Z \cap (\mathbb{A}^1 \times F)$  is equidimensional over  $\mathbb{A}^1$  and  $m$  is flat and equidimensional of relative dimension 1 (hence are  $m \times \tau$  and  $\tilde{m}$ ). The map  $\tilde{m}$  is the identity on the  $\square^n$  factor, thus for  $Z \subset \mathbb{A}^1 \times \square^n$  as above and a codimension 1 face  $F$  of  $\square^n$ ,  $\tilde{m}^{-1}(Z)$  satisfies

$$\tilde{m}^{-1}(Z) \cap (\mathbb{A}^1 \times \square^1 \times F) = \tilde{m}^{-1}(Z \cap (\mathbb{A}^1 \times F))$$

which makes  $\tilde{m}^*$  into a morphism of complexes.

Moreover, assuming that the fiber of  $Z$  at 0 is empty, as  $\tilde{m}$  restricted to

$$\{0\} \times \square^1 \times \square^n$$

factors through the inclusion  $\{0\} \times \square^n \longrightarrow \mathbb{A}^1 \times \square^n$ , the intersection

$$\tilde{m}^{-1}(Z) \cap (\{0\} \times \square^1 \times \square^n)$$

is empty. Hence the fiber of  $\tilde{m}^{-1}(Z)$  over  $\{0\} \times \square^1$  by  $p_{\mathbb{A}^1 \times \square^1}$  is empty and the same holds for the fiber over  $\{0\}$  by  $p_{\mathbb{A}^1} \circ p_{\mathbb{A}^1 \times \square^1}$ .

Now, let  $Z$  be an irreducible subvariety of  $\mathbb{A}^1 \times \square^1 \times \square^n$  such that for any face  $F$  of  $\square^n$

$$Z \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1 \times \square^1$$

is equidimensional of relative dimension  $\dim(F) - p$ . Let  $F'$  be a face of

$$\square^{n+1} = \square^1 \times \square^n.$$

The face  $F'$  is either of the form  $\square^1 \times F$  or of the form  $\{\varepsilon\} \times F$  with  $F$  a face of  $\square^n$  and  $\varepsilon \in \{0, \infty\}$ . If  $F'$  is of the first type, as

$$Z \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1 \times \square^1$$

is equidimensional and, as  $\mathbb{A}^1 \times \square^1 \longrightarrow \mathbb{A}^1$  is equidimensional of relative dimension 1, the projection

$$Z \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1$$

is equidimensional of relative dimension

$$\dim(F) - p + 1 = \dim(F') - p.$$

If  $F'$  is of the second type, by symmetry of the role of 0 and  $\infty$ , we can assume that  $\varepsilon = 0$ . Then the intersection

$$Z \cap (\mathbb{A}^1 \times \{0\} \times F)$$

is nothing but the fiber of  $Z \cap (\mathbb{A}^1 \times \square^1 \times F)$  over  $\mathbb{A}^1 \times \{0\}$ . Hence, it has pure dimension  $\dim(F) - p + 1$ .

Moreover, denoting with a subscript the fiber, the composition

$$Z \cap (\mathbb{A}^1 \times \{0\} \times F) = (Z \cap (\mathbb{A}^1 \times \square^1 \times F))_{\mathbb{A}^1 \times \{0\}} \longrightarrow \mathbb{A}^1 \times \{0\} \longrightarrow \mathbb{A}^1$$

is equidimensional of relative dimension

$$\dim(F) - p = \dim(F') - p.$$

This shows that  $h_{\mathbb{A}^1, n}^p$  gives a well defined morphism and that it preserves the fiber at a point  $x$  in  $\mathbb{A}^1$ ; in particular, if  $Z$  has an empty fiber at 0, so does  $h_{\mathbb{A}^1, n}^p(Z)$ .

Finally, the last part of the proposition is deduced from the fact that  $\theta$  exchanges the role of 0 and 1.  $\square$

**REMARK 3.11.** We have remarked that  $\tilde{m}$  sends cycles with empty fiber at 0 to cycles with empty fiber at any point in  $\{0\} \times \square^1$ . Similarly  $\tilde{m}$  sends cycles with empty fiber at 0 to cycles that also have an empty fiber at any point in  $\mathbb{A}^1 \times \{\infty\}$ .

From the proof of Levine's Proposition 4.2 in [Lev94], we deduce that  $\mu^*$  gives a homotopy between  $p_0^* \circ i_0^*$  and  $\text{id}$  where  $i_0$  is the zero section  $\{0\} \rightarrow \mathbb{A}^1$  and  $p_0$  the projection onto the point  $\{0\}$ .

**PROPOSITION 3.12.** *Notations are the ones from Proposition 3.10 above. Let  $i_0$  (resp.  $i_1$ ) be the inclusion of 0 (resp. 1) in  $\mathbb{A}^1$ :*

$$i_0 : \{0\} \longrightarrow \mathbb{A}^1, \quad i_1 : \{1\} \longrightarrow \mathbb{A}^1.$$

*Let  $p_0$  and  $p_1$  be the corresponding projections  $p_\varepsilon : \mathbb{A}^1 \longrightarrow \{\varepsilon\}$  for  $\varepsilon = 0, 1$ .*

*Then  $\mu^*$  provides a homotopy between*

$$p_0^* \circ i_0^* \text{ and } \text{id} : \mathcal{N}_{\mathbb{A}^1}^{eq, \bullet} \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}$$

*and similarly  $\nu^*$  provides a homotopy between*

$$p_1^* \circ i_1^* \text{ and } \text{id} : \mathcal{N}_{\mathbb{A}^1}^{eq, \bullet} \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}.$$

*In other words, one has*

$$\partial_{\mathbb{A}^1} \circ \mu^* + \mu^* \circ \partial_{\mathbb{A}^1} = \text{id} - p_0^* \circ i_0^* \quad \text{and} \quad \partial_{\mathbb{A}^1} \circ \nu^* + \nu^* \circ \partial_{\mathbb{A}^1} = \text{id} - p_1^* \circ i_1^*.$$

The proposition follows from computing the different compositions involved and the relation between the differential on  $\mathcal{N}_{\mathbb{A}^1 \times \square^1}^{eq, \bullet}$  and the one on  $\mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}$  via the map  $h_{\mathbb{A}^1, n}^p$ .



PROOF. We denote by  $i_{0,\square}$  and  $i_{\infty,\square}$  the zero section and the infinity section  $\mathbb{A}^1 \rightarrow \mathbb{A}^1 \times \square^1$ . The action of  $\theta$  only exchanges the role of 0 and 1 in  $\mathbb{A}^1$ , hence it is enough to prove the statement for  $\mu^*$ . As previously, in order to obtain the proposition for  $\mathcal{N}_{\mathbb{A}^1}^{eq,k}(p)$ , it is enough to work on the generators of  $\mathcal{Z}_{eq}^p(\mathbb{A}^1, n)$  with  $n = 2p - k$ .

By the previous proposition 3.10, the morphism  $\tilde{m}^*$  commutes with the differential on  $\mathcal{Z}_{eq}^p(\mathbb{A}^1, \bullet)$  and on  $\mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, \bullet)$ . As the morphism  $\mu^*$  is defined by  $\mu^* = h_{\mathbb{A}^1, n}^p \circ \tilde{m}^*$ , the proof relies on computing  $\partial_{\mathbb{A}^1} \circ h_{\mathbb{A}^1, n}^p$ . Let  $Z$  be a generator of  $\mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n)$ . In particular,

$$Z \subset \mathbb{A}^1 \times \square^1 \times \square^n$$

and  $h_{\mathbb{A}^1, n}^p(Z)$  is also given by  $Z$  but viewed in

$$\mathbb{A}^1 \times \square^{n+1}.$$

The differentials denoted by  $\partial_{\mathbb{A}^1}^{n+1}$  on  $\mathcal{Z}_{eq}^p(\mathbb{A}^1, n+1)$  and  $\partial_{\mathbb{A}^1 \times \square^1}^n$  on  $\mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n)$  are both given by intersections with the codimension 1 faces but the first  $\square^1$  factor in  $\square^{n+1}$  gives two more faces and introduces a change of sign. Namely, using an extra subscript to indicate in which cycle groups the intersections take place, one has:

$$\begin{aligned} \partial_{\mathbb{A}^1}^{n+1}(h_{\mathbb{A}^1, n}^p(Z)) &= \sum_{i=1}^{n+1} (-1)^{i-1} (\partial_{i, \mathbb{A}^1}^0(Z) - \partial_{i, \mathbb{A}^1}^\infty(Z)) \\ &= \partial_{1, \mathbb{A}^1}^0(Z) - \partial_{1, \mathbb{A}^1}^\infty(Z) - \sum_{i=2}^{n+1} (-1)^{i-2} (\partial_{i, \mathbb{A}^1}^0(Z) - \partial_{i, \mathbb{A}^1}^\infty(Z)) \\ &= i_{0, \square}^*(Z) - i_{\infty, \square}^*(Z) - \sum_{i=1}^n (-1)^{i-1} (\partial_{i+1, \mathbb{A}^1}^0(Z) - \partial_{i+1, \mathbb{A}^1}^\infty(Z)). \end{aligned}$$

Hence one gets

$$\begin{aligned} \partial_{\mathbb{A}^1}^{n+1}(h_{\mathbb{A}^1, n}^p(Z)) &= i_{0, \square}^*(Z) - i_{\infty, \square}^*(Z) \\ &\quad - \sum_{i=1}^n (-1)^{i-1} \left( h_{\mathbb{A}^1, n-1}^p(\partial_{i, \mathbb{A}^1 \times \square^1}^0(Z) - \partial_{i, \mathbb{A}^1 \times \square^1}^\infty(Z)) \right) \end{aligned}$$

which can be written has

$$\partial_{\mathbb{A}^1}^{n+1}(h_{\mathbb{A}^1, n}^p(Z)) = i_{0, \square}^*(Z) - i_{\infty, \square}^*(Z) - h_{\mathbb{A}^1, n-1}^p \circ \partial_{\mathbb{A}^1 \times \square^1}^n(Z).$$

Thus one can compute  $\partial_{\mathbb{A}^1} \circ \mu^* + \mu^* \circ \partial_{\mathbb{A}^1}$  on  $\mathcal{Z}_{eq}^p(\mathbb{A}^1, n)$  as

$$\begin{aligned} \partial_{\mathbb{A}^1} \circ \mu^* + \mu^* \circ \partial_{\mathbb{A}^1} &= \partial_{\mathbb{A}^1} \circ h_{\mathbb{A}^1, n}^p \circ \tilde{m}^* + h_{\mathbb{A}^1, n-1}^p \circ \tilde{m}^* \circ \partial_{\mathbb{A}^1} \\ &= i_{0, \square}^* \circ \tilde{m}^* - i_{\infty, \square}^* \circ \tilde{m}^* - h_{\mathbb{A}^1, n-1}^p \circ \partial_{\mathbb{A}^1} \circ \tilde{m}^* \\ &\quad + h_{\mathbb{A}^1, n-1}^p \circ \partial_{\mathbb{A}^1} \circ \tilde{m}^* \\ &= i_{0, \square}^* \circ \tilde{m}^* - i_{\infty, \square}^* \circ \tilde{m}^*. \end{aligned}$$

The morphism  $i_{\infty, \square}^* \circ \tilde{m}^*$  is induced by

$$\begin{array}{ccccccc} \mathbb{A}^1 & \xrightarrow{i_{\infty, \square}} & \mathbb{A}^1 \times \square^1 & \xrightarrow{\tau} & \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1 \\ x & \longmapsto & (x, \infty) & \longmapsto & (x, 0) & \longmapsto & 0 \end{array}$$

which factors through

$$\begin{array}{ccccc} \mathbb{A}^1 & \xrightarrow{i_{\infty, \square}} & \mathbb{A}^1 \times \square^1 & \xrightarrow{\tau} & \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1 \\ \downarrow p_0 & & & & & & \downarrow \text{id}_{\mathbb{A}^1} \\ \{0\} & \xrightarrow{i_0} & & & & & \mathbb{A}^1 \end{array}$$

Thus,

$$i_{\infty, \square}^* \circ \tilde{m}^* = (i_0 \circ p_0)^* = p_0^* \circ i_0^*.$$

Similarly  $i_{0, \square}^* \circ \tilde{m}^*$  is induced by

$$\begin{array}{ccccc} \mathbb{A}^1 & \xrightarrow{i_{0, \square}} & \mathbb{A}^1 \times \square^1 & \xrightarrow{\tau} & \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1 \\ x & \longmapsto & (x, 0) & \longmapsto & (x, 1) & \longmapsto & x \end{array}$$

which factors through  $\text{id}_{\mathbb{A}^1} : \mathbb{A}^1 \rightarrow \mathbb{A}^1$  and one has

$$i_{0, \square}^* \circ \tilde{m}^* = \text{id}$$

which concludes the proof of the proposition.  $\square$

**3.3. Weight 1, weight 2 and polylogarithm cycles.** For now on, we set  $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ .

3.3.1. *Two weight 1 cycles generating the  $H^1$ .* As mentioned before, there is a decomposition of  $H^1(\mathcal{N}_X^\bullet(p))$  as

$$H^1(\mathcal{N}_X^\bullet(p)) \simeq H^1(\mathcal{N}_{\mathbb{Q}}^\bullet(p)) \oplus H^0(\mathcal{N}_{\mathbb{Q}}^\bullet(p-1)) \otimes \mathbb{Q}\mathcal{L}_0 \oplus H^0(\mathcal{N}_{\mathbb{Q}}^\bullet(p-1)) \otimes \mathbb{Q}\mathcal{L}_1$$

and  $\mathcal{L}_0$  and  $\mathcal{L}_1$  (which are in weight 1 and degree 1) generates the  $H^*(\mathcal{N}_X^\bullet)$  relatively to  $H^*(\mathcal{N}_{\mathbb{Q}}^\bullet)$ . Explicit expression for  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are given below.

In Example 3.4, a cycle  $\overline{\mathcal{L}_0}$  was constructed using the graph of  $t \mapsto t$  from  $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ . Taking its restriction to  $X \times \square^1$ , and using the same convention, one gets a cycle

$$(5) \quad \mathcal{L}_0 = [t; t] \subset X \times \square^1, \quad \mathcal{L}_0 \in \mathcal{N}_X^1(1).$$

Similarly, using the graph of  $t \mapsto 1-t$ , one gets

$$(6) \quad \mathcal{L}_1 = [t; 1-t] \subset X \times \square^1, \quad \mathcal{L}_1 \in \mathcal{N}_X^1(1).$$

One notices that the cycles  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are both equidimensional over  $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$  but not equidimensional over  $\mathbb{A}^1$ .

Moreover, as

$$\mathcal{L}_0 \cap (X \times \{\varepsilon\}) = \mathcal{L}_0 \cap (\mathbb{P}^1 \setminus \{0, 1, \infty\} \times \{\varepsilon\}) = \emptyset$$

for  $\varepsilon = 0, \infty$ , the above intersection tells us that  $\partial(\mathcal{L}_0) = 0$ . Similarly, one shows that  $\partial(\mathcal{L}_1) = 0$ . Thus  $\mathcal{L}_0$  and  $\mathcal{L}_1$  give two well defined classes in  $H^1(\mathcal{N}_X^\bullet(1))$ .

In order to show that they are non-trivial and that they give the above decomposition of the  $H^1(\mathcal{N}_X^1)$ , one shows that, in the localization sequence, their images under the boundary map

$$H^1(\mathcal{N}_X^\bullet(1)) \xrightarrow{\delta} H^0(\mathcal{N}_{\{0\}}^\bullet(0)) \oplus H^0(\mathcal{N}_{\{1\}}^\bullet(0))$$

are non-zero. It is enough to treat the case of  $\mathcal{L}_0$ . Recall that  $\overline{\mathcal{L}_0}$  is the closure of  $\mathcal{L}_0$  in  $\mathbb{A}^1 \times \square^1$  and is given by the parametrized cycle

$$\overline{\mathcal{L}_0} = [t; t] \subset \mathbb{A}^1 \times \square^1.$$

Its intersection with the face  $u_1 = 0$  is of codimension 1 in  $\mathbb{A}^1 \times \{0\}$  and the intersection with  $u_1 = \infty$  is empty. Hence  $\overline{\mathcal{L}_0}$  is admissible.

Thus, considering the definition of  $\delta$ ,  $\delta(\mathcal{L}_0)$  is given by the intersection of the differential of  $\overline{\mathcal{L}_0}$  with  $\{0\}$  and  $\{1\}$  on the first and second factor, respectively. The above discussion on the admissibility of  $\overline{\mathcal{L}_0}$  tells us that  $\delta(\mathcal{L}_0)$  is non-zero on the factor  $\mathbb{H}^0(\mathcal{N}_{\{0\}}^\bullet(0))$  and 0 on the other factor as the admissibility condition is trivial for  $\mathcal{N}_{\{0\}}^\bullet(0)$  and the restriction of  $\overline{\mathcal{L}_0}$  to 1 is empty. The situation is reverse for  $\mathcal{L}_1$  using its closure  $\overline{\mathcal{L}_1}$  in  $\mathbb{A}^1 \times \square^1$ .

Hence, even if the differentials of  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are 0 in  $\mathcal{N}_X^\bullet$ , the differentials of their closure in  $\mathbb{A}^1$  are non-zero in  $\mathcal{N}_{\mathbb{A}^1}^\bullet$  and have a particular behavior when multiplied by an equidimensional cycle (see Lemma 3.13 below and Equation (11) for an example). We consider here only equidimensional cycles as it is needed to work with such cycles in order to pull-back by the multiplication. We use below notations of propositions 3.10 and 3.12.

LEMMA 3.13. *Let  $C$  be an element in  $\mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}$ , then*

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_0})C = C|_{t=0} \quad \text{and} \quad \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_1})C = C|_{t=1}$$

where the notation  $C|_{t=0}$  (resp.  $C|_{t=1}$ ) denotes, as in Definition 3.3 the (image under the projector  $\mathcal{Alt}$  of the) fiber at 0 (resp. 1) of the irreducible closed subvarieties composing the formal sum that defines  $C$ .

PROOF. It is enough to assume that  $C$  is given by  $C = \mathcal{Alt}(Z)$  where  $Z$  is an irreducible closed subvariety of  $\mathbb{A}^1 \times \square^n$  such that for any face  $F$  of  $\square^n$ , the intersection  $Z \cap (X \times F)$  is empty or the restriction of  $p_1 : \mathbb{A}^1 \times \square^n \rightarrow \mathbb{A}^1$  to

$$Z \cap (\mathbb{A}^1 \times F) \rightarrow \mathbb{A}^1$$

is equidimensional of relative dimension  $\dim(F) - p$ .

Remark that for  $\varepsilon = 0, 1$  the cycle  $\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_\varepsilon})$  is given by the point

$$\{\varepsilon\} \in \mathbb{A}^1$$

which is of codimension 1 in  $\mathbb{A}^1$ . In order to compute the product  $\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_\varepsilon})C$ , one considers first the product in  $\mathbb{A}^1 \times \mathbb{A}^1 \times \square^n$ :

$$\{\varepsilon\} \times Z \subset \mathbb{A}^1 \times \mathbb{A}^1 \times \square^n.$$

Let  $\Delta$  denote the image of the diagonal  $\mathbb{A}^1 \rightarrow \mathbb{A}^1 \times \mathbb{A}^1$ . The equidimensionality of  $Z$  insures that for any face  $F$  of  $\square^n$

$$(\{\varepsilon\} \times Z) \cap (\Delta \times F) \simeq (Z \cap (\{\varepsilon\} \times \square^n)) \cap (\mathbb{A}^1 \times F)$$

is of codimension  $p + 1$ . Thus the product  $\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_\varepsilon})C$  is simply the image under  $\mathcal{Alt}$  of

$$Z \cap (\{\varepsilon\} \times \square^n) = Z|_{t=\varepsilon} \subset \mathbb{A}^1 \times \square^n.$$

□

3.3.2. *A weight 2 example: the Totaro cycle.* One considers the linear combination

$$b = \mathcal{L}_0 \cdot \mathcal{L}_1 \in \mathcal{N}_X^2(2).$$

It is given as a parametrized cycle by

$$b = [t; t, 1 - t] \subset X \times \square^2$$

or in terms of defining equations by

$$T_1 V_1 - U_1 T_2 = 0 \quad \text{and} \quad U_1 V_2 + U_2 V_1 = V_1 V_2$$

where  $T_1$  and  $T_2$  denote the homogeneous coordinates on  $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$  and  $U_i, V_i$  the homogeneous coordinates on each factor  $\square^1 = \mathbb{P}^1 \setminus \{1\}$  of  $\square^2$ . One sees that the intersection of  $b$  with faces  $U_i$  or  $V_i = 0$  for  $i = 1, 2$  is empty because  $T_1$  and  $T_2$  are different from 0 in  $X$  and because  $U_i$  is different from  $V_i$  in  $\square^1$ . Thus it tells us that

$$\partial(b) = 0.$$

Now, let  $\bar{b}$  denote the algebraic closure of  $b$  in  $\mathbb{A}^1 \times \square^2$ . As previously, its expression as parametrized cycle is

$$\bar{b} = \overline{\mathcal{L}_0 \mathcal{L}_1} = [t; t, 1 - t] \subset \mathbb{A}^1 \times \square^2$$

and the intersection with  $\mathbb{A}^1 \times F$  for any codimension 1 face  $F$  of  $\square^2$  is empty. Writing, as before,  $\partial_{\mathbb{A}^1}$  for the differential in  $\mathcal{N}_{\mathbb{A}^1}$ , one has  $\partial_{\mathbb{A}^1}(\bar{b}) = 0$ .

As  $\overline{\mathcal{L}_0}$  (resp.  $\overline{\mathcal{L}_1}$ ) is equidimensional over  $\mathbb{A}^1 \setminus \{0\}$  (resp. over  $\mathbb{A}^1 \setminus \{1\}$ ), the cycle  $\bar{b}$  is equidimensional over  $\mathbb{A}^1 \setminus \{0, 1\}$ . Moreover, as  $\overline{\mathcal{L}_0}$  (resp.  $\overline{\mathcal{L}_1}$ ) has an empty fiber at 1 (resp. at 0),  $\bar{b}$  has empty fiber at both 0 and 1. So  $\bar{b}$  is equidimensional over  $\mathbb{A}^1$  with empty fibers at 0 and 1. Following notations of Proposition 3.12, one defines two elements in  $\mathcal{N}_{\mathbb{A}^1}^1(2)$  by pull back by the multiplication ( resp. twisted multiplication):

$$(7) \quad \overline{\mathcal{L}_{01}} = \mu^*(\bar{b}) \quad \text{and} \quad \overline{\mathcal{L}_{01}^1} = \nu^*(\bar{b}).$$

One also defines their restrictions to  $X$

$$(8) \quad \mathcal{L}_{01} = j^*(\overline{\mathcal{L}_{01}}) \quad \text{and} \quad \mathcal{L}_{01}^1 = j^*(\overline{\mathcal{L}_{01}^1}).$$

Now, direct application of Proposition 3.12 shows that

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{01}}) = -\mu^*(\partial_{\mathbb{A}^1}(\bar{b})) + \bar{b} - p_0^* \circ i_0^*(b) = -0 + \overline{\mathcal{L}_0 \mathcal{L}_1} - 0$$

because  $\bar{b}$  has empty fiber at 0 and is 0 under  $\partial_{\mathbb{A}^1}$ . More generally, as  $j^*$  is a morphism of c.d.g.a., Proposition 3.12 gives the following.

LEMMA 3.14. *Cycles  $\mathcal{L}_{01}$ ,  $\overline{\mathcal{L}_{01}}$ ,  $\mathcal{L}_{01}^1$  and  $\overline{\mathcal{L}_{01}^1}$  satisfy the following properties*

- (1)  $\mathcal{L}_{01}$  and  $\mathcal{L}_{01}^1$  (resp.  $\overline{\mathcal{L}_{01}}$  and  $\overline{\mathcal{L}_{01}^1}$ ) are equidimensional over  $X$ , that is elements in  $\mathcal{N}_X^{eq, 1}(2)$  (resp. equidimensional over  $\mathbb{A}^1$ ).
- (2) They satisfy the following differential equations

$$\partial(\mathcal{L}_{01}) = \partial(\mathcal{L}_{01}^1) = b = \mathcal{L}_0 \mathcal{L}_1$$

$$\text{and } \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{01}}) = \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{01}^1}) = \bar{b} = \overline{\mathcal{L}_0 \mathcal{L}_1}.$$

- (3) By the definition given in Equation (8), the cycle  $\overline{\mathcal{L}_{01}}$  (resp.  $\overline{\mathcal{L}_{01}^1}$ ) extends  $\mathcal{L}_{01}$  (resp.  $\mathcal{L}_{01}^1$ ) over  $\mathbb{A}^1$  and has an empty fiber at 0 (resp. at 1).

Moreover, one can explicitly compute the two pull-backs and obtain parametric representations

$$(9) \quad \mathcal{L}_{01} = [t; 1 - \frac{t}{x}, x, 1 - x], \quad \mathcal{L}_{01}^1 = [t; \frac{x-t}{x-1}, x, 1 - x].$$

The multiplication map inducing  $\mu^*$  is given by

$$\mathbb{A}^1 \times \square^1 \times \square^2 \longrightarrow \mathbb{A}^1 \times \square^2, \quad [t; u_1, u_2, u_3] \longmapsto [\frac{t}{1-u_1}; u_2, u_3].$$

In order to compute the pull-back, one should remark that if  $u = 1 - t/x$  then

$$\frac{t}{1-u} = x.$$

Computing the pull-back by  $\mu^*$  is then just rescaling the new  $\square^1$  factor which arrives in first position. The case of  $\nu^*$  is similar but using the fact that for  $u = \frac{x-t}{x-1}$  one has

$$\frac{t-u}{1-u} = x.$$

REMARK 3.15. The cycle  $\mathcal{L}_{01}$  is nothing but Totaro's cycle [Tot92], already described in [BK94, Blo91].

Moreover,  $\mathcal{L}_{01}$  corresponds to the function  $t \mapsto Li_2^{\mathbb{C}}(t)$  as shown in [BK94].

One recovers the value  $\zeta(2)$  by specializing at  $t = 1$  using the extension of  $\mathcal{L}_{01}$  to  $\mathbb{A}^1$ .

3.3.3. *Polylogarithm cycles.* By induction one can build cycles  $Li_n^{cy} = \mathcal{L}_{0\dots 01}$  ( $n-1$  zeros and one 1). We define  $Li_1^{cy}$  to be equal to  $\mathcal{L}_1$ .

LEMMA 3.16. *For any integer  $n \geq 2$  there exists an equidimensional cycle over  $X$ ,  $Li_n^{cy}$  in  $\mathcal{N}_{X^{eq,1}}^1(n) \subset \mathcal{N}_X^1(n)$  satisfying*

- (1) *There is an equidimensional cycle over  $\mathbb{A}^1$ ,  $\overline{Li}_n^{cy}$  in  $\mathcal{N}_{\mathbb{A}^1}^{eq,1}(n)$ , such that  $Li_n^{cy} = j^*(\overline{Li}_n^{cy})$  (it has in particular a well defined fiber at 1).*
- (2) *The cycle  $\overline{Li}_n^{cy}$  has empty fiber at 0.*
- (3) *The cycles  $Li_n^{cy}$  and  $\overline{Li}_n^{cy}$  satisfy the differential equations*

$$\partial(Li_n^{cy}) = \mathcal{L}_0 \cdot Li_{n-1}^{cy} \quad \text{and} \quad \partial(\overline{Li}_n^{cy}) = \overline{\mathcal{L}}_0 \cdot \overline{Li}_{n-1}^{cy}.$$

- (4)  *$Li_n^{cy}$  is explicitly given as a parametrized cycle by*

$$[t; 1 - \frac{t}{x_{n-1}}, x_{n-1}, 1 - \frac{x_{n-1}}{x_{n-2}}, x_{n-2}, \dots, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \subset X \times \square^{2n-1}.$$

PROOF. For  $n = 2$ , we have already defined  $Li_2^{cy} = \mathcal{L}_{01}$  satisfying the expected properties.

Assume that one has built the cycles  $Li_k^{cy}$  for  $2 \leq k < n$ . One considers in  $\mathcal{N}_{\mathbb{A}^1}^2(n)$  the product

$$\overline{b} = \overline{\mathcal{L}}_0 \cdot \overline{Li}_{n-1}^{cy} = [t; t, 1 - \frac{t}{x_{n-2}}, x_{n-2}, 1 - \frac{x_{n-2}}{x_{n-3}}, x_{n-3}, \dots, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1].$$

As  $\overline{\mathcal{L}}_0$  is equidimensional over  $\mathbb{A}^1 \setminus \{0\}$  and as  $\overline{Li}_{n-1}^{cy}$  is equidimensional over  $\mathbb{A}^1$ ,  $\overline{b}$  is equidimensional over  $\mathbb{A}^1 \setminus \{0\}$ . Moreover, as  $\overline{Li}_{n-1}^{cy}$  has empty fiber at 0,  $\overline{b}$  is equidimensional over  $\mathbb{A}^1$  with empty fiber at 0.

Computing the differential with the Leibniz rule and Lemma 3.13, one gets

$$\partial_{\mathbb{A}^1}(\bar{b}) = \overline{\text{Li}_{n-1}^{cy}}|_{t=0} - \overline{\mathcal{L}_0} \cdot \overline{\mathcal{L}_0} \cdot \overline{\text{Li}_{n-2}^{cy}} = 0.$$

One concludes using Proposition 3.12. The same argument used to obtain the parametrized representation for  $\mathcal{L}_{01}$  at Equation (9) shows that

$$\text{Li}_n^{cy} = [t; 1 - \frac{t}{x_{n-1}}, x_{n-1}, 1 - \frac{x_{n-1}}{x_{n-2}}, x_{n-2}, \dots, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \subset \mathbb{A}^1 \times \square^{2n-1}.$$

□

REMARK 3.17.

- One retrieves the expression given in [BK94].
- Moreover,  $\text{Li}_n^{cy}$  corresponds to the function  $t \mapsto \text{Li}_n^{\mathbb{C}}(t)$  as shown in [BK94] (or in [GGL09]).
- $\overline{\mathcal{L}_0}$  having an empty fiber at 1, one can also pull-back by the twisted multiplication and obtain similarly cycles  $\mathcal{L}_{0\dots 01}^1$  satisfying  $\partial(\mathcal{L}_{0\dots 01}^1) = \partial(\mathcal{L}_{0\dots 01})$ . In some sense, they correspond to  $\mathcal{L}_{0\dots 01} - p^* \circ i_1^*(\mathcal{L}_{0\dots 01})$  which in terms of integrals corresponds to  $\text{Li}_n^{\mathbb{C}}(t) - \zeta(n)$ .

### 3.4. Some higher weight examples for multiple polylogarithm cycles.

3.4.1. *Weight 3.* The cycle  $\mathcal{L}_{01}$  was defined previously, so was the cycle  $\mathcal{L}_{001} = \text{Li}_3^{cy}$  by considering the product

$$b = \mathcal{L}_0 \cdot \mathcal{L}_{01}.$$

Now, in weight 3, one could also consider the product

$$(10) \quad \mathcal{L}_{01} \cdot \mathcal{L}_1 \in \mathcal{N}_X^2(3).$$

However the above product does not lead by similar arguments to a new cycle. Before explaining how to follow the strategy used in weight 2 and for the polylogarithms in order to obtain another weight 3 cycle, the author would like to spend a little time on the obstruction occurring with the product in Equation (10) as it enlightens in particular the need of the cycle  $\mathcal{L}_{01}^1$  previously built.

Thus let  $b = \mathcal{L}_{01} \cdot \mathcal{L}_1$  be the above product in  $\mathcal{N}_X^2(3)$ , given as a parametrized cycle by

$$b = [t; 1 - \frac{t}{x_1}, x_1, 1 - x_1, 1 - t] \subset X \times \square^4.$$

From this expression, one sees that  $b$  is admissible and that  $\partial(b) = 0$  because  $t \in X$  can not be equal to 1.

Let  $\bar{b}$  be the closure of the defining cycle of  $b$  in  $\mathbb{A}^1 \times \square^4$ , that is the image under the projector  $\text{Alt}$  of

$$\left\{ (t, 1 - \frac{t}{x_1}, x_1, 1 - x_1, 1 - t) \text{ such that } t \in \mathbb{A}^1, x_1 \in \mathbb{P}^1 \right\} \cap \mathbb{A}^1 \times \square^4.$$

Let  $u_i$  denote the coordinate on the  $i$ -th factor  $\square^1$ . As most of the intersections of  $\bar{b}$  with face  $\mathbb{A}^1 \times F$  are empty, in order to prove that  $\bar{b}$  is admissible and gives an element in  $\mathcal{N}_{\mathbb{A}^1}^2(3)$ , it is enough to check the (co)dimension condition on the three faces :  $u_1 = 0$ ,  $u_4 = 0$  and  $u_1 = u_4 = 0$ . The intersection of  $\bar{b}$  with the face  $u_1 = u_4 = 0$  is empty as  $u_2 \neq 1$ . The intersection  $\bar{b}$  with the face defined by  $u_1 = 0$  or  $u_4 = 0$  is 1 dimensional and so of codimension 3 in  $\mathbb{A}^1 \times F$ .

Computing the differential in  $\mathcal{N}_{\mathbb{A}^1}^\bullet$ , using Lemma 3.13 or the fact that the intersection with  $u_1 = 0$  is killed by the projector  $\mathcal{A}lt$ , gives

$$(11) \quad \partial_{\mathbb{A}^1}(\bar{b}) = \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{01}\mathcal{L}_1}) = -\overline{\mathcal{L}_{01}}|_{t=1} \neq 0$$

and the homotopy trick used previously will not work as it relies (partly) on beginning with a cycle  $\bar{b}$  satisfying  $\partial_{\mathbb{A}^1}(\bar{b}) = 0$ .

In order to bypass this, one could introduce the constant cycle  $\mathcal{L}_{01}(1) = p^* \circ i_1^*(\overline{\mathcal{L}_{01}})$  and consider the linear combination

$$(12) \quad \bar{b} = (\overline{\mathcal{L}_{01}} - \overline{\mathcal{L}_{01}}(1)) \cdot \overline{\mathcal{L}_1} \in \mathcal{N}_{\mathbb{A}^1}^2(3).$$

and its equivalent in  $\mathcal{N}_X^2(3)$ . Now, the correction by  $-\overline{\mathcal{L}_{01}}(1) \cdot \overline{\mathcal{L}_1}$  insures that  $\partial_{\mathbb{A}^1}(\bar{b}) = 0$ .

However, it is still not good enough as the use of the homotopy property for the pull-back by the multiplication requires to work with equidimensional cycles which is not the case for  $\bar{b}$  (the problem comes from the fiber at 1).

The fact that  $\overline{\mathcal{L}_1}$  is not equidimensional over  $\mathbb{A}^1$  but equidimensional on  $\mathbb{A}^1 \setminus \{1\}$  requires to multiply it by a cycle with an empty fiber at 1 which insures that the fiber of the product at 1 is empty. Thus one considers the product in  $\mathcal{N}_{\mathbb{A}^1}^{eq, 2}(3)$

$$\bar{b} = \overline{\mathcal{L}_{01}^1} \overline{\mathcal{L}_1} = -\overline{\mathcal{L}_1} \overline{\mathcal{L}_{01}^1}$$

which has an empty fiber at 0 and 1. Moreover the Leibniz rule and Lemma 3.13 imply that

$$\partial_{\mathbb{A}^1}(\bar{b}) = \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{01}^1}) \overline{\mathcal{L}_1} - \overline{\mathcal{L}_{01}^1} \partial(\overline{\mathcal{L}_1}) = \overline{\mathcal{L}_1} \overline{\mathcal{L}_0} \overline{\mathcal{L}_1} - \overline{\mathcal{L}_{01}^1}|_{t=1} = 0.$$

Thus one defines

$$(13) \quad \overline{\mathcal{L}_{011}} = \mu^*(\overline{\mathcal{L}_{01}^1} \overline{\mathcal{L}_1}) \quad \text{and} \quad \overline{\mathcal{L}_{011}^1} = \nu^*(\overline{\mathcal{L}_{01}^1} \overline{\mathcal{L}_1})$$

and their restrictions to  $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$$(14) \quad \mathcal{L}_{011} = j^*(\overline{\mathcal{L}_{011}}) \quad \text{and} \quad \mathcal{L}_{011}^1 = j^*(\overline{\mathcal{L}_{011}^1}).$$

As previously, propositions 3.12 and 3.10 insure the following.

LEMMA 3.18. *The cycles  $\mathcal{L}_{011}$ ,  $\overline{\mathcal{L}_{011}}$ ,  $\mathcal{L}_{011}^1$  and  $\overline{\mathcal{L}_{011}^1}$  satisfy the following properties*

- (1)  $\mathcal{L}_{011}$  and  $\mathcal{L}_{011}^1$  (resp.  $\overline{\mathcal{L}_{011}}$  and  $\overline{\mathcal{L}_{011}^1}$ ) are in  $\mathcal{N}_X^{eq, 1}(2)$  (resp. in  $\mathcal{N}_{\mathbb{A}^1}^{eq, 1}(2)$ ).
- (2) They satisfy the following differential equations

$$\partial(\mathcal{L}_{011}) = \partial(\mathcal{L}_{011}^1) = \mathcal{L}_{01}^1 \mathcal{L}_1 = -\mathcal{L}_1 \mathcal{L}_{01}^1$$

$$\text{and } \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{011}}) = \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{011}^1}) = \overline{\mathcal{L}_{01}^1} \overline{\mathcal{L}_1}.$$

- (3) The cycle  $\overline{\mathcal{L}_{011}}$  (resp.  $\overline{\mathcal{L}_{011}^1}$ ) has an empty fiber at 0 (resp. at 1).

3.4.2. *Weight 4.* In weight 4 the first linear combination appears. The situation in weight 4 is given by the following Lemma

LEMMA 3.19. *Let  $W$  be one of the Lyndon words 0001, 0011 or 0111. There exist cycles  $\mathcal{L}_W, \mathcal{L}_W^1$  in  $\mathcal{N}_X^{eq, 1}(4)$  and cycles  $\overline{\mathcal{L}_W}, \overline{\mathcal{L}_W^1}$  in  $\mathcal{N}_{\mathbb{A}^1}^{eq, 1}(4)$  which satisfy the following properties*

- (1)  $\mathcal{L}_W = j^*(\overline{\mathcal{L}_W})$  and  $\mathcal{L}_W^1 = j^*(\overline{\mathcal{L}_W^1})$
- (2)  $\overline{\mathcal{L}_W}$  (resp.  $\overline{\mathcal{L}_W^1}$ ) has an empty fiber at 0 (resp. at 1)

(3) Cycles  $\mathcal{L}_W$  and  $\mathcal{L}_W^1$  for  $W = 0001, 0011$  and  $0111$  satisfy the following differential equations derived from the differential equations satisfied by  $\overline{\mathcal{L}_W}$  and  $\overline{\mathcal{L}_W^1}$

$$(15) \quad \partial(\mathcal{L}_{0001}) = \partial(\mathcal{L}_{0001}^1) = \mathcal{L}_0 \mathcal{L}_{0001},$$

$$(16) \quad \partial(\mathcal{L}_{0011}) = \partial(\mathcal{L}_{0011}^1) = \mathcal{L}_0 \mathcal{L}_{011} + \mathcal{L}_{001}^1 \mathcal{L}_1 - \mathcal{L}_{01} \mathcal{L}_{01}^1$$

and

$$(17) \quad \partial(\mathcal{L}_{0111}) = \partial(\mathcal{L}_{0111}^1) = \mathcal{L}_{011}^1 \mathcal{L}_1.$$

PROOF. The proof goes as before as the main difficulty is to “guess” the differential equations. The case of  $\mathcal{L}_{0001} = \text{Li}_4^{cy}$  and  $\mathcal{L}_{0001}^1$  has already been treated in Lemma 3.16 and the remark afterward. The case of  $\mathcal{L}_{0111}$  and  $\mathcal{L}_{0111}^1$  is extremely similar to the case of  $\mathcal{L}_{011}$ . We will only describe the case of  $\mathcal{L}_{0011}$ . Let  $\bar{b}$  be the element in  $\mathcal{N}_{\mathbb{A}^1}^2(4)$  defined by:

$$\bar{b} = \overline{\mathcal{L}_0 \mathcal{L}_{011}} + \overline{\mathcal{L}_{001}^1 \mathcal{L}_1} - \overline{\mathcal{L}_{01} \mathcal{L}_{01}^1}.$$

All the cycles involved are equidimensional over  $\mathbb{A}^1 \setminus \{0, 1\}$ . As the products in the above equation always involve a cycle with empty fiber at 0 and one with empty fiber at 1, the product has empty fiber at 0 and 1 and is equidimensional over  $\mathbb{A}^1$ .

This shows that  $\bar{b}$  is equidimensional over  $\mathbb{A}^1$  with empty fiber at 0 and 1. One computes  $\partial_{\mathbb{A}^1}(\bar{b})$  using the Leibniz rule, Lemma 3.13 and the previously obtained differential equations:

$$\partial_{\mathbb{A}^1}(\bar{b}) = -\overline{\mathcal{L}_0 \mathcal{L}_{01}^1 \mathcal{L}_1} + \overline{\mathcal{L}_0 \mathcal{L}_{01} \mathcal{L}_1} - \overline{\mathcal{L}_0 \mathcal{L}_1 \mathcal{L}_{01}^1} + \overline{\mathcal{L}_{01} \mathcal{L}_0 \mathcal{L}_1} = 0$$

One can thus define

$$\overline{\mathcal{L}_{0011}} = \mu^*(\bar{b}) \quad \text{and} \quad \overline{\mathcal{L}_{0011}^1} = \nu^*(\bar{b})$$

and conclude with propositions 3.12 and 3.10.  $\square$

3.4.3. *General statement and a weight 5 example.* In weight 5 there are six Lyndon words and the combinatorics of equation (ED-T) leads to six cycles with empty fiber at 0 and six cycles with empty fiber at 1. The general statement proved in [Sou12] is given below.

THEOREM 3.20. *For any Lyndon word  $W$  of length  $p$  greater or equal to 2, there exist two cycles  $\mathcal{L}_W$  and  $\mathcal{L}_W^1$  in  $\mathcal{N}_X^1(p)$  such that :*

- $\mathcal{L}_W, \mathcal{L}_W^1$  are elements in  $\mathcal{N}_X^{eq,1}(p)$ .
- There exist cycles  $\overline{\mathcal{L}_W}, \overline{\mathcal{L}_W^1}$  in  $\mathcal{N}_{\mathbb{A}^1}^{eq,1}(p)$  such that

$$\mathcal{L}_W = j^*(\overline{\mathcal{L}_W}) \quad \text{and} \quad \mathcal{L}_W^1 = j^*(\overline{\mathcal{L}_W^1}).$$

- The restriction of  $\overline{\mathcal{L}_W}$  (resp.  $\overline{\mathcal{L}_W^1}$ ) to the fiber  $t = 0$  (resp.  $t = 1$ ) is empty.
- The cycle  $\mathcal{L}_W$  satisfies the equation

$$(18) \quad \partial(\mathcal{L}_W) = \sum_{U < V} a_{U,V}^W \mathcal{L}_U \mathcal{L}_V + \sum_{U,V} b_{U,V}^W \mathcal{L}_U \mathcal{L}_V^1$$

and resp.  $\mathcal{L}_W^1$  satisfies

$$(19) \quad \partial(\mathcal{L}_W^1) = \sum_{0 < U < V} a'_{U,V} \mathcal{L}_U^1 \mathcal{L}_V^1 + \sum_{U,V} b'_{U,V} \mathcal{L}_U \mathcal{L}_V^1 + \sum_V a'_{0,V} \mathcal{L}_0 \mathcal{L}_V$$



and the same holds for their extensions  $\overline{\mathcal{L}_W}$  and  $\overline{\mathcal{L}_W^1}$  to  $N_{\mathbb{A}^1}^{eq,1}$ . In the above equations  $U$  and  $V$  are Lyndon words of smaller length than  $W$  and the coefficients  $a_{U,V}^W, b_{U,V}^W, a'_{U,V}$  and  $b'_{U,V}$  are integers derived from equation (ED-T).

REMARK 3.21. Without giving a proof which works by induction on the length of  $W$ , the author would like to stress that the construction of the cycles  $\overline{\mathcal{L}_W}$  (resp.  $\overline{\mathcal{L}_W^1}$ ) relies on a geometric argument that has already been described and used here: the pull-back by the (twisted) multiplication  $\mu^*$  (resp.  $\nu^*$ ) gives a homotopy between the identity and  $p^* \circ i_0^*$  (resp.  $p^* \circ i_1^*$ ). Thus, defining

$$\overline{A_W} = \sum_{U < V} a_{U,V}^W \overline{\mathcal{L}_U \mathcal{L}_V} + \sum_{U,V} b_{U,V}^W \overline{\mathcal{L}_U \mathcal{L}_V^1}$$

and

$$\overline{A_W^1} = \sum_{0 < U < V} a'_{U,V} \overline{\mathcal{L}_U^1 \mathcal{L}_V^1} + \sum_{U,V} b'_{U,V} \overline{\mathcal{L}_U \mathcal{L}_V^1} + \sum_V a'_{0,V} \overline{\mathcal{L}_0 \mathcal{L}_V},$$

the cycle  $\overline{\mathcal{L}_W}$  and  $\overline{\mathcal{L}_W^1}$  are defined by

$$(20) \quad \overline{\mathcal{L}_W} = \mu^*(\overline{A_W}) \quad \text{and} \quad \overline{\mathcal{L}_W^1} = \nu^*(\overline{A_W^1}).$$

The fact that  $\overline{A_W}$  (resp.  $\overline{A_W^1}$ ) is equidimensional over  $\mathbb{A}^1$  with empty fiber at 0 (resp. 1) is essentially a consequence of the induction. The main problem is to show that  $\partial_{\mathbb{A}^1}(\overline{A_W}) = \partial_{\mathbb{A}^1}(\overline{A_W^1}) = 0$  which in [Sou12] is deduced after a long preliminary work from the combinatorial situation given by the trees  $T_{W^*}$ .

In weight 5 appears the need of two distinct differential equations and the first example with coefficient different from  $\pm 1$ .

EXAMPLE 3.22. The two cycles associated to the Lyndon word 01011 satisfy

$$(21) \quad \partial(\mathcal{L}_{01011}) = -\mathcal{L}_{01} \mathcal{L}_{011} - \mathcal{L}_1 \mathcal{L}_{0011}^1 - 2\mathcal{L}_{011} \mathcal{L}_{01}^1$$

$$(22) \quad \partial(\mathcal{L}_{01011}^1) = \mathcal{L}_{01}^1 \mathcal{L}_{011}^1 - \mathcal{L}_{011} \mathcal{L}_{01}^1 - \mathcal{L}_{01} \mathcal{L}_{011}^1 - \mathcal{L}_1 \mathcal{L}_{0011}^1.$$

The factor 2 in the last term of  $\partial(\mathcal{L}_{01011})$  is related to the factor 2 appearing in  $d_{cy}(T_{01011^*})$  presented in Equation (4). The term

$$2\mathcal{L}_{011} \mathcal{L}_0 \mathcal{L}_1$$

which is equal to  $\partial(-2\mathcal{L}_{011} \mathcal{L}_{01}^1)$  cancels with one term in  $-\mathcal{L}_0 \mathcal{L}_1 \mathcal{L}_{011}$  coming from  $\partial(-\mathcal{L}_{01} \cdot \mathcal{L}_{011})$  and one term in  $\mathcal{L}_1 \mathcal{L}_0 \mathcal{L}_{011}$  coming from  $\partial(\mathcal{L}_1 \cdot \mathcal{L}_{0011})$ . The whole computation can in fact be done over  $\mathbb{A}^1$  and  $\overline{\mathcal{L}_{01011}}$  is defined as previously as the pull-back by  $\mu^*$  of

$$\overline{b} = -\overline{\mathcal{L}_{01} \mathcal{L}_{011}} - \overline{\mathcal{L}_1 \mathcal{L}_{0011}^1} - 2\overline{\mathcal{L}_{011} \mathcal{L}_{01}^1}.$$

The cycle  $\mathcal{L}_{01011}$  is then its restriction to  $X$ . The above linear combination has an empty fiber at 0 (which allows the use of  $\mu^*$ ). However its fiber at 1 is nonempty and given by

$$-\overline{\mathcal{L}_{01}}|_{t=1} \overline{\mathcal{L}_{011}}|_{t=1}$$

and its pull-back by the twisted multiplication  $\nu^*$  satisfies

$$\partial_{\mathbb{A}^1}(\nu^*(\overline{b})) = \overline{b} + p^* \circ i_1^*(\overline{b}) \neq \overline{b}.$$

That is why we have introduced the linear combination

$$\mathcal{L}_{01}^1 \mathcal{L}_{011}^1 - \mathcal{L}_{011} \mathcal{L}_{01}^1 - \mathcal{L}_{01} \mathcal{L}_{011}^1 - \mathcal{L}_1 \mathcal{L}_{0011}^1$$

whose extension to  $\mathbb{A}^1$  has empty fiber at 1 (but not at 0). This allows us to define

$$\overline{\mathcal{L}_{01011}^1} = \nu^* (\mathcal{L}_{01}^1 \mathcal{L}_{011}^1 - \mathcal{L}_{011} \mathcal{L}_{01}^1 - \mathcal{L}_{01} \mathcal{L}_{011}^1 - \mathcal{L}_1 \mathcal{L}_{0011}^1).$$

#### 4. Parametric and combinatorial representation for the cycles: trees with colored edges

One can give a combinatorial approach to describe cycles  $\mathcal{L}_W$  and  $\mathcal{L}_W^1$  as parametrized cycles using trivalent trees with two types of edge.

DEFINITION 4.1. Let  $\mathcal{T}^{\parallel}$  be the  $\mathbb{Q}$  vector space spanned by rooted trivalent trees such that

- the edges can be of two types:  $|$  or  $\parallel$ ;
- the root vertex is decorated by  $t$ ;
- other external vertices are decorated by 0 or 1.

We say that such a tree is a rooted colored tree or simply a colored tree.

We define two bilinear maps  $\mathcal{T}^{\parallel} \otimes \mathcal{T}^{\parallel} \longrightarrow \mathcal{T}^{\parallel}$  as follows on the colored trees:

- Let  $T_1 \frown T_2$  be the colored tree given by joining the two roots of  $T_1$  and  $T_2$  and adding a new root and a new edge of type  $|$  :

$$T_1 \frown T_2 = \begin{array}{c} \circlearrowleft^t \\ | \\ \bullet \\ \begin{array}{cc} \text{---} \bullet & \text{---} \bullet \\ T_1 & T_2 \end{array} \end{array}$$

where the dotted edges denote either type of edges.

- Let  $T_1 \parallel T_2$  be the colored tree given by joining the two roots of  $T_1$  and  $T_2$  and adding a new root and a new edge of type  $\parallel$  :

$$T_1 \parallel T_2 = \begin{array}{c} \circlearrowleft^t \\ \parallel \\ \bullet \\ \begin{array}{cc} \text{---} \bullet & \text{---} \bullet \\ T_1 & T_2 \end{array} \end{array}$$

where the dotted edges denote either type of edges.

DEFINITION 4.2. Let  $\mathfrak{T}_0$  and  $\mathfrak{T}_1$  be the colored trees defined by

$$\mathfrak{T}_0 = \begin{array}{c} \circlearrowleft^t \\ \parallel \\ \bullet \\ 0 \end{array} \quad \text{and} \quad \mathfrak{T}_1 = \begin{array}{c} \circlearrowleft^t \\ | \\ \bullet \\ 1 \end{array}.$$

For any Lyndon word  $W$  of length greater or equal to 2, let  $\mathfrak{T}_W$  (resp.  $\mathfrak{T}_W^1$ ) be the linear combination of colored trees given by

$$\mathfrak{T}_W = \sum_{U < V} a_{U,V}^W \mathfrak{T}_U \frown \mathfrak{T}_V + \sum_{U,V} b_{U,V}^W \mathfrak{T}_U \parallel \mathfrak{T}_V,$$

and respectively by

$$\mathfrak{T}_W^1 = \sum_{0 < U < V} a'_{U,V} \mathfrak{T}_U^1 \parallel \mathfrak{T}_V^1 + \sum_{U,V} b'_{U,V} \mathfrak{T}_U \parallel \mathfrak{T}_V^1 + \sum_V a'_{0,V} \mathfrak{T}_0 \parallel \mathfrak{T}_V.$$

where the coefficients appearing are the ones from Theorem 3.20.

To a colored tree  $T$  with  $p$  external leaves and a root, one associates a function  $f_T : X \times (\mathbb{P}^1)^{p-1} \longrightarrow X \times (\mathbb{P}^1)^{2p-1}$  as follows :

- Endow  $T$  with its natural order as trivalent tree.
- This induces a numbering of the edges of  $T : (e_1, e_2, \dots, e_{2p-1})$ .
- The edges being oriented away from the root, the numbering of the edges induces a numbering of the vertices  $(v_1, v_2, \dots, v_{2p})$  such that the root is  $v_1$ .
- Associate variables  $x_1, \dots, x_{p-1}$  to each internal vertices such that the numbering of the variables is opposite to the order induced by the numbering of the vertices (first internal vertex has variable  $x_{p-1}$ , second internal vertex has variable  $x_{p-2}$  and so on).
- For each edge  $e_i = \begin{array}{c} a \bullet \\ \vdots \\ b \end{array}$  oriented from  $a$  to  $b$ , define a function

$$f_i(a, b) = \begin{cases} 1 - \frac{a}{b} & \text{if } e_i \text{ is of type } |, \\ \frac{b-a}{b-1} & \text{if } e_i \text{ is of type } \parallel. \end{cases}$$

- Finally  $f_T : X \times (\mathbb{P}^1)^{p-1} \longrightarrow X \times (\mathbb{P}^1)^{2p-1}$  is defined by

$$f_T(t, x_1, \dots, x_{p-1}) = (t, f_1, \dots, f_{2p-1}).$$

Let  $\Gamma(T)$  be the intersection of the image of  $f_T$  with  $X \times \square^{2p-1}$ . One extends the definition of  $\Gamma$  to  $\mathcal{T}^\parallel$  by linearity and thus obtains a *twisted forest cycling map* similar to the one defined by Gangl, Goncharov and Levin in [GGL09].

The map  $\Gamma$  satisfies:

- $\text{Alt}(\Gamma(\mathfrak{T}_0)) = \mathcal{L}_0$  and  $\text{Alt}(\Gamma(\mathfrak{T}_1)) = \mathcal{L}_1$ .
- For any Lyndon word of length  $p \geq 2$ ,

$$\text{Alt}(\Gamma(\mathfrak{T}_W)) = \mathcal{L}_W \quad \text{and} \quad \text{Alt}(\Gamma(\mathfrak{T}_W^1)) = \mathcal{L}_W^1.$$

The fact that  $\Gamma(\mathfrak{T}_0)$  (resp.  $\Gamma(\mathfrak{T}_1)$ ) is the graph of  $t \mapsto t$  (resp.  $t \mapsto 1-t$ ) follows from the definition. Thus one already has  $\Gamma(\mathfrak{T}_0)$  (resp.  $\Gamma(\mathfrak{T}_1)$ ) in  $\mathcal{Z}_{eq}^1(X, 1)$  and

$$\text{Alt}(\Gamma(\mathfrak{T}_0)) = \mathcal{L}_0 \quad \text{and} \quad \text{Alt}(\Gamma(\mathfrak{T}_1)) = \mathcal{L}_1.$$

Then the above property is deduced by induction. Recall that the defining equation (20) for the cycle  $\overline{\mathcal{L}}_W$  is

$$\overline{\mathcal{L}}_W = \mu^* \left( \sum_{U < V} a_{U,V}^W \overline{\mathcal{L}}_U \overline{\mathcal{L}}_V + \sum_{U,V} b_{U,V}^W \overline{\mathcal{L}}_U \overline{\mathcal{L}}_V^1 \right).$$

As already remarked in Example 9, in order to compute the pull-back by  $\mu^*$  one sets the former parameter  $t$  to a new variable  $x_n$  and parametrizes the new  $\square^1$  factor arriving in first position by  $1 - \frac{t}{x_n}$ ;  $t$  is again the parameter over  $X$  or  $\mathbb{A}^1$  depending if one considers cycles over  $\mathbb{A}^1$  or their restriction to  $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ . Thus the expression of  $\mathcal{L}_W$ , restriction of  $\overline{\mathcal{L}}_W$  to  $X$  is exactly given by

$$\mathcal{L}_W = \text{Alt}(\Gamma(\mathfrak{T}_W)).$$

The case of  $\nu^*$  is similar but parametrizing the new  $\square^1$  factor by  $\frac{x_n-t}{x_n-1}$ .

For the previously built examples, we give below the corresponding colored trees and expressions as parametrized cycles (omitting the projector  $\text{Alt}$ ). We also recall the corresponding differential equations as given by Theorem 3.20.

EXAMPLE 4.3 (Weight 1).

$$\mathfrak{T}_0 = \begin{array}{c} \circ \\ | \\ 0 \end{array}, \quad \mathfrak{T}_1 = \begin{array}{c} \circ \\ | \\ 1 \end{array} \quad \text{and} \quad \partial(\mathcal{L}_0) = \partial(\mathcal{L}_1) = 0.$$

We recall below how cycles  $\mathcal{L}_0$  and  $\mathcal{L}_1$  are expressed in terms of parametrized cycles:

$$\mathcal{L}_0 = [t; t] \subset X \times \square^1 \quad \text{and} \quad \mathcal{L}_1 = [t; 1-t] \subset X \times \square^1.$$

EXAMPLE 4.4 (Weight 2).

$$\mathfrak{T}_{01} = \begin{array}{c} \circ \\ / \backslash \\ 0 \quad 1 \end{array}, \quad \mathfrak{T}_{01}^1 = \begin{array}{c} \circ \\ | \\ \begin{array}{c} / \backslash \\ 0 \quad 1 \end{array} \end{array} \quad \text{and} \quad \partial(\mathcal{L}_{01}) = \partial(\mathcal{L}_{01}^1) = \mathcal{L}_0 \mathcal{L}_1$$

We have seen in Equation (9) that cycles  $\mathcal{L}_{01}$  and  $\mathcal{L}_{01}^1$  are given (in  $X \times \square^3$ ) by

$$\mathcal{L}_{01} = [t; 1 - \frac{t}{x_1}, x_1, 1 - x_1] \quad \text{and} \quad \mathcal{L}_{01}^1 = [t; \frac{x_1 - t}{x_1 - 1}, x_1, 1 - x_1].$$

EXAMPLE 4.5 (Weight 3).

$$\partial(\mathcal{L}_{001}) = \partial(\mathcal{L}_{001}^1) = \mathcal{L}_0 \mathcal{L}_{01}, \quad \partial(\mathcal{L}_{011}) = \partial(\mathcal{L}_{011}^1) = -\mathcal{L}_1 \mathcal{L}_{01}^1.$$

$$\mathfrak{T}_{001} = \begin{array}{c} \circ \\ / \backslash \\ \begin{array}{c} / \backslash \\ 0 \quad 1 \end{array} \\ / \backslash \\ 0 \quad 1 \end{array}, \quad \mathfrak{T}_{001}^1 = \begin{array}{c} \circ \\ | \\ \begin{array}{c} / \backslash \\ 0 \quad 1 \end{array} \\ / \backslash \\ 0 \quad 1 \end{array}, \quad \mathfrak{T}_{011} = - \begin{array}{c} \circ \\ / \backslash \\ \begin{array}{c} / \backslash \\ 1 \quad 0 \end{array} \\ / \backslash \\ 1 \quad 0 \end{array}, \quad \mathfrak{T}_{011}^1 = - \begin{array}{c} \circ \\ | \\ \begin{array}{c} / \backslash \\ 1 \quad 0 \end{array} \\ / \backslash \\ 1 \quad 0 \end{array}$$

The corresponding expression as parametrized cycles are given below (following our “twisted forest cycling map”):

$$\begin{aligned} \mathcal{L}_{001} &= [t; 1 - \frac{t}{x_2}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \subset X \times \square^5, \\ \mathcal{L}_{001}^1 &= [t; \frac{x_2 - t}{x_2 - 1}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \subset X \times \square^5 \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{011} &= -[t; 1 - \frac{t}{x_2}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1] \subset X \times \square^5, \\ \mathcal{L}_{011}^1 &= -[t; \frac{x_2 - t}{x_2 - 1}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1] \subset X \times \square^5 \end{aligned}$$

EXAMPLE 4.6 (Weight 4). The differential equations satisfied by the weight 4 cycles are:

$$\begin{aligned} \partial(\mathcal{L}_{0001}) &= \partial(\mathcal{L}_{0001}^1) = \mathcal{L}_0 \mathcal{L}_{001} \\ \partial(\mathcal{L}_{0011}) &= \partial(\mathcal{L}_{0011}^1) = \mathcal{L}_0 \mathcal{L}_{011} - \mathcal{L}_1 \mathcal{L}_{001}^1 - \mathcal{L}_{01} \mathcal{L}_{01}^1 \\ \partial(\mathcal{L}_{0111}) &= \partial(\mathcal{L}_{0111}^1) = -\mathcal{L}_1 \mathcal{L}_{011}^1 \end{aligned}$$

The corresponding colored trees are given by:

$$\mathfrak{T}_{0001} = \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ 0 \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 0 \quad 1 \end{array}, \quad \mathfrak{T}_{0001}^1 = \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ 0 \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 0 \quad 1 \end{array}, \quad \mathfrak{T}_{0111} = \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ 1 \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 0 \quad 1 \end{array}, \quad \mathfrak{T}_{0111}^1 = \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ 1 \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 0 \quad 1 \end{array},$$

$$\mathfrak{T}_{0011} = - \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ 0 \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 1 \quad 1 \end{array} - \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ 1 \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 0 \quad 1 \end{array} - \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 0 \quad 1 \end{array}$$

and

$$\mathfrak{T}_{0011}^1 = - \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ 0 \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 1 \quad 1 \end{array} - \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ 1 \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 0 \quad 1 \end{array} - \begin{array}{c} t \odot \\ | \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \quad \swarrow \quad \searrow \\ \quad 0 \quad 1 \end{array}.$$

The expressions as parametrized cycles of  $\mathcal{L}_{0001}$ ,  $\mathcal{L}_{0001}^1$ ,  $\mathcal{L}_{0111}$  and  $\mathcal{L}_{0111}^1$  are given below (in  $X \times \square^7$ ):

$$\mathcal{L}_{0001} = [t; 1 - \frac{t}{x_3}, x_3, 1 - \frac{x_3}{x_2}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1],$$

$$\mathcal{L}_{0001}^1 = [t; \frac{x_3 - t}{x_3 - 1}, x_3, 1 - \frac{x_3}{x_2}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1],$$

$$\mathcal{L}_{0111} = [t; 1 - \frac{t}{x_3}, 1 - x_3, \frac{x_2 - x_3}{x_2 - 1}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1],$$

$$\mathcal{L}_{0111}^1 = [t; \frac{x_3 - t}{x_3 - 1}, 1 - x_3, \frac{x_2 - x_3}{x_2 - 1}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1],$$

while the expressions for  $\mathcal{L}_{0011}$  and  $\mathcal{L}_{0011}^1$  involved linear combinations:

$$\begin{aligned} \mathcal{L}_{0011} = & - [t; 1 - \frac{t}{x_3}, x_3, 1 - \frac{x_3}{x_2}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1] \\ & - [t; 1 - \frac{t}{x_3}, 1 - x_3, \frac{x_2 - x_3}{x_2 - 1}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \\ & - [t; 1 - \frac{t}{x_3}, 1 - \frac{x_3}{x_2}, x_2, 1 - x_2, \frac{x_1 - x_3}{x_1 - 1}, x_1, 1 - x_1] \end{aligned}$$

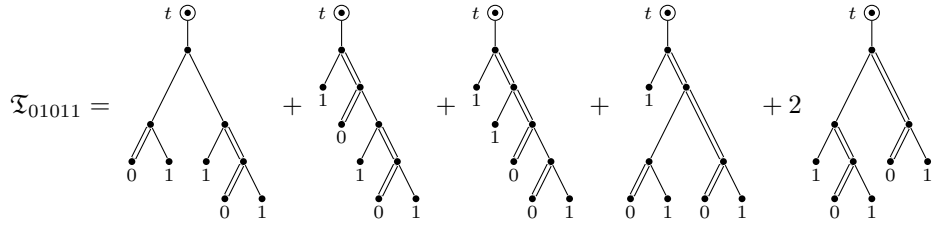
and

$$\begin{aligned} \mathcal{L}_{0011}^1 = & - [t; \frac{x_3 - t}{x_3 - 1}, x_3, 1 - \frac{x_3}{x_2}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1] \\ & - [t; \frac{x_3 - t}{x_3 - 1}, 1 - x_3, \frac{x_2 - x_3}{x_2 - 1}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \\ & - [t; \frac{x_3 - t}{x_3 - 1}, 1 - \frac{x_3}{x_2}, x_2, 1 - x_2, \frac{x_1 - x_3}{x_1 - 1}, x_1, 1 - x_1]. \end{aligned}$$

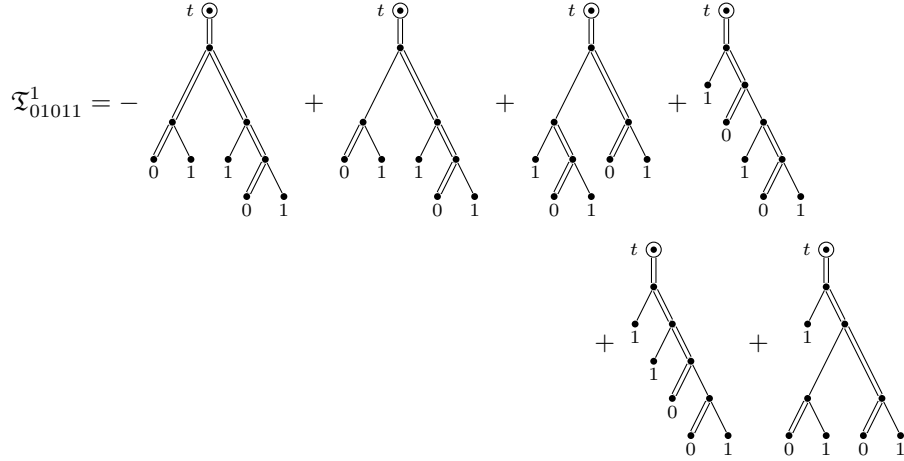
EXAMPLE 4.7 (Weight 5). The differential equations satisfied by  $\mathcal{L}_{01011}$  and  $\mathcal{L}_{01011}^1$  are:

$$\begin{aligned}\partial(\mathcal{L}_{01011}) &= -\mathcal{L}_{01} \cdot \mathcal{L}_{011} - \mathcal{L}_1 \mathcal{L}_{0011}^1 - 2\mathcal{L}_{011} \mathcal{L}_{01}^1 \\ \partial(\mathcal{L}_{01011}^1) &= \mathcal{L}_{01}^1 \cdot \mathcal{L}_{011}^1 - \mathcal{L}_{011} \cdot \mathcal{L}_{01}^1 - \mathcal{L}_{01} \cdot \mathcal{L}_{011}^1 - \mathcal{L}_1 \cdot \mathcal{L}_{0011}^1.\end{aligned}$$

The corresponding colored trees are given by:



and



The corresponding expression as parametrized cycles are given below (in  $X \times \square^9$ ):

$$\begin{aligned}\mathcal{L}_{01011} &= [t; 1 - \frac{t}{x_4}, 1 - \frac{x_4}{x_3}, x_3, 1 - x_3, 1 - \frac{x_4}{x_2}, 1 - x_2, \frac{x_1 - x_2}{x_2 - 1}, x_1, 1 - x_1] \\ &+ [t; 1 - \frac{t}{x_4}, 1 - x_4, \frac{x_3 - x_4}{x_3 - 1}, x_3, 1 - \frac{x_3}{x_2}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1] \\ &+ [t; 1 - \frac{t}{x_4}, 1 - x_4, \frac{x_3 - x_4}{x_3 - 1}, 1 - x_3, \frac{x_2 - x_3}{x_2 - 1}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \\ &+ [t; 1 - \frac{t}{x_4}, 1 - x_4, \frac{x_3 - x_4}{x_3 - 1}, 1 - \frac{x_3}{x_2}, x_2, 1 - x_2, \frac{x_1 - x_3}{x_1 - 1}, x_1, 1 - x_1] \\ &+ 2[t; 1 - \frac{t}{x_4}, 1 - \frac{x_4}{x_3}, 1 - x_3, \frac{x_2 - x_3}{x_2 - 1}, x_2, 1 - x_2, \frac{x_1 - x_4}{x_1 - 1}, x_1, 1 - x_1],\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{01011}^1 = & -[t, \frac{x_4 - t}{x_4 - 1}, \frac{x_3 - x_4}{x_3 - 1}, x_3, 1 - x_3, \frac{x_2 - x_4}{x_2 - 1}, 1 - x_2, \frac{x_1 - x_2}{x_2 - 1}, x_1, 1 - x_1] \\
& + [t, \frac{x_4 - t}{x_4 - 1}, 1 - \frac{x_4}{x_3}, x_3, 1 - x_3, \frac{x_2 - x_4}{x_2 - 1}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1] \\
& + [t, \frac{x_4 - t}{x_4 - 1}, 1 - \frac{x_4}{x_3}, 1 - x_3, \frac{x_2 - x_3}{x_2 - 1}, x_2, 1 - x_2, \frac{x_1 - x_4}{x_1 - 1}, x_1, 1 - x_1] \\
& + [t, \frac{x_4 - t}{x_4 - 1}, 1 - x_4, \frac{x_3 - x_4}{x_3 - 1}, x_3, 1 - \frac{x_3}{x_2}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1] \\
& + [t, \frac{x_4 - t}{x_4 - 1}, 1 - x_4, \frac{x_3 - x_4}{x_3 - 1}, 1 - x_3, \frac{x_2 - x_3}{x_2 - 1}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \\
& + [t, \frac{x_4 - t}{x_4 - 1}, 1 - x_4, \frac{x_3 - x_4}{x_3 - 1}, 1 - \frac{x_3}{x_2}, x_2, 1 - x_2, \frac{x_1 - x_3}{x_1 - 1}, x_1, 1 - x_1].
\end{aligned}$$

## 5. Bar construction settings

In the cycle motives setting, a motive over  $X$  is a comodule on the  $H^0$  of the bar construction over  $\mathcal{N}_X^\bullet$  modulo shuffle products. For more details, one can look at the works of Bloch and Kriz [BK94], Spitzweck [Spi, Spi01] (i.e. as presented in [Lev05]) and Levine [Lev11].

In this context, the cycles constructed above, which are expected to correspond to multiple polylogarithms (as outlined in Section 6), induce elements in this  $H^0$  and naturally gives rise to an associated comodule, thus to mixed Tate motives corresponding to multiple polylogarithms.

Before, giving explicit expressions for the induced elements in the bar construction, the beginning of the section is devoted to a short review of the bar construction.

**5.1. Bar construction.** As there does not seem to exist a global sign convention for the various operations on the bar construction, the main definitions in the cohomological setting are recalled below following the (homological) description given in [LV12].

Let  $A$  be a commutative differential graded algebra (c.d.g.a.) with augmentation  $\varepsilon : A \rightarrow \mathbb{Q}$ , with product  $\mu_A$  and let  $A^+$  be the augmentation ideal  $A^+ = \ker(\varepsilon)$ . Note again that *commutative* in this context stand for *graded commutative*.

In order to understand the sign convention below and the “bar grading”, one should think of the bar construction as built on the tensor coalgebra over the shifted (suspended) graded vector space  $A^+[1]$ .

**DEFINITION 5.1.** The bar construction  $B(A)$  over  $A$  is the tensor coalgebra over the suspension of  $A^+$ .

- In particular, as vector space  $B(A)$  is given by :

$$B(A) = T(A^+) = \bigoplus_{n \geq 0} (A^+)^{\otimes n}.$$

- A homogeneous element  $\mathbf{a}$  of tensor degree  $n$  is denoted using the *bar* notation, that is

$$\mathbf{a} = [a_1 | \dots | a_n]$$

and its degree is

$$\deg_B(\mathbf{a}) = \sum_{i=1}^n (\deg_A(a_i) - 1).$$

- The coalgebra structure comes from the natural deconcatenation coproduct, that is

$$\Delta([a_1 | \dots | a_n]) = \sum_{i=0}^n [a_1 | \dots | a_i] \otimes [a_{i+1} | \dots | a_n].$$

REMARK 5.2. This construction can be seen as a simplicial total complex associated to the complex  $A$  (Cf. [BK94]). The augmentation makes it possible to use directly  $A^+$  without referring to the tensor coalgebra over  $A$  and without the need of killing the degeneracies.

However this simplicial presentation usually masks the need of working with the shifted complex.

We associate to any bar element  $[a_1 | \dots | a_n]$  the function  $\eta(i)$  giving its “partial” degree

$$\eta(i) = \sum_{k=1}^i (\deg_A(a_k) - 1).$$

The original differential  $d_A$  induces a differential  $D_1$  on  $B(A)$  given by

$$D_1([a_1 | \dots | a_n]) = - \sum_{i=1}^n (-1)^{\eta(i-1)} [a_1 | \dots | d_A(a_i) | \dots | a_n]$$

where the initial minus sign comes from the fact that the differential on the shifted complex  $A[1]$  is  $-d_A$ . Moreover, the multiplication on  $A$  induces another differential  $D_2$  on  $B(A)$  given by

$$D_2([a_1 | \dots | a_n]) = - \sum_{i=1}^{n-1} (-1)^{\eta(i)} [a_1 | \dots | \mu_A(a_i, a_{i+1}) | \dots | a_n]$$

where the signs are coming from Koszul commutation rules (due to the shifting). One checks that the two differentials anticommute providing  $B(A)$  with a total differential.

DEFINITION 5.3. The total differential on  $B(A)$  is defined by

$$d_{B(A)} = D_1 + D_2.$$

The last structure arising with the bar construction is the graded shuffle product

$$[a_1 | \dots | a_n] \amalg [a_{n+1} | \dots | a_{n+m}] = \sum_{\sigma \in sh(n,m)} (-1)^{\varepsilon_{gr}(\sigma)} [a_{\sigma(1)}, \dots, a_{\sigma(n+m)}]$$

where  $sh(n, m)$  denotes the permutation of  $\{1, \dots, n+m\}$  such that if  $1 \leq i < j \leq n$  or  $n+1 \leq i < j \leq n+m$  then  $\sigma(i) < \sigma(j)$ . The sign is the graded signature of the permutation (for the degree in  $A^+[1]$ ) given by

$$\varepsilon_{gr}(\sigma) = \sum_{\substack{i < j \\ \sigma(i) > \sigma(j)}} (\deg_A(a_i) - 1)(\deg_A(a_j) - 1).$$

With these definitions, one can explicitly check the following



PROPOSITION 5.4. *Let  $A$  be a (Adams/weight graded) c.d.g.a. The operations  $\Delta$ ,  $d_{B(A)}$  and  $\mathfrak{m}$  together with the obvious unit and counit give  $B(A)$  a structure of (Adams graded) commutative graded differential Hopf algebra.*

*In particular, these operations induce on  $H^0(B(A))$ , and more generally on  $H^*(B(A))$ , a (Adams graded) commutative Hopf algebra structure. This (Adams graded) algebra is cohomologically graded in the case of  $H^*(B(A))$  and cohomologically graded concentrated in degree 0 in the case of  $H^0(B(A))$ .*

We recall that the set of indecomposable elements of an augmented c.d.g.a. is defined as the augmentation ideal  $I$  modulo products, that is  $I/I^2$ . Applying a general fact about Hopf algebras, the coproduct structure on  $H^0(B(A))$  induces a coLie algebra structure on its set of indecomposable elements.

**5.2. Bar elements.** Considering the bar construction over  $\mathcal{N}_X^\bullet$ , part of the issue is to associate to any cycle  $\mathcal{L}_W$  and  $\mathcal{L}_W^1$  a corresponding element in  $H^0(B(\mathcal{N}_X^\bullet))$ .

As the weight 1 cycles  $\mathcal{L}_0$  and  $\mathcal{L}_1$  have 0 differential in  $\mathcal{N}_X^\bullet$ , there are obvious corresponding bar elements:

$$(23) \quad \mathcal{L}_0^B = [\mathcal{L}_0] \quad \text{and} \quad \mathcal{L}_1^B = [\mathcal{L}_1].$$

Let  $\mathcal{M}_X$  denote the indecomposable elements of  $H^0(B(\mathcal{N}_X^\bullet))$  and let  $\tau$  be the morphism exchanging the two factors of  $H^0(B(\mathcal{N}_X^\bullet)) \otimes H^0(B(\mathcal{N}_X^\bullet))$ . We denote by

$$d_\Delta = \frac{1}{2}(\Delta - \tau\Delta)$$

the differential on the coLie algebra  $\mathcal{M}_X$  induced by the coproduct on  $H^0(B(\mathcal{N}_X^\bullet))$ . In general, one should have the following.

CLAIM. *For any Lyndon word  $W$  (of length greater or equal to 2), there exist elements  $\mathcal{L}_W^B$  and  $\mathcal{L}_W^{1,B}$  in  $B(\mathcal{N}_X^\bullet)$  of bar degree 0 satisfying:*

- *Let  $d_B$  denotes the total bar differential  $d_B = d_{B(\mathcal{N}_X^\bullet)}$ . Then one has:*

$$d_B(\mathcal{L}_W^B) = d_B(\mathcal{L}_W^{1,B}) = 0.$$

- *The tensor degree 1 part of  $\mathcal{L}_W^B$  (resp.  $\mathcal{L}_W^{1,B}$ ) is given by  $[\mathcal{L}_W]$  (resp.  $[\mathcal{L}_W^1]$ ).*
- *The elements  $\mathcal{L}_W^B$  (resp.  $\mathcal{L}_W^{1,B}$ ) satisfy the differential equation (18) (resp. (19)) in  $\mathcal{M}_X$ . That is*

$$d_\Delta(\mathcal{L}_W^B) = - \left( \sum_{U < V} a_{U,V}^W \mathcal{L}_U^B \mathcal{L}_V^B + \sum_{U,V} b_{U,V}^W \mathcal{L}_U^B \mathcal{L}_V^{1,B} \right) \in \mathcal{M}_X \wedge \mathcal{M}_X$$

and

$$d_\Delta(\mathcal{L}_W^{1,B}) = - \left( \sum_{0 < U < V} a'_{U,V}{}^W \mathcal{L}_U^{1,B} \mathcal{L}_V^{1,B} + \sum_{U,V} b'_{U,V}{}^W \mathcal{L}_U^B \mathcal{L}_V^{1,B} + \sum_V a'_{0,V} \mathcal{L}_0^B \mathcal{L}_V^B \right) \in \mathcal{M}_X \wedge \mathcal{M}_X$$

where the overall minus sign is due to shifting reasons.

The obstruction for proving the general statement lies in the control of the global combinatorics relating  $D_1$ ,  $D_2$  and the two systems of differential equations.

Below, one finds some elements  $\mathcal{L}_W^B$  and  $\mathcal{L}_W^{1,B}$  corresponding to the previously described examples together with some relations among those elements. Once the element  $\mathcal{L}_W^B$  are explicitly described, it is a straightforward computation to check that it lies in the kernel of  $d_B$  and this verification will be omitted.

Note that all cycles  $\mathcal{L}_W$  and  $\mathcal{L}_W^1$  are in cohomological degree 1, that is in  $\mathcal{N}_X^1$ . Thus, signs appearing in the operations on the bar construction are much simpler as all terms in  $\deg_A(a_i) - 1$  are 0.

EXAMPLE 5.5 (Weight 2). Cycles  $\mathcal{L}_{01}$  and  $\mathcal{L}_{01}^1$  satisfy  $\partial(\mathcal{L}_{01}) = \partial(\mathcal{L}_{01}^1) = \mathcal{L}_0\mathcal{L}_1$ . Thus one can define

$$(24) \quad \mathcal{L}_{01}^B = [\mathcal{L}_{01}] - \frac{1}{2}([\mathcal{L}_0|\mathcal{L}_1] - [\mathcal{L}_1|\mathcal{L}_0]) \quad \text{and} \quad \mathcal{L}_{01}^{1,B} = [\mathcal{L}_{01}^1] - \frac{1}{2}([\mathcal{L}_0|\mathcal{L}_1] - [\mathcal{L}_1|\mathcal{L}_0]).$$

Remark that, looking at things modulo products, that is in  $\mathcal{M}_X$ , the tensor degree 2 involves some choices. Instead of

$$-\frac{1}{2}([\mathcal{L}_0|\mathcal{L}_1] - [\mathcal{L}_1|\mathcal{L}_0]),$$

we could have used

$$-[\mathcal{L}_0|\mathcal{L}_1] \quad \text{or} \quad [\mathcal{L}_1|\mathcal{L}_0]$$

and obtained the same elements in  $\mathcal{M}_X$  as

$$-\frac{1}{2}([\mathcal{L}_0|\mathcal{L}_1] - [\mathcal{L}_1|\mathcal{L}_0]) = -[\mathcal{L}_0|\mathcal{L}_1] + \frac{1}{2}\mathcal{L}_0^B \boxtimes \mathcal{L}_1^B = [\mathcal{L}_1|\mathcal{L}_0] - \frac{1}{2}\mathcal{L}_0^B \boxtimes \mathcal{L}_1^B.$$

The above choice reflects in some sense that there is no preferred choice for either

$$\partial(\mathcal{L}_{01}) = \partial(\mathcal{L}_{01}^1) = \mathcal{L}_0\mathcal{L}_1 \quad \text{or} \quad \partial(\mathcal{L}_{01}) = \partial(\mathcal{L}_{01}^1) = -\mathcal{L}_1\mathcal{L}_0.$$

Recall that we have defined a cycle  $\mathcal{L}_{01}(1)$  in  $\mathcal{N}_X^1$  by

$$\mathcal{L}_{01}(1) = j^*(p^* \circ i_1^*(\overline{\mathcal{L}_{01}}))$$

Building the cycle  $\mathcal{L}_{011}$ , we have introduced the cycle  $\mathcal{L}_{01}^1$  instead of using the difference  $\mathcal{L}_{01} - \mathcal{L}_{01}(1)$  in order to keep working with equidimensional cycles. The ‘‘correspondence’’

$$\mathcal{L}_{01}^1 \leftrightarrow \mathcal{L}_{01} - \mathcal{L}_{01}(1)$$

becomes an equality in  $H^0(B(\mathcal{N}_X^\bullet))$ .

More precisely, using either the commutation of the above morphisms with the differential or the expression of  $\mathcal{L}_{01}(1)$  as parametrized cycle, one sees that  $\partial(\mathcal{L}_{01}(1)) = 0$  and one defines

$$\mathcal{L}_{01}^B(1) = [\mathcal{L}_{01}(1)].$$

A direct computation shows that

$$\mathcal{L}_{01}(1) = [t; 1 - \frac{1}{x_1}, x_1, 1 - x_1].$$

Now, from the expressions of  $\mathcal{L}_{01}$ ,  $\mathcal{L}_{01}(1)$  and  $\mathcal{L}_{01}^1$  as parametrized cycles, one checks that in  $\mathcal{N}_X^1$

$$\mathcal{L}_{01} - \mathcal{L}_{01}(1) = \mathcal{L}_{01}^1 + \partial(C_{01})$$

where  $C_{01}$  is the element of  $\mathcal{N}_X^0$  defined by

$$C_{01} = -[t; y, \frac{y - \frac{x_1 - t}{x_1}}{y - \frac{x_1 - 1}{x_1}}, x_1, 1 - x_1] \subset X \times \square^4.$$

The bar element  $C_{01}^B = [C_{01}]$  is of bar degree  $-1$  and gives in  $B(\mathcal{N}_X^\bullet)$

$$\mathcal{L}_{01}^B - \mathcal{L}_{01}^B(1) = \mathcal{L}_{01}^{1,B} - d_B(C_{01}^B)$$

and thus, the equality  $\mathcal{L}_{01}^B - \mathcal{L}_{01}^B(1) = \mathcal{L}_{01}^{1,B}$  in the  $H^0$ .

For these weight 2 examples, computing the deconcatenation coproduct is trivial and gives the expected relation

$$d_\Delta(\mathcal{L}_{01}^B) = d_\Delta(\mathcal{L}_{01}^{1,B}) = -\mathcal{L}_0^B \wedge \mathcal{L}_1^B.$$

Finally the motive corresponding to  $\mathcal{L}_{01}$  is the comodule generated by  $\mathcal{L}_{01}^B$ , that is the subvector space of  $\mathcal{M}_X$  spanned by  $\mathcal{L}_{01}^B$ ,  $\mathcal{L}_0^B$  and  $\mathcal{L}_1^B$ .

EXAMPLE 5.6 (Weight 3). The differentials of  $\mathcal{L}_{001}$ ,  $\mathcal{L}_{001}^1$ ,  $\mathcal{L}_{011}$ ,  $\mathcal{L}_{011}^1$  allow us to easily write down the corresponding tensor degree 1 and 2. The expressions below try to keep a symmetric presentation for the part in tensor degree 3.

In the equations below, cycles  $\mathcal{L}_W$  are simply denoted by  $W$  and cycles  $\mathcal{L}_W^1$  simply by  $\overline{W}$ . We will also use this abuse of notation later on in weight 4. One defines

$$\begin{aligned} \mathcal{L}_{001}^B &= [001] - \frac{1}{2}([0|01] - [01|0]) + \frac{1}{4}([0|0|1] - [0|1|0] + [1|0|0]), \\ \mathcal{L}_{001}^{1,B} &= [\overline{001}] - \frac{1}{2}([0|01] - [01|0]) + \frac{1}{4}([0|0|1] - [0|1|0] + [1|0|0]) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{011}^B &= [011] - \frac{1}{2}([\overline{01}1] - [1\overline{01}]) + \frac{1}{4}([0|1|1] - [1|0|1] + [1|1|0]), \\ \mathcal{L}_{011}^{1,B} &= [\overline{011}] - \frac{1}{2}([\overline{01}1] - [1\overline{01}]) + \frac{1}{4}([0|1|1] - [1|0|1] + [1|1|0]). \end{aligned}$$

As the cycles  $\mathcal{L}_{001}$  and  $\mathcal{L}_{001}^1$  (resp.  $\mathcal{L}_{011}$  and  $\mathcal{L}_{011}^1$ ) differ only by their first  $\square^1$  factors, the arguments used to compare  $\mathcal{L}_{01}^B$  and  $\mathcal{L}_{01}^{1,B}$  apply here and give:

$$\mathcal{L}_{001}^B - \mathcal{L}_{001}^B(1) = \mathcal{L}_{001}^{1,B} \quad \text{and} \quad \mathcal{L}_{011}^B - \mathcal{L}_{011}^B(1) = \mathcal{L}_{011}^{1,B} \in \mathcal{M}_X.$$

The ‘‘correction’’ cycles giving the explicit relations between the cycles are

$$C_{001} = -[t; s, \frac{s - \frac{x_2 - t}{x_2}}{s - \frac{x_2 - 1}{x_2}}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1]$$

and

$$C_{011} = [t; s, \frac{s - \frac{x_2 - t}{x_2}}{s - \frac{x_2 - 1}{x_2}}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1].$$

Now, computing the reduced coproduct  $\Delta' = \Delta - 1 \otimes \text{id} - \text{id} \otimes 1$  of  $\mathcal{L}_{001}^B$  gives:

$$\begin{aligned} \Delta'(\mathcal{L}_{001}^B) &= -\frac{1}{2}([0] \otimes [01] - [01] \otimes [0]) \\ &\quad + \frac{1}{4}([0] \otimes [0|1] - [0] \otimes [1|0] + [1] \otimes [0|0] \\ &\quad \quad \quad + [0|0] \otimes [1] - [0|1] \otimes [0] + [1|0] \otimes [0]). \end{aligned}$$

As  $[0|0] = 1/2\mathcal{L}_0^B \sqcup \mathcal{L}_0^B$ , one has modulo products

$$\Delta'(\mathcal{L}_{001}^B) = -\frac{1}{2} \left( [0] \otimes \left( [01] - \frac{1}{2}([0|1] - [1|0]) \right) - \left( [01] - \frac{1}{2}([0|1] - [1|0]) \right) \otimes [0] \right).$$

Similar computations apply to  $\mathcal{L}_{001}^{1,B}$ ,  $\mathcal{L}_{011}^B$  and  $\mathcal{L}_{011}^{1,B}$  and give in  $\mathcal{M}_X \wedge \mathcal{M}_X$ :

$$d_\Delta(\mathcal{L}_{001}^B) = d_\Delta(\mathcal{L}_{001}^{1,B}) = -\mathcal{L}_0^B \wedge \mathcal{L}_{01}^B \quad \text{and} \quad d_\Delta(\mathcal{L}_{011}^B) = d_\Delta(\mathcal{L}_{011}^{1,B}) = -\mathcal{L}_{01}^{1,B} \wedge \mathcal{L}_1^B.$$

One should remark that the equality  $\mathcal{L}_{01}^{1,B} = \mathcal{L}_{01}^B - \mathcal{L}_{01}^B(1)$  in the  $H^0$  implies

$$(25) \quad d_\Delta(\mathcal{L}_{011}^B) = -(\mathcal{L}_{01}^B - \mathcal{L}_{01}^B(1)) \wedge \mathcal{L}_1^B$$

which is (up to a global minus sign) the equation satisfied by  $T_{011^*}$  as shown at Example 2.13.

Finally, the corresponding comodules giving motives associated to the cycle  $\mathcal{L}_{001}$  and  $\mathcal{L}_{011}$  are the subvector spaces of  $\mathcal{M}_X$  generated respectively by

$$\langle \mathcal{L}_{001}^B, \mathcal{L}_{01}^B, \mathcal{L}_0^B, \mathcal{L}_1^B \rangle$$

and

$$\langle \mathcal{L}_{011}^B, \mathcal{L}_{01}^B, \mathcal{L}_{01}^B(1), \mathcal{L}_0^B, \mathcal{L}_1^B \rangle.$$

The above arguments apply similarly in weight 4. Hence, we will describe below the case of  $\mathcal{L}_{0011}$  as it gives a ‘‘preview’’ of the combinatorial difficulties related to the bar construction context.

EXAMPLE 5.7 (Weight 4:  $\mathcal{L}_{0011}^B$ ). We give below an element  $\mathcal{L}_{0011}^B$  in the bar construction with zero differential and with tensor degree 1 part equal to  $\mathcal{L}_{0011}$ :

$$\begin{aligned} \mathcal{L}_{0011}^B = & [0011] - \frac{1}{2} ([0|011] - [0|011] + [\overline{001}|1] - [1|\overline{001}] - [01|\overline{01}] + [\overline{01}|01]) \\ & + \frac{1}{4} (-[0|1|\overline{01}] + [0|1|01] - [01|0|1] + [\overline{01}|0|1] - [\overline{01}|1|0] + [01|1|0] \\ & - [1|0|01] + [1|0|\overline{01}] + [1|\overline{01}|0] + [1|01|0] + [0|\overline{01}|1] + [0|01|1]) \\ & - \frac{1}{2} ([0|0|1|1] - [1|1|0|0]). \end{aligned}$$

Identifying the reduced coproduct of  $\mathcal{L}_{0011}^B$  with

$$(26) \quad -\frac{1}{2} \left( \mathcal{L}_0^B \otimes \mathcal{L}_{011}^B - \mathcal{L}_{011}^B \otimes \mathcal{L}_0^B + \mathcal{L}_{001}^{1,B} \otimes \mathcal{L}_1^B - \mathcal{L}_1^B \otimes \mathcal{L}_{001}^{1,B} \right. \\ \left. - \mathcal{L}_{01}^B \otimes \mathcal{L}_{01}^{1,B} + \mathcal{L}_{01}^{1,B} \otimes \mathcal{L}_{01}^B \right)$$

is more difficult than in the previous cases. First of all, one remarks that in the above expression the terms in  $(\mathcal{N}_X^1)^{\otimes 2} \otimes (\mathcal{N}_X^1)^{\otimes 2}$  are coming only from  $-\mathcal{L}_{01}^B \otimes \mathcal{L}_{01}^{1,B}$  and  $\mathcal{L}_{01}^{1,B} \otimes \mathcal{L}_{01}^B$  and cancel each other. Thus the expression (26) has no term in  $(\mathcal{N}_X^1)^{\otimes 2} \otimes (\mathcal{N}_X^1)^{\otimes 2}$ . On the other hand, the terms in  $(\mathcal{N}_X^1)^{\otimes 2} \otimes (\mathcal{N}_X^1)^{\otimes 2}$  coming from  $\Delta'(\mathcal{L}_{0011}^B)$  are given by

$$\begin{aligned} -\frac{1}{2} ([0|0] \otimes [1|1] - [1|1] \otimes [0|0]) = \\ -\frac{1}{8} (\mathcal{L}_0^B \text{ III } \mathcal{L}_0^B \otimes \mathcal{L}_1^B \text{ III } \mathcal{L}_1^B - \mathcal{L}_1^B \text{ III } \mathcal{L}_1^B \otimes \mathcal{L}_0^B \text{ III } \mathcal{L}_0^B) \end{aligned}$$

and thus are zero in  $\mathcal{M}_X \wedge \mathcal{M}_X$ .

The terms in  $\mathcal{N}_X^1 \otimes \mathcal{N}_X^1$  in the above expression (26) obviously agree with the corresponding terms of  $\Delta'(\mathcal{L}_{0011}^B)$  as the term of tensor degree 2 of  $\mathcal{L}_{0011}^B$  is written down that way.

Computations below are done in  $B(\mathcal{N}_X^\bullet) \otimes B(\mathcal{N}_X^\bullet)$ . They will induce the expected relation in  $\mathcal{M}_X \wedge \mathcal{M}_X$  after going to the  $H^0$  and taking the quotient modulo shuffle product. Let  $\pi_n : B(\mathcal{N}_X^\bullet) \rightarrow (\mathcal{N}_X^\bullet)^{\otimes n}$  be the projection to the  $n$ -th tensor factor. From the above discussion it is enough to compute  $\Delta'(\pi_3(\mathcal{L}_{0011}^B))$  and part of  $\Delta'(\pi_4(\mathcal{L}_{0011}^B))$ .

First the definition of the coproduct gives

$$\begin{aligned} 4\Delta'(\pi_3(\mathcal{L}_{0011}^B)) &= ([0] \otimes [\overline{01}|1] + [0|01] \otimes [1] - [01] \otimes [0|1] - [0|1] \otimes [\overline{01}]) \\ &\quad + (-[0] \otimes [1|\overline{01}] - [01|0] \otimes [1] + [01] \otimes [1|0] + [1|0] \otimes [\overline{01}]) \\ &\quad + (-1) ([\overline{01}|1] \otimes [0] + [1] \otimes [0|01] - [0|1] \otimes [01] - [\overline{01}] \otimes [0|1]) \\ &\quad + (-1) (-[1|\overline{01}] \otimes [0] - [1] \otimes [01|0] + [1|0] \otimes [01] + [\overline{01}] \otimes [1|0]) \\ &\quad + ([0] \otimes ([01|1] + [1|01]) + ([01|1] + [1|01]) \otimes [0]) \\ &\quad + ([1] \otimes ([\overline{01}|0] + [0|\overline{01}]) + ([\overline{01}|0] + [0|01]) \otimes [0]). \end{aligned}$$

The four first lines of the above equality correspond to 4 times the terms of Equation (26) in  $\mathcal{N}_X^\bullet \otimes \mathcal{N}_X^{\bullet \otimes 2} \oplus \mathcal{N}_X^{\bullet \otimes 2} \otimes \mathcal{N}_X^\bullet$ . The two last lines can be written as

$$[0] \otimes ([01] \# [1]) + ([01] \# [1]) \otimes [0] + [0] \otimes ([\overline{01}] \# [0]) + ([\overline{01}] \# [0]) \otimes [1]$$

which covers many terms of a shuffle between terms in the  $H^0$  of the bar construction. As an example,

$$[0] \otimes ([01] \# [1])$$

covers many terms of

$$[\mathcal{L}_0^B] \otimes (\mathcal{L}_{01}^B \# \mathcal{L}_1^B) = [0] \otimes \left( [01] \# [1] - \frac{1}{2}([0|1] - [1|0]) \# [1] \right).$$

Computing the reduced coproduct of  $\pi_4(\mathcal{L}_{0011}^B)$  gives

$$\begin{aligned} \Delta'(\pi_4(\mathcal{L}_{0011}^B)) &= \left(-\frac{1}{2}\right) \left(\frac{1}{4}\right) (4[0] \otimes [0|1|1] + 4[0|0] \otimes [1|1] + 4[0|0|1] \otimes [1] \\ &\quad - 4[1] \otimes [1|0|0] + 4[1|1] \otimes [0|0] - 4[1|1|0] \otimes [0]). \end{aligned}$$

where the factors  $1/4$  and  $4$  make it easier to relate  $\Delta'(\pi_4(\mathcal{L}_{0011}^B))$  with shuffle products and the corresponding terms in Equation (26).

We have already remarked that the terms  $[0|0] \otimes [1|1]$  and  $[1|1] \otimes [0|0]$  in the equation above can be expressed as shuffles. The four other terms are similar. Hence we will only discuss the case of  $[0] \otimes [0|1|1]$ . One can write

$$\begin{aligned} 4[0|1|1] &= 2[0|1|1] + 2[1|1|0] + 2[0|1|1] - 2[1|1|0] \\ &= [0|1|1] + [1|1|0] - [1|0|1] + [0|1|1] + [1|0|1] + [1|1|0] + 2[0|1|1] - 2[1|1|0]. \end{aligned}$$

Now, one remarks that

$$[0|1|1] + [1|0|1] + [1|1|0] = \frac{1}{2}[0] \# [1] \# [1] = \frac{1}{2}\mathcal{L}_0^B \# \mathcal{L}_1^B \# \mathcal{L}_1^B$$

and that the tensor degree 3 part of  $-\frac{1}{2}\mathcal{L}_{01}^B \# \mathcal{L}_1^B$  is equal to

$$-\frac{1}{2} \frac{-1}{2} ([0|1] \# [1] - [1|0] \# [1]) = \frac{1}{4} (2[0|1|1] - 2[1|1|0]).$$

Then one can conclude that

$$\begin{aligned}
 \Delta'(\mathcal{L}_{0011}^B) = & -\frac{1}{2} \left( \mathcal{L}_0^B \otimes \mathcal{L}_{011}^B - \mathcal{L}_{011}^B \otimes \mathcal{L}_0^B + \mathcal{L}_{001}^{1,B} \otimes \mathcal{L}_1^B - \mathcal{L}_1^B \otimes \mathcal{L}_{001}^{1,B} \right. \\
 & \left. - \mathcal{L}_{01}^B \otimes \mathcal{L}_{01}^{1,B} + \mathcal{L}_{01}^{1,B} \otimes \mathcal{L}_{01}^B \right) \\
 & + \frac{1}{4} \left( \mathcal{L}_{01}^B \text{ III } \mathcal{L}_1^B \otimes \mathcal{L}_0^B + \mathcal{L}_{01}^{1,B} \text{ III } \mathcal{L}_0^B \otimes \mathcal{L}_1^B + \mathcal{L}_0^B \otimes \mathcal{L}_{01}^B \text{ III } \mathcal{L}_1^B \right. \\
 & \left. + \mathcal{L}_1^B \otimes \mathcal{L}_{01}^{1,B} \text{ III } \mathcal{L}_0^B \right) \\
 & - \frac{1}{16} \left( -\mathcal{L}_0^B \text{ III } \mathcal{L}_1^B \text{ III } \mathcal{L}_1^B \otimes \mathcal{L}_0^B - \mathcal{L}_0^B \text{ III } \mathcal{L}_0^B \text{ III } \mathcal{L}_1^B \otimes \mathcal{L}_1^B \right. \\
 & \left. + \mathcal{L}_0^B \otimes \mathcal{L}_0^B \text{ III } \mathcal{L}_1^B \text{ III } \mathcal{L}_1^B + \mathcal{L}_1^B \otimes \mathcal{L}_0^B \text{ III } \mathcal{L}_0^B \text{ III } \mathcal{L}_1^B \right).
 \end{aligned}$$

In  $\mathcal{M}_X \wedge \mathcal{M}_X$ , one simply gets

$$(27) \quad d_\Delta(\mathcal{L}_{0011}^B) = - \left( \mathcal{L}_0^B \wedge \mathcal{L}_{011}^B + \mathcal{L}_{001}^{1,B} \wedge \mathcal{L}_1^B - \mathcal{L}_{01}^B \wedge \mathcal{L}_{01}^{1,B} \right)$$

and using the relations between  $\mathcal{L}_W^B$  and  $\mathcal{L}_W^{1,B}$ , one recovers (up to a global minus sign) the differential equation associated to  $T_{0011}$ .

$$(28) \quad d_\Delta(\mathcal{L}_{0011}^B) = - \left( \mathcal{L}_0^B \wedge \mathcal{L}_{011}^B + (\mathcal{L}_{001}^B - \mathcal{L}_{001}^B(1)) \wedge \mathcal{L}_1^B + \mathcal{L}_{01}^B \wedge \mathcal{L}_{01}^B(1) \right).$$

The associated motive is as above the sub-vector space of  $\mathcal{M}_X$  generated by

$$\mathcal{L}_0^B, \mathcal{L}_{001}^{1,B}, \mathcal{L}_{0011}^B, \mathcal{L}_{01}^B, \mathcal{L}_{011}^B, \mathcal{L}_{01}^{1,B}, \mathcal{L}_1^B.$$

**5.3. Goncharov's motivic coproduct.** In this subsection, we would like to illustrate how the differential equation satisfied by the elements  $\mathcal{L}_W^B$ , written using its ‘‘tree differential form’’ (that is using the elements  $\mathcal{L}_U^B(1)$  instead of the elements  $\mathcal{L}_U^{1,B}$ ), gives another expression for Goncharov's motivic coproduct.

Work of Levine [Lev11] insures that the above differential coincides with Goncharov's motivic coproduct for motivic iterated integrals (modulo products). We will not review this theory here but only recall some of the needed properties satisfied by Goncharov's motivic iterated integrals [Gon05]. A short exposition of the combinatorics involved is also recalled in [GGL09][Section 8].

For our purpose, it is enough to consider motivic iterated integrals as degree  $n$  generating elements  $I(a_0; a_1, \dots, a_n; a_{n+1})$  of a Hopf algebra with  $a_i$  in  $\mathbb{A}^1(\mathbb{Q})$ . They are subject to the following relations.

**Path composition:** for  $x$  in  $\mathbb{A}^1(\mathbb{Q})$ , one has

$$I(a_0; a_1, \dots, a_n; a_{n+1}) = \sum_{k=0}^n I(a_0; a_1, \dots, a_k; x) I(x; a_{k+1}, \dots, a_n; a_{n+1}).$$

**Inversion:** which relates  $I(a_0; a_1, \dots, a_n; a_{n+1})$  and  $I(a_{n+1}; a_n, \dots, a_1; a_0)$

$$I(a_{n+1}; a_n, \dots, a_1; a_0) = (-1)^n I(a_0; a_1, \dots, a_n; a_{n+1}).$$

**Unit and neutral identities:**

$$\text{for } a \neq b \quad I(a; b) = 1 \quad \text{and} \quad I(a_0; a_1, \dots, a_n; a_0) = 0.$$

**Rescaling:** If  $a_{n+1}$  and at least one of the  $a_i$  is not zero then

$$I(0; a_1, \dots, a_n, a_{n+1}) = I(0; a_1/a_{n+1}, \dots, a_n/a_{n+1}, 1).$$

**Regularization:**

$$I(0; 1; 1) = I(0; 0; 1) = 0.$$

The product is given by the shuffle relations

$$I(a; a_1, \dots, a_n; b)I(a; a_{n+1}, \dots, a_{n+m}; b) = \sum_{\sigma \in sh(n, m)} I(a; a_{\sigma(1)}, \dots, a_{\sigma(n+m)}; b)$$

where  $sh(n, m)$  denotes the set of permutations preserving the order of the ordered subset  $\{1, \dots, n\}$  and  $\{n+1, \dots, n+m\}$ . Such a motivic iterated integral corresponds formally to the iterated integral

$$\int_{\Delta_{a_0, a_{n+1}}} \frac{dt}{t - a_1} \wedge \dots \wedge \frac{dt}{t - a_n}$$

with  $\Delta_{a_0, a_{n+1}}$  the image of the standard simplex induced by a path from  $a_0$  to  $a_{n+1}$ . The above relations reflect the relations satisfied by the integrals.

The coproduct is given by the formula

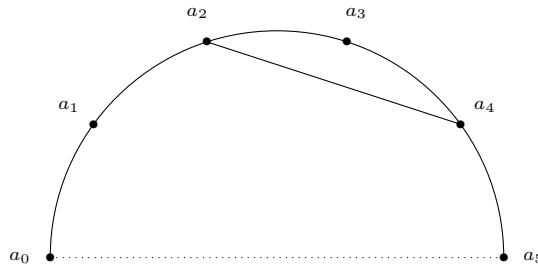
$$\begin{aligned} \Delta^M(I(a_0; a_1 \dots, a_n; a_{n+1})) = \\ \sum_{\substack{(a_{k_1}, \dots, a_{k_r}) \\ \{k_1, \dots, k_r\} \subset \{1, \dots, n\}}} I(a_0; a_{k_1}, \dots, a_{k_r}; a_{n+1}) \otimes \prod_{l=0}^r I(a_{k_l}; a_{k_{l+1}}, \dots, a_{k_{l+1}-1}; a_{k_{l+1}}) \end{aligned}$$

with the convention that  $r$  runs from 0 to  $n$  and that  $k_0 = 0$  and  $k_{r+1} = n+1$ .

Now, considering the reduced coproduct  $\Delta'^M = \Delta^M - (1 \otimes \text{id} + \text{id} \otimes 1)$  on the space of indecomposable elements (that is modulo products), the above formula reduces to

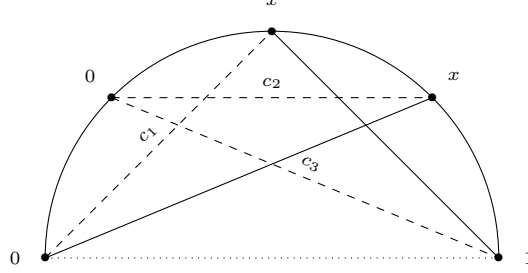
$$\begin{aligned} \Delta'^M(I(a_0; a_1 \dots, a_n; a_{n+1})) = \\ \sum_{\substack{k < l \\ k \neq l-1}} I(a_0; a_1, \dots, a_k, a_l, a_{l+1}, \dots, a_n; a_{n+1}) \otimes I(a_k; a_{k+1}, \dots, a_{l-1}, a_l). \end{aligned}$$

This formula can be pictured placing the  $a_i$  on a semicircle in the order dictated by their indices. Then a term in the above sums corresponds to a non-trivial chord between two vertices:



Considering the relation between multiple polylogarithms and iterated integrals, we want to relate our expression of the differential of  $\mathcal{L}_{011}^B$  at  $t$  to the reduced coproduct for the motivic iterated integral  $I(0; 0, x, x; 1)$  for  $x = t^{-1}$ . From

the semicircle representation, one sees that there are five terms to consider:



However, the chord  $c_1$  gives a zero term modulo products as  $I(0; x, x; 1) = \frac{1}{2}I(0; x; 1)I(0; x; 1)$  and chord  $c_2$  and  $c_3$  give terms equal to 0 using the regularization relations. There are thus only two terms to consider

$$I(0; 0, x; 1) \otimes I(x; x; 1) \quad \text{and} \quad I(0; x; 1) \otimes I(0; 0, x; x).$$

Using path composition, inversion and regularization relations, in the set of indecomposable elements, one has

$$I(x; x; 1) = I(0; x; 1) + I(x; x; 0) = I(0; x; 1) - I(0; x; x) = I(0; x; 1).$$

Thus the first term equals

$$I(0; 0, x; 1) \otimes I(x; x; 1) = I(0; 0, x; 1) \otimes I(0; x; 1).$$

From the rescaling relation, the second term equals

$$I(0; x; 1) \otimes I(0; 0, 1; 1)$$

and one can write modulo products

$$(29) \quad \Delta'^M(I(0; 0, x, x; 1)) = I(0; 0, x; 1) \otimes I(0; x; 1) + I(0; x; 1) \otimes I(0; 0, 1; 1).$$

Keeping in mind that, for  $x = t^{-1}$ ,  $I(0; x; 1)$  corresponds to the fiber at  $t$  of  $\mathcal{L}_1^B$  ( $t \neq 1$ ) and that  $I(0; 0, x; 1)$  corresponds to the fiber at  $t$  of  $\mathcal{L}_{01}^B$  (any  $t$ ), the above formula (29) corresponds exactly to Equation (25):

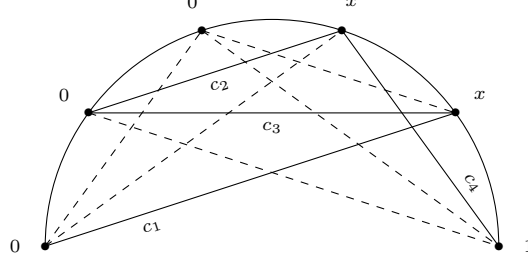
$$d_\Delta(\mathcal{L}_{011}^B) = -(\mathcal{L}_{01}^B - \mathcal{L}_{01}^B(1)) \wedge \mathcal{L}_1^B = -(\mathcal{L}_{01}^B \wedge \mathcal{L}_1^B + \mathcal{L}_1^B \wedge \mathcal{L}_{01}^B(1)).$$

The fact that, as in the above example, Goncharov's motivic iterated integrals corresponding to multiple polylogarithms in one variable, satisfy the tree differential equations (ED-T) is easily checked for Lyndon words with one 1 (that is for the classical polylogarithms) but can also be checked for some Lyndon words with two 1's or more. It seems to be a general behavior. However, the above example and the example of  $I(0; 0, 0, x, x; 1)$  corresponding to  $\mathcal{L}_{0011}^B$  below show that it involves using all the relations in order to pass from Goncharov's formula to the shape of Equation (ED-T).

The case of  $\mathcal{L}_{0011}^B$  involves more computations but works essentially as the case of  $\mathcal{L}_{011}^B$ . The reduced coproduct for  $I(0; 0, 0, x, x; 1)$  modulo products gives nine



terms corresponding to the nine chords below:



The five dashed chords give terms equal to 0 for one of the following reasons:  $I(a; \dots; a) = 0$ , regularization relations or shuffle relations. Hence we are left with four terms. The chord  $c_1$  gives, using the rescaling relation,

$$I(0; x; 1) \otimes I(0; 0, 0, 1; 1)$$

corresponding to  $\mathcal{L}_1^B \wedge \mathcal{L}_{001}^B(1)$ . The chord  $c_2$  gives a term in

$$I(0; 0, x, x; 1) \otimes I(0; 0; x)$$

corresponding to  $\mathcal{L}_0^B \wedge \mathcal{L}_{011}^B$ . The chord  $c_3$  gives, using the rescaling relation, a term in

$$I(0; 0, x; 1) \otimes I(0; 0, x; x) = I(0; 0, x; 1) \otimes I(0; 0, 1; 1)$$

corresponding to  $\mathcal{L}_{01}^B \wedge \mathcal{L}_{01}^B(1)$ . Finally the chord  $c_4$  gives, using the path composition and regularization relations

$$\begin{aligned} I(0; 0, 0, x; 1) \otimes I(x; x; 1) &= I(0; 0, 0, x; 1) \otimes I(0; x; 1) + I(0; 0, 0, x; 1) \otimes I(x; x; 0) \\ &= I(0; 0, 0, x; 1) \otimes I(0; x; 1). \end{aligned}$$

Finally,  $\Delta'^M(I(0; 0, 0, x, x; 1))$  can be written as

$$(30) \quad \begin{aligned} \Delta'^M(I(0; 0, 0, x, x; 1)) &= I(0; 0, 0, x, x; 1) \otimes I(0; 0; x) + I(0; 0, 0, x; 1) \otimes I(0; x; 1) + \\ &\quad I(0; x; 1) \otimes I(0; 0, 0, 1; 1) + I(0; 0, x; 1) \otimes I(0; 0, 1; 1). \end{aligned}$$

This expression corresponds to Equation (28):

$$d_\Delta(\mathcal{L}_{0011}^B) = -(\mathcal{L}_0^B \wedge \mathcal{L}_{011}^B + (\mathcal{L}_{001}^B - \mathcal{L}_{001}^B(1)) \wedge \mathcal{L}_1^B + \mathcal{L}_{01}^B \wedge \mathcal{L}_{01}^B(1)).$$

## 6. Integrals and multiple zeta values

We present here a sketch of how to associate an integral to cycles  $\mathcal{L}_{01}$ ,  $\mathcal{L}_{01}^1$  and  $\mathcal{L}_{011}$ . The author will directly follow the algorithm described in [GGL09][Section 9] and put in detailed practice in [GGL07]. There will be no general review of the direct Hodge realization from Bloch-Kriz motives [BK94][Section 8 and 9]. Gangl, Goncharov and Levin's construction seems to consist in setting particular choices of representatives in the intermediate Jacobians for their algebraic cycles.

However, the goal of this paper is not to formalize such an idea. That is why computations below are only outlined. In particular, the lack of precise knowledge of the ‘‘algebraico-topological cycle algebra’’ described in [GGL09] makes it difficult to control how ‘‘negligible’’ cycles are killed looking at the  $H^0$  of its bar construction.

**6.1. An integral associated to  $\mathcal{L}_{01}$  and  $\mathcal{L}_{01}^1$ .** We recall the parametrized cycle expression for  $\mathcal{L}_{01}$ :

$$\mathcal{L}_{01} = [t; 1 - \frac{t}{x_1}, x_1, 1 - x_1] \subset X \times \square^3.$$

One wants to bound  $\mathcal{L}_{01}$  by an algebraic-topological cycle in a larger bar construction (not described here) introducing topological variables  $s_i$  in real simplices

$$\Delta_s^n = \{0 \leq s_1 \leq \dots \leq s_n \leq 1\}.$$

Let  $d^s : \Delta_s^n \rightarrow \Delta_s^{n-1}$  denotes the simplicial differential

$$d^s = \sum_{k=0}^n (-1)^k i_k^*$$

where  $i_k : \Delta_s^{n-1} \rightarrow \Delta_s^n$  is given by the face  $s_k = s_{k+1}$  in  $\Delta_s^n$  with the usual conventions for  $k = 0, n$ .

Defining

$$C_{01}^{s,1} = -[t; 1 - \frac{s_2 t}{x_1}, x_1, 1 - x_1]$$

for  $s_2$  going from 0 to 1, one sees that  $d^s(C_{01}^{s,1}) = \mathcal{L}_{01}$  as  $s_2 = 0$  implies that the first cubical coordinate is 1. The algebraic boundary  $\partial$  of  $C_{01}^{s,1}$  is given by the intersection with the faces of  $\square^3$ :

$$\partial(C_{01}^{s,1}) = -[t; s_2 t, 1 - s_2 t] \subset X \times \square^2.$$

This cycle is part of the boundary of a larger ‘‘simplicial’’ algebraic cycle

$$C_{01}^{s,2} = -[t; s_2 t, 1 - s_1 t].$$

Computing the simplicial differential of  $C_{01}^{s,2}$  gives

$$d^s(C_{01}^{s,2}) = [t; s_2 t, 1 - s_2 t] - [t; t, 1 - s_1 t] \subset X \times \square^2$$

with  $0 \leq s_2 \leq 1$  in the first term and  $0 \leq s_1 \leq 1$  in the second term.

Note that the cycle  $[t; t, 1 - s_1 t]$  is negligible as it is a product

$$[t; t, 1 - s_1 t] = \mathcal{L}_0 [t; 1 - s_1 t]$$

and thus can be canceled in the bar construction setting as the multiplicative boundary of

$$-[\mathcal{L}_0 [t, 1 - s_1 t]].$$

Thus, up to negligible terms,

$$(d^s + \partial)(C_{01}^{s,1} + C_{01}^{s,2}) = \mathcal{L}_{01}.$$

Now, we fix the situation at the fiber  $t_0$  and following Gangl, Goncharov and Levin, we associate to the algebraic cycle  $\mathcal{L}_{01}|_{t=t_0}$  the integral  $I_{01}(t_0)$  of the standard volume form

$$\frac{1}{(2i\pi)^2} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

over the simplex given by  $C_{01}^{s,2}$ . That is :

$$\begin{aligned} I_{01}(t_0) &= -\frac{1}{(2i\pi)^2} \int_{0 \leq s_1 \leq s_2 \leq 1} \frac{ds_2}{s_2} \wedge \frac{-t_0 ds_1}{1 - t_0 s_1} \\ &= \frac{-1}{(2i\pi)^2} \int_{0 \leq s_1 \leq s_2 \leq 1} \frac{ds_1}{t_0^{-1} - s_1} \wedge \frac{ds_2}{s_2} = \frac{-1}{(2i\pi)^2} Li_2^{\mathbb{C}}(t_0). \end{aligned}$$

In particular, this expression is valid for  $t_0 = 1$ , as is the cycle  $\mathcal{L}_{01}|_{t=1}$ , and gives  $-1/(2\pi^2)\zeta(2)$ .

Before presenting the weight 3 example of  $\mathcal{L}_{011}$ , we describe shortly below the situation for  $\mathcal{L}_{01}^1$ . In the bar construction the element  $\mathcal{L}_{01}^{1,B}$  is equal to the difference  $\mathcal{L}_{01}^B - \mathcal{L}_{01}^B(1)$ . Associating an integral to  $\mathcal{L}_{01}^1$  works in the same way as the cycle  $\mathcal{L}_{01}$  but it also reflects the correspondence with  $\mathcal{L}_{01}^B - \mathcal{L}_{01}^B(1)$ .

The expression of  $\mathcal{L}_{01}^1$  in terms of parametrized cycle is given by

$$\mathcal{L}_{01}^1 = [t; \frac{x_1 - t}{x_1 - 1}, x_1, 1 - x_1]$$

and can be bounded using the ‘‘simplicial’’ algebraic cycle

$$C_{01}^{s,1} = -[t; \frac{x_1 - s_2 t}{x_1 - s_2}, x_1, 1 - x_1].$$

Now, the algebraic boundary of  $C_{01}^{s,1}$  gives two terms

$$\partial(C_{01}^{s,1}) = -[t; s_2 t, 1 - s_2 t] + [t; s_2, 1 - s_2].$$

Then one defines  $C_{01}^{s,2}$  for simplicial variables  $0 \leq s_1 \leq s_2 \leq 1$  as

$$C_{01}^{s,2} = -[t; s_2 t, 1 - s_1 t] + [t; s_2, 1 - s_1]$$

whose simplicial boundary cancels  $\partial(C_{01}^{s,1})$  up to negligible cycles. Again, fixing a fiber  $t_0$  the integral associated to  $\mathcal{L}_{01}^1|_{t=t_0}$  is the integral of the standard volume form over the  $C_{01}^{s,2}$ :

$$I_{01}(t_0) = -\frac{1}{(2i\pi)^2} \left( \int_{0 \leq s_1 \leq s_2 \leq 1} \frac{ds_2}{s_2} \wedge \frac{-t_0 ds_1}{1 - t_0 s_1} + \int_{0 \leq s_1 \leq s_2 \leq 1} \frac{ds_2}{s_2} \wedge \frac{-ds_1}{1 - s_1} \right).$$

This expression is exactly the difference

$$I_{01}(t_0) = \frac{-1}{(2i\pi)^2} (Li_2^{\mathbb{C}}(t_0) - Li_2^{\mathbb{C}}(1)).$$

**6.2. An integral associated to  $\mathcal{L}_{011}$ .** Let’s recall the expression of  $\mathcal{L}_{011}$  as parametrized cycle:

$$\mathcal{L}_{011} = -[t; 1 - \frac{t}{x_2}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1].$$

As previously, one wants to bound  $\mathcal{L}_{011}$  by an algebraic-topological cycle. Hence we define

$$C_{011}^{s,1} = [t; 1 - \frac{s_3 t}{x_2}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1]$$

for  $s_3$  going from 0 to 1. Then  $d^s(C_{011}^{s,1}) = \mathcal{L}_{011}$  as  $s_3 = 0$  implies that the first cubical coordinate is 1.

Now the algebraic boundary  $\partial$  of  $C_{011}^{s,1}$  is given by the intersection with the codimension 1 faces of  $\square^5$ :

$$\partial(C_{011}^{s,1}) = [t; 1 - s_3 t, \frac{x_1 - s_3 t}{x_1 - 1}, x_1, 1 - x_1].$$

We can again bound this cycle by introducing a new simplicial variable  $0 \leq s_2 \leq s_3$  and the cycle

$$C_{011}^{s,2} = [t; 1 - s_3 t, \frac{x_1 - s_2 t}{x_1 - s_2/s_3}, x_1, 1 - x_1].$$

The intersection with the face of the simplex  $\{0 \leq s_2 \leq s_3 \leq 1\}$  given by  $s_2 = 0$  leads to an empty cycle (as one cubical coordinate equals 1) while the intersection with face  $s_3 = 1$  leads to a “negligible” cycle. Thus, the simplicial boundary of  $C_{011}^{s,2}$  satisfies (up to a negligible term)

$$d^s(C_{011}^{s,2}) = -\partial(C_{011}^{s,1}) = -[t; 1 - s_3t, \frac{x_1 - s_3t}{x_1 - 1}, x_1, 1 - x_1].$$

Its algebraic boundary is given by

$$\partial(C_{011}^{s,2}) = -[t; 1 - s_3t, s_2t, 1 - s_2t] + [t; 1 - s_3t, \frac{s_2}{s_3}, 1 - \frac{s_2}{s_3}].$$

Finally, we introduce a last simplicial variable  $0 \leq s_1 \leq s_2$  and a purely topological cycle

$$\widetilde{C}_{011}^{s,3} = -[t; 1 - s_3t, s_2t, 1 - s_1t] + [t; 1 - s_3t, \frac{s_2}{s_3}, 1 - \frac{s_1}{s_3}]$$

whose simplicial differential is (up to negligible terms) given in one hand by the face  $s_1 = s_2$ :

$$[t; 1 - s_3t, s_2t, 1 - s_2t] - [t; 1 - s_3t, \frac{s_2}{s_3}, 1 - \frac{s_2}{s_3}]$$

which is equal to  $-\partial(C_{011}^{s,2})$ ; and in the other hand by the face  $s_2 = s_3$ :

$$-[t; 1 - s_3t, s_3t, 1 - s_1t].$$

In order to cancel this extra term, we defined  $C_{011}^{s,3}$  by

$$C_{011}^{s,3} = \widetilde{C}_{011}^{s,3} + [t; 1 - s_2t, s_3t, 1 - s_1t]$$

whose algebraic boundary is 0 (up to negligible terms).

Finally one has

$$(d^s + \partial)(C_{011}^{s,1} + C_{011}^{s,2} + C_{011}^{s,3}) = \mathcal{L}_{011}$$

up to negligible terms.

Now, we fix the situation at the fiber  $t_0$  and following Gangl, Goncharov and Levin, we associate to the algebraic cycle  $\mathcal{L}_{011}|_{t=t_0}$  the integral  $I_{011}(t_0)$  of the standard volume form

$$\frac{1}{(2i\pi)^3} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \frac{dz_3}{z_3}$$

over the simplex given by  $C_{011}^{s,3}$ . That is :

$$\begin{aligned} I_{011}(t_0) = & -\frac{1}{(2i\pi)^3} \int_{0 \leq s_1 \leq s_2 \leq s_3 \leq 1} \frac{t_0 ds_3}{1 - t_0 s_3} \wedge \frac{ds_2}{s_2} \wedge \frac{t_0 ds_1}{1 - t_0 s_1} \\ & + \frac{1}{(2i\pi)^3} \int_{0 \leq s_3 \leq 1} \frac{t_0 ds_3}{1 - t_0 s_3} \int_{0 \leq s_1 \leq s_2 \leq 1} \frac{ds_2}{s_2} \wedge \frac{ds_1}{1 - s_1} \\ & + \frac{1}{(2i\pi)^3} \int_{0 \leq s_1 \leq s_2 \leq s_3 \leq 1} \frac{t_0 ds_2}{1 - t_0 s_2} \wedge \frac{ds_3}{s_3} \wedge \frac{t_0 ds_1}{1 - t_0 s_1}. \end{aligned}$$

Taking care of the change of sign due to the numbering, the first term in the above sum is (for  $t_0 \neq 0$  and up to the factor  $(2i\pi)^{-3}$ ) equal to

$$Li_{1,2}^{\mathbb{C}}(t_0) = \int_{0 \leq s_1 \leq s_2 \leq s_3 \leq 1} \frac{ds_1}{t_0^{-1} - s_1} \wedge \frac{ds_2}{s_2} \wedge \frac{ds_3}{t_0^{-1} - s_3}$$

while (up to the same multiplicative factor) the second term is equal to

$$-Li_1^{\mathbb{C}}(t_0)Li_2^{\mathbb{C}}(1)$$

and the third term is equal to

$$Li_{2,1}^{\mathbb{C}}(t_0).$$

Globally the integral is well defined for  $t_0 = 0$  and, which is the interesting part, also for  $t_0 = 1$  as the divergencies as  $t_0$  goes to 1 cancel each other in the above sum. A simple computation and the shuffle relation for  $Li_1^{\mathbb{C}}(t_0)Li_2^{\mathbb{C}}(t_0)$  show that the integral associated to the fiber of  $\mathcal{L}_{011}$  at  $t_0 = 1$  is given by

$$(2i\pi)^3 I_{011}(1) = -Li_{2,1}^{\mathbb{C}}(1) = -\zeta(2, 1).$$

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