Yet another short proof of Bourgain's distortion estimate for embedding of trees into uniformly convex Banach spaces

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Yet another short proof of Bourgain's distorsion estimate

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We use a self-improvement argument to give a very short and elementary proof of the result of Bourgain saying that regular trees do not admit bi-Lipschitz embeddings into uniformly convex Banach spaces.

Let T_n be the binary rooted tree of depth n and let $c_B(A)$ denote the distorsion of the metric space A in B, that is to say the infimum of all numbers D such that there is a number s and a map $\varphi : A \to B$ such that

$$sd(x,y) \leqslant d(\varphi(x),\varphi(y)) \leqslant sDd(x,y)$$

for all $x, y \in A$.

The modulus of (uniform) convexity $\delta_X(\varepsilon)$ of a Banach space X with norm $|\cdot|$ is defined as

$$\inf \left\{ 1 - \left| \frac{x+y}{2} \right| \middle| |x| = |y| = 1 \text{ and } |x-y| \geqslant \varepsilon \right\}$$

for $\varepsilon \in (0,2]$. The space X is said to be uniformly convex of type $p \geqslant 2$ if $\delta_X(\varepsilon) \geqslant c\varepsilon^p$ for some $c > 0$. Note that in particular, for $p \in (1,\infty)$ the L^p spaces are uniformly convex of type $\max(2,p)$.

Our main goal is to prove the following as simply as possible.

Theorem 1 (Bourgain)**.** *If X is uniformly convex of type p then*

$$c_X(T_n) \gtrsim (\ln n)^{\frac{1}{p}}.$$

Several proofs of this result have been given over the years, see notably [1, 3, 4]. As we discovered after writing a first draft of this paper, the method we use is very close to Johnson and Schechtman's proof of the distorsion estimate for diamond graphs [2]. However, it seems not to have been noticed before that this method gives such an effective proof of Bourgain's estimate.

Proof. The first step is similar to previous proofs, notably the one by Matoušek's [3]. Let Y be the four-vertices tree with one root a_0 which has one child a_1 and two grand-children a_2, a_2'.

Lemma 2. *There is a constant $K = K(X)$ such that if $\varphi : Y \to X$ is D-Lipschitz and distance non-decreasing, then either*

$$|\varphi(a_0) - \varphi(a_2)| \leqslant 2(D - \frac{K}{D^{p-1}})$$

or

$$|\varphi(a_0) - \varphi(a_2')| \leqslant 2(D - \frac{K}{D^{p-1}})$$

We provide the proof below for the sake of completeness.

Let now $\varphi : T_n \to X$ a D-Lipschitz, distance non-decreasing map. By the lemma, the root a_0 has at least two grand-children a_2^i $(i = 1, 2)$ such that

$$2 \leqslant |\varphi(a_0) - \varphi(a_2^i)| \leqslant 2f(D)$$

where $f(D) = D - K/D^{p-1}$. Applying the lemma again, each of a_2^i also has two grand-children satisfying similar inequalities, and we can apply he same reasoning every other generation. Restricting φ to these vertices, we get an embedding of $T_{\lfloor \frac{n}{2} \rfloor}$ whose distorsion is at most $f(D)$.

We can iterate these restrictions $\lfloor \log_2(n) \rfloor$ times to get an embedding of T_1 whose distorsion is $f^{\lfloor \log_2(n) \rfloor}(D)$. This must be at least 1 and $f^{D^p/K}(D) < 1$, so that

$$\log_2(n) \lesssim D^p$$

which is Theorem 1. □

Remark 3. Working out the constants gives the more precise result that

$$c_X(T_n) \geqslant \left(\frac{pc}{2}\right)^{\frac{1}{p}} (\log_2 n)^{\frac{1}{p}} + \text{l.o.t.} \qquad (1)$$

where c can be replaced by $\liminf \delta_X(\varepsilon)\varepsilon^{-p}$. In particular

$$c_{\ell^2}(T_n) \geqslant \frac{1}{2\sqrt{2}} (\log_2 n)^{\frac{1}{2}} + \text{l.o.t.}$$

Proof of Lemma 2. Assume $\varphi(a_0) = 0$ and let $x_1 = \varphi(a_1)$, $x_2 = \varphi(a_2)$ and $x_2' = \varphi(a_2')$.

Suppose that $|x_2| \geqslant 2(D - \eta)$ for some η to be chosen afterward; then by the triangle inequality, $|x_1|$ and $|x_2 - x_1|$ are at least $D - 2\eta$.

Define $v = \frac{|x_1|}{|x_2 - x_1|}(x_2 - x_1)$; then

$$|x_1 + v - x_2| = ||x_1| - |x_2 - x_1|| \leqslant 2\eta$$

and

$$|x_1 + v| \geqslant |x_2| - |x_1 + v - x_2| \geqslant 2D - 4\eta.$$

The vectors $x_1/|x_1|$ and $v/|x_1|$ have unit norm and their average has norm at least $1 - 2\eta/D$; letting $\varepsilon = (2\eta/cD)^{\frac{1}{p}}$ the convexity assumption therefore yields $|x_1 - v| \leqslant \varepsilon D$. It follows that

$$|2x_1 - x_2| \leqslant |x_1 + v - x_2| + |x_1 - v| \leqslant 2\eta + \varepsilon D.$$

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Suppose that also $|x_2'| \geqslant 2(D-\eta)$; then the same reasoning yields the same estimate on $|x_2' - 2x_1|$ so that

$$|x_2 - x_2'| \leqslant 4\eta + 2D\left(\frac{2\eta}{cD}\right)^{\frac{1}{p}}$$

Now we can choose $\eta = K/D^{p-1}$ with K small enough to ensure that the above inequality reads $|x_2 - x_2'| < 2$. This contradicts the hypothesis that φ is distance non-decreasing, therefore as desired $|x_2|$ or $|x_2'|$ must be smaller than $2(D-\eta)$. □

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